Lecture 16: Moving to $\mathbb{R}^N$

With essentially no new ideas and the introduction of only slightly uglier notation, we will transfer most of the contents of §§7–15 to $\mathbb{R}^N$.

If $f \in L^1(\mathbb{R}^N;\mathbb{C})$, its Fourier transform is the function

$$\hat{f}(\boldsymbol{\xi}) = \int e^{\imath(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N}} f(\mathbf{x})\,d\mathbf{x},$$

and, using exactly the same arguments as we did when $N=1$, one can easily show that $\|\hat{f}\|_{\mathrm{u}} \le \|f\|_{L^1(\lambda_{\mathbb{R}^N};\mathbb{C})}$, $\hat{f}$ is continuous and that if $f \in C^1(\mathbb{R}^N;\mathbb{C}) \cap L^1(\lambda_{\mathbb{R}^N};\mathbb{C})$ and $f' \in L^1(\lambda_{\mathbb{R}^N};\mathbb{C})$, then $\widehat{\partial_{x_j} f}(\boldsymbol{\xi}) = -\imath\xi_j \hat{f}(\boldsymbol{\xi})$ for $1 \le j \le N$, from which it follows that $\hat{f}(\boldsymbol{\xi}) \longrightarrow 0$ as $|\boldsymbol{\xi}| \to \infty$.

To develop an inversion formula, one introduces the functions

$$g_t(\mathbf{x}) = (2\pi t)^{-\frac{N}{2}} e^{-\frac{|\mathbf{x}|^2}{2t}},$$

uses Fubini's theorem to check that $\widehat{g_t}(\boldsymbol{\xi}) = e^{-\frac{t|\boldsymbol{\xi}|^2}{2}}$, and proceeds as before to see first that

$$\int g_t(\mathbf{x}-\mathbf{y}) f(\mathbf{y})\,d\mathbf{y} = (2\pi)^{-N} \int e^{-\frac{t|\boldsymbol{\xi}|^2}{2}} e^{-\imath(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N}} \hat{f}(\boldsymbol{\xi})\,d\boldsymbol{\xi},$$

and then that, as $t \searrow 0$,

$$(2\pi)^{-N} \int e^{-\frac{t|\boldsymbol{\xi}|^2}{2}} e^{-\imath(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N}} \hat{f}(\boldsymbol{\xi})\,d\mathbf{x} \text{ converges to } \begin{cases} f & \text{in } L^1(\lambda_{\mathbb{R}^N};\mathbb{C}) \\ f(\mathbf{x}) & \text{if } f \text{ is continuous at } \mathbf{x}. \end{cases}$$

The normalized Hermite functions on $\mathbb{R}^N$ are indexed by $\mathbf{m} = (m_1,\dots,m_N) \in \mathbb{N}^N$ and defined by

$$\tilde{h}_{\mathbf{m}}(\mathbf{x}) = \tilde{h}_{n_1}(x_1)\cdots\tilde{h}_{n_N}(x_N).$$

By standard results about products of Hilbert spaces, one knows that they form an orthonormal basis in $L^2(\lambda_{\mathbb{R}^N};\mathbb{C})$. In addition, if

$$\mathcal{H} = |\mathbf{x}|^2 - \Delta = \sum_{j=1}^N \big(x_j^2 - \partial_{x_j}^2\big),$$

then

$$\mathcal{H}\tilde{h}_{\mathbf{m}} = \mu_{\mathbf{m}}\tilde{h}_{\mathbf{m}} \text{ where } \mu_{\mathbf{m}} = \sum_{j=1}^N \mu_{m_j}$$

and

$$(\tilde{h}_{\mathbf{m}})^\wedge = \imath^{\|\mathbf{m}\|_1}(2\pi)^{\frac{N}{2}}\tilde{h}_{\mathbf{m}} \text{ where } \|\mathbf{m}\|_1 = \sum_{j=1}^N m_j.$$

