Lecture 14: Tempered Distributions

Schwartz developed the theory of distribution in order to provide a mathematically rigorous way to describe the sort of generalized functions that appear in the work by Boole and Heaviside in connection with applications of the Laplace transform to ordinary differential equations, and those that were somewhat later introduced by Sobolev for applications to partial differential equations. What Schwartz realized is that generalized functions should be thought of in terms of their *action* (i.e., their $L^2(\lambda_{\mathbb{R}};\mathbb{C})$ inner product with) on smooth functions rather than their value (which won't exist in general) at points.

To make that idea mathematically precise, he said a generalized function, which he called a *distribution*, should be a continuous linear functional on a topological vector space of smooth functions. One of the spaces Schwartz considered is $C_c^\infty(\mathbb{R};\mathbb{C})$, but the appropriate topology on that space is rather cumbersome (for instance, elements don't have countable neighborhood bases). A second, and much more tractable, choice is $\mathscr{S}(\mathbb{R};\mathbb{C})$. Because elements of $\mathscr{S}(\mathbb{R};\mathbb{C})$ need not have compact support, an element of its dual space must satisfy restricted growth conditions and is therefore called a *tempered distribution*.

Recall that the dual space X^* of a topological vector space X over $\mathbb{C}$ is the space of continuous, $\mathbb{C}$-valued linear functions on X. When, like $\mathscr{S}(\mathbb{R};\mathbb{C})$, all the elements of X have a countable neighborhood basis, a linear function Λ on X is an element of X^* if $\Lambda x_n \longrightarrow \Lambda x$ whenever $x_n \longrightarrow x$ in X. Because we want to think of elements of $\mathscr{S}(\mathbb{R};\mathbb{C})^*$ as generalized functions which act via their inner product with elements of $\mathscr{S}(\mathbb{R};\mathbb{C})$, we will use letters like u to denote elements of $\mathscr{S}(\mathbb{R};\mathbb{C})^*$ and write their action on $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{C})$ as $\langle \varphi, u\rangle$.

Lemma 14.1. *For each $u \in \mathscr{S}(\mathbb{R};\mathbb{C})^*$ there is an $m \geq 0$ and a $C \in (0,\infty)$ such that*

$$|\langle \varphi, u\rangle| \leq C\|\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})} \text{ for all } \varphi \in \mathscr{S}(\mathbb{R};\mathbb{C}).$$

Proof. Because sets of the form $\left\{\|\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})} \leq r\right\}$ form a neighborhood basis for $\mathbf{0}$ in $\mathscr{S}(\mathbb{R};\mathbb{C})$, there is an $m \geq 0$ and $r > 0$ such that $|\langle \varphi, u\rangle| \leq 1$ when $\|\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})} \leq r$. Hence $|\langle \varphi, u\rangle| \leq r^{-1}\|\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}$. □

Simple as it is, Lemma 14.1 has many consequences. For example, it allows us to say that

$$\langle \varphi, u\rangle = \sum_{k=0}^{\infty} (\varphi, \tilde{h}_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\langle \tilde{h}_k, u\rangle, \tag{14.1}$$

where the series is absolutely convergent. Indeed, if $|\langle \varphi, u\rangle| \leq C\|\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}$, then $|\langle \tilde{h}_k, u\rangle| \leq C\mu_k^m$, and so, since $|(\varphi, \tilde{h}_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}| = \mu_k^{-n}|(\mathcal{H}^n\varphi, \tilde{h}_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}|$ for all $n \geq 0$, the series $\sum_{k=0}^\infty (\varphi, \tilde{h}_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\langle \tilde{h}_k, u\rangle$ is absolutely convergent. Hence, if $\varphi_n = \sum_{k=0}^n (\varphi, \tilde{h}_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}$, then $\varphi_n \longrightarrow \varphi$ in $\mathscr{S}(\mathbb{R};\mathbb{C})$ and therefore

$$\begin{aligned}\langle \varphi, u\rangle &= \lim_{n\to\infty}\langle \varphi_n, u\rangle = \lim_{n\to\infty}\sum_{k=0}^{n} (\varphi, \tilde{h}_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\langle \tilde{h}_k, u\rangle \\ &= \sum_{k=0}^{\infty} (\varphi, \tilde{h}_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\langle \tilde{h}_k, u\rangle.\end{aligned}$$

Obviously, given a measurable function $f:\mathbb{R}\longrightarrow\mathbb{C}$ with at most polynomial growth, one can think of it as the element of $f\lambda_{\mathbb{R}}\in\mathscr{S}(\mathbb{R};\mathbb{C})^*$ given by $\langle\varphi,f\lambda_{\mathbb{R}}\rangle=\int\varphi\bar{f}\,d\lambda_{\mathbb{R}}$, and in this way $\mathscr{S}(\mathbb{R};\mathbb{C})$ can be thought of as a subset of $\mathscr{S}(\mathbb{R};\mathbb{C})^*$. Although the distribution corresponding to f is $f\lambda_{\mathbb{R}}$, it is conventional to denote it by f instead, and we will adopt this convention.

