Define $\mathbf{M}_1(\mathbb{R}^N)$ to be the set of Borel probability measures on $\mathbb{R}^N$. Clearly $\mathbf{M}_1(\mathbb{R}^N)$ is a convex subset of $\mathscr{S}(\mathbb{R}^N;\mathbb{C})^*$, but it is a subset that possesses properties that are not shared by most other elements of $\mathscr{S}(\mathbb{R}^N;\mathbb{C})^*$, and the topology of $\mathscr{S}(\mathbb{R}^N;\mathbb{C})^*$ does not take full advantage of those properties. There are three stronger topologies that recommend themselves. Namely: the *uniform* topology, which is the one for which[7]

$$\|\mu-\nu\|_{\text{var}} \equiv \sup\bigl\{|\langle\varphi,\mu-\nu\rangle| : \varphi \text{ a Borel measurable function with } \|\varphi\|_{\text{u}}=1\bigr\}$$

is the metric; the *strong* for which sets of the form

$$S(\mu,r;\varphi_1,\dots,\varphi_n)=\bigl\{\nu : |\langle\varphi_m,\nu-\mu\rangle|<r \text{ for } 1\le m\le n\bigr\},$$

where φ_m's are bounded Borel measurable $\mathbb{R}$-valued functions on $\mathbb{R}^N$, are a neighborhood basis for μ; and the *weak* for which sets of the $S(\mu,r;\varphi_1,\dots,\varphi_n)$ are a neighborhood basis for μ, only now with the restriction that φ_m's must be continuous as well as bounded.

Obviously, the strength of the uniform topology is greater than that of the strong topology, which is stronger than the weak topology, which, at first sight, looks stronger than the one which $\mathbf{M}_1(\mathbb{R}^N)$ inherits as a subset of $\mathscr{S}(\mathbb{R}^N;\mathbb{C})^*$. Each of them has its virtues and flaws. The uniform topology admits a metric and is the strong topology on the dual space of the Banach space $C_0(\mathbb{R}^N;\mathbb{R})$ with the uniform topology; the strong topology is not separable and points don't have countable neighborhood bases; as we will show below, the weak topology is both separable and admits a metric, and it is the one which is most useful in practice.

In what follows, we will study some of the properties and applications of the weak topology.

Lemma 17.1. *The sets $S(\mu,r;\varphi_1,\dots,\varphi_n)$ with $\varphi_1,\dots,\varphi_n\in C_c^\infty(\mathbb{R}^N;\mathbb{R})$ are a neighborhood basis at μ for the weak topology.*

Proof. We begin by showing if that $\varphi\in C_b^\infty(\mathbb{R};\mathbb{C})$ with $\|\varphi\|_{\text{u}}=1$ and $r>0$, then there exist $\varphi_1,\varphi_2\in C_c^\infty(\mathbb{R}^N;\mathbb{C})$ such that

$$\bigl\{\nu : |\langle\varphi_1,\nu-\mu\rangle|\vee|\langle\varphi_2,\nu-\mu\rangle|<\tfrac{r}{4}\bigr\}\subseteq\{\nu : |\langle\varphi,\nu-\mu\rangle|<r\}.$$

To this end, choose $R>0$ so that $\mu\bigl(B(\mathbf{0},R)\bigr)>1-\frac{r}{4}$, and take $\eta\in C^\infty(\mathbb{R}^N;\mathbb{R})$ so that $\eta=1$ on $\overline{B(\mathbf{0},R)}$ and $\nu=0$ off $B(\mathbf{0},R+1)$. Then

$$|\langle\varphi,\nu-\mu\rangle|\le|\langle\eta\varphi,\nu-\mu\rangle|+|\langle(1-\eta)\varphi,\nu-\mu\rangle|$$

and

$$\begin{aligned}|\langle(1-\eta)\varphi,\nu-\mu\rangle|&\le\langle 1-\eta,\mu\rangle+\langle 1-\eta,\nu\rangle\\&\le 2\langle 1-\eta,\mu\rangle+|\langle 1-\eta,\nu-\mu\rangle|=2\langle 1-\eta,\mu\rangle+|\langle\eta,\nu-\mu\rangle|.\end{aligned}$$

