Lecture 13: Schwartz Test Functions

In this section we will study a space of functions introduced by Laurent Schwartz[6] and used by him to construct the class of distributions discussed in the next section.

The function space alluded to above is denoted by $\mathscr{S}(\mathbb{R};\mathbb{C})$ and consists of functions $\varphi \in C^\infty(\mathbb{R};\mathbb{C})$ with the property that $x \rightsquigarrow x^k\partial^\ell\varphi(x)$ is bounded for all $k,\ \ell \in \mathbb{N}$. Obviously, $\mathscr{S}(\mathbb{R};\mathbb{C})$ is a vector space. In addition, it is closed under differentiation as well as products with smooth functions which, together with all their derivatives, have at most polynomial growth (i.e., grow no faster than some power of $(1+x^2)$). Thus the Hermite functions are all in $\mathscr{S}(\mathbb{R};\mathbb{C})$. Finally, since, for $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{C})$,

$$\int |\varphi(x)|^p\,dx \le \|(1+x^2)\varphi\|_{\mathrm{u}}^p \int (1+x^2)^{-p}\,dx,$$

$\mathscr{S}(\mathbb{R};\mathbb{C}) \subseteq \bigcap_{p\in[1,\infty]} L^p(\lambda_{\mathbb{R}};\mathbb{C})$.

There is an obvious notion of convergence for sequences in $\mathscr{S}(\mathbb{R};\mathbb{C})$. Namely, define the norms

$$\|\varphi\|_{\mathrm{u}}^{(k,\ell)} = \|x^k\partial^\ell\varphi\|_{\mathrm{u}}$$

for $k,\ \ell\in\mathbb{N}$, and say that $\varphi_j \longrightarrow \varphi$ in $\mathscr{S}(\mathbb{R};\mathbb{C})$ if $\lim_{n\to\infty}\|\varphi_j-\varphi\|_{\mathrm{u}}^{(k,\ell)} = 0$ for all $k,\ \ell\in\mathbb{N}$. The corresponding topology is the one for which G is open if and only if for each $\varphi\in G$ there an $m\in\mathbb{N}$ and $r>0$ such that

$$\left\{\psi:\ \|\psi-\varphi\|_{\mathrm{u}}^{(m)} < r\right\} \subseteq G,$$

where

$$\|\cdot\|_{\mathrm{u}}^{(m)} \equiv \sum_{\substack{k,\ell\in\mathbb{N}\\ k+\ell\le m}} \|\cdot\|_{\mathrm{u}}^{(j,\ell)}.$$

We will now develop a more convenient description of the topology on $\mathscr{S}(\mathbb{R};\mathbb{C})$, one that shows that $\mathscr{S}(\mathbb{R};\mathbb{C})$ shares many properties with Hilbert spaces. Define the operator $\mathcal{H}$ on $\mathscr{S}(\mathbb{R};\mathbb{C})$ into itself by

$$\mathcal{H}\varphi = x^2\varphi - \partial^2\varphi.$$

Since (cf. (11.1)) $\mathcal{H} = (2a_+a_- + \mathbf{1})$,

$$\mathcal{H}\tilde{h}_k = \mu_k\tilde{h}_k \quad \text{where } \mu_k = 2k+1, \tag{13.1}$$

and so we can define operators $\mathcal{H}^s$ for any $s\in\mathbb{R}$ by

$$\mathcal{H}^s\varphi = \sum_{m=0}^\infty \mu_m^s(\varphi,\tilde{h}_m)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\tilde{h}_m.$$

For each $m\ge 0$, set

$$\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C}) = \left\{\varphi\in L^2(\lambda_{\mathbb{R}};\mathbb{C}):\ \sum_{k=1}^\infty \mu_k^m\left|(\varphi,\tilde{h}_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\right|^2 < \infty\right\},$$

[6] There are many books in which Schwartz's theory is presented, but his own original treatment in *Théorie des distributions, I* published in 1950 by Hermann, Paris remains one of the best accounts.

and define

$$(\varphi,\psi)_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}=\sum_{k=0}^{\infty}\mu_k^m(\varphi,\tilde h_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}(\tilde h_k,\psi)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}=\big(\varphi,\mathcal{H}^m\psi\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}$$
$$\|\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}=(\varphi,\varphi)^{\frac12}_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}=\big(\varphi,\mathcal{H}^m\varphi\big)^{\frac12}.$$

Clearly $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$ is a vector space for which $(\varphi,\psi)_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}$ is an inner product. Below we will show below that it is a separable Hilbert space.

