The convolution product turns $\mathbf{M}_1(\mathbb{R}^N)$ into a commutative ring in which $\delta_{\mathbf{0}}$ is the identity. A $\mu \in \mathbf{M}_1(\mathbb{R}^N)$ is said to be *infinitely divisible* in this ring if, for each $n \geq 1$, there exists a $\mu_{\frac{1}{n}} \in \mathbf{M}_1(\mathbb{R}^N)$ such that

$$\mu = \mu_{\frac{1}{n}}^{*n} \equiv \underbrace{\mu_{\frac{1}{n}} * \cdots * \mu_{\frac{1}{n}}}_{n \text{ times}},$$

and the set $\mathcal{I}(\mathbb{R}^N)$ of infinitely divisible measures is an important source of building blocks for constructions in probability theory.

For probabililists, an element of $\mathcal{I}(\mathbb{R}^N)$ is the distribution of a random variable which, for each $n \geq 1$, can be written as the sum of n identically distributed random variables. Using commutativity, it is easy to check that set $\mathcal{I}(\mathbb{R}^N)$ of infinitely divisible measures is a subring of $\mathbf{M}_1(\mathbb{R}^N)$.

A famous theorem of Lévy and A. Khinchine describes the characteristic function of any element of $\mathcal{I}(\mathbb{R}^N)$. Namely, $\mu \in \mathcal{I}(\mathbb{R}^N)$ if and only if

$$\begin{aligned} \hat{\mu}(\boldsymbol{\xi}) = \exp\Bigg(& i(\mathbf{b}, \boldsymbol{\xi})_{\mathbb{R}^N} - \frac{1}{2}(\boldsymbol{\xi}, A\boldsymbol{\xi})_{\mathbb{R}^N} \\ & + \int \Big(e^{i(\boldsymbol{\xi}, y)_{\mathbb{R}^N}} - 1 - i\mathbf{1}_{B(0,1)}(y)(\boldsymbol{\xi}, y)_{\mathbb{R}^N} \Big) M(dy) \Bigg), \end{aligned} \tag{19.1}$$

for some $\mathbf{b} \in \mathbb{R}^N$, non-negative definite, symmetric $A \in \mathrm{Hom}(\mathbb{R}^N; \mathbb{R}^N)$, and Borel measure M on $\mathbb{R}^N$ such that $M(\{0\}) = 0$ and $\int \frac{|\mathbf{y}|^2}{1+|\mathbf{y}|^2} M(d\mathbf{y}) < \infty$. The expression in (19.1) is called the *Lévy–Khinchine formula*, a measure M satisfying the stated conditions is called a *Lévy measure*, and the triple $(\mathbf{b}, A, M)$ is called a *Lévy system*. It is clear that if the right hand side of (19.1) is a characteristic function for every Lévy system, then these are characteristic functions of infinitely divisible laws. Indeed, if μ corresponds to (b, A, M) and $\mu_{\frac{1}{n}}$ corresponds to $\left(\frac{\mathbf{b}}{n}, \frac{A}{n}, \frac{M}{n}\right)$, then $\hat{\mu} = (\widehat{\mu_{\frac{1}{n}}})^n$.

Proving that the function $f_{(\mathbf{b},A,M)}$ on the right hand side of (19.1) is a characteristic function is a relatively easy. To wit, $f_{(0,\mathbf{I},0)} = \hat{\gamma}$, where γ is the standard Gaussian measure on $\mathbb{R}^N$, and so it is easy to check that $f_{\mathbf{b},A,0}$ is the characteristic function of the distribution of $\mathbf{x} \rightsquigarrow \mathbf{b} + A^{\frac{1}{2}}\mathbf{x}$ under γ. Also, if the Lévy measure M is finite and π_M is the Poisson measure given by

$$\pi_M = e^{-M(\mathbb{R}^N)} \sum_{n=0}^{\infty} \frac{M^{*n}}{n!}, \tag{19.2}$$

then

$$\widehat{\pi_M}(\boldsymbol{\xi}) = e^{-M(\mathbb{R}^N)} \sum_{n=0}^{\infty} \frac{\hat{M}(\boldsymbol{\xi})^n}{n!} = e^{-M(\mathbb{R}^N) + \hat{M}(\boldsymbol{\xi})} = \exp\left(\int \left(e^{\imath(\boldsymbol{\xi}, \mathbf{y})_{\mathbb{R}^N}} - 1 \right) M(d\mathbf{y}\right),$$