Thus

$$|\langle(1-\eta)\varphi,\nu-\mu\rangle|\le|\langle\eta\varphi,\nu-\mu\rangle|+2\mu\bigl(B(\mathbf{0},R)^{\complement}\bigr)+|\langle\eta,\nu-\mu\rangle|,$$

and so

$$\bigl\{\nu : |\langle\eta\varphi,\nu-\mu\rangle|\vee|\langle\eta,\nu-\mu\rangle|<\tfrac{r}{4}\bigr\}\subseteq\{\nu : |\langle\varphi,\nu-\mu\rangle|<r\}.$$

[7] We will continue to use $\langle\varphi,\mu\rangle$ to denote the integral with respect to μ of a function φ, even if $\varphi\notin\mathscr{S}(\mathbb{R}^N;\mathbb{C})$. Also, $\langle\varphi,\nu-\mu\rangle\equiv\langle\varphi,\nu\rangle-\langle\varphi,\mu\rangle$.

In view of the preceding, it suffices to show that if $\varphi \in C_c(\mathbb{R}^N;\mathbb{C})$ with $\|\varphi\|_u = 1$ and $r > 0$, then there exists a $\psi \in C_c^\infty(\mathbb{R}^N;\mathbb{C})$ such

$$|\langle \psi, \nu - \mu \rangle| < \tfrac{r}{3} \implies |\langle \varphi, \nu - \mu \rangle| < r.$$

To this end, simply choose $\psi \in C_c^\infty(\mathbb{R}^N;\mathbb{C})$ so that $\|\varphi - \psi\|_u < \frac{r}{3}$, and check that this ψ will serve. □

As Lemma 17.1 makes clear, what we are calling the weak topology on $\mathbf{M}_1(\mathbb{R}^N)$ is what a functional analyst would call the weak* topology on the dual space $C_0(\mathbb{R}^N;\mathbb{R})^*$ of the Banach space $C_0(\mathbb{R}^N;\mathbb{R})$ with the uniform norm. Indeed, the Riesz representation theorem allows one to identify $C_0(\mathbb{R}^N;\mathbb{R})$ with the space of finite signed Borel measures on $\mathbb{R}^N$, and so $\mathbf{M}_1(\mathbb{R}^N)$ can be thought of as a convex subset of the unit ball in $C_0(\mathbb{R}^N;\mathbb{R})^*$, in which case Lemma 17.1 shows that the weak topology on $\mathbf{M}_1(\mathbb{R}^N)$ is the topology $\mathbf{M}_1(\mathbb{R}^N)$ inherits as a subset from the weak* topology on $C_0(\mathbb{R}^N;\mathbb{R})^*$.

Theorem 17.2. *The weak topology on $\mathbf{M}_1(\mathbb{R}^N)$ is a separable, metric topology.*

Proof. Let $\{\varphi_k : k \ge 1\}$ be a dense subset of $C_c(\mathbb{R}^N;\mathbb{R})$, and define

$$\rho(\mu,\nu) = \sum_{k=1}^\infty \frac{|\langle \varphi_k, \nu - \mu \rangle|}{2^k(1 + |\langle \varphi_k, \nu - \mu \rangle|)}.$$

Using Lemma 17.1, it is easy to check that φ is a metric for the weak topology on $\mathbf{M}_1(\mathbb{R}^N)$.

