Define $T : L^2(\gamma;\mathbb{C}) \longrightarrow L^2(\lambda_{\mathbb{R}};\mathbb{C})$ so that $T\varphi(x) = \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}} \varphi(2^{\frac{1}{2}}x)$, and check that

$$\|T\varphi\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} = \|\varphi\|_{L^2(\gamma;\mathbb{C})} \text{ and } T^{-1}f(x) = \pi^{\frac{1}{4}} e^{\frac{x^2}{4}} f(2^{-\frac{1}{2}}x).$$

Thus T is an isometric isomorphism from $L^2(\gamma;\mathbb{C})$ onto $L^2(\lambda_{\mathbb{R}};\mathbb{C})$.

Set $h_m = TH_m$ and $\tilde{h}_m = h_m = T\tilde{H}_m$. Then, because $\{\tilde{H}_m : m \geq 0\}$ is an orthonormal basis in $L^2(\gamma;\mathbb{C})$, $\{\tilde{h}_m : m \geq 0\}$ is an orthonormal bases in $L^2(\lambda_{\mathbb{R}};\mathbb{C})$.

Assuming that $\varphi \in C^1(\mathbb{R};\mathbb{C})$, it easy to show that

$$TA_{\pm}\varphi = a_{\pm}T\varphi \text{ where } a_{\pm} = 2^{-\frac{1}{2}}(x\mathbf{1} \mp \partial_x)$$

and therefore that

$$a_+h_m = h_{m+1} \text{ and } a_-h_m = mh_{m-1}. \tag{11.1}$$

Theorem 11.1. *For all $m \geq 0$, $\widehat{h_m} = (2\pi)^{\frac{1}{2}} \imath^m h_m$.*

Proof. Certainly $\widehat{h_0} = (2\pi)^{\frac{1}{2}} h_0$. Assuming that $\widehat{h_m} = (2\pi)^{\frac{1}{2}}\imath^m h_m$, use integration by parts to see that

$$\begin{aligned}\widehat{h_{m+1}}(\xi) &= \int e^{\imath\xi x} a_+ h_m(x)\,dx = \int x e^{\imath\xi x} h_m(x)\,dx + \imath\xi\widehat{h}_m(\xi) \\ &= -\imath(\widehat{h_m})' + \imath\xi\widehat{h}_m(\xi) = (2\pi)^{\frac{1}{2}}\imath^{m+1} a_+ h_m(\xi) = (2\pi)^{\frac{1}{2}}\imath^{m+1} h_{m+1}(\xi).\end{aligned}$$

□

Corollary 11.2. *For all $m \geq 0$,*

$$\begin{aligned}&\|\tilde{h}_m\|_{L^1(\lambda_{\mathbb{R}};\mathbb{C})} \leq (2\pi)^{\frac{1}{2}}(m+1)^{\frac{1}{2}}, \|\tilde{h}_m\|_{\mathrm{u}} \leq (m+1)^{\frac{1}{2}} \text{ and} \\ &\|x\tilde{h}_m\|_{\mathrm{u}} \vee \|\partial\tilde{h}_m\|_{\mathrm{u}} \leq 2m + \frac{3}{2}.\end{aligned} \tag{11.2}$$

Proof. Since $\|\tilde{h}_0\|_{L^1(\lambda_{\mathbb{R}};\mathbb{C})} = 2^{\frac{1}{2}}\pi^{\frac{1}{4}}$, $\|\tilde{h}_0\|_{\mathrm{u}} \leq \pi^{-\frac{1}{4}}$, and

$$\pi^{\frac{1}{4}}\left\|(\tilde{h}_0)'\right\|_{\mathrm{u}} = \sup_{x\geq 0} x e^{-\frac{x^2}{2}} = e^{-\frac{1}{2}},$$

there is nothing to do when $m = 0$.

Now assume that $m \geq 1$. Using the facts that $xh_m(x) - h_m' = 2^{\frac{1}{2}}h_{m+1}$ and $xh_m + h_m' = 2^{\frac{1}{2}}mh_{m-1}$, one sees that

$$\begin{aligned}x\tilde{h}_m(x) &= \frac{m^{\frac{1}{2}}\tilde{h}_{m-1}(x) + (m+1)^{\frac{1}{2}}\tilde{h}_{m+1}(x)}{2^{\frac{1}{2}}} \\ (\tilde{h}_m)'(x) &= \frac{m^{\frac{1}{2}}\tilde{h}_{m-1}(x) - (m+1)^{\frac{1}{2}}\tilde{h}_{m+1}(x)}{2^{\frac{1}{2}}}.\end{aligned} \tag{11.3}$$

Hence,

$$\int x^2\tilde{h}_m(x)^2\,dx = m + \tfrac{1}{2},$$

and so

$$\int (1+x^2)\tilde{h}_m(x)^2\,dx = m + \tfrac{3}{2},$$

which, by Schwarz's inequality, means that

$$\|\tilde{h}_m\|_{L^1(\lambda_{\mathbb{R}};\mathbb{C})} = \int (1+x^2)^{-\frac{1}{2}}(1+x^2)^{\frac{1}{2}}\tilde{h}_m(x)^2\,dx \leq \pi^{\frac{1}{2}}\left(m+\tfrac{3}{2}\right)^{\frac{1}{2}} \leq (2\pi)^{\frac{1}{2}}(m+1)^{\frac{1}{2}}.$$

Because $(\tilde{h}_m)^\wedge = (2\pi)^{\frac{1}{2}} \imath^m \tilde{h}_m$,

$$\|\tilde{h}_m\|_{\mathrm{u}} = (2\pi)^{-\frac{1}{2}} \big\|(\tilde{h}_m)^\wedge\big\|_{\mathrm{u}} \le (2\pi)^{-\frac{1}{2}} \|\tilde{h}_m\|_{L^1(\lambda_{\mathbb{R}};\mathbb{C})} \le (m+1)^{\frac{1}{2}}.$$