and so $\widehat{\pi_M} = f_{(\mathbf{b}_M, 0, M)}$, where $b_M = \int_{B(0,1)} \mathbf{y}\, M(d\mathbf{y})$. Hence, when M is finite, $f_{(\mathbf{b},A,M)}$ is the characteristic function of $\gamma_{\mathbf{b}-\mathbf{b}_M, A} * \pi_M$. Finally, for general Lévy measures M, set $M_k(d\mathbf{y}) = \mathbf{1}_{[\frac{1}{k},\infty)}(|\mathbf{y}|)M(dy)$. Then M_k is finite, and so $f_{(\mathbf{b},A,M_k)}$ is a characteristic function. Therefore, since $f_{(\mathbf{b},A,M_k)} \longrightarrow f_{(\mathbf{b},A,M)}$ uniformly on compact subsets, Theorem 18.2 says that $f_{(\mathbf{b},A,M)}$ is a characteristic function.

There are no easy proofs that the characteristic function of any $\mu \in \mathcal{I}(\mathbb{R}^N)$ is given by (19.1). The first step is to show that if $\mu \in \mathcal{I}(\mathbb{R}^N)$, then there is a unique $\ell \in C(\mathbb{R}^N;\mathbb{C})$ such that $\ell(\mathbf{0}) = 0$, $\frac{|\ell(\boldsymbol{\xi})|}{1+|\boldsymbol{\xi}|^2}$ is bounded, and $\hat{\mu}(\boldsymbol{\xi}) = e^{\ell(\boldsymbol{\xi})}$. Showing that ℓ exists and is unique comes down to showing that $\hat{\mu}$ never vanishes. To do that, choose $r > 0$ so that $|1-\hat{\mu}(\boldsymbol{\xi})| \le \frac{1}{2}$ when $|\boldsymbol{\xi}| \le r$. Then there is an ℓ for which $\ell(0) = 0$, $|\ell(\xi)| \le 2$, and $\hat{\mu}(\boldsymbol{\xi}) = e^{\ell(\boldsymbol{\xi})}$ if $|\boldsymbol{\xi}| \le r$. Using $\log z = -\sum_{n=1}^\infty \frac{(1-z)^n}{n}$ when $|1-z| < 1$, one sees that $|\ell(\boldsymbol{\xi})| \le 2$ for $|\boldsymbol{\xi}| < r$.

Since $\widehat{\mu_{\frac{1}{n}}}(\xi)^n = \hat{\mu}(\boldsymbol{\xi})$, $\widehat{\mu_{\frac{1}{n}}}(\boldsymbol{\xi}) \neq 0$ when $|\boldsymbol{\xi}| \le r$, and so, by uniqueness, it must be that $\widehat{\mu_{\frac{1}{n}}(\boldsymbol{\xi})} = e^{\frac{\ell(\boldsymbol{\xi})}{n}}$ for $|\boldsymbol{\xi}| \le r$, and therefore $|1-\widehat{\mu_{\frac{1}{n}}}(\boldsymbol{\xi})| \le \frac{2}{n}$ when $|\boldsymbol{\xi}| \le r$. Hence, by (18.2), for any $R > 0$,

$$\mu_{\frac{1}{n}}\big(\{\mathbf{y} : |\mathbf{y}| \ge R\}\big) \le \frac{2N}{ns(rN^{-\frac{1}{2}}R)},$$

and so

$$|1-\widehat{\mu_{\frac{1}{n}}(\boldsymbol{\xi})}| \le \int \big|1-e^{\imath(\boldsymbol{\xi},\mathbf{y})}\big|\,\mu_{\frac{1}{n}}(d\mathbf{y}) \le |\boldsymbol{\xi}|R + 2\mu_{\frac{1}{n}}\big(\{\mathbf{y} : |\mathbf{y}| \ge R\}\big) \le |\boldsymbol{\xi}|R + \frac{2N}{ns(rN^{-\frac{1}{2}}R)}.$$

Given $\boldsymbol{\xi} \neq \mathbf{0}$, take $R = \frac{1}{4r|\boldsymbol{\xi}|}$, choose n so that $\frac{2N}{ns(rN^{-\frac{1}{2}}R)} \le \frac{1}{4}$, and conclude that $|1-\widehat{\mu_{\frac{1}{n}}}(\boldsymbol{\xi})| \le \frac{1}{2}$ and therefore $|\hat{\mu}(\boldsymbol{\xi})| \ge 2^{-n}$. This proves that $\hat{\mu}$ never vanishes and therefore that $\hat{\mu} = e^{\ell}$. In addition, by using the fact that $\lim_{t\searrow} \frac{s(t)}{t^2} = \frac{1}{6}$, the preceding line of reasoning shows that there is a $C < \infty$ such that $\big|1-e^{\frac{\ell(\boldsymbol{\xi})}{n}}\big| \le \frac{1}{2}$ when $n \ge C|\boldsymbol{\xi}|^2$, and therefore $\frac{|\ell(\boldsymbol{\xi})|}{1+|\boldsymbol{\xi}|^2}$ is bounded.

