## Lecture 15: Extending Continuous Operators on $\mathscr{S}(\mathbb{R};\mathbb{C})$ to $\mathscr{S}(\mathbb{R};\mathbb{C})^*$

The extension that we made of the operators $\mathcal{H}^s$ to $\mathscr{S}(\mathbb{R};\mathbb{C})^*$ is a special case of the fact that many continuous linear operator on $\mathscr{S}(\mathbb{R};\mathbb{C})$ determine a unique continuous operator from $\mathscr{S}(\mathbb{R};\mathbb{C})^*$ itself. To explain how this done, suppose that $A$ is a continuous operator on $\mathscr{S}(\mathbb{R};\mathbb{C})$ and let $A^*$ be its formal adjoint. That is, for $\psi \in \mathscr{S}(\mathbb{R};\mathbb{C})$ define $A^*\psi \in \mathscr{S}(\mathbb{R};\mathbb{C})^*$ so that

$$\langle \varphi, A^*\psi\rangle = (A\varphi,\psi)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}.$$

Of course, when, like $\mathcal{H}^s$, $A$ on $\mathscr{S}(\mathbb{R};\mathbb{C})$ is symmetric with respect to the $L^2$-inner product, $A^* = A$.

**Theorem 15.1.** *Let $A$ be a continuous operator on $\mathscr{S}(\mathbb{R};\mathbb{C})$, assume that $A^*$ maps $\mathscr{S}(\mathbb{R};\mathbb{C})$ continuously into itself, and define $Au$ for $u \in \mathscr{S}(\mathbb{R};\mathbb{C})^*$ so that*

$$\langle \varphi, Au\rangle = \langle A^*\varphi, u\rangle.$$

*Then $A$ is the unique extention of $A$ as a continuous operator on $\mathscr{S}(\mathbb{R};\mathbb{C})^*$ into itself.*

*Proof.* Because $A^*$ maps $\mathscr{S}(\mathbb{R};\mathbb{C})$ continuously into itself, for each $m \ge 0$ there exists an $n \ge 0$ and $C < \infty$ such that $\|A^*\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})} \le C\|\varphi\|_{\mathscr{S}^{(n)}(\mathbb{R};\mathbb{C})}$, and therefore, if $u \in \mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})$, then

$$|\langle\varphi, Au\rangle| = |\langle A^*\varphi, u\rangle| \le \|A^*\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}\|u\|_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})} \le C\|\varphi\|_{\mathscr{S}^{(n)}(\mathbb{R};\mathbb{C})}\|u\|_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})}.$$

Hence $\|Au\|_{\mathscr{S}^{(-n)}(\mathbb{R};\mathbb{C})} \le C\|u\|_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})}$, and so $A$ maps $\mathscr{S}(\mathbb{R};\mathbb{C})^*$ continuously into itself. Furthermore, since $\mathscr{S}(\mathbb{R};\mathbb{C})$ is dense in $\mathscr{S}(\mathbb{R};\mathbb{C})^*$ and $\langle\varphi, A\psi\rangle = \big(A^*\varphi,\psi\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}$ for $\psi \in \mathscr{S}(\mathbb{R};\mathbb{C})$, $A$ is the one and only continuous extension to $\mathscr{S}(\mathbb{R};\mathbb{C})^*$ of $A \restriction \mathscr{S}(\mathbb{R};\mathbb{C})$. □

Given a continuous operator $A$ on $\mathscr{S}(\mathbb{R};\mathbb{C})^*$ and $m,n \in \mathbb{Z}$

$$\|A\|_{\mathscr{S}^{(n)}(\mathbb{R};\mathbb{C})\to\mathscr{S}(\mathbb{R};\mathbb{C})(m)} = \sup\big\{|Au\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})} : \|u\|_{\mathscr{S}^{(n)}(\mathbb{R};\mathbb{C})} = 1\big\}.$$