We will need to know that $\mathscr{S}(\mathbb{R};\mathbb{C})$ is dense in $\mathscr{S}(\mathbb{R};\mathbb{C})^*$. To see that it is, let $u\in\mathscr{S}(\mathbb{R};\mathbb{C})^*$, and set

$$\psi_n=\sum_{k=0}^{n}\overline{\langle\tilde{h}_k,u\rangle}\tilde{h}_k.$$

Clearly $\psi_n\in\mathscr{S}(\mathbb{R};\mathbb{C})$, and, for each $\varphi\in\mathscr{S}(\mathbb{R};\mathbb{C})$,

$$\begin{aligned}(\varphi,\psi_n)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}&=\sum_{k=0}^{n}(\varphi,\tilde{h}_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\overline{(\psi_n,\tilde{h}_k)}_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\\&=\sum_{k=0}^{n}(\varphi,\tilde{h}_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\langle\tilde{h}_k,u\rangle\longrightarrow\sum_{k=0}^{\infty}(\varphi,\tilde{h}_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\langle\tilde{h}_k,u\rangle=\langle\varphi,u\rangle.\end{aligned}$$

The importance of this density result is that it tells us how to extend continuous operators like $\mathcal{H}^s$ as continuous operators on $\mathscr{S}(\mathbb{R};\mathbb{C})^*$. Namely, because $(\varphi,\mathcal{H}^s\psi)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}=(\mathcal{H}^s\varphi,\psi)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}$ for $\varphi,\psi\in\mathscr{S}(\mathbb{R};\mathbb{C})$ and $\mathscr{S}(\mathbb{R};\mathbb{C})$ is dense in $\mathscr{S}(\mathbb{R};\mathbb{C})^*$, the one and only continuous extension of $\mathcal{H}^s$ to $\mathscr{S}(\mathbb{R};\mathbb{C})^*$ is given by

$$\langle\varphi,\mathcal{H}^s u\rangle\equiv\langle\mathcal{H}^s\varphi,u\rangle. \tag{14.2}$$

Since $\mathscr{S}(\mathbb{R};\mathbb{C})$ can be written as the intersection of the spaces $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$, $\mathscr{S}(\mathbb{R};\mathbb{C})^*$ must be able to be written as the union of the spaces $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})^*$. Of course, because $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$ is a Hilbert space, Riesz's theorem provides an isomorphism between $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})^*$ and $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$. However, in order to be consistent with the idea that $\langle\varphi,u\rangle$ is a generalization of the L^2 inner product, this is not the way we will think about $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})^*$. Instead, we want to identify $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})^*$ as the Hilbert space

$$\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})=\left\{u\in\mathscr{S}(\mathbb{R};\mathbb{C})^*:\sum_{k=0}^{\infty}\mu_k^{-m}|\langle\tilde{h}_k,u\rangle|^2<\infty\right\}$$

with inner product

$$(u,v)_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})}=\sum_{k=0}^{\infty}\mu_k^{-m}\langle\tilde{h}_k,u\rangle\overline{\langle\tilde{h}_k,v\rangle}.$$

Recall that if X is a Banach space and $\Lambda\in X^*$, then $\|\Lambda\|_{X^*}=\sup\{|\Lambda(x)|:\|x\|_X=1\}$. Thus

$$\|u\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})^*}=\sup\{|\langle\varphi,u\rangle|:\|\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}=1\}.$$

Theorem 14.2. *For each $m\geq 0$, $\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})$ is a separable Hilbert space and*

$$\begin{aligned}u\in\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})&\iff\mathcal{H}^{-\frac{m}{2}}u\in L^2(\lambda_{\mathbb{R}};\mathbb{C})\ \&\ |\mathcal{H}^{-\frac{m}{2}}u\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}=\|u\|_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})}\\&\iff u\in\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})^*.\end{aligned}$$

Moreover, if $u\in\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})$, then $\|u\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})^*}=\|u\|_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})}$ and therefore

$$|\langle\varphi,u\rangle|\leq\|\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}\|u\|_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})}. \tag{14.3}$$