Knowing that $\widehat{\mu_{\frac{1}{n}}} = e^{\frac{\ell}{n}}$ and that ℓ has at most quadratic growth, one knows that

$$\ell(\xi) = \lim_{n\to\infty} n\big(\widehat{\mu_{\frac{1}{n}}}(\xi) - 1\big).$$

Thinking of ℓ as a tempered distribution, the challenge is to describe the distribution of which it is the Fourier transform. Thus, set $u = \check{\ell}$. Then

$$\begin{aligned}
(2\pi)^N\langle\varphi, u\rangle = \langle\hat{\varphi}, \ell\rangle &= \lim_{n\to\infty} n\int \hat{\varphi}(\boldsymbol{\xi})\left(\int \big(e^{-\imath(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N}} - 1\big)\mu_{\frac{1}{n}}(d\mathbf{x})\right)d\boldsymbol{\xi} \\
&= \lim_{n\to\infty} n\int\left(\int \big(e^{-\imath(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N}} - 1\big)\hat{\varphi}(\boldsymbol{\xi})\big)d\boldsymbol{\xi}\right)\mu_{\frac{1}{n}}(d\mathbf{x}) \\
&= (2\pi)^N \lim_{n\to\infty} n\int \big(\varphi(\mathbf{x}) - \varphi(\mathbf{0})\big)\mu_{\frac{1}{n}}(d\mathbf{x}),
\end{aligned}$$

and so

$$\langle\varphi, u\rangle = \lim_{n\to\infty} n\int \big(\varphi(\mathbf{x}) - \varphi(\mathbf{0})\big)\mu_{\frac{1}{n}}(d\mathbf{x}).$$

In particular, u satisfies the obvious $\mathbb{R}^N$ analog of the minimum principle in (14.4). Thus, by the $\mathbb{R}^N$-analog of Theorem 14.7, we know that

$$\begin{aligned}
\langle\varphi, u\rangle = \frac{1}{2}\sum_{i,j=1}^N A_{i,j}\partial_{x_i}\partial_{x_j}\varphi(\mathbf{0}) + \sum_{i=1} b_i\partial_{x_i}\varphi(0) \\
+ \int \Big(\varphi(y) - \varphi(\mathbf{0}) - \mathbf{1}_{B(0,1)}(y)\big(y, \nabla\varphi(\mathbf{0})\big)_{\mathbb{R}^N}\Big)M(dy),
\end{aligned}$$

where $(\mathbf{b}, A, M)$ is a Lévy system.

To compute the Fourier transform of u, introduce the operator

$$\mathcal{L}_{(\mathbf{b},A,M)}\varphi(\mathbf{x}) = \frac{1}{2}\sum_{i,j=1}^{N} A_{i,j}\partial_{x_i}\partial_{x_j}\varphi(\mathbf{x}) + \sum_{i=1}^{N} b_i\partial_{x_i}\varphi(\mathbf{x})$$
$$+ \int \Big(\varphi(\mathbf{x}+\mathbf{y}) - \varphi(\mathbf{x}) - \big(\mathbf{b}, \nabla\varphi(\mathbf{x})\big)_{\mathbb{R}^N}\Big) M(d\mathbf{y}).$$

What we have shown is that $\langle \varphi, u\rangle = \mathcal{L}_{(\mathbf{b},A,M)}\varphi(\mathbf{0})$. Using $\widehat{\partial_{x_j}\varphi}(\boldsymbol{\xi}) = -\imath\xi_j\hat{\varphi}(\boldsymbol{\xi})$ and Fubini's theorem, one sees that

$$\widehat{\mathcal{L}_{(\mathbf{b},A,M)}\varphi}(\boldsymbol{\xi}) = \hat{\varphi}(\boldsymbol{\xi})\ell_{(\mathbf{b},A,M)}(-\boldsymbol{\xi}),$$

where

$$\ell_{(\mathbf{b},A,M)}(\boldsymbol{\xi}) = \log f_{(\mathbf{b},A,M)}$$
$$= -\tfrac{1}{2}\big(\boldsymbol{\xi}, A\boldsymbol{\xi}\big) + \imath(\mathbf{b},\boldsymbol{\xi})_{\mathbb{R}^N} + \int \Big(e^{\imath(\boldsymbol{\xi},\mathbf{y})} - 1 - \imath\mathbf{1}_{B(\mathbf{0},1)}(\mathbf{y})\big(\boldsymbol{\xi},\mathbf{y}\big)\Big) M(d\mathbf{y}).$$