Finally, the estimates in (11.2) can be used to show that

$$\|\tilde{h}_{\mathbf{m}}\|_{L^1(\lambda_{\mathbb{R}^N};\mathbb{C})} \le \left(\prod_{j=1}^N \big(2\pi(m_j+1)\big)\right)^{\frac{1}{2}}, \ \|\tilde{h}_{\mathbf{m}}\|_{\mathrm{u}} \le \left(\prod_{j=1}^N (m_j+1)\right)^{\frac{1}{2}} \text{ and}$$
$$\|x_j\tilde{h}_{\mathbf{m}}\|_{\mathrm{u}} \vee \|\partial_{x_j}\tilde{h}_{\mathbf{m}}\|_{\mathrm{u}} \le 2^N \prod_{j=1}^N (m_j+1). \tag{16.1}$$

Therefore, exactly the same reasoning as we used in §12 shows that the Fourier transform can be extended to $L^2(\lambda_{\mathbb{R}^N};\mathbb{C})$ as a continuous operator that satisfies

$$\widehat{\partial_{x_j} f} = -\imath \xi_j \hat{f} \text{ if } f \in C^1(\mathbb{R}^N;\mathbb{C}) \text{ and } f \text{ and } \partial_{x_j} f \text{ are in } L^1(\lambda_{\mathbb{R}^N};\mathbb{C}) \cup L^2(\mathbb{R}^N;\mathbb{C}).$$

and the Parseval equality

$$\big(\hat{f},\hat{g}\big)_{L^2(\lambda_{\mathbb{R}^N};\mathbb{C})} = (2\pi)^N (f,g)_{L^2(\lambda_{\mathbb{R}^N};\mathbb{C})} \text{ for } f,g \in L^2(\lambda_{\mathbb{R}};\mathbb{C}).$$

The Schwartz test function space $\mathscr{S}(\mathbb{R}^N;\mathbb{C})$ for $\mathbb{R}^N$ is defined as the space of $\varphi \in C^\infty(\mathbb{R}^N;\mathbb{C})$ with the property that $\|x_i^k \partial_{x_j}^\ell \varphi\|_{\mathrm{u}} < \infty$ for all $1 \le i,j \le N$ and $k,\ell \in \mathbb{N}$. Again one introduces the operators

$$\mathcal{H}^s \varphi = \sum_{\mathbf{k}\in\mathbb{N}^N} \mu_{\mathbf{k}}^s (\varphi, \tilde{h}_{\mathbf{k}})_{L^2(\lambda_{\mathbb{R}^N};\mathbb{C})} \tilde{h}_{\mathbf{k}}$$

and defines the norms

$$\|\varphi\|_{\mathrm{u}}^{(m)} = \sum_{\substack{1\le i,j\le N\\ k+\ell\le m}} \|x_i \partial_{x_j}\varphi\|_{\mathrm{u}}$$

and

$$\|\varphi\|_{\mathscr{S}(\mathbb{R};\mathbb{C})^{(m)}(\mathbb{R}^N;\mathbb{C})} = \sum_{\mathbf{k}\in\mathbb{N}^N} \mu_{\mathbf{k}}^m |(\varphi,\tilde{h}_{\mathbf{k}})_{L^2(\lambda_{\mathbb{R}^N};\mathbb{C})}|^2,$$

and the spaces

$$\mathscr{S}^{(m)}(\mathbb{R}^N;\mathbb{C}) = \{\varphi \in L^2(\lambda_{\mathbb{R}^N};\mathbb{C}) : \|\varphi\|_{\mathscr{S}(\mathbb{R};\mathbb{C})^{(m)}} < \infty\}.$$

Clearly, if $\varphi \in C^m(\mathbb{R}^N;\mathbb{C})$, then $\|\varphi\|_{\mathscr{S}(\mathbb{R};\mathbb{C})^{(m)}} = \|\mathcal{H}^{\frac{m}{2}}\varphi\|_{L^2(\lambda_{\mathbb{R}^N};\mathbb{C})}$.