Proof. If $u \in \mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})$, then

$$|\langle \varphi, \mathcal{H}^{-\frac{m}{2}} u\rangle| = \left|\sum_{k=0}^{\infty} \mu_k^{-\frac{m}{2}} (\varphi, \tilde{h}_k)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} \langle \tilde{h}_k, u\rangle\right| \le \|\varphi\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} \|u\|_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})},$$

and so $\mathcal{H}^{-\frac{m}{2}} u \in L^2(\lambda_{[0,1)};\mathbb{C})$ and $\|\mathcal{H}^{-\frac{m}{2}} u\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} = \|u\|_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})}$. Conversely, if $\mathcal{H}^{-\frac{m}{2}} u \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$, then

$$\|u\|^2_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})} = \sum_{k=0}^{\infty} \mu_k^{-m} |\langle \tilde{h}_k, u\rangle|^2 = \sum_{k=0}^{\infty} |\langle \tilde{h}_k, \mathcal{H}^{-\frac{m}{2}} u\rangle|^2 = \|\mathcal{H}^{-\frac{m}{2}} u\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}.$$

To prove the second equivalence, first suppose that $u \in \mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})^*$. Then, since $\|\mathcal{H}^{-\frac{m}{2}} \varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})} = \|\varphi\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}$,

$$\begin{aligned} |\langle \varphi, \mathcal{H}^{-\frac{m}{2}} u\rangle| &= |\langle \mathcal{H}^{-\frac{m}{2}} \varphi, u\rangle| \\ &\le \|\mathcal{H}^{-\frac{m}{2}} \varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})} \|u\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})^*} = \|u\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})^*} \|\varphi\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}, \end{aligned}$$

and so $\mathcal{H}^{-\frac{m}{2}} u \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$ and $\|u\|_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})} \le \|u\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})^*}$. Conversely, if $u \in \mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})$, set $f = \mathcal{H}^{-\frac{m}{2}} u$, then

$$\begin{aligned} |\langle \varphi, u\rangle| &= |(\mathcal{H}^{\frac{m}{2}} \varphi, f)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}| \\ &\le \|\mathcal{H}^{\frac{m}{2}} \varphi\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} \|f\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} = \|u\|_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})} \|\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}, \end{aligned}$$

and so $u \in \mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})^*$ and $\|u\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})^*} \le \|u\|_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})}$. □

By combining Lemma 14.1 and Theorem 14.2, we know that

$$\mathscr{S}(\mathbb{R};\mathbb{C})^* = \bigcup_{m=0}^{\infty} \mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C}).$$

Theorem 14.3. *If $u \in \mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})$ is non-negative in the sense that $\langle \varphi, u\rangle \ge 0$ whenever $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{C})$ is non-negative, then there exists a Borel measure μ on $\mathbb{R}$ such that*

$$\int (1+x^2)^{-\frac{m+3}{2}} \mu(dx) < \infty \text{ and } \langle \varphi, \mu\rangle = \int \varphi\, d\mu.$$

Conversely, if μ is a Borel measure on $\mathbb{R}$ satisfying

$$\int (1+x^2)^{-\frac{m}{2}} \mu(dx) < \infty$$

and $u \in \mathscr{S}(\mathbb{R};\mathbb{C})^*$ is defined by $\langle \varphi, u\rangle = \int \varphi\, d\mu$, then $u \in \mathscr{S}^{(-m-3)}(\mathbb{R};\mathbb{C})$.

Proof. Assume that $u \in \mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})$ is non-negative. Choose $\eta \in C^\infty\big(\mathbb{R};[0,1]\big)$ so that $\eta = 1$ on $[-1,1]$ and $\eta = 0$ off $[-2,2]$, set $\eta_R(x) = \eta\big(\frac{x}{R}\big)$ for $R \ge 1$, and define $u_R \in \mathscr{S}(\mathbb{R};\mathbb{C})^*$ by $\langle \varphi, u_R\rangle = \langle \eta_R \varphi, u\rangle$. Given an $\mathbb{R}$-valued $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{C})$, $\|\varphi\|_{\mathrm{u}} \eta_R \pm \varphi \eta_R \ge 0$, and therefore $|\langle \varphi, u_R\rangle| \le \|\varphi\|_{\mathrm{u}} \langle \eta_R, u\rangle$. Thus there is a unique extension of $\varphi \rightsquigarrow \langle \varphi, u_R\rangle$ as a continuous, non-negative linear functional on $C\big([-2R;2R],\mathbb{R}\big)$, which, by the Riesz representation theorem, means that there is a finite Borel measure μ_R on $\mathbb{R}$ such that $\langle \varphi, u_R\rangle = \int \varphi\, d\mu_R$. In particular, $\mu_R(\mathbb{R}) = \langle \eta_R, u\rangle \le \|\eta_R\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})} \|u\|_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})}$. Since $\|\eta_R\|^2_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})} = \big(\eta_R, \mathcal{H}^m \eta_R\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}$

and $\mathcal{H}^m\eta_R$ is a linear combinations of terms of the form $\frac{x^k}{R^\ell}\eta^{(\ell)}\big(\frac{x}{R}\big)$, where $0 \le k+\ell \le 2m$, there exists a $C<\infty$ such that

$$\left(\int \eta_R(x)\mathcal{H}^m\eta_R(x)\,dx\right)^{\frac12} \le CR^{m+\frac12},$$

and so $\mu_R(\mathbb{R}) \le C\|u\|_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})}R^{m+\frac12}$.