Hence, by Parseval's indentity,

$$\langle \hat{\varphi}, \ell\rangle = (2\pi)^N\langle \varphi, u\rangle = (2\pi)^N\mathcal{L}_{(\mathbf{b},A,M)}(\mathbf{0}) = \langle \hat{\varphi}, \ell_{(\mathbf{b},A,M)}(\boldsymbol{\xi})\rangle,$$

and so $\ell = \ell_{(\mathbf{b},A,M)}$.

We will now use (19.1) to prove some properties of the associated measures based on properties of the Lévy system. Use $\mu_{(\mathbf{b},A,M)} \in \mathscr{S}(\mathbb{R}^N;\mathbb{C})^*$ to denote the probability measure of which $f_{(\mathbf{b},A,M)}$ is the Fourier transform, and set $\mu_t = \mu_{(t\mathbf{b},tA,tM)}$ for $t > 0$. Then

$$(2\pi)^N\partial_t\langle \varphi, \mu_t\rangle = \langle \hat{\varphi}, \ell_{(\mathbf{b},A,M)}f_{(t\mathbf{b},tA,tM)}\rangle = (2\pi)^N\langle \mathcal{L}_{(\mathbf{b},A,M)}\varphi, \mu_t\rangle.$$

That is, we have shown that

$$\partial_t\langle \varphi, \mu_{(t\mathbf{b},tA,tM)}\rangle = \langle \mathcal{L}_{(\mathbf{b},A,M)}\varphi, \mu_{(t\mathbf{b},tA,tM)}\rangle. \tag{19.3}$$

Theorem 19.1. *If either A is non-degenerate or $M(G) > 0$ for all non-empty open sets $G \subseteq \mathbb{R}^N \setminus \{\mathbf{0}\}$, then $\mu_{(\mathbf{b},A,M)}(G) > 0$ for all non-empty open sets $G \subseteq \mathbb{R}^N$.*

Proof. First observe that $\mu_{(\mathbf{b},A,M)} = \delta_{\mathbf{b}} * \mu_{(\mathbf{0},A,M)}$, and therefore we can assume that $\mathbf{b} = \mathbf{0}$. Next note that $\mu_{(\mathbf{0},A,M)} = \gamma_A * \mu_{(\mathbf{0},0,M)}$ where γ_A is the distribution of $x \rightsquigarrow A^{\frac{1}{2}}x$ under γ, and so, if A is non-degenerate and therefore γ_A has a strictly positive density, $\mu_{(\mathbf{0},A,M)}$ does also.

Now assume that $\mathbf{b} = 0$, $A = 0$, and $M(G) > 0$ for all open $\emptyset \neq G \subseteq \mathbb{R}^N \setminus \{0\}$. Given an open $G \neq \emptyset$, choose an $\eta \in C^\infty\big(\mathbb{R}^N;[0,1]\big)$ which is strictly positive on G and vanishes off of G. Then

$$\mathcal{L}_{(\mathbf{0},0,M)}\eta(\mathbf{x}) = \int \Big(\eta(\mathbf{x}+\mathbf{y}) - \eta(\mathbf{x}) - \mathbf{1}_{B(\mathbf{0},1)}(\mathbf{y})\big(\nabla\eta(\mathbf{x}),\mathbf{y}\big)_{\mathbb{R}^N}\Big) M(d\mathbf{y})$$
$$= \int \eta(\mathbf{x}+\mathbf{y})\, M(d\mathbf{y}) > 0$$

if $\mathbf{x} \notin G$. Hence, if $f(t) = \langle \eta, \mu_{(\mathbf{0},0,tM)}\rangle$, then $f \geq 0$ and, by (19.3), $\mu_{(\mathbf{0},0,tM)}(G) = 0 \implies f'(t) > 0$. But $\mu_{(\mathbf{0},0,tM)}(G) = 0$ also implies that $f(t) = 0$, which, by the first derivative test, is possible only if $f'(t) = 0$. Hence $f(t) > 0$ for all $t > 0$, and so $\mu_{(\mathbf{0},0,M)}(G) > 0$. □