The argument given in the proof of Theorem 15.1 shows that, for $m,n \in \mathbb{N}$,

$$\|A\|_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})\to\mathscr{S}^{(-n)}(\mathbb{R};\mathbb{C})} = \|A^*\|_{\mathscr{S}^{(n)}(\mathbb{R};\mathbb{C})\to\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}. \tag{15.1}$$

The Fourier transform is a particularly important operator on $\mathscr{S}(\mathbb{R};\mathbb{C})^*$, and its adjoint is given by $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{C}) \longmapsto \check{\varphi} \in \mathscr{S}(\mathbb{R};\mathbb{C})$. Hence

$$\langle\varphi, u\rangle = \langle\check{\varphi}, u\rangle,$$

and, since $\|\check{\varphi}\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})} = (2\pi)^{\frac{1}{2}}\|\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}$ for all $m \ge 0$, (15.1) says that $\|\hat{u}\|_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})} = (2\pi)^{\frac{1}{2}}\|u\|_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})}$ for all $m \ge 0$. In addition,

$$\langle\varphi, u\rangle = (2\pi)^{-1}\langle(\hat{\varphi})^\wedge, u\rangle = (2\pi)^{-1}\langle\hat{\varphi},\hat{u}\rangle,$$

which gives an extension of Parseval's identity to the Fourier transform on $\mathscr{S}(\mathbb{R};\mathbb{C})^*$. Further, because $\hat{\varphi}$ that adjoint of $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{C}) \longmapsto \check{\varphi} \in \mathscr{S}(\mathbb{R};\mathbb{C})$, $\langle\varphi,\hat{u}\rangle = \langle\hat{\varphi}, u\rangle$ and therefore

$$\langle\varphi, (\hat{u})^\wedge\rangle = \langle\hat{\varphi},\hat{u}\rangle = 2\pi\langle\varphi, u\rangle,$$

similarly, $\langle\varphi, (\check{u})^\vee\rangle = 2\pi\langle\varphi, u\rangle$. Hence we have proved the Fourier inversion formula

$$(\hat{u})^\wedge = 2\pi u = (\check{u})^\vee.$$

Computing Fourier transforms can be hard! Among those that are easy are those of $\partial^\ell\delta_a$ and, for $f\in L^1(\lambda_{\mathbb{R}};\mathbb{C})$, $\widehat{f}$. Indeed,

$$\langle\varphi,\hat\delta_a\rangle=\check\varphi(a)=\int e^{-\imath ax}\varphi(x)\,dx=\langle\varphi,e_a\rangle,\quad\text{where } e_a(x)=e^{\imath ax}.$$

Hence, $\widehat{\partial^\ell\delta_a}=(-\imath\xi)^\ell e_a$. To compute $\hat f$ when $f$ is thought of as the distribution $f\lambda_R$, note that

$$\langle\check\varphi,f\rangle=\int\bar f(\xi)\Big(\int e^{-\imath\xi x}\varphi(x)\,dx\Big)\,d\xi=\int\varphi(x)\overline{\hat f(x)}\,dx=\langle\varphi,\hat f\rangle,$$

and therefore $\widehat{f\lambda_{\mathbb{R}}}$ equals the distribution is $\hat f\lambda_{\mathbb{R}}$. When $f\in L^2(\lambda_{\mathbb{R}};\mathbb{C})$, one uses the fact that, as $R\to\infty$, $\mathbf{1}_{[-R,R]}f\longrightarrow f$ in $\mathscr{S}(\mathbb{R};\mathbb{C})^*$ and therefore $\widehat{f}=\hat f$ where $\hat f=\lim_{R\to\infty}\widehat{f_R}$ is the $L^2$-Fourier transform of $f$. Similarly, when $\mu$ is a finite Borel measure on $\mathbb{R}$, $\hat\mu$ as a distribution is equal to the function $\hat\mu$ given by

