Lecture 21: The Hilbert Transform

A key fact about $G_{i,j}^{(N)}$ is that it is a homogeneous function of order N whose integral over $\mathbb{S}^{N-1}$ is 0. That is, it is a function of the form

$$k(x) = \frac{\Omega(x)}{|x|^N}$$

where $\Omega(rx) = \Omega(x)$ for all $r > 0$ and $\int_{\mathbb{S}^{N-1}} \Omega(\omega)\,\lambda_{\mathbb{S}^{N-1}}(d\omega) = 0$. Such functions are called *Calderòn–Zygmund* kernels because Calderòn and Zygmund were able to prove a large number of deep results about convolution with respect to them. In particular (cf. (23.2) below), they showed that, in great generality, for each $p \in (1,\infty)$ there is a constant C_p, depending on N and Ω, such that $\|\varphi * k\|_{L^p(\lambda_{\mathbb{R}^N};\mathbb{C})} \le C_p\|\varphi\|_{L^p(\lambda_{\mathbb{R}^N};\mathbb{C})}$.

When $N = 1$ there is, up to a multiple constant, only one C-K kernel, namely, the function $h(x) = \frac{1}{\pi x}$. Convolution with respect to h was studied originally by Hilbert and has been known as the *Hilbert transform* ever since. A seminal observation of Hilbert is that, even though $h \notin L^1(\lambda_{\mathbb{R}};\mathbb{C})$, this transform is a bounded mapping of $L^2(\lambda_{\mathbb{R}};\mathbb{C})$ into itself. Indeed, thinking of h as a tempered distribution, we showed in (6.2) that $\hat{h}(\xi) = \imath\operatorname{sgn}(\xi)$. Thus, we know that $\|\varphi * h\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} \le \|\varphi\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}$.

In order to prove the estimate for $p \ne 2$, I will use an beautiful approach that I think was introduced by M. Riesz and is closely related to the ideas we used to compute $\hat{h}$. Recall the functions $p_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2}$ and $q_y = \frac{1}{\pi}\frac{x}{x^2+y^2}$ which are, respectively, the real and imaginary parts of $\frac{\imath}{z}$ when $z = x + \imath y$. Next, set $h_y(x) = \mathbf{1}_{[y,\infty)}(x)h(x)$, and observe that $\|h_y - q_y\|_{L^1(\lambda_{\mathbb{R}};\mathbb{C})} = \|h_1 - q_1\|_{L^1(\lambda_{\mathbb{R}};\mathbb{C})} \le \frac{2}{\pi}$, and therefore $\|\varphi * h_y - \varphi * q_y\|_{L^p(\lambda_{\mathbb{R}};\mathbb{C})} \le \frac{2}{\pi}\|\varphi\|_{L^p(\lambda_{\mathbb{R}};\mathbb{C})}$. Thus, showing that $\sup_{y>0}\|\varphi * q_y\|_{L^p(\lambda_{\mathbb{R}};\mathbb{C})} \le C_p\|\varphi\|_{L^p(\lambda_{\mathbb{R}};\mathbb{C})}$ for some $C_p < \infty$ will show that

$$\sup_{y>0}\|\varphi * h_y\|_{L^p(\lambda_{\mathbb{R}};\mathbb{C})} \le C_p\|\varphi\|_{L^p(\lambda_{\mathbb{R}};\mathbb{C})} \text{ for some other } C_p < \infty.$$

The advantage that q_y has over h_y is its connection to analytic functions. Namely, since $\frac{\imath}{z} = p_y(x) + \imath q_y(x)$

$$f(z) = \varphi * p_y(x) + \imath\varphi * q_y(x) = \frac{\imath}{\pi}\int \frac{\varphi(\xi)}{x + \imath y - \xi}\,d\xi.$$

Further, because $\|p_y\|_{L^p(\lambda_R;\mathbb{C})} = 1$, $\|\varphi * p_y\|_{L^p(\lambda_{\mathbb{R}};\mathbb{C})} \le \|\varphi\|_{L^p(\lambda_R;\mathbb{C})}$, and Riesz's idea is to use these observations to control $\|\varphi * q_y\|_{L^p(\lambda_{\mathbb{R}};\mathbb{C})}$ in terms of $\|\varphi * p_y\|_{L^p(\lambda_R;\mathbb{C})}$. To do so he needed the fact that, for each $n \ge 1$ there exist finite constants A_n and B_n such that

$$(\mathfrak{Im}\zeta)^{2n} \le A_n\mathfrak{Re}\zeta^n + B_n(\mathfrak{Re}\zeta)^{2n} \text{ for } \zeta \in \mathbb{C}. \tag{$*$}$$

Proving $(*)$ comes down to showing that $\cos^{2n}\theta \le A_n\cos 2n\theta + B_n\sin^{2n}\theta$ for $\theta \in [-\pi,\pi]$. Clearly, if $\theta \in \left[-\frac{\pi}{8n}, \frac{\pi}{8n}\right] \cup \left[\frac{7\pi}{8}, \frac{9\pi}{8}\right]$, A_n can be chosen so the $A_n\cos 2n\theta$ dominates $\cos^{2n}\theta$; and for θ not in those intervals, B_n can be chosen so that $B_n\sin^{2n}\theta$ dominates $\cos^{2n}\theta - A_n\cos 2n\theta$.

With the preceding at hand, we know that

$$\int \big(\varphi * q_y(x)\big)^{2n}\,dx \le A_n\mathfrak{Re}\left(\int f(x+\imath y)^{2n}\,dx\right)\,dx + B_n\int\big(\varphi * p_y(x)\big)^{2n}\,dx.$$

What Riesz saw is that he could use Cauchy's theorem to prove that the integral of $x \rightsquigarrow f(x+\imath y)^{2n}$ is independent of $y > 0$. Indeed, consider the rectangle $\{z = x + \imath y : |x| \le R \;\&\; y_1 \le y \le y_2\}$. Cauchy's theorem says that the contour integral of f^{2n} around the boundary is 0. In addition, as $R \to \infty$, since $\varphi \in \mathscr{S}(\mathbb{R}^2;\mathbb{C})$, the contribution to the integral from the vertical parts of the boundary tends to 0, and so the integrals over the horizontal parts are equal. Finally, as $y \nearrow \infty$, $\int f(x+\imath y)^{2n}\,dx \longrightarrow 0$, and so we now know that

$$\|\varphi * q_y\|_{L^{2n}(\lambda_{\mathbb{R}};\mathbb{C})} \le B_n^{\frac{1}{2n}} \|\varphi\|_{L^{2n}(\lambda_{\mathbb{R}};\mathbb{C})}.$$

Hence, we have proved that, for each $n \ge 1$ there is a $C_{2n} < \infty$ such that

$$\sup_{y>0} \|\varphi * h_y\|_{L^{2n}(\lambda_{\mathbb{R}};\mathbb{C})} \le C_{2n}\|\varphi\|_{L^p(\lambda_{\mathbb{R}};\mathbb{C})}. \tag{21.1}$$