To prove separability, define D to be the set of measures $\sum_{m=1}^n a_m \delta_{\mathbf{x}_m}$, where $n \ge 1$, the a_m's are non-negative rational numbers whose sum is 1, and the $\mathbf{x}_m$'s are elements of $\mathbb{R}^N$ with rational coordinates. Clearly D is countable. Therefore it suffices to show that, for each $\mu \in \mathbf{M}_1(\mathbb{R}^N)$, each cellection $\{\varphi_1, \dots, \varphi_\ell\} \subseteq C_b(\mathbb{R}^N;\mathbb{R})$, and $\epsilon > 0$, there is a $\nu \in D$ such that $\max_{1 \le k \le \ell} |\langle \varphi_k, \nu - \mu \rangle| < \epsilon$. Further, we need do so only for φ_k's and a μ which are supported on a ball $B(\mathbf{0}, R)$. Given such a φ_k's and μ, choose $r > 0$ so that $\max_{1 \le k \le \ell} |\varphi_k(\mathbf{y}) - \varphi_k(\mathbf{x})| < \frac{\epsilon}{2}$ if $|\mathbf{y} - \mathbf{x}| < r$. Next, cover $\overline{B(\mathbf{0}, R)}$ with balls $B(\mathbf{x}_m, r)$, where $1 \le m \le n$, each $\mathbf{x}_m \in B(\mathbf{0}, R)$ and has rational coordinates, and define $A_1 = B(\mathbf{x}_1, r)$ and $A_m = B(x_m, r) \setminus \bigcup_{k=1}^{m-1} A_k$ for $2 \le m \le n$. Finally, choose non-negative, rational numbers $a_1, \dots, a_n$ so that

$$\max_{1 \le k \le \ell} \|\varphi_k\|_u \sum_{m=1}^n |a_m - \mu(A_m)| < \frac{\epsilon}{2}$$

and $\sum_{m=1}^n a_m = 1$, and take $\nu = \sum_{m=1}^n a_m \delta_{x_m}$. Then, for $1 \le k \le \ell$,

$$|\langle \varphi_k, \mu - \nu \rangle| \le \sum_{m=1}^n \int_{A_m} |\varphi_k(\mathbf{x}) - \varphi(\mathbf{x}_m)|\, d\mu + \|\varphi_k\|_u \sum_{m=1} |\mu(A_m) - a_m| < \epsilon.$$

□

We will use the notation $\mu_n \xrightarrow{\mathrm{w}} \mu$ to mean that $\mu_n \longrightarrow$ in the weak topology on $\mathbf{M}_1(\mathbb{R}^N)$.

Theorem 17.3. *Given $\{\mu_n : n \ge 1\} \cup \{\mu\} \subseteq \mathbf{M}_1(\mathbb{R}^N)$, the following are equivalent:*

(i) $\mu_n \xrightarrow{\mathrm{w}} \mu$.

(ii) $|\langle \varphi, \mu_n - \mu \rangle| \longrightarrow 0$ *for all* $\varphi \in C_c^\infty(\mathbb{R}^N;\mathbb{R})$.

(**iii**) *For all closed sets* $F \subseteq \mathbb{R}^N$, $\overline{\lim}_{n\to\infty} \mu_n(F) \le \mu(F)$.

(**iv**) *For all open sets* $G \subseteq \mathbb{R}^N$, $\underline{\lim}_{n\to\infty} \mu_n(G) \ge \mu(G)$.

(**v**) *For all upper continuous functions* $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ *that are bounded above,* $\overline{\lim}_{n\to\infty}\langle f, \mu_n\rangle \le \langle f, \mu\rangle$.

(**vi**) *For all lower continuous functions* $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ *that are bounded below,* $\underline{\lim}_{n\to\infty}\langle f, \mu_n\rangle \ge \langle f, \mu\rangle$.

Finally, if $\Gamma \in \mathcal{B}$ *and the boundary* $\partial\Gamma$ *has* μ*-measure* 0*, then* $\mu_n \xrightarrow{\text{w}} \mu \implies \mu(\Gamma) = \lim_{n\to\infty} \mu_n(\Gamma)$.