Lemma 13.1. *For each $m\ge 0$,*

$$\|x\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}\vee\|\partial\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}\le\|\varphi\|_{\mathscr{S}^{(m+1)}(\mathbb{R};\mathbb{C})}.$$

Proof. By the first part of (11.3),

$$\begin{aligned}\|x\varphi\|^2_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}&=\sum_{k=0}^{\infty}\mu_k^m|(x\varphi,\tilde h_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}|^2\\&\le\sum_{k=0}^{\infty}k\mu_k^m|(\varphi,\tilde h_{k-1})_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}|^2+\sum_{k=1}^{\infty}(k+1)\mu_k^m|(\varphi,\tilde h_{k+1})_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}|^2\\&=|(\varphi,\tilde h_0)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}|^2+\sum_{k=1}^{\infty}\mu_k^{m+1}|(\varphi,\tilde h_m)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}|^2=\|\varphi\|^2_{\mathscr{S}^{(m+1)}(\mathbb{R};\mathbb{C})}.\end{aligned}$$

Using the second part of (11.3) and the same argument, one can show that $\|\partial\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}$ $\le\|\varphi\|_{\mathscr{S}^{(m+1)}(\mathbb{R};\mathbb{C})}$. □

Theorem 13.2. *For each $m\in\mathbb{N}$, $\mathscr{S}(\mathbb{R};\mathbb{C})$ is a dense subset of $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$. In addition, for each $m\ge 0$, there exists a $K_m\in(0,\infty)$ such that*

$$\|\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}\le K_m\|\varphi\|_{\mathrm{u}}^{(m+1)}\tag{13.2}$$

and

$$\|\varphi\|_{\mathrm{u}}^{(m)}\le K_m\|\varphi\|_{\mathscr{S}^{(m+3)}(\mathbb{R};\mathbb{C})}.\tag{13.3}$$

for all $\varphi\in\mathscr{S}(\mathbb{R};\mathbb{C})$. Thus $\varphi_n\longrightarrow\varphi$ in $\mathscr{S}(\mathbb{R};\mathbb{C})$ if and only if

$$\lim_{n\to\infty}\|\varphi_n-\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}=0$$

for all $m\in\mathbb{N}$. In particular, for each $\varphi\in\mathscr{S}(\mathbb{R};\mathbb{C})$,

$$\sum_{k=0}^{n}(\varphi,\tilde h_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\tilde h_k\longrightarrow\varphi\ \textit{in}\ \mathscr{S}(\mathbb{R};\mathbb{C})\ \textit{as}\ n\to\infty.$$

Proof. Since $\mathcal{H}\restriction\mathscr{S}(\mathbb{R};\mathbb{C})$ is a symmetric operator, (13.1) implies that

$$\mu_k^m(\varphi,\tilde h_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}=(\varphi,\mathcal{H}^m\tilde h_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}=(\mathcal{H}^m\varphi,\tilde h_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})},$$

for all $\varphi\in\mathscr{S}(\mathbb{R};\mathbb{C})$ and $m\ge 0$, from which it is clear that $\mathscr{S}(\mathbb{R};\mathbb{C})\subseteq\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$ for all $m\ge 0$. Moreover, since, for each $\varphi\in\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$,

$$\mathscr{S}(\mathbb{R};\mathbb{C})\ni\sum_{k=0}^{n}(\varphi,\tilde h_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\tilde h_k\longrightarrow\varphi\ \text{in}\ \mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})\ \text{as}\ n\to\infty,$$

$\mathscr{S}(\mathbb{R};\mathbb{C})$ is dense in $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$.