Note that $R \le R' \implies \mu_{R'}\restriction[-R,R] = \mu_R\restriction[-R,R]$, and therefore there is a Borel measure μ on $\mathbb{R}$ such that $\mu\restriction[-R,R] = \mu_R\restriction[-R,R]$ for all $R\ge 1$. Furthermore

$$\begin{aligned}\int(1+x^2)^{-\frac{m+3}{2}}\,\mu(dx) &= \sum_{n=0}^\infty\int_{[-n,n]}(1+|x|^2)^{-\frac{m+3}{2}}\,\mu_n(dx)\\ &\le C\|u\|_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})}\sum_{n=0}^\infty\frac{(n+1)^{m+\frac12}}{(1+n^2)^{\frac{m+3}{2}}} = 2^{\frac{2m+1}{4}}C\|u\|_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})}\sum_{n=0}^\infty\frac{1}{1+n^2}<\infty.\end{aligned}$$

Finally,

$$\langle\varphi,u\rangle = \lim_{R\to\infty}\langle\eta_R\varphi,u\rangle = \lim_{R\to\infty}\int\eta_R\varphi\,d\mu = \int\varphi\,d\mu.$$

Conversely, suppose that μ is a Borel measure on $\mathbb{R}$ and that

$$C\equiv\int(1+x^2)^{-\frac{m}{2}}\,d\mu(dx)<\infty.$$

Clearly $\varphi\rightsquigarrow\int\varphi\,d\mu$ determines a distribution u. In fact, by (13.3),

$$|\langle\varphi,u\rangle| \le C\|(1+x^2)^{\frac{m}{2}}\varphi\|_{\mathrm{u}} \le C\|(1+|x|)^m\varphi\|_{\mathrm{u}} \le CK_m\|\varphi\|_{\mathscr{S}^{(m+3)}(\mathbb{R};\mathbb{C})},$$

and therefore $u\in\mathscr{S}^{(-m-3)}(\mathbb{R};\mathbb{C})$. □

As a consequence of Theorem 14.3, we know that for any measurable $f:\mathbb{R}\longrightarrow\mathbb{C}$ for which there exists an $m\in\mathbb{Z}$ such that

$$\int(1+x^2)^{-\frac{m}{2}}|f(x)|\,dx<\infty,$$

there is a distribution $f\in\mathscr{S}^{(-m-3)}(\mathbb{R};\mathbb{C})$ such that

$$\langle\varphi,f\rangle = \int\varphi(x)\bar f(x)\,dx.$$

The following generalizes the preceding observation.

Theorem 14.4. *Let μ be a Borel measure on $\mathbb{R}$, and assume that*

$$M_\mu\equiv\int(1+x^2)^{-\frac{m}{2}}\,\mu(dx)<\infty.$$

If $f\in L^p(\mu;\mathbb{C})$, then there is a distribution $f\mu$ given by

$$\varphi\in\mathscr{S}(\mathbb{R};\mathbb{C})\longmapsto\int\varphi\bar f\,d\mu\in\mathbb{C}.$$

Moreover, if $m_p=\min\{n:\ m\le 2p'n\}$, where p' is the Hölder conjugate of p, then $f\mu\in\mathscr{S}^{(-m_p-3)}(\mathbb{R};\mathbb{C})$ and

$$\|f\mu\|_{\mathscr{S}^{(-m_p-3)}(\mathbb{R};\mathbb{C})} \le K_{m_p}M_\mu^{\frac{1}{p'}}\|f\|_{L^p(\mu;\mathbb{C})}.$$

Proof. By Hölder's inequality,

$$\left|\int \varphi \bar{f}\, d\mu\right| \le \|f\|_{L^p(\mu;\mathbb{C})}\|\varphi\|_{L^{p'}(\mu;\mathbb{C})}.$$