Theorem 19.2. *If $N = 1$, then $\mu_{(b,A,M)}\big((-\infty,0)\big) = 0$ if and only if*

$$A = 0,\ M\big((-\infty,0)\big) = 0,\ \text{and} \int_{|y|<1} y\, M(dy) \le b. \tag{19.4}$$

Proof. Observe that,

$$\{\mathbf{x} \in \mathbb{R}^n : x_j < 0 \text{ for } 1 \le j \le n|\} \subseteq \left\{\mathbf{x} \in \mathbb{R}^n : \sum_{j=1}^n x < 0\right\},$$

and therefore $\mu_{\frac{1}{n}}\big((-\infty,0)\big)^n \le \mu^{*n}\big((-\infty,0)\big)$ for any $\mu \in \mathbf{M_1}(\mathbb{R})$.

Now assume that $\mu_{(\mathbf{b},A,M)}\big((-\infty,0)\big) = 0$. Since $\mu_{(\mathbf{b},A,M)} = \gamma_A * \mu_{(b,0,M)}$ and $\gamma_A(G) > 0$ for all open $G \ne \emptyset$ unless $A = 0$, it follows that $A = 0$. Next observe that $f_{(b,0,M)}$ has a bounded analytic extension to $\{\zeta \in \mathbb{C} : \mathfrak{Re}\zeta < 0\}$, and therefore $M\big((-\infty,0)\big)$ must be 0. Finally, to prove the inequality in (19.4), set $\mu_{\frac{1}{n}} = \mu_{(\frac{b}{n},0,\frac{M}{n})}$. Since $\mu_1 = \mu_{\frac{1}{n}}^{*n}$, the observation above shows that $\mu_{\frac{1}{n}}\big((-\infty,0)\big) = 0$, and therefore, if $\varphi \ge 0$ on $[0,\infty)$ and $\varphi(0) = 0$, then, by (19.3),

$$\mathcal{L}_{(b,0,M)}\varphi(0) = \lim_{n\to\infty} n\big(\langle\varphi,\mu_{\frac{1}{n}}\rangle - \varphi(0)\big) \ge 0,$$

and so

$$b\varphi'(0) + \int \big(\varphi(y) - \mathbf{1}_{(-1,1)}(y)y\varphi'(0)\big)\, M(dy) \ge 0.$$

Now choose $\eta \in C^\infty\big(\mathbb{R};[0,1]\big)$ so that $\eta = 1$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$ and $\eta = 0$ off $(-1,1)$, and, for $r \in (0,1)$, set $\varphi_r(x) = y\eta_r(y)$ where $\eta_r(y) = \eta\big(\frac{y}{r}\big)$. By the preceding applied to φ_r,

$$b - \int \big(\mathbf{1}_{(-1,1)}(y) - \eta_r(y)\big)y\, M(dy) \ge 0,$$

and so

$$\int_{(r,1)} yM(dy) \le b \text{ for all } r \in (0,1).$$

Finally, assume that (19.4) holds, and set $M_r(dy) = \mathbf{1}_{[r,\infty)}(y)\, M(dy)$ and $b_r = b - \int y\, M_r(dy)$ for $r > 0$. Then (19.4) holds for $(b,0,M_r)$ and (cf. (19.2)) $\mu_{(b,0,M_r)} = \delta_{b_r} * \pi_{M_r}$, from which it is clear that $\mu_{(b,0,M_r)}\big((-\infty,0)\big) = 0$. Therefore, since $\mu_{(b,0,M_r)} \xrightarrow{\mathrm{w}} \mu_{(b,0,M)}$, $\mu_{(b,0,M)}\big((-\infty,0)\big) = 0$. □

Exercise 19.3. If M is symmetric, show that the integral in (19.1) can be replaced by

$$\int \big(\cos(\xi,\mathbf{y})_{\mathbb{R}^N} - 1\big)\, M(d\mathbf{y}).$$

If M is invariant under orthogonal transformations, show that the integral in (19.1) is equal to

$$\int_{\mathbb{S}^{N-1}} \big(\cos(\mathbf{e},\omega)_{\mathbb{R}^N} - 1\big)\, \lambda_{\mathbb{S}^{N-1}}(d\omega)|\xi|^\alpha,$$

where $\mathbf{e} \in \mathbb{S}^{N-1}$ and $\alpha \in (0,2)$. In particular, by combining this with part (**iii**) of Exercise 4.3, conclude that $e^{-|\xi|^\alpha}$ is a characteristic function if and only if $\alpha \in [0,2]$.