$$\hat\mu(\xi)=\int e^{\imath\xi x}\,\mu(dx). \tag{15.2}$$

Trickier is the computation of the Fourier transform of distributions like $\log|x|$. One way to do so is to observe that $\partial\log|x|=\frac{1}{x}$ and first compute $\widehat{x^{-1}}$. For that purpose, set $f_y(x)=\frac{x}{x^2+y^2}$ for $y>0$, and observe that, as $y\searrow 0$, $f_y\longrightarrow x^{-1}$ and therefore $\widehat{f_y}\longrightarrow\widehat{x^{-1}}$ in $\mathscr{S}(\mathbb{R};\mathbb{C})^*$. Next observe that observe that, by (7.11),

$$\widehat{f_y}(\xi)=\lim_{R\to\infty}\int_{-R}^R\frac{xe^{\imath\xi x}}{x^2+y^2}\,dx=\imath\lim_{R\to\infty}\int_{-R}^R\frac{x\sin x}{x^2+y^2}\,dx=\imath\pi\mathrm{sgn}(\xi)e^{-y|\xi|}.$$

Hence

$$\widehat{x^{-1}}=\imath\pi\mathrm{sgn}. \tag{15.3}$$

Knowing (15.3) one might expect that one can use $\widehat{\partial u}=-\imath\xi\hat u$ to compute $\widehat{\log|x|}$. However to do so it is necessary to confront a technical difficulty. Namely, $\frac{\imath\pi\mathrm{sgn}(\xi)}{-\imath\xi}=-\frac{\pi}{|\xi|}$, and $|\xi|^{-1}$ is *not* a distribution. On the other hand,

$$\varphi\rightsquigarrow\int\frac{\varphi(\xi)-\varphi(0)e^{-\frac{\xi^2}{2}}}{|\xi|}\,d\xi$$

is a distribution. Thus, to overcome the problem, set $u=\log|x|$ and write

$$\langle\varphi,\hat u\rangle=\langle\varphi-\varphi(0)\widehat{g_1},\hat u\rangle+\varphi(0)\langle\widehat{g_1},\hat u\rangle.$$

and note that $\langle\widehat{g_1},\hat u\rangle=2\pi\int g_1(x)\log|x|\,dx$. At the same time,

$$\begin{aligned}\langle\varphi-\varphi(0)\widehat{g_1},\hat u\rangle&=\left\langle\frac{\varphi-\varphi(0)e^{-\frac{\xi^2}{2}}}{\imath\xi},-\imath\xi\hat u\right\rangle\\&=\left\langle\frac{\varphi-\varphi(0)e^{-\frac{\xi^2}{2}}}{\imath\xi},\widehat{\partial u}\right\rangle=-\pi\left\langle\frac{\varphi-\varphi(0)e^{-\frac{\xi^2}{2}}}{|\xi|},\lambda_{\mathbb{R}}\right\rangle.\end{aligned}$$

Hence

$$\langle\varphi,\widehat{\log|x|}\rangle=-\pi\int\frac{\varphi(\xi)-\varphi(0)e^{-\frac{\xi^2}{2}}}{|\xi|}\,d\xi+2\pi\varphi(0)\int g_1(x)\log|x|\,dx.$$

Next, consider a differential operator $L = \sum_{j=0}^J a_j \partial^j$ where $\{a_0, \dots, a_J\} \subseteq C^\infty(\mathbb{R}; \mathbb{C})$ and all the $a_j$'s and their derivatives have at most polynomial growth. Then

$$L^*\varphi = \sum_{j=0}^{J} (-1)^j \partial^j (a_j \varphi).$$

Since it a obvious that $\partial^j$ maps $\mathscr{S}^{(m)}(\mathbb{R}; \mathbb{C})$ continuously into $\mathscr{S}^{(m+j)}(\mathbb{R}; \mathbb{C})$, to see that $L^*$ is continuous we need the following.