To complete the proof, use the second part of (11.3) plus the preceding to see that

$$\begin{aligned}\big\|\partial\tilde{h}_m\big\|_{\mathrm{u}} &\le \big(m^{\frac{1}{2}}\|\tilde{h}_{m-1}\|_{\mathrm{u}} + (m+1)^{\frac{1}{2}}\|\tilde{h}_{m+1}\|_{\mathrm{u}}\big)\\ &\le \big(m + (m+1)^{\frac{1}{2}}(m+2)^{\frac{1}{2}}\big) \le 2m + \frac{3}{2}.\end{aligned}$$

The same argument, only this time using the first part of (11.3), proves the same estimate for $\|x\tilde{h}_m\|_{\mathrm{u}}$. □

The kernel which plays the role for the Hermite functions that the Ornstein–Uhlenbeck kernel (cf. (9.2)) $p(t,x,y)$ plays for the Hermite polynomial is

$$\begin{aligned}q(t,x,y) &= 2^{\frac{1}{2}} e^{-\frac{t}{2}} e^{-\frac{x^2}{2}} p\big(2t, 2^{\frac{1}{2}}x, 2^{\frac{1}{2}}y\big) e^{\frac{y^2}{2}}\\ &= (2\pi\sinh t)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2\tanh t} + \frac{xy}{\sinh t} - \frac{y^2}{2\tanh t}\right).\end{aligned} \tag{11.4}$$

Observe that $q(t,x,\,\cdot\,) \in L^p(\lambda_{\mathbb{R}};\mathbb{C})$ for all $p \in [1,\infty]$ and that

$$e^{\frac{t}{2}} \int q(t,x,y) f(y)\,dy = e^{-\frac{x^2}{2}} \int p(2t, 2^{\frac{1}{2}}x, y) e^{\frac{y^2}{4}} f\big(2^{-\frac{1}{2}}y\big)\,dy = \big(TP_{2t}T^{-1}f\big)(x).$$

Hence, the operator Q_t given by

$$Q_t f(x) = \int q(t,x,y) f(y)\,dy$$

is well defined on $L^2(\lambda_{\mathbb{R}};\mathbb{C})$ and is equal to $e^{-\frac{t}{2}} TP_{2t}T^{-1}$. In particular, by (9.10),

$$e^{\frac{t}{2}} \|Q_t f\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} = \|P_{2t}T^{-1}f\|_{L^2(\gamma;\mathbb{C})} \le \|T^{-1}f\|_{L^2(\gamma;\mathbb{C})} = \|f\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})},$$

and, by (9.11)

$$\begin{aligned}\Big\|e^{\frac{t}{2}} Q_t f - f\Big\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} &= \big\|T(P_{2t}T^{-1}f - T^{-1}f)\big\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\\ &= \big\|P_{2t}T^{-1}f - T^{-1}f\big\|_{L^2(\gamma;\mathbb{C})} \longrightarrow 0 \text{ as } t \searrow 0.\end{aligned}$$

Hence

$$\|Q_t f\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} \le e^{-\frac{t}{2}} \|f\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} \text{ and } \lim_{t\searrow 0} \|Q_t f - f\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} = 0.$$

In addition, by (10.3), $Q_t h_m = e^{-\frac{t}{2}} TP_{2t}H_m = e^{-(m+\frac{1}{2})t} h_m$.

Theorem 11.3. *If $f \in L^1(\lambda_{\mathbb{R}};\mathbb{C}) \cup L^2(\lambda_{\mathbb{R}};\mathbb{C})$, then*

$$\int q(t,x,y) f(y)\,dy = e^{-\frac{1}{2}} \sum_{m=0}^{\infty} e^{-mt} (f, \tilde{h}_m)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} \tilde{h}_m \textit{ for } t > 0,$$

where the convergence of the series is absolute uniformly for $x \in \mathbb{R}$.

Proof. First observe that, by the estimates in Corollary 11.2, the series is absolutely convergent uniformly in $x \in \mathbb{R}$ and that both sides are continuous as functions of $f \in L^1(\mathbb{R};\mathbb{C})$ or of $f \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$. In particular, it suffices to prove the equality when $f \in C_{\mathrm{c}}(\mathbb{R};\mathbb{C})$.

Given $f \in C_c(\mathbb{R};\mathbb{C})$, set $f_n = \sum_{m=0}^{n}(f,\tilde{h}_m)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\tilde{h}_m$. Then

$$\int q(t,x,y)f_n(y)\,dy = e^{-\frac{t}{2}}\sum_{m=0}^{n} e^{-mt}(f,\tilde{h}_m)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\tilde{h}_m(x).$$

Because $q(t,x,\,\cdot\,) \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$ and $f_n \longrightarrow f$ in $L^2(\lambda_{\mathbb{R}};\mathbb{C})$, the left hand side converges to $\int q(t,x,y)f(y)\,dy$. □

Exercise 3.3: Define the *Mehler kernel* $M(\theta,x,y)$ for $(\theta,x,y) \in (0,1)\times\mathbb{R}\times\mathbb{R}$ by

$$M(\theta,x,y) = \big(2\pi(1-\theta^2)\big)^{-\frac{1}{2}}\exp\left(-\frac{\theta^2x^2 - 2\theta xy + \theta^2y^2}{1-\theta^2}\right),$$

and show that

$$M(\theta,x,y) = \sum_{m=0}^{\infty}\theta^m\frac{H_m(x)H_m(y)}{m!},$$

where the series converges uniformly for (x,y) in compact subsets. This famous equation is known as *Mehler's formula*.

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