Lecture 12: The Fourier Transform for $L^2(\lambda_{\mathbb{R}};\mathbb{C})$

The basic goal here is to extend the Fourier transform on $L^1(\mathbb{R};\mathbb{C})\cap L^2(\lambda_{\mathbb{R}};\mathbb{C})$ as a bounded operation on $L^2(\lambda_{\mathbb{R}};\mathbb{C})$ into $L^2(\lambda_{\mathbb{R}};\mathbb{C})$. We will then examine some of the fundamental properties of this extension.

Lemma 12.1. *If $f\in L^1(\mathbb{R};\mathbb{C})$, then*

$$
\begin{aligned}
&\frac{e^{-\frac{\xi^2\tanh t}{2}}}{(2\pi\cosh t)^{\frac12}}\int e^{\frac{\imath\xi x}{\cosh t}}e^{-\frac{x^2\tanh t}{2}}f(x)\,dx \\
&\qquad\qquad = e^{-\frac{t}{2}}\sum_{m=0}^{\infty}(\imath e^{-t})^m(f,\tilde h_m)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\tilde h_m(\xi)
\end{aligned}
\tag{12.1}
$$

for $(t,\xi)\in(0,\infty)\times\mathbb{R}$.

Proof. Since both sides of (12.1) are continuous functions of $f\in L^1(\mathbb{R};\mathbb{C})$, we may and will assume that $f\in C_{\mathrm{c}}(\mathbb{R};\mathbb{C})$.

Set $\mathbb{D}=\{\zeta\in\mathbb{C}:\,|\zeta|<1\ \&\ \zeta\notin(-1,0]\}$, and define $\alpha_\pm(\zeta)=\frac12\big(\frac1\zeta\mp\zeta\big)$ for $\zeta\in\mathbb{D}$. Next, for fixed $\xi\in\mathbb{R}$ and all $\zeta\in\mathbb{D}$, define

$$
\Phi(\zeta)=\big(2\pi\alpha_+(\zeta)\big)^{-\frac12}e^{-\frac{\alpha_-(\zeta)}{2\alpha_+(\zeta)}\xi^2}\int e^{\frac{\xi x}{\alpha_+(\zeta)}}e^{-\frac{\alpha_-(\zeta)}{2\alpha_+(\zeta)}x^2}f(x)\,dx
$$

and

$$
\Psi(\zeta)=\zeta^{\frac12}\sum_{m=0}^{\infty}\zeta^m(f,\tilde h_m)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\tilde h_m(\xi),
$$

and observe that both Φ and Ψ are analytic functions on $\mathbb{D}$. Furthermore, since $\alpha_+(e^{-t})=\sinh t$ and $\alpha_-(e^{-t})=\cosh t$, Lemma 11.3 says that $\Phi=\Psi$ on $(0,1)$, and therefore, by analytic continuation, $\Phi=\Psi$ on $\mathbb{D}$. In particular, $\Phi\big(\imath e^{-t}\big)=\Psi\big(\imath e^{-t}\big)$. Finally, because $\alpha_+\big(\imath e^{-t}\big)=\frac{\cosh t}{\imath}$ and $\alpha_-\big(\imath e^{-t}\big)=\frac{\sinh t}{\imath}$, one sees that the left hand and right sides of (12.1) equal, respectively $\imath^{\frac12}\Phi\big(\imath e^{-t}\big)$ and $\imath^{\frac12}\Psi\big(\imath e^{-t}\big)$. □

Theorem 12.2. *If $f\in L^1(\mathbb{R};\mathbb{C})\cap L^2(\mathbb{R};\mathbb{C})$, then*

$$
\hat f=(2\pi)^{\frac12}\sum_{m=0}^{\infty}\imath^m(f,\tilde h_m)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\tilde h_m
\tag{12.2}
$$

almost everywhere.