Proof. We already proved in Lemma 17.1 the equivalence of (**i**) and (**ii**), and the equivalence of (**iii**) and (**iv**) as well as that of (**v**) and (**vi**) is obvious. In addition, it is clear that (**v**) together with (**vi**) implies (**i**). Thus, we need only check that (**i**) implies (**iii**) and that (**iv**) implies (**vi**).

Assume that $\mu_n \xrightarrow{\text{w}} \mu$. Given a closed set F, define $\varphi_k(x) = 1 - \left(\frac{|x-F|}{1+|x-F|}\right)^{\frac{1}{k}}$. Then $\varphi_k \in C\big(\mathbb{R}^N; [0,1]\big)$ and $\varphi_k \searrow \mathbf{1}_F$ as $k \to \infty$. Hence, for all k,

$$\langle \varphi_k, \mu\rangle = \lim_{n\to\infty} \langle \varphi_k, \mu_n\rangle \ge \overline{\lim_{n\to\infty}}\, \mu_n(F),$$

and so $\mu(F) = \lim_{k\to\infty}\langle \varphi_k, \mu\rangle \ge \overline{\lim}_{n\to\infty} \mu_n(F)$. Thus (**i**) $\implies$ (**iii**).

In proving that (**iv**) implies (**vi**), it suffices to handle f's which are positive as well as lower semicontinuous. Given such an f, define

$$f_k = \sum_{j=1}^{\infty} \frac{j \wedge 4^k}{2^k} \mathbf{1}_{I_{j,k}} \circ f = \frac{1}{2^k} \sum_{j=1}^{4^k} \mathbf{1}_{J_{j,k}} \circ f,$$

where

$$I_{j,k} = \left(\frac{j}{2^k}, \frac{j+1}{2^k}\right] \text{ and } J_{j,k} = \left(\frac{j}{2^k}, \infty\right).$$

Then $0 \le f_k \nearrow f$ as $k \to \infty$. In addition, because f is lower semicontinuous, the sets $G_{j,k} = \{x : f(x) \in J_{j,k}\}$ are open. Hence, if (**iv**) holds, then, for all k,

$$\langle f_k, \mu\rangle \le \underline{\lim_{n\to\infty}}\, \langle f_k, \mu_n\rangle \le \underline{\lim_{n\to\infty}}\, \langle f, \mu_n\rangle,$$

and so

$$\langle f, \mu\rangle = \lim_{k\to\infty} \langle f_k, \mu\rangle \le \underline{\lim_{n\to\infty}}\, \langle f, \mu_n\rangle.$$

To prove the concluding assertion, assume $\mu_n \xrightarrow{\text{w}} \mu$ and that $\mu(\partial\Gamma) = 0$. Set $G = \mathring{\Gamma}$ and $F = \bar{\Gamma}$. Then

$$\mu(\Gamma) = \mu(G) \le \underline{\lim_{n\to\infty}}\, \mu_n(G) \le \underline{\lim_{n\to\infty}}\, \mu_n(\Gamma)$$

and

$$\mu(\Gamma) = \mu(F) \ge \overline{\lim_{n\to\infty}}\, \mu_n(F) \ge \overline{\lim_{n\to\infty}}\, \mu_n(\Gamma),$$

and so $\mu(\Gamma) = \lim_{n\to\infty} \mu_n(\Gamma)$. □

Another useful fact about weak convergence is the following.

