Lecture 22: Interpolation

Although (21.1) is already significant, one should suspect that a similar estimate holds for all $p \in (0,\infty)$, not just even integers. However, because Riesz needed f^p to be an analytic function, he needed p to be an integer; and because he needed $(\mathfrak{Re} f)^p$ to be non-negative, he needed it to be an even integer. It was to overcome this problem that he proved a powerful general result, known as an *interpolation* theorem, that can be viewed as an operator theoretic analog of Hölder's equality. The following version and proof of his result is due to G. Thorin.

Theorem 22.1. *Given a σ-finite measure space $(E,\mathcal{F},\mu)$ and numbers $1 \le p_0,\, p_1,\, q_0,\, q_1 \le \infty$ with $p_0 \wedge p_1 < \infty$, assume that T is a linear operator on $L^{p_1}(\mu;\mathbb{C}) \cap L^{p_2}(\mu;\mathbb{C})$ into $L^{q_1}(\mu;\mathbb{C}) \cap L^{q_2}(\mu;\mathbb{C})$ satisfying*

$$\|Tf\|_{L^{q_j}(\mu;\mathbb{C})} \le M_j \|f\|_{L^{p_j}(\mu;\mathbb{C})} \text{ for } j \in \{0,1\},$$

where $M_0 \vee M_1 < \infty$. Then, for each $\theta \in [0,1]$

$$\|Tf\|_{L^{q_\theta}(\mu;\mathbb{C})} \le M_1^{1-\theta} M_2^{\theta} \|f\|_{L^{p_\theta}(\mu;\mathbb{C})},$$

where $p_\theta = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Thorin's proof of Theorem 22.1 requires to following simple version, due to Hadamard and known as the *three lines theorem*, of the Phragmen–Lindelöf theorem.

Lemma 22.2. *Suppose that F is a bounded continuous function on the closed strip $S = \{z \in \mathbb{C} : \mathfrak{Re} z \in [0,1]\}$ which is analytic on the interior of S. If $|F(\imath y)| \le m_0$ and $|F(1+\imath y)| \le m_1$ for all $y \in \mathbb{R}$, then $|F(z)| \le m_0^{1-x} m_1^{x}$ for $z = x + \imath y \in S$.*

Proof. By replacing F with $\frac{F(z)}{m_0^{1-z} m_1^{z}}$, one can reduce to the case when $m_0 = m_1 = 1$, in which case one needs to show that $|F(z)| \le 1$ for $z \in S$. Thus we will assume that $m_0 = m_1 = 1$ and will prove that $|F| \le 1$.

If $\lim_{|y|\to\infty} \sup_{x\in[0,1]} |F(x+\imath y)| = 0$, then the maximum principle for analytic functions says that

$$\begin{aligned}\sup_{\substack{z\in S\\ |\mathfrak{Im} z|\le R}} |F(z)| &= \sup\bigl\{|F(x+\imath y)| : (x,y) \in \bigl(\{0,1\}\times[-R,R]\bigr) \cup (0,1)\times\{-R,R\}\bigr\} \\ &\longrightarrow \sup_{y\in\mathbb{R}}\{|F(\imath y) \vee |F(1+\imath y)|\} \le 1.\end{aligned}$$

Even if $F(x+\imath y)$ doesn't tend to 0 as $|y| \to \infty$, for each $n \ge 1$, the function $F_n(z) = e^{\frac{z^2-1}{n}} F(z)$ does. In addition, $|F_n(\imath y)| \vee |F_n(1+\imath y)| \le 1$, and so $|F_n(z)| \le 1$. Now let $n \to \infty$. $\square$

Proof of Theorem 22.1. Without loss in generality, we will assume that $p_0 \le p_1$. Also, q' will be used to denote the Hölder conjugate of $q \in [1,\infty]$.

The first step is to check that it suffices to prove that

$$\left|\int g(\xi) Tf(\xi)\,\mu(d\xi)\right| \le M_0^{1-\theta} M_1^{\theta} \tag{$*$}$$

for simple functions f and g satisfying $\|f\|_{L^{p_\theta}(\mu;\mathbb{C})} = 1$ and $\|g\|_{L^{q'_\theta}(\mu;\mathbb{C})} = 1$. Indeed, $\|Tf\|_{L^{q_\theta}(\mu;\mathbb{C})}$ equals the supremum of $\left|\int gTf\,d\mu\right|$ over simple functions g with $\|g\|_{L^{q'_\theta}(\mu;\mathbb{C})} = 1$, and, if $p_1 < \infty$, then, for any $f \in L^{p_0}(\mu;\mathbb{C}) \cap L^{p_1}(\mu;\mathbb{C})$, we can choose simple function f_n such that $f_n \longrightarrow f$ in both $L^{p_0}(\mu;\mathbb{C})$ and in $L^{p_1}(\mu;\mathbb{C})$. Hence, if $(*)$ holds for simple functions, then, by Hölder's inequality,