**Lemma 15.2.** *Let $f \in C^\infty(\mathbb{R}; \mathbb{R})$, and assume that for each $m \geq 0$ there exists an $k_m \geq 0$ such that*

$$F_m \equiv \max_{1 \leq j \leq m} \sup_{x \in \mathbb{R}} \frac{|\partial^j f(x)|}{1 + |x|^{k_m}} < \infty.$$

*Then, for each $m \geq 0$,*

$$\|\varphi f\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})} \leq 2K_m F_m \|\varphi\|_{\mathscr{S}^{(m+k_m)}(\mathbb{R};\mathbb{C})}.$$

*Proof.* By Exercise 3.5 with $n = 0$, it is sufficient for us to show that for each $k, \ell \in \mathbb{N}$ with $k + \ell \leq m$, there is a $c_{k,\ell}$ such that

$$\|x^k \partial^\ell (\varphi\psi)\|_{L^2(\lambda_\mathbb{R};\mathbb{C})} \leq c_{k,\ell} \|\psi\|_{\mathscr{S}^{(m+3)}(\mathbb{R};\mathbb{C})} \|\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}.$$

To this end, remember that

$$\partial^\ell(\varphi\psi) = \sum_{j=0}^{\ell} \binom{\ell}{j} \partial^j \varphi \partial^{\ell-j} \psi,$$

and

$$\|x^k \partial^j \varphi \partial^{\ell-j} f\|_{L^2(\lambda_\mathbb{R};\mathbb{C})} \leq F_m \big\|(1 + |x|^{k_m}) x^k \partial^j \varphi\|_{L^2(\lambda_\mathbb{R};\mathbb{C})} \leq 2F_m \|\varphi\|_{\mathscr{S}^{(m+k_m)}(\mathbb{R};\mathbb{C})}\big\|.$$

□

Knowing the result in Lemma , it is clear that $L^*$ maps $\mathscr{S}^{(m)}(\mathbb{R}; \mathbb{C})$ continuously into $\mathscr{S}^{(m+J)}(\mathbb{R}; \mathbb{C})$ for each $m \geq 0$. Using this fact, it is easy to check that $\widehat{\partial u} = \imath\xi\hat{u}$. Indeed, both sides of the equation are continuous functions of $u \in \mathscr{S}(\mathbb{R}; \mathbb{C})^*$, and the equation holds when $u \in \mathscr{S}(\mathbb{R}; \mathbb{C})$.

Another important operation is convolution. That is, given $\psi \in \mathscr{S}(\mathbb{R}; \mathbb{C})$, consider the operator $\mathcal{C}_\psi$ on $\mathscr{S}(\mathbb{R}; \mathbb{C})$ given by $\mathcal{C}_\psi \eta = \eta * \psi$. Because $\widehat{\eta * \psi} = \hat{\eta}\hat{\psi}$, Lemma 15 guarantees that $\mathcal{C}_\psi$ maps $\mathscr{S}^{(m)}(\mathbb{R}; \mathbb{C})$ continuously into itself for all $m \geq 0$. In addition,

$$\langle \varphi, \psi * \eta \rangle = \iint \varphi(x) \bar{\psi}(x - y) \bar{\eta}(y)\, dx dy = \iint \varphi(x + y) \bar{\psi}(x) \bar{\eta}(y)\, dx dy = \langle \mathcal{C}_\psi^* \varphi, \eta \rangle$$

where

$$\mathcal{C}_\psi^* \varphi(y) = \int \varphi(x + y) \bar{\psi}(x)\, dx.$$

Since $\widehat{\mathcal{C}_\psi^* \varphi}(\xi) = \hat{\varphi}(\bar{\psi})^\vee$, Lemma 15 again guarantees that, for all $m \geq 0$, $\mathcal{C}_\psi^*$ maps $\mathscr{S}^{(m)}(\mathbb{R}; \mathbb{C})$ continuously into itself, and so $\mathcal{C}_\psi$ has a unique extention to $\mathscr{S}(\mathbb{R}; \mathbb{C})^*$, and this extention is a continuous map of $\mathscr{S}^{(m)}(\mathbb{R}; \mathbb{C})$ into itself for all $m \in \mathbb{Z}$.