Next observe that there exist constants $c^{(m)}_{j,\ell}\in\mathbb{R}$ such that

$$\big(x^2-\partial^2\big)^m\varphi=\sum_{\substack{k,\ell\in\mathbb{N}\\ k+\ell\le 2m}} c^{(m)}_{j.\ell}x^k\partial^\ell\varphi,$$

and use integration by parts to see that

$$\big(\varphi,x^k\partial^\ell\varphi\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}=(-1)^{\ell'}\big(\partial^{\ell'}(x^{k'}\varphi),x^{k-k'}\partial^{\ell-\ell'}\varphi\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})},$$

where

$$k'=\begin{cases}\frac{k}{2} & \text{if } k \text{ is even}\\ \frac{k-1}{2} & \text{if } k \text{ is odd}\end{cases}\quad\text{and}\quad \ell'=\begin{cases}\frac{\ell}{2} & \text{if } \ell \text{ is even}\\ \frac{\ell+1}{2} & \text{if } \ell \text{ is odd.}\end{cases}$$

Hence there exist constants $b^{(m)}_{(k_1,\ell_1),(k_2,\ell_2)}\in\mathbb{R}$ such that

$$\begin{aligned}\big(\varphi,\mathcal{H}^m\varphi\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}&\le\sum_{\substack{(k_1,\ell_1),(j_2,\ell_2)\in\mathbb{N}^2\\ (k_1+\ell_1)\vee(j_2+\ell_2)\le m}}\big|b^{(m)}_{(k_1,\ell_1),(k_2,\ell_2)}\big(x^{k_1}\partial^{\ell_1}\varphi,x^{k_2}\partial^{\ell_2}\varphi\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\big|\\ &\le\sum_{\substack{(k_1,\ell_1),(k_2,\ell_2)\in\mathbb{N}^2\\ (k_1+\ell_1)\vee(k_2+\ell_2)\le m}}|b^{(m)}_{(k_1,\ell_1),(k_2,\ell_2)}|\big\|x^{k_1}\partial^{\ell_1}\varphi\big\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\big\|x^{k_2}\partial^{\ell_2}\varphi\big\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}.\end{aligned}$$

Finally, observe that

$$\begin{aligned}\|x^k\partial^\ell\varphi\|^2_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}&=\int(1+x^2)^{-1}\big|(1+x^2)^{\frac12}x^k\partial^\ell\varphi(x)\big|^2\,dx\\ &\le\pi\big(\|x^k\partial^\ell\varphi\|^2_{\mathrm{u}}+\|x^{k+1}\partial^\ell\varphi\|^2_{\mathrm{u}}\big),\end{aligned}$$

and combine this with the preceding to arrive at (13.2).

To prove (13.3), begin by making repeated application of Lemma 13.1 to show that

$$\|x^k\partial^\ell\varphi\|_{\mathscr{S}^{(3)}(\mathbb{R};\mathbb{C})}\le\|\varphi\|_{\mathscr{S}^{(k+\ell+3)}(\mathbb{R};\mathbb{C})}.$$

Thus, if we show that there is a $K\in(0,\infty)$ such that

$$\|\varphi\|_{\mathrm{u}}\le K\|\varphi\|_{\mathscr{S}^{(3)}(\mathbb{R};\mathbb{C})},\tag{$*$}$$

then

$$\|x^k\partial^\ell\varphi\|_{\mathrm{u}}\le K\|x^k\partial^\ell\varphi\|_{\mathscr{S}^{(3)}(\mathbb{R};\mathbb{C})}\le K\|\varphi\|_{\mathscr{S}^{(k+\ell+3)}(\mathbb{R};\mathbb{C})},$$

in which case we would know that $\|\varphi\|^{(m)}_{\mathrm{u}}\le\frac{Km(m-1)}{2}\|\varphi\|_{\mathscr{S}^{(m+3)}(\mathbb{R};\mathbb{C})}$. To prove ($*$), use the estimate in (11.2) to see that

$$\begin{aligned}\|\varphi\|_{\mathrm{u}}&\le\sum_{k=0}^\infty|(\varphi,\tilde h_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}|\|\tilde h_k\|_{\mathrm{u}}\\ &\le\sum_{k=0}^\infty(k+1)^{\frac12}|(\varphi,\tilde h_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}|=\sum_{k=0}^\infty\left(\frac{k+1}{\mu_k^3}\right)^{\frac12}\mu_k^{\frac32}|(\varphi,\tilde h_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}|\\ &\le\left(\sum_{k=0}^\infty\frac{k+1}{\mu_k^3}\right)^{\frac12}\left(\sum_{k=0}^\infty\mu_k^3|(\varphi,\tilde h_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}|^2\right)^{\frac12}=K\|\varphi\|_{\mathscr{S}^{(3)}(\mathbb{R};\mathbb{C})}.\end{aligned}$$