At the same time,

$$\begin{aligned}\|\varphi\|_{L^{p'}(\mu;\mathbb{C})} &\le \left(\int (1+x^2)^{-\frac{m}{2}}(1+x^2)^{\frac{m}{2}}|\varphi(x)|^{p'}\,\mu(dx)\right)^{\frac{1}{p'}}\\ &\le M_\mu^{\frac{1}{p'}}\left\|(1+x^2)^{\frac{m}{2p'}}\varphi\right\|_{\mathrm{u}} \le K_{m_p}M_\mu^{\frac{1}{p'}}\|\varphi\|_{\mathscr{S}^{(m_p+3)}(\mathbb{R};\mathbb{C})}.\end{aligned}$$

Hence,

$$|\langle\varphi, f\mu\rangle| \le K_{m_p}M_\mu^{\frac{1}{p'}}\|f\|_{L^p(\mu;\mathbb{C})}\|\varphi\|_{\mathscr{S}^{(m_p+3)}(\mathbb{R};\mathbb{C})}.$$

□

Loosely related to the preceding is the following theorem of Schwartz. Given a $u \in \mathscr{S}(\mathbb{R};\mathbb{C})^*$, its *support* is the smallest closed set F such that $\langle\varphi, u\rangle = 0$ for all φ that are 0 on $F^\complement$. Equivalently, $\langle\varphi_1, u\rangle = \langle\varphi_2, u\rangle$ if $\varphi_1 = \varphi_2$ on an open set containing F.

Theorem 14.5. *If $u \in \mathscr{S}^{(-n+1)}(\mathbb{R};\mathbb{C})$, then u is supported on $\{0\}$ if and only if*

$$u = \sum_{m=0}^{n} a_m \partial^m \delta_0$$

for some $\{a_0, \dots, a_n\} \subseteq \mathbb{C}$.

Proof. The sufficiency statement is trivial. To prove the necessity assertion, first note that, by Theorem 13.2, there is a $C \in [0,\infty)$ such that $|\langle\varphi, u\rangle| \le C\|\varphi\|_{\mathrm{u}}^{(n)}$. Next, choose $\eta \in C^\infty\big(\mathbb{R};[0,1]\big)$ so that $\eta = 1$ on $[-1,1]$ and $\eta = 0$ off of $[-2,2]$, and define $\eta_r(x) = \eta\big(\frac{x}{r}\big)$ for $r \in (0,1]$. Because 0 is the support of u, $\langle\varphi, u\rangle = \langle\eta_r\varphi, u\rangle$ for all $r \in (0,1]$. In particular, this means that

$$|\langle\varphi, u\rangle| \le C\sum_{\ell=0}^{n}\|\eta_r\varphi^{(\ell)}\|_{\mathrm{u}}$$

for some other $C < \infty$.

We will now show that $\langle\varphi, u\rangle = 0$ if $\varphi(x) = x^{n+1}\eta(x)\psi(x)$ for some $\psi \in C^\infty(\mathbb{R};\mathbb{C})$. To this end, set $\varphi_r(x) = x^{n+1}\eta_r(x)\psi(x)$, and note $\langle\varphi, u\rangle = \langle\varphi_r, u\rangle$ for all $r \in (0,1]$. Next observe that $\partial^\ell\varphi_r$ is a linear combination of terms of the form

$$x^{n+1-i}r^{-j}\eta^{(j)}\big(\tfrac{x}{r}\big)\psi^{(k)}(x) = x^{n+1-i-j}\left(\frac{x}{r}\right)^j\eta^{(j)}\big(\tfrac{x}{r}\big)\psi^{(k)}(x)$$

where $i + j + k = \ell$. Since

$$\left|x^{n+1-i-j}\left(\frac{x}{r}\right)^j\eta^{(j)}\big(\tfrac{x}{r}\big)\psi^{(k)}(x)\right| \le r^{n+1-i-j}\|x^j\eta^{(j)}\|_{\mathrm{u}}\|\psi^{(k)}\|_{\mathrm{u}},$$

$\lim_{r\searrow 0}\|\varphi_r^{(\ell)}\|_{\mathrm{u}} = 0$ for $\ell \le n$, and so

$$\langle\varphi, u\rangle = \lim_{r\searrow 0}\langle\varphi_r, u\rangle = 0.$$

Now let $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{C})$ and use Taylor's theorem to write

$$\varphi(x) = \sum_{m=0}^{n} \frac{\varphi^{(m)}(0)}{m!} x^m + \frac{x^{n+1}}{n!} \int_0^1 (1-t)^n \varphi^{(n+1)}(tx)\, dt.$$

Set $\psi(x) = \frac{1}{n!}\int_0^1 (1-t)^n \varphi^{(n+1)}(tx)\, dt$, and apply the preceding to see that $\langle x^{n+1}\eta\psi, u\rangle = 0$ and therefore that

$$\langle \varphi, u\rangle = \langle \eta\varphi, u\rangle = \sum_{m=0}^{n} \frac{\varphi^{(m)}(0)}{m!} \langle x^m \eta, u\rangle.$$

Hence

$$u = \sum_{m=0}^{n} \frac{(-1)^m \langle x^m \eta, u\rangle}{m!} \partial^m \delta_0.$$

□

The next result characterizes distributions $u \in \mathscr{S}(\mathbb{R};\mathbb{C})^*$ which satisfy the *minimum principle*

(14.4) $\qquad \langle \varphi, u\rangle \ge 0$ if $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{R})$ and $\varphi(0) = \min\{\varphi(x) : x \in \mathbb{R}\}$.