In order to gain a better understanding of $\mathcal{C}_\psi$, we need to use the translation maps $\tau_x : \mathscr{S}(\mathbb{R}; \mathbb{C}) \longrightarrow \mathscr{S}(\mathbb{R}; \mathbb{C})$ defined in Exercise 13.10, and define $\psi * u(x) = \langle \tau_{-x}\psi, u \rangle$ for $u \in \mathscr{S}(\mathbb{R}; \mathbb{C})^*$ and $x \in \mathbb{R}$.

**Theorem 15.3.** *For $\psi \in \mathscr{S}(\mathbb{R};\mathbb{C})$ and $u \in \mathscr{S}(\mathbb{R};\mathbb{C})$, $\psi * u = \mathcal{C}_\psi u$, $\widehat{\psi * u} = \hat{\psi}\hat{u}$, and $\psi * u = (2\pi)^{-1}(\hat{\psi}\hat{u})^\vee$ .*

*Proof.* Since $\mathcal{C}_\psi \eta = \psi * \eta$ and $\widehat{\psi * \eta} = \hat{\psi}\hat{\eta}$ when $\eta \in \mathscr{S}(\mathbb{R};\mathbb{C})$ and $\mathcal{C}_\psi$ is a continuous operator on $\mathscr{S}(\mathbb{R};\mathbb{C})^*$, it suffices to to show that $u \rightsquigarrow \psi * u$ is a continuous of $\mathscr{S}(\mathbb{R};\mathbb{C})^*$ into itself. To this end, note that, by Exercise 13.10, $x \rightsquigarrow \tau_{-x}\psi$ is a continuous map of $\mathscr{S}(\mathbb{R};\mathbb{C})$ into itself and therefore that $\psi * u$ is a continuous function. In addition, by Theorem 13.2 and that exercise,

$$\begin{aligned}|\psi * u(x)| &\le \|\tau_x\psi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}\|u\|_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})} \le K_m\|\tau_x\psi\|_{\mathrm{u}}^{(m+1)}\|u\|_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})}\\ &\le K_m 2^{(m+1)}(|x|\vee 1)^{m+1}\|\psi\|_{\mathrm{u}}^{(m+1)}\|u\|_{\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})},\end{aligned}$$

and so $\psi * u \in \mathscr{S}(\mathbb{R};\mathbb{C})^*$. Finally, if $u_n \longrightarrow u$ in $\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})$, then $\psi * u_n(x) \longrightarrow \psi * u(x)$ for each $x$ and

$$\sup_{n\ge 1}\sup_{x\in\mathbb{R}} \frac{|\psi * u_n(x)|}{(1+|x|)^{m+1}} < \infty.$$

Hence, by Lebesgue's dominated convergence theorem, for each $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{C})$,

$$\langle \varphi, \psi * u_n\rangle = \int \varphi(x)\overline{\psi * u_n(x)}\,dx = \int \varphi(x)\overline{\psi * u(x)}\,dx = \langle \varphi, \psi * u\rangle.$$

□

A simple, but typical, application of these results is to the ordinary differential equation $\lambda u - u'' = \mu$, where $\lambda > 0$ and $\mu$ is a finite Borel measure on $\mathbb{R}$. The solution $u$ to this equation describes the electric potential along a wire produced by a charge distribution $\mu$ when the wire has resistance that is a linear function of the potential. To solve this equation, assume that $u \in \mathscr{S}(\mathbb{R};\mathbb{C})^*$, and take the Fourier transform of both sides. Then $\lambda\hat{u} + \xi^2\hat{u} = \hat{\mu}$, and so $\hat{u} = \frac{\hat{\mu}}{\lambda+\xi^2}$. Next observe (cf. (7.5)) that $\frac{1}{\lambda+\xi^2} = \widehat{G_\lambda}$, where

$$G_\lambda(x) = \frac{1}{2\lambda^{\frac12}}e^{-\lambda^{\frac12}|x|}.$$

Even though $G_\lambda \notin \mathscr{S}(\mathbb{R};\mathbb{C})$, the function $x \rightsquigarrow G_\lambda * \mu(x) = \int G_\lambda(x-y)\,\mu(dy)$ is an element of $L^1(\lambda_{\mathbb{R}};\mathbb{C})$ and therefore of $\mathscr{S}(\mathbb{R};\mathbb{C})^*$. In addition, by Fubini's theorem, $\widehat{G_\lambda * \mu} = \widehat{G_\lambda}\hat{\mu}$, and therefore

$$u(x) = \frac{1}{2\lambda^{\frac12}}\int e^{-\lambda^{\frac12}|x-y|}\,\mu(dy).$$