where $K=\left(\sum_{k=0}^\infty\frac{k+1}{\mu_k^3}\right)^{\frac12}$. □

As a consequence of Theorem 13.2, we know that

$$\rho_{\mathscr{S}}(\varphi,\psi) \equiv \sum_{m=0}^{\infty} \frac{1}{2^{k+1}} \frac{\|\varphi-\psi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}}{1+\|\varphi-\psi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}}$$

is a metric for the topology on $\mathscr{S}(\mathbb{R};\mathbb{C})$. In addition, $\mathscr{S}(\mathbb{R};\mathbb{C}) = \bigcap_{m=0}^{\infty} \mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$, and so we can learn about properties of $\mathscr{S}(\mathbb{R};\mathbb{C})$ by understanding those of the $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$'s.

For each $m \geq 0$, let $\mathfrak{s}^{(m)}(\mathbb{N};\mathbb{C})$ be the space of functions $s:\mathbb{N}\longrightarrow\mathbb{C}$ such that

$$\|s\|_{\mathfrak{s}^{(m)}(\mathbb{N};\mathbb{C})} \equiv \left(\sum_{k=0}^{\infty} \mu_k^m |s(k)|^2\right)^{\frac{1}{2}} < \infty,$$

and define

$$(s,t)_{\mathfrak{s}^{(m)}(\mathbb{N};\mathbb{C})} = \sum_{k=0}^{\infty} \mu_k^m s(k)\overline{t(k)} \text{ for } s,\, t \in \mathfrak{s}^{(m)}(\mathbb{N};\mathbb{C}),$$

Clearly each $\mathfrak{s}^{(m)}(\mathbb{N};\mathbb{C})$ is a vector space with inner product $(s,t)_{\mathfrak{s}^{(m)}(\mathbb{N};\mathbb{C})}$. Finally, set $\mathfrak{s}(\mathbb{N};\mathbb{C}) = \bigcap_{m=0}^{\infty} \mathfrak{s}^{(m)}(\mathbb{N};\mathbb{C})$, and turn $\mathfrak{s}(\mathbb{N};\mathbb{C})$ into a metric space with metric

$$\rho_{\mathfrak{s}}(s,t) \equiv \sum_{m=0}^{\infty} \frac{1}{2^{m+1}} \frac{\|t-s\|_{\mathfrak{s}^{(m)}(\mathbb{N};\mathbb{C})}}{1+\|t-s\|_{\mathfrak{s}^{(m)}(\mathbb{N};\mathbb{C})}}.$$

The following corollary is essentially a reformulation of the results in Theorem 13.2. It is the analogue for $\mathscr{S}(\mathbb{R};\mathbb{C})$ of the fact that every separable Hilbert space is isomorphic to $\ell^2(\mathbb{N};\mathbb{C})$.

Corollary 13.3. *Define the map* $S: L^2(\lambda_{\mathbb{R}};\mathbb{C}) \longrightarrow \ell^2(\mathbb{N};\mathbb{C})$ *by*

$$[S(\varphi)](k) = (\varphi, \tilde{h}_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}.$$

Then, for each $m \geq 0$, $S \restriction \mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$ *is an isometric isomorphism from* $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$ *onto* $\mathfrak{s}^{(m)}(\mathbb{N};\mathbb{C})$, *and so* $S \restriction \mathscr{S}(\mathbb{R};\mathbb{C})$ *is isometric homeomorphism from* $\mathscr{S}(\mathbb{R};\mathbb{C})$ *onto* $\mathfrak{s}(\mathbb{N};\mathbb{C})$.

Corollary 13.3 means that any topological property of $\mathfrak{s}^{(m)}(\mathbb{N};\mathbb{C})$ or $\mathfrak{s}(\mathbb{N};\mathbb{C})$ is a property of $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$ or $\mathscr{S}(\mathbb{R};\mathbb{C})$, and the following lemma facilitates the study of such properties.