In preparation for the proof of the characterization, I have to introduce the following partition of unity for $\mathbb{R} \setminus \{\mathbf{0}\}$. Choose $\psi \in C^\infty\big(\mathbb{R};[0,1]\big)$ so that ψ has compact support in $(0,2) \setminus \overline{(0,\frac14)}$ and $\psi(y) = 1$ when $\frac12 \le |y| \le 1$, and set $\psi_m(y) = \psi(2^m y)$ for $m \in \mathbb{Z}$. Then, if $y \in \mathbb{R}$ and $2^{-m-1} \le |y| \le 2^{-m}$, $\psi_m(y) = 1$ and $\psi_n(y) = 0$ unless $-m-2 \le n \le -m+1$. Hence, if $\Psi(y) = \sum_{m\in\mathbb{Z}} \psi_m(y)$ for $y \in \mathbb{R}\setminus\{0\}$, then Ψ is a smooth function with values in $[1,4]$; and therefore, for each $m \in \mathbb{Z}$, the function χ_m given by $\chi_m(0) = 0$ and $\chi_m(y) = \frac{\psi_m(y)}{\Psi(y)}$ for $y \in \mathbb{R}\setminus\{0\}$ is a smooth, $[0,1]$-valued function that vanishes off of $(0,2^{-m+1}) \setminus \overline{(0,2^{-m-2})}$. In addition, for each $y \in \mathbb{R}\setminus\{0\}$, $\sum_{m\in\mathbb{Z}} \chi_m(y) = 1$ and $\chi_m(y) = 0$ unless $2^{-m-2} \le |y| \le 2^{-m+1}$.

Lemma 14.6. *If $u \in \mathscr{S}(\mathbb{R};\mathbb{R})$ satisfies* (14.4)*, then there exists a unique Borel measure M on $\mathbb{R}$ such that $M(\{0\}) = 0$, $\int \frac{y^2}{1+y^2}\, M(dy) < \infty$, and*

$$\langle \varphi, u\rangle = \int \varphi(y)\, M(dy)$$

if φ, φ', and φ'' vanish at 0.

Proof. Referring to the partition of unity described above, define $\Lambda_m\varphi = \langle \chi_m\varphi, u\rangle$ for $\varphi \in C^\infty\big(\overline{(0,2^{-m+1})} \setminus (0,2^{-m-2});\mathbb{R}\big)$, where

$$\chi_m\varphi(y) = \begin{cases} \chi_m(y)\varphi(y) & \text{if } 2^{-m-2} \le |y| \le 2^{-m+1} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly Λ_m is linear. In addition, if $\varphi \ge 0$, then $\chi_m\varphi \ge 0 = \chi_m\varphi(0)$, and so, by (14.4), $\Lambda_m\varphi \ge 0$. Similarly, for any $\varphi \in C^\infty\big(\overline{(0,2^{-m+1})} \setminus (0,2^{-m-2});\mathbb{R}\big)$, $\|\varphi\|_{\mathrm{u}}\chi_m \pm \chi_m\varphi \ge 0 = \big(\|\varphi\|_{\mathrm{u}}\chi_m \pm \chi_m\varphi\big)(0)$, and therefore $|\Lambda_m\varphi| \le K_m\|\varphi\|_{\mathrm{u}}$, where $K_m = \langle \chi_m, u\rangle$. Hence, Λ_m admits a unique extension as a continuous linear functional on $C\big(\overline{(0,2^{-m+1})} \setminus (0,2^{-m-2});\mathbb{R}\big)$ that is non-negativity preserving and has norm K_m; and so, by the Riesz representation theorem, we know that there is a unique non-negative Borel measure M_m on $\mathbb{R}$ such that M_m is supported on $\overline{(0,2^{-m+1})} \setminus (0,2^{-m-2})$, $K_m = M_m(\mathbb{R})$, and $\langle \chi_m\varphi, u\rangle = \int_{\mathbb{R}} \varphi(y)\, M_m(dy)$ for all $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{R})$.