It is an instructive exercise to check that this $u$ is a solution. To this end, first use Exercise 3.6 below to see that $u'$ is the function

$$u'(x) = \frac{\lambda^{\frac12}}{2}\int \mathrm{sgn}(y-x)e^{-\lambda^{\frac12}|x-y|}\,dy.$$

Thus

$$\begin{aligned}\langle \varphi, u''\rangle = -\langle \varphi', u'\rangle &= \int \varphi'(x)\left(\frac12\int \mathrm{sgn}(x-y)e^{-\lambda^{\frac12}|x-y|}\varphi'(y)\,\mu(dy)\right)dx\\ &= \int\left(\frac12\int \mathrm{sgn}(x-y)e^{-\lambda^{\frac12}|x-y|}\varphi'(x)\,dx\right)\mu(dy).\end{aligned}$$

Next note that

$$\int \operatorname{sgn}(x-y)e^{-\lambda^{\frac{1}{2}}|x-y|}\varphi'(x)\,dx = \int_y^\infty e^{\lambda^{\frac{1}{2}}(y-x)}\varphi'(x)\,dx - \int_{-\infty}^y e^{\lambda^{\frac{1}{2}}(x-y)}\varphi'(x)\,dx$$

$$= -\varphi(y) + \lambda^{\frac{1}{2}}\int_y^\infty e^{\lambda^{\frac{1}{2}}(y-x)}\,dx - \varphi(y) + \lambda^{\frac{1}{2}}\int_{-\infty}^y e^{\lambda^{\frac{1}{2}}(x-y)}\,dx = -2\varphi(y) + 2\lambda u(y),$$

and therefore $\langle\varphi, u''\rangle = -\langle\varphi,\mu\rangle + \lambda\langle\varphi, u\rangle$, which means that $\lambda u - u'' = \mu$.

**Exercise 15.4.** This exercise deals with the special case when an element of $\mathscr{S}(\mathbb{R};\mathbb{C})^*$ is given by a Borel measure $\mu$.

(**i**) Show that $\psi * \mu$ equals the function

$$x \in \mathbb{R} \longmapsto \int \psi(x-y)\,\mu(dy) \in \mathbb{C}.$$

(**ii**) If $\mu$ is finite, show that $\hat{\mu}$ equals the function

$$\xi \in \mathbb{R} \longmapsto \hat{\mu}(\xi) \equiv \int e^{\imath\xi x}\,\mu(dx) \in \mathbb{C}$$

and that $\hat{\mu} \in C_{\mathrm{b}}(\mathbb{R};\mathbb{C})$ with norm $\|\hat{u}\|_{\mathrm{u}} = \mu(\mathbb{R})$.

**(iii)** If $\int(1+x^2)^{\frac{m}{2}}\,\mu(dx) < \infty$ for some $m \geq 0$, show that $\hat{\mu} \in C^m_{\mathrm{b}}(\mathbb{R};\mathbb{C})$ and that

$$\|\partial^k\hat{\mu}\|_{\mathrm{u}} \leq \int |x|^k\,\mu(dx) \text{ for } 0 \leq k \leq m.$$

(**iv**) Assume that $\int |x|^k\,\mu(dx) < \infty$ for all $k \in \mathbb{N}$, and show that $\psi * \mu$ is an element of $\mathscr{S}(\mathbb{R};\mathbb{C})$ for all $\psi \in \mathscr{S}(\mathbb{R};\mathbb{C})$.

**Hint**: Show that $\widehat{\psi * \mu}$ is an element of $\mathscr{S}(\mathbb{R};\mathbb{C})$.

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