Theorem 17.4. *Assume that* $\mu_n \xrightarrow{\text{w}} \mu$*, let* $\psi \in C\big(\mathbb{R}^N; [0,\infty)\big)$ *be an element of* $L^1(\mu; \mathbb{R})$ *as well as of* $\bigcap_{n=1}^{\infty} L^1(\mu_n; \mathbb{R})$*. Then* $\langle \psi, \mu\rangle \le \underline{\lim}_{n\to\infty}\langle \psi, \mu_n\rangle$*. In addition, if* $\{\varphi_n : n \ge 1\} \subseteq C(\mathbb{R}^N; \mathbb{R})$*,* $|\varphi_n| \le \psi$ *for all* $n \ge 1$*, and* $\langle \psi, \mu_n\rangle \longrightarrow \langle \psi, \mu\rangle$*, then* $\langle \varphi_n, \mu\rangle \longrightarrow \langle \varphi, \mu\rangle$ *if* $\varphi_n \longrightarrow \varphi$ *uniformly on compact subsets.*

Proof. Clearly,

$$\langle \psi \wedge R, \mu\rangle = \lim_{n\to\infty} \langle \psi \wedge R, \mu_n\rangle \le \varliminf_{n\to\infty} \langle \psi, \mu_n\rangle$$

for all $R > 0$, and so $\langle \psi, \mu\rangle \le \varliminf_{n\to\infty}\langle \psi, \mu_n\rangle$.

Now suppose that $\langle \psi, \mu_n\rangle \longrightarrow \langle \psi, \mu\rangle$, that $|\varphi_n| \le \psi$, and that $\varphi_n \longrightarrow \varphi$ uniformly on compact subsets. Clearly

$$|\langle \varphi_n, \mu_n\rangle - \langle \varphi, \mu\rangle| \le |\langle \varphi_n - \varphi, \mu_n\rangle| + |\langle \varphi, \mu - \mu_n\rangle|.$$

For each $R > 0$, choose $\eta_R \in C^\infty\big(\mathbb{R}^N; [0,1]\big)$ so that $\eta_R = 1$ on $\overline{B(\mathbf{0}, R)}$ and $\eta_R = 0$ off $B(\mathbf{0}, R+1)$. Then, for each $R > 0$,

$$\begin{aligned}
&\varlimsup_{n\to\infty} |\langle \varphi_n - \varphi, \mu_n\rangle| \\
&\le \varlimsup_{n\to\infty} \sup_{|x|\le R+1} |\varphi_n(x) - \varphi(x)|\langle \eta_R, \mu_n\rangle + \varlimsup_{n\to\infty} |\langle (1-\eta_R)(\varphi_n - \varphi), \mu_n\rangle| \\
&\le 2 \varlimsup_{n\to\infty} \langle (1-\eta_R)\psi, \mu_n\rangle = 2\langle (1-\eta_R)\psi, \mu\rangle,
\end{aligned}$$

and, by Lebesgue's dominated convergence theorem, the last expression tends to 0 as $R \to \infty$. Similarly,

$$\begin{aligned}
&\varlimsup_{n\to\infty} |\langle \varphi, \mu_n - \mu\rangle| \\
&\le \varlimsup_{n\to\infty} |\langle \eta_R\varphi, \mu_n - \mu\rangle| + \varlimsup_{n\to\infty} \langle (1-\eta_R)\psi, \mu_n\rangle + \langle (1-\eta_R)\psi, \mu\rangle = 2\langle (1-\eta_R)\psi, \mu\rangle,
\end{aligned}$$

and so $\varlimsup_{n\to\infty} |\langle \varphi, \mu_n - \mu\rangle| = 0$. □

We will next investigate when a subset of $\mathbf{M}_1(\mathbb{R}^N)$ is relatively compact. Because the unit ball in the dual space of a Banach is compact in the weak* topology, a careless functional analyst might think that $\mathbf{M}_1(\mathbb{R}^N)$ is itself compact. However, although $\mathbf{M}_1(\mathbb{R}^N)$ is closed in the strong topology on $C_0(\mathbb{R}^N;\mathbb{R})^*$, it is *not* closed in the weak* topology. Indeed, the sequence $\{\delta_n : n \ge 1\} \subseteq \mathbf{M}_1(\mathbb{R})$ is weak* convergent to measure whose total mass is 0, which is not an element of $\mathbf{M}_1(\mathbb{R})$. As this example indicates, in order for the weak* limit of a sequence $\{\mu_n : n \ge 1\} \subseteq \mathbf{M}_1(\mathbb{R}^N)$ to be in $\mathbf{M}_1(\mathbb{R}^N)$, one needs to know that the mass of the μ_n's is not escaping to infinity. With that in mind, we will say that a subset A of $\mathbf{M}_1(\mathbb{R}^N)$ is *tight* if, for each $\epsilon \in (0,1)$, there exists an $R \in [0,\infty)$ such that