Now define the Borel measure M on $\mathbb{R}$ by $M = \sum_{m\in\mathbb{Z}} M_m$. Clearly, $M(\{0\}) = 0$. In addition, if $\varphi \in C_c^\infty\big(\mathbb{R}\setminus\{0\};\mathbb{R}\big)$, then there is an $n \in \mathbb{Z}$ such that $\chi_m\varphi \equiv 0$ unless $|m| \le n$. Thus,

$$\begin{aligned}\langle\varphi,u\rangle &= \sum_{m=-n}^{n} A(\chi_m\varphi) = \sum_{m=-n}^{n}\int_{\mathbb{R}}\varphi(y)\,M_m(dy)\\ &= \int_{\mathbb{R}^N}\left(\sum_{m=-n}^{n}\chi_m(y)\varphi(y)\right)M(dy) = \int_{\mathbb{R}^N}\varphi(y)\,M(dy),\end{aligned}$$

and therefore

$$\langle\varphi,u\rangle = \int_{\mathbb{R}}\varphi(y)\,M(dy) \tag{14.5}$$

for $\varphi \in C_c^\infty\big(\mathbb{R}\setminus\{\mathbf{0}\};\mathbb{R}\big)$.

Before taking the next step, observe that, as an application of (14.4), if $\varphi_1,\varphi_2 \in \mathscr{S}(\mathbb{R};\mathbb{R})$, then

$$\varphi_1 \le \varphi_2 \text{ and } \varphi_1(0) = \varphi_2(0) \implies \langle\varphi_1,u\rangle \le \langle\varphi_2,u\rangle. \tag{$*$}$$

Indeed, by linearity, this reduces to the observation that, by (14.4), if $\varphi \in \mathbb{D}$ is non-negative and $\varphi(0) = 0$, then $\langle\varphi,u\rangle \ge 0$.

With these preparations, we can show that, for any $\varphi \in \mathbb{D}$,

$$\varphi \ge 0 = \varphi(\mathbf{0}) \implies \int_{\mathbb{R}}\varphi(y)\,M(d\mathbf{y}) \le \langle\varphi,u\rangle. \tag{$**$}$$

To check this, apply $(*)$ to $\varphi_n = \sum_{m=-n}^{n}\chi_m\varphi$ and φ, and use (14.5) together with the monotone convergence theorem to conclude that

$$\int_{\mathbb{R}}\varphi(y)\,M(dy) = \lim_{n\to\infty}\int_{\mathbb{R}}\varphi_n(y)\,M(dy) = \lim_{n\to\infty}\langle\varphi_n,u\rangle \le \langle\varphi,u\rangle.$$

Now let $\eta \in C^\infty\big(\mathbb{R};[0,1]\big)$ satisfy $\eta = 0$ on $[-1,1]$ and $\eta = 0$ off $(-2,2)$, and set $\eta_R(y) = \eta(R^{-1}y)$ for $R > 0$. By $(**)$ with $\varphi(y) = |y|^2\eta(y)$ we know that

$$\int_{\mathbb{R}}|y|^2\eta(y)\,M(dy) \le \langle\varphi,u\rangle < \infty.$$

At the same time, by (14.5) and $(*)$,

$$\int_{\mathbb{R}^N}\big(1-\eta(y)\big)\eta_R(y)\,M(dy) \le \langle(\mathbf{1}-\eta),u\rangle$$

for all $R > 0$, and therefore, by Fatou's Lemma,

$$\int_{\mathbb{R}}\big(1-\eta(y)\big)\,M(dy) \le \langle(\mathbf{1}-\eta),u\rangle < \infty.$$

Hence, I have proved that

$$\int_{\mathbb{R}}\frac{y^2}{1+y^2}\,M(dy) < \infty. \tag{14.6}$$

We are now in a position to show that (14.5) continues to hold for any $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{R})$ that vanishes along with its first and second order derivatives at 0. To this end, first suppose that φ vanishes in a neighborhood of 0. Then, for each $R>0$, (14.5) applies to $\eta_R\varphi$, and so

$$\int_{\mathbb{R}} \eta_R(y)\varphi(y)\,M(dy) = \langle \eta_R\varphi, u\rangle = \langle \varphi, u\rangle + \langle (\mathbf{1}-\eta_R)\varphi, u\rangle.$$

Since $(1-\eta_R)\varphi \longrightarrow 0$ in $\mathscr{S}(\mathbb{R};\mathbb{R})$ as $R\to\infty$ and φ is M-integrable, Lebesgue's dominated convergence theorem implies that,

$$\langle \varphi, u\rangle = \lim_{R\to\infty}\int_{\mathbb{R}} \eta_R(y)\varphi(y)\,M(dy) = \int_{\mathbb{R}} \varphi(y)\,M(dy).$$

