20. Singular Integral Operators

The classic *Poisson problem* is that of finding, for a given a function φ, a solution u to the equation $\Delta u = -\varphi$ in $\mathbb{R}^N$, and one of the questions that arises is determining how properties of the function φ are reflected by the solution u. In particular, one wants to know whether second order derivatives of u can be estimated in terms of φ. When $N = 1$, this problem doesn't arise because $-\varphi$ is the second derivative of u. However, when $N \geq 2$, it is not at all clear to what extent the entire Hessian matrix of u is controlled by its trace.

To address this question, it is best to begin by giving an integral representation of the solution u. Depending on dimension, u is given by

$$u(\mathbf{x}) = \int G_0^{(N)}(\mathbf{x}-\mathbf{y})\varphi(\mathbf{y})\,d\mathbf{y},$$

where $G_0^{(N)}$ is the (cf. § 16) Green's function for the Laplacian in $\mathbb{R}^N$:

$$G_0^{(N)}(\mathbf{x}) = \begin{cases} \frac{1}{\pi}\log|\mathbf{x}| & \text{if } N = 2 \\ \frac{1}{(N-2)\omega_{N-1}|\mathbf{x}|^{N-2}} & \text{if } N \geq 3. \end{cases}$$

Thus

$$\partial_{x_i}\partial_{x_j}u(x) = \int G_{i,j}^{(N)}(x-y)\varphi(y)\,dy$$

where

$$G_{i,j}^{(N)}(\mathbf{x}) = \frac{1}{\omega_{N-1}|\mathbf{x}|^N}\int\left(-\delta_{i,j} + N\frac{x_i x_j}{|\mathbf{x}|^2}\right). \tag{20.1}$$

Because $G_{i,j}^{(N)}$ is not an integrable function, one has take care when interpreting convolution with it. On the other hand, since $G_0^{(N)} \in \mathscr{S}(\mathbb{R}^N;\mathbb{C})^*$, so is is $G_{i,j}^{(N)}$, and therefore $\varphi * G_{i,j}^{(N)}$ makes perfectly good sense when $\varphi \in \mathscr{S}(\mathbb{R}^N;\mathbb{C})$. The question then is whether, using this interpretation, one can derive estimates.

Before getting into the details, it is important to know what sort of estimates are possible. In particular, because $G_{i,j}^{(N)}$ is neither integrable nor bounded, one should not expect that convolution with it will map either $L^1(\lambda_{\mathbb{R}^N};\mathbb{C})$ or $L^\infty(\lambda_{\mathbb{R}^N};\mathbb{C})$ into itself. Even so, it turns out (cf. (24.2) below) that it maps $L^p(\lambda_{\mathbb{R}^N};\mathbb{C})$ boundedly into itself when $p \in (1,\infty)$, and what follows is one way to prove that.

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