Proof. Because $f\in L^1(\mathbb{R};\mathbb{C})$, the left hand side of (12.1) tends pointwise to $(2\pi)^{-\frac12}\hat f$ as $t\searrow 0$, and because $f\in L^2(\lambda_{\mathbb{R}};\mathbb{C})$, the right hand side tends in $L^2(\lambda_{\mathbb{R}};\mathbb{C})$ to the series on the right hand side of (12.2). □

As a consequence of Theorem 12.2, we know that $\|\hat f\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}=(2\pi)^{\frac12}\|f\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}$ for $f\in L^1(\lambda_{\mathbb{R}};\mathbb{C})\cap L^2(\lambda_{\mathbb{R}};\mathbb{C})$. Hence the map $f\in L^1(\lambda_{\mathbb{R}};\mathbb{C})\cap L^2(\lambda_{\mathbb{R}};\mathbb{C})\rightsquigarrow\hat f$ admits a unique continuous extension as a linear map with norm $(2\pi)^{\frac12}$ from $L^2(\lambda_{\mathbb{R}};\mathbb{C})$ into $L^2(\lambda_{\mathbb{R}};\mathbb{C})$, and (12.2) continuous to hold for this extension.

Define $\breve f(x)=f(-x)$, and observe that $\breve h_m=(-1)^m h_m$, $(\breve f,g)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}=(f,\breve g)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}$, and $\breve{\hat f}=\widehat{\breve f}$. In addition, by Fubini's theorem,

$$
\big(\hat\varphi,\tilde h_m\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}=\int\!\!\int e^{\imath\xi x}\varphi(x)\tilde h_m(\xi)\,dxd\xi=\int\varphi(x)\hat{\tilde h}_m(x)\,dx=(2\pi)^{\frac12}\imath^m(\varphi,\tilde h_m)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})},
$$

and so, for $f, g \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$,

$$\begin{aligned}\big(\hat f, \hat g\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} &= \sum_{m=0}^{\infty}\big(\hat f, \tilde h_m\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\overline{\big(\hat g, \tilde h_m\big)}_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\\ &= 2\pi\sum_{m=0}^{\infty}\big(f, \tilde h_m\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\overline{\big(g, \tilde h_m\big)}_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} = 2\pi(f,g)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})},\end{aligned}$$

which means that *Parseval's identity*

$$(\hat f, \hat g)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} = 2\pi\big(f, g\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} \tag{12.3}$$

holds. Finally, set $\check f = \check{\hat f}$, and check that

$$\big(\check f, \tilde h_m\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} = \big(\hat f, \check{\tilde h}_m\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} = (-1)^m\big(\hat f, \tilde h_m\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} = (2\pi)^{\frac12}(-\imath)^m\big(f, \tilde h_m\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}$$

Hence

$$\big((\hat f)^{\vee}, \tilde h_m\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} = (2\pi)^{\frac12}(-\imath)^m\big(\hat f, \tilde h_m\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} = 2\pi(f, \tilde h_m)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}.$$

Similarly, $\big((\check f)^{\wedge}, \tilde h_m\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} = 2\pi\big(f, \tilde h_m\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}$, and so we have proved the *Fourier inversion formula*

$$\big(\hat f\big)^{\vee} = 2\pi f = (\check f)^{\wedge}. \tag{12.4}$$

It is important to keep in mind that $\hat f$ is not given by a Lebesgue integral for $f \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$ unless $f \in L^1(\lambda_{\mathbb{R}};\mathbb{C})$ as well. On the other hand, because $f_R \equiv \mathbf{1}_{[-R,R]}f \in L^1(\lambda_{\mathbb{R}};\mathbb{C}) \cap L^2(\lambda_{\mathbb{R}};\mathbb{C})$ and $f_R \longrightarrow f$ in $L^2(\lambda_{\mathbb{R}};\mathbb{C})$,

$$\hat f(\xi) = \lim_{R\to\infty}\int_{-R}^{R} e^{\imath\xi x} f(x)\,dx,$$

where the convergence is in $L^2(\lambda_{\mathbb{R}};\mathbb{C})$.

Exercise 12.3. Define $\mathcal{F}f(\xi) = \hat f(2\pi\xi)$, and show that $\mathcal{F}$ is an orthogonal operator on $L^2(\lambda_{\mathbb{R}};\mathbb{C})$. Further, show that if $\mathcal{F}^*$ is the adjoint of $\mathcal{F}$, then equals $\mathcal{F}^{-1}f = \mathcal{F}^*f = \mathcal{F}\check f = (\mathcal{F}f)^{\cup}$.