$$\inf_{\mu\in A} \mu\big(\overline{B(\mathbf{0}, R)}\big) \ge 1 - \epsilon.$$

Theorem 17.5. *A subset $A \subseteq \mathbf{M}_1(\mathbb{R}^N)$ is relatively compact in the weak topology if and only if it is tight.*

Proof. Assume that A is tight, and let $\{\mu_n : n \ge 1\} \subseteq \mathbf{M}_1(\mathbb{R}^N)$. As pointed out above, there is a subsequence of $\{\mu_n : n \ge 1\} \subseteq \mathbf{M}_1(\mathbb{R}^N)$ which is weak* convergent in $C_0(\mathbb{R}^N;\mathbb{R})^*$ to a $\nu \in C_0(\mathbb{R}^N;\mathbb{R})^*$ which is a non-negative measure with total mass less than or equal to 1, and so, without loss in generality, we will assume that $\{\mu_n : n \ge 1\}$ is weak* convergent to ν. In order to check that $\nu(\mathbb{R}^N) = 1$, for any $\epsilon \in (0,1)$ choose R so that $\inf_{n\ge 1} \mu_n\big(\overline{B(\mathbf{0}, R)}\big) \ge 1 - \epsilon$, and choose $\eta \in C\big(\mathbb{R}^N; [0,1]\big)$ so that $\eta = 1$ on $\overline{B(\mathbf{0}, R)}$ and $\eta = 0$ off $B(\mathbf{0}, R+1)$. Then

$$\nu(\mathbb{R}^N) \ge \nu\big(\overline{B(\mathbf{0}, R+1)}\big) \ge \langle \eta, \nu\rangle = \lim_{n\to\infty} \langle \eta, \mu_n\rangle \ge \varlimsup_{n\to\infty} \mu_n\big(\overline{B(\mathbf{0}, R)}\big) \ge 1 - \epsilon,$$

and so $\nu(\mathbb{R}^N)$ must be 1.

Conversely, suppose that $A \subseteq \mathbf{M}_1(\mathbb{R}^N)$ is relatively compact in the weak topology. If A were not tight, then there would exist a $\theta \in [0,1)$ and, for each $n \geq 1$, a $\mu_n \in A$ such that $\mu_n\big(\overline{B(\mathbf{0},n)}\big) \leq \theta$, and, because A is relatively compact, we could assume that $\mu_n \xrightarrow{\mathrm{w}} \mu$ for some $\mu \in \mathbf{M}_1(\mathbb{R}^N)$. But if $\eta_m \in C\big(\mathbb{R}^N;[0,1]\big)$ equals 1 on $\overline{B(\mathbf{0},m)}$ and 0 off of $B(\mathbf{0},m+1)$, that would mean that, for all $m \geq 1$,

$$\mu\big(B(\mathbf{0},m)\big) \leq \langle \eta_m, \mu\rangle = \lim_{n\to\infty} \langle \eta_m, \mu_n\rangle \leq \varliminf_{n\to\infty} \mu_n\big(B(\mathbf{0},n)\big) \leq \theta,$$

and so $\mu(\mathbb{R}^N)$ would have to be less than or equal to $\theta < 1$. □

Exercise 17.6. Show that $\mu_n \xrightarrow{\mathrm{w}} \mu$ if and only if $\mu_n \longrightarrow \mu$ in $\mathscr{S}(\mathbb{R}^N;\mathbb{C})^*$.

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