Using the estimates in (16.1) and the reasoning in Lemma 13.1 and Theorem 13.2, one sees that, for each m there is a $K_m \in (0,\infty)$ such that

$$\|\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R}^N;\mathbb{C})} \le K_m \|\varphi\|_{\mathrm{u}}^{(m+N)}$$

and

$$\|\varphi\|_{\mathrm{u}}^{(m)} \le K_m \|\varphi\|_{\mathscr{S}^{(m+3N)}(\mathbb{R}^N;\mathbb{C})}.$$

Hence, $\mathscr{S}(\mathbb{R}^N;\mathbb{C}) = \bigcap_{m=0}^\infty \mathscr{S}^{(m)}(\mathbb{R}^N;\mathbb{C})$ and $\mathscr{S}(\mathbb{R}^N;\mathbb{C})^*$ can be identified as the union $\bigcup_{m=0}^\infty \mathscr{S}^{(-m)}(\mathbb{R}^N;\mathbb{C})$ where $\mathscr{S}^{(-m)}(\mathbb{R}^N;\mathbb{C})$ is the analog for $N \ge 2$ of $\mathscr{S}^{(-m)}(\mathbb{R};\mathbb{C})$ for $N=1$. Further, the obvious analogs of Theorems 14.3 and 14.5 hold. In proving the analogs of Theorems 14.5 and 14.7, one needs to use the $\mathbb{R}^N$ version of Taylor's theorem which says that

$$\varphi(\mathbf{x}) = \sum_{m=0}^n \sum_{\|\mathbf{k}\|_1=m} \frac{\partial^{\mathbf{k}}\varphi(0)}{\mathbf{k}!}\mathbf{x}^{\mathbf{k}} + \frac{1}{n!}\sum_{\|\mathbf{k}\|_1=n+1} \binom{n+1}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \int_0^1 (1-t)^n \partial^{\mathbf{k}}\varphi(t\mathbf{x})\,dt,$$

where $\mathbf{k}! = \prod_{j=1}^N k_j$, $\mathbf{x}^{\mathbf{k}} = \prod_{j=1}^N x_j^{k_j}$, $\partial^{\mathbf{k}} = \prod_{j=1}^N \partial_{x_j}^{k_j}$, and $\binom{n+1}{\mathbf{k}}$ is the multinomial coefficient $\frac{(n+1)!}{\mathbf{k}!}$.

Once one has the preceding, it should be clear how to extend continuous operators on $\mathscr{S}(\mathbb{R}^N;\mathbb{C})$ to continuous operators on $\mathscr{S}(\mathbb{R}^N;\mathbb{C})^*$. In particular, both the Fourier transform and convolution have such extensions.

To demonstrate its use, consider again the example discussed at the end of §15, only now its analog $au - \Delta u = \mu$ in $\mathbb{R}^N$, where $\lambda > 0$ and μ is a finite Borel measure on $\mathbb{R}^N$. Just as before, the Fourier transform of this equation lead to the conclusion that $\hat{u} = \frac{\hat{\mu}}{\lambda+|\boldsymbol{\xi}|^2}$. To find the function G_λ of which $(\lambda + |\boldsymbol{\xi}|^2)^{-1}$ is the Fourier transform, note that

$$\frac{1}{\lambda + |\boldsymbol{\xi}|^2} = \int_0^\infty e^{-t(a+|\boldsymbol{\xi}|)^2}\,dt = \int_0^\infty e^{-\lambda t}\widehat{g_{2t}}(\boldsymbol{\xi})\,dt,$$

from which it follows that

$$G_\lambda(\mathbf{x}) = \int_0^\infty e^{-\lambda t} g_{2t}(\mathbf{x})\,dt = (4\pi)^{-\frac{1}{2}} \int_0^\infty t^{-\frac{N}{2}} e^{-\lambda t - \frac{|\mathbf{x}|^2}{4}}\,dt.$$

The function G_λ is a Bessel function, and a more explicit expression for it is easy to obtain only when N is odd. For example, when $N = 1$, we already knew that $G_\lambda(x) = \frac{1}{2\lambda^{-\frac{1}{2}}} e^{-\lambda^{\frac{1}{2}}|x|}$, and when $N = 3$, after differentiating (7.6) with respect to x, one sees that

$$G_\lambda(\mathbf{x}) = \frac{e^{-\lambda^{\frac{1}{2}}|\mathbf{x}|}}{2\pi|\mathbf{x}|}.$$