Lemma 13.4. *Let* $\{\alpha_k : k \geq 0\} \subseteq (0,\infty)$, *and define the measure* ν *on* $\mathbb{N}$ *by* $\nu(\{k\}) = \alpha_k$. *Then* $L^2(\nu;\mathbb{C})$ *is a separable Hilbert space. In addition, a set* $B \subseteq L^2(\nu;\mathbb{C})$ *is relatively compact if and only if* B *is bounded and tight in the sense that*

$$\lim_{K\to\infty} \sup_{s\in B} \sum_{k>K} \alpha_k |s(k)|^2 = 0.$$

Proof. Since the L^2-space for any measure on a countably generated σ-algebra is a separable Hilbert space, $L^2(\nu;\mathbb{C})$ is a separable Hilbert space.

Since $L^2(\nu;\mathbb{C})$ is complete, to prove that a bounded, tight subset B is relatively compact it suffices to show that B is totally bounded (i.e., for every $r>0$ there is a finite cover of B by balls of radius r with centers in B). To that end, let $r>0$ be given, and choose K so that

$$\sup_{s\in B} \sum_{k>K} \alpha_k |s(k)|^2 < \frac{r^2}{4}.$$

Next, note that $\{(s(0),\dots,s(K)) : s \in B\}$ is a bounded subset of $\mathbb{C}^{K+1}$ and therefore totally bounded there. Hence there exists a finite set $\{s_j : 1 \le j \le J\} \subseteq B$ such that, for each $s \in B$,

$$\min_{1\le j\le J} \sum_{k=0}^{K} \alpha_k |s(k) - s_j(k)|^2 < \frac{r^2}{2},$$

which means that, for each $s \in B$ there exists a $1 \le j \le J$ such that

$$\|s - s_j\|^2_{L^2(\nu;\mathbb{C})} = \sum_{k=0}^{K} \alpha_k |s(k) - s_j(k)|^2 + \sum_{k>K} \alpha_k |s(k) - s_j(k)|^2 \le r^2.$$

Finally, suppose that B is relatively compact. Certainly it is bounded. To see that it must be tight, suppose it were not. Then there would exist an $\epsilon > 0$ such that, for each $K \in \mathbb{N}$,

$$\sup_{s\in B} \sum_{k>K} \alpha_k |s(k)|^2 > \epsilon.$$

Thus we could find a sequence $\{s_K : K \ge 0\} \subseteq B$ with the property that $\sum_{k>K} \alpha_k |s_K(k)|^2 \ge \epsilon$, and, because B is relatively compact, we could choose it to be a sequence which converges to some $t \in L^2(\nu;\mathbb{C})$. But this would mean that

$$\sum_{k>K} \alpha_k |t(k)|^2 \ge \sum_{k>K} \alpha_k |s_K(k)|^2 - \|t - s_K\|^2_{L^2(\nu;\mathbb{C})} \ge \frac{\epsilon}{2}$$

for sufficient large K, and that would mean the t can't be in $L^2(\nu;\mathbb{C})$. □

Say that $B \subseteq \mathscr{S}(\mathbb{R};\mathbb{C})$ is *bounded* in $\mathscr{S}(\mathbb{R};\mathbb{C})$ if

$$\sup_{\varphi\in B} \|\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})} < \infty \text{ for each } m \ge 0.$$

Theorem 13.5. *$\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$ is a separable Hilbert space for each $m \ge 0$, and $\mathscr{S}(\mathbb{R};\mathbb{C})$ is a complete separable metric space. Moreover, a subset $B \subseteq \mathscr{S}(\mathbb{R};\mathbb{C})$ is relatively compact if and only if it is bounded in $\mathscr{S}(\mathbb{R};\mathbb{C})$.*

Proof. By Lemma 13.4 applied with $\alpha_k = \mu_k^m$, we know that each of the spaces $\mathfrak{s}^{(m)}(\mathbb{N};\mathbb{C})$ is a separable Hilbert space, and therefore, by Corollary 13.3, so is each $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$. Thus, since $\mathscr{S}(\mathbb{R};\mathbb{C})$ is dense in every $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$, we can find a sequence $\{\varphi_n : n \ge 1\} \subseteq \mathscr{S}(\mathbb{R};\mathbb{C})$ which is dense in $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$ for all $m \ge 0$. Since this means that

$$\inf_{n\ge 1} \|\varphi - \varphi_n\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})} = 0 \text{ for all } \varphi \in \mathscr{S}(\mathbb{R};\mathbb{C}) \text{ and } m \ge 0,$$

it follows that

$$\inf_{n\ge 1} \rho_{\mathscr{S}(\mathbb{R};\mathbb{C})}(\varphi,\varphi_n) = 0 \text{ for all } \varphi \in \mathscr{S}(\mathbb{R};\mathbb{C}).$$

That is, $\{\varphi_n : n \ge 1\}$ is dense in $\mathscr{S}(\mathbb{R};\mathbb{C})$, and so $\mathscr{S}(\mathbb{R};\mathbb{C})$ is separable.