We still have to replace the assumption that φ vanishes in a neighborhood of 0 by the assumption that it vanishes to second order there. For this purpose, first note that, by (14.6), φ is certainly M-integrable, and therefore

$$\int_{\mathbb{R}^N} \varphi(y)\,M(dy) = \lim_{r\searrow 0}\langle (\mathbf{1}-\eta_r)\varphi, u\rangle = \langle \varphi, u\rangle - \lim_{r\searrow 0}\langle \eta_r\varphi, u\rangle.$$

By our assumptions about φ at 0, we can find a $C<\infty$ such that $|\eta_r\varphi(y)| \le Cry^2\eta(y)$ for all $r\in(0,1]$. Hence, by $(*)$ and the M-integrability of $y^2\eta(y)$, there is a $C'<\infty$ such that $\langle \eta_r\varphi, u\rangle \le C'r$ for small $r>0$, and therefore $\langle \eta_r\varphi, u\rangle \longrightarrow 0$ as $r\searrow 0$. □

Theorem 14.7. *If $u\in\mathscr{S}(\mathbb{R};\mathbb{R})$ satisfies* (14.4)*, then there exist an $a\ge 0$, a $b\in\mathbb{R}$, and Borel measure M on $\mathbb{R}$ such that $M(\{0\})=0$,* (14.6) *holds, and*

$$\langle \varphi, u\rangle = \tfrac{a}{2}\varphi''(0) + b\varphi'(0) + \int \big(\varphi(y)-\varphi(0)-\mathbf{1}_{[0,1]}(y)\varphi'(0)y\big)M(dy).$$

In fact, M is determined by

$$\langle \varphi, u\rangle = \int \varphi(y)\,M(dy) \text{ if } \varphi\in C^\infty_{\mathrm{c}}\big(\mathbb{R}\setminus\{0\}\big),$$

and, for any $\eta\in C^\infty\big(\mathbb{R};[0,1]\big)$ which is 1 on $[-1,1]$ and 0 off $(-2,2)$

$$a = \langle y^2\eta^2, u\rangle - \int y^2\eta(y)^2\,M(dy)$$

and

$$b = \langle y\eta, y\rangle - \int y\big(\eta(y)-\mathbf{1}_{[0,1]}(y)\big)\,M(dy).$$

Proof. Let η be as in the statement, and define

$$\psi(y) = \varphi(y)-\varphi(0)-\varphi'(0)y\eta(y)-\tfrac12\varphi''(0)y^2\eta(y)^2.$$

Then ψ vanishes to second order at 0, and so, by Lemma 14.6, $\langle \psi, u\rangle = \int \psi(y)\,M(dy)$. Hence,

$$\begin{aligned}\langle \varphi, u\rangle &= \varphi'(0)\langle y\eta, u\rangle + \tfrac12\varphi''(0)\langle y^2\eta^2, u\rangle \\ &\quad + \int \big(\varphi(y)-\varphi(0)-\varphi'(0)y\eta(y)-\tfrac12\varphi''(0)y^2\eta(y)^2\big)M(dy),\end{aligned}$$

and so

$$\begin{aligned}\langle \varphi, u\rangle &= \varphi'(0)\langle y\eta, u\rangle - \tfrac12\varphi''(0)\left(\langle y^2\eta, u\rangle - \int y^2\eta(y)^2\,M(dy)\right) \\ &\quad + \int \big(\varphi(y)-\varphi(0)-\varphi'(0)y\eta(y)\big)M(dy).\end{aligned}$$

Finally, because $y\big(\eta(y)-\mathbf{1}_{[-1.1]}(y)\big)$ vanishes on $[-1,1]$ and is therefore M-integrable, we can replace $\varphi'(0)\langle y\eta, u\rangle$ by

$$\varphi'(0)\Big(\langle y\eta, u\rangle - \int y\big(\eta(y) - \mathbf{1}_{[-1,1]}(y)\big)\Big)M(dy)$$

and $\int\big(\varphi(y) - \varphi(0) - \varphi'(0)y\eta(y)\big)M(dy)$ by

$$\int\big(\varphi(y) - \varphi(0) - \varphi'(0)y\mathbf{1}_{[-1,1]}(y)\big)M(dy).$$

□

Exercise 14.8. Let $f \in C^1_{\mathrm{b}}(\mathbb{R};\mathbb{C})$, set $u = f(|x|)$, and show that $u' = \mathrm{sgn}(x)f'(|x|)$. Next assume that $f \in C^2_{\mathrm{b}}(\mathbb{R};\mathbb{C})$, and show that $u'' = f'(|x|)\delta_0 + f''(|x|)$.