In any case, it is clear that $G_\lambda \in L^1(\lambda_{\mathbb{R}^N};\mathbb{C})$ and therefore that

$$u(\mathbf{x}) = \int G_\lambda(\mathbf{x} - \mathbf{y})\,\mu(d\mathbf{y}).$$

The *Poisson problem* $\Delta u = -\mu$ is a closely related to the preceding. The Fourier equivalent equation is $|\boldsymbol{\xi}|^2\hat{u} = \hat{\mu}$, which means that $\hat{u} = \frac{\hat{\mu}}{|\boldsymbol{\xi}|^2}$, and so one has to figure out for which μ's $\frac{\hat{\mu}}{|\boldsymbol{\xi}|^2} \in \mathscr{S}(\mathbb{R}^N;\mathbb{C})^*$. A further complication is that, even if a solution exists, it will not be unique. Indeed, given any solution u, $u + v$ will also be a solution for any $v \in \mathscr{S}(\mathbb{R}^N;\mathbb{C})^*$ which is *harmonic* (i.e., $\Delta v = 0$). Notice that if v is harmonic, then $|\boldsymbol{\xi}|^2\hat{v} = 0$, and therefore $\{\mathbf{0}\}$ is the support of $\hat{v}$, which, by Theorem 14.5 means that $\hat{v}$ is a linear combination of derivatives of δ_0 and therefore that v is a polynomial. Thus, $v \in \mathscr{S}(\mathbb{R};\mathbb{C})^*$ is harmonic harmonic if and only if $v = ax + b$, but when $N \geq 2$ there are harmonic polynomials of all orders. For example, the real part of any complex polynomial will be a harmonic element of $\mathscr{S}(\mathbb{R}^2;\mathbb{C})$.

As for the question of existence of solutions, when $N = 1$ one can check that if $\int(1 + |x|)\,\mu(x) < \infty$, then $x \rightsquigarrow \int (x - y)^-\,\mu(dx)$ is a solution. When $N = 2$, one can use Green's formula and the divergence theorem to show that

$$\int \Delta\varphi(\mathbf{x}) \log|\mathbf{x} - \mathbf{y}|\,d\mathbf{x} = 2\pi\varphi(\mathbf{y})$$

foor $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{C})$, and therefore, if $G_0(\mathbf{y}) \equiv -\frac{1}{2\pi}\log|\mathbf{y}|$ and

$$\iint \big|G_0(\mathbf{x} - \mathbf{y})\big|\,d\mathbf{x}\mu(d\mathbf{y}) < \infty, \tag{$*$}$$

then $G_0 * \mu$ is a solution. When $N \geq 3$, one should look for the tempered distribution of which $|\boldsymbol{\xi}|^{-2}$ is the Fourier transform. To that end, observe that

$$\frac{1}{|\boldsymbol{\xi}|^2} = \int_0^\infty e^{-t|\boldsymbol{\xi}|^2}\,dt = \int_0^\infty \widehat{g_{2t}}(\boldsymbol{\xi})\,dt,$$

and so $|\boldsymbol{\xi}|^{-2}$ is the Fourier transform of

$$G_0(\mathbf{x}) = (4\pi)^{-\frac{N}{2}} \int_0^\infty t^{-\frac{N}{2}} e^{-\frac{|\mathbf{x}|^2}{4t}}\,dt = \frac{1}{4\pi^{\frac{N}{2}}|\mathbf{x}|^{N-2}} \int_0^\infty t^{\frac{N}{2}-2} e^{-t}\,dt = \frac{\Gamma\big(\frac{N-2}{2}\big)}{4\pi^{\frac{N}{2}}|\mathbf{x}|^{N-2}},$$

where Γ is Euler's gamma function. Because $\Gamma\left(\frac{N}{2}\right) = \frac{N-2}{2}\Gamma\left(\frac{N-2}{2}\right)$ and $\frac{2\Gamma\left(\frac{N}{2}\right)}{\pi^{\frac{N}{2}}}$ is the area ω_{N-1} of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$, we have that

$$G_0(\mathbf{x}) = \frac{1}{(N-2)\omega_{N-1}|\mathbf{x}|^{N-2}}.$$

Thus, $G_0 * \mu$ is a solution if $(*)$ holds. The function G_0 is called the *Green's function* for the Laplacian in $\mathbb{R}^N$.

Exercise 16.1. Show that if f is an entire function on $\mathbb{C}$ (i.e., an analytic function there), then, as a function on $\mathbb{R}^2$ it is tempered distribution if and only if it is a polynomial. Conclude that if an entire function is not a polynomial, then it grows at infinity faster that any power of z.

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