To see that $\mathscr{S}(\mathbb{R};\mathbb{C})$ is complete, first use Lemma 13.4 and Corollary 13.3 to see that each $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$ is complete. Now suppose that $\{\varphi_n : n \ge 1\} \subseteq \mathscr{S}(\mathbb{R};\mathbb{C})$ is $\rho_{\mathscr{S}(\mathbb{R};\mathbb{C})}$-Cauchy convergent. Then it is $\|\cdot\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}$-Cauchy convergent for each $m \ge 0$, and therefore it is convergent in each $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$ to some element of $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$. But if $\varphi_n \longrightarrow \varphi$ in $\mathscr{S}^{(m+1)}(\mathbb{R};\mathbb{C})$, then $\varphi_n \longrightarrow \varphi$ in $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$, and so there is a unique $\varphi \in \bigcap_{m=0}^{\infty} \mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$ to which $\{\varphi_n : n \ge 1\}$ converges in $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$ for all $m \ge 0$. Therefore $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{C})$ and $\lim_{n\to\infty} \rho_{\mathscr{S}}(\varphi,\varphi_n) = 0$.

Finally, suppose that $B \subseteq \mathscr{S}(\mathbb{R};\mathbb{C})$ is relatively compact in $\mathscr{S}(\mathbb{R};\mathbb{C})$. Because B is then relatively compact in each $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$ and therefore bounded there, it is a bounded subset of $\mathscr{S}(\mathbb{R};\mathbb{C})$. Conversely, if B is bounded in $\mathscr{S}(\mathbb{R};\mathbb{C})$, in order to show that it is relatively compact in $\mathscr{S}(\mathbb{R};\mathbb{C})$ we need only show that it is totally bounded there. To that end, first observe it is bounded in each $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$. Thus, by Lemma 13.4 and Corollary 13.3, we will know that it is relatively compact in $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$ if

$$\lim_{K\to\infty}\sup_{\varphi\in B}\sum_{k>K}\mu_k^m|(\varphi,\tilde{h}_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}|^2 = 0. \tag{$*$}$$

But

$$\sum_{k>K}\mu_k^m|(\varphi,\tilde{h}_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}|^2 \le \frac{1}{\mu_{K+1}}\|\varphi\|^2_{\mathscr{S}^{(m+1)}(\mathbb{R};\mathbb{C})},$$

and so, since B is bounded in $\mathscr{S}^{(m+1)}(\mathbb{R};\mathbb{C})$, $(*)$ holds. To complete the proof that B $\rho_{\mathscr{S}}$-totally bounded, let $r > 0$ be given, and choose m so that $2^{-m} < \frac{r}{2}$. Next, using the fact that B is relatively compact in $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$, choose $\{\varphi_j : 1 \le j \le J\} \subseteq B$ so that

$$\sup_{\varphi\in B}\min_{1\le j\le J}\|\varphi-\varphi_j\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})} < \frac{r}{2},$$

and conclude that

$$B \subseteq \bigcup_{j=1}^{J}\{\varphi : \rho_{\mathscr{S}(\mathbb{R};\mathbb{C})}(\varphi,\varphi_j) < r\}.$$

□

The assertion in the following is one of the many virtues possessed by $\mathscr{S}(\mathbb{R};\mathbb{C})$.

Theorem 13.6. *The map $\varphi \rightsquigarrow \hat{\varphi}$ is an isomorphism from $\mathscr{S}(\mathbb{R};\mathbb{C})$ onto itself, and, for each $m \ge 0$, $\|\hat{\varphi}\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})} = (2\pi)^{\frac{1}{2}}\|\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}$.*

Proof. We already know that the Fourier transform is an isomorphism of $L^2(\lambda_{\mathbb{R}};\mathbb{C})$ onto $L^2(\lambda_{\mathbb{R}};\mathbb{C})$. In addition, by Theorem 12.2, $(\hat{\varphi},\tilde{h}_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} = (2\pi)^{\frac{1}{2}}\imath^k(\varphi,\tilde{h}_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}$, and so

$$\|\hat{\varphi}\|^2_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})} = 2\pi\sum_{k=0}^{\infty}\mu_k^m|(\varphi,\tilde{h}_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}|^2 = 2\pi\|\varphi\|^2_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}.$$

□

Exercise 13.7. Show that for each $(m,n) \in \mathbb{N}^2$ there is a $C_{n,m} \in (0,\infty)$ such that

$$\frac{1}{C_{n,m}}\max_{\substack{k,\ell\in\mathbb{N}\\ k+\ell\le m}}\|x^k\partial^\ell\varphi\|_{\mathscr{S}^{(n)}(\mathbb{R};\mathbb{C})} \le \|\varphi\|_{\mathscr{S}^{(n+m)}(\mathbb{R};\mathbb{C})} \le C_{n,m}\max_{\substack{k,\ell\in\mathbb{N}\\ k+\ell\le m}}\|x^k\partial^\ell\varphi\|_{\mathscr{S}^{(n)}(\mathbb{R};\mathbb{C})}.$$

Hint: In proving the upper bound, consider using the equation

$$\left(a_+^n\varphi,\tilde{h}_{k+n}\right)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} = \frac{(k+n)!}{n!}\left(\varphi,\tilde{h}_k\right)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}.$$

.

Exercise 13.8. Let $\{\varphi_n : n \geq 1\}$ be a bounded sequence in $\mathscr{S}(\mathbb{R};\mathbb{C})$ such that $\lim_{n\to\infty} \varphi_n(x)$ exists for each $x \in \mathbb{R}$. Show that there is a $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{C})$ such that $\varphi_n \longrightarrow \varphi$ in $\mathscr{S}(\mathbb{R};\mathbb{C})$.

Hint: Use Theorem 13.5.

Exercise 13.9. This exercise deals with the relationship between various function spaces.

(**i**) Show that $C_c^\infty(\mathbb{R};\mathbb{C})$ is a dense subset of $\mathscr{S}(\mathbb{R};\mathbb{C})$

(**ii**) Set

$$C_0(\mathbb{R};\mathbb{C}) = \left\{ f \in C(\mathbb{R};\mathbb{C}) : \lim_{|x|\to\infty} f(x) = 0 \right\}.$$

Show that $C_0(\mathbb{R};\mathbb{C})$ with the uniform norm is a Banach space in which both C_c^∞ and $\mathscr{S}(\mathbb{R};\mathbb{C})$ are dense subsets.

Exercise 13.10. For $x \in \mathbb{R}$ and $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{C})$, define $\tau_x\varphi(y) = \varphi(x+y)$. Show that $\tau_x\varphi \in \mathscr{S}(\mathbb{R};\mathbb{C})$ and that $\|\tau_x\varphi\|_{\mathrm{u}}^{(m)} \leq 2^m(|x| \vee 1)^m \|\varphi\|_{\mathrm{u}}^{(m)}$ for all $m \geq 0$. In addition, show that

$$\|\tau_{x_2}\varphi - \tau_{x_1}\varphi\|_{\mathrm{u}}^{(m)} \leq 2^m \big(|x_1| \vee |x_2| \vee 1\big)^m \|\varphi\|_{\mathrm{u}}^{(m+1)} |x_2 - x_1|.$$

Hint: To prove the first estimate, check that

$$|y^k \partial^\ell \tau_x\varphi(y)| \leq \begin{cases} (2|x|)^m \big|(\partial^\ell\varphi)(x+y)\big| & \text{if } |y| \leq 2|x| \\ 2^m \big|(x+y)^k (\partial^\ell\varphi)(x+y)\big| & \text{if } |y| \geq 2|x|. \end{cases}$$

To prove the second estimate, assume that $x_1 \leq x_2$, note that

$$\tau_{x_2}\varphi - \tau_{x_1}\varphi = \int_{x_1}^{x_2} \tau_t\varphi'\, dt,$$

and therefore that

$$\|\tau_{x_2}\varphi - \tau_{x_1}\varphi\|_{\mathrm{u}}^{(m)} \leq \int_{x_1}^{x_2} \|\tau_t\varphi'\|_{\mathrm{u}}^{(m)}\, dt.$$

Finally, apply the first estimate.

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