$$\begin{aligned}\|Tf\|_{L^{q_\theta}(\mu;\mathbb{C})} &\le \|T(f_n-f)\|_{L^{q_\theta}(\mu;\mathbb{C})} + \|Tf_n\|_{L^{q_\theta}(\mu;\mathbb{C})}\\ &\le \|T(f_n-f)\|^{1-\theta}_{L^{q_0}(\mu;\mathbb{C})}\|T(f_n-f)\|^{\theta}_{L^{q_1}(\mu;\mathbb{C})} + M_0^{1-\theta}M_2^{\theta}\|f_n\|_{L^{p_\theta}(\mu;\mathbb{C})}\\ &\le M_0^{1-\theta}M_1^{\theta}\big(2\|f_n-f\|^{1-\theta}_{L^{p_0}(\mu;\mathbb{C})}\|f_n-f\|^{\theta}_{L^{p_1}(\mu;\mathbb{C})} + \|f\|_{L^{p_\theta}(\mu;\mathbb{C})}\big),\end{aligned}$$

from which the required estimate follows when $n\to\infty$. When $p_1=\infty$, one can choose the f_n's so that they converge to f in $L^{p_1}(\mu;\mathbb{C})$ and are uniformly bounded and thereby use the preceding to get the desired result.

Turning to the proof of $(*)$, let $\theta\in(0,1)$ and determine p and q by $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$ and $\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$. Next, define $p(z)$ and $q(z)$ for (cf. Lemma 22.2) $z\in S$ so that $\frac{1}{p(z)}=\frac{1-z}{p_0}+\frac{z}{p_1}$ and $\frac{1}{q'(z)}=\frac{1-z}{q_0'}+\frac{z}{q_1'}$. Given simple functions

$$f=\sum_{m=1}^n a_m\mathbf{1}_{\Gamma_m} \text{ and } g=\sum_{m=1}^n b_m\mathbf{1}_{\Delta_m} \text{ with } \|f\|_{L^p(\mu;\mathbb{C})}=1 \text{ and } \|g\|_{L^{q'}(\mu;\mathbb{C})}=1,$$

define $f_z=|f|^{\frac{p}{p(z)}}\frac{f}{|f|}$ and $g_z=|g|^{\frac{q'}{q'(z)}}\frac{g}{|g|}$, where $\frac{h(\xi)}{|h(\xi)|}$ is taken to be equal 0 if $h(\xi)=0$. Then

$$f_z=\sum_{m=1}^n|a_m|^{\frac{p}{p(z)}}\frac{a_m}{|a_m|}\mathbf{1}_{\Gamma_m} \text{ and } g_z=\sum_{m=1}^n|b_m|^{\frac{q'}{q'(z)}}\frac{b_m}{|b_m|}\mathbf{1}_{\Delta_m}.$$

Now define

$$F(z)=\int g_z(\xi)Tf_z(\xi)\,\mu(d\xi)=\sum_{k,\ell=1}^n|a_k|^{\frac{p}{p(z)}}\frac{a_k}{|a_k|}|b_\ell|^{\frac{q'}{q'(z)}}\frac{b_\ell}{|b_\ell|}\int_{\Delta_\ell}T\mathbf{1}_{\Gamma_k}(\xi)\,\mu(d\xi).$$

Then F is a bounded continuous function on S that is analytic function on the interior of S, and so, by Lemma 22.2,

$$|F(\theta)|\le m_0^{1-\theta}m_1^{\theta} \text{ where } m_0=\sup_{y\in\mathbb{R}}|F(\imath y)| \text{ and } m_1=\sup_{y\in\mathbb{R}}|F(1+\imath y)|.$$

Thus, what remains is to check that $m_0\le M_0$ and $m_1\le M_1$. But, by Hölder's inequality,

$$|F(\imath y)|\le\|g_{\imath y}\|_{L^{q_0'}(\mu;\mathbb{C})}\|Tf_{\imath y}\|_{L^{q_0}(\mu;\mathbb{C})}\le M_0\|g_{\imath y}\|_{L^{q_0'}(\mu;\mathbb{C})}\|f_{\imath y}\|_{L^{p_0}(\mu;\mathbb{C})},$$

and

$$\|f_{\imath y}\|^{p_0}_{L^{p_0}(\mu;\mathbb{C})}=\sum_{m=1}^n\big||a_m|^{\frac{p}{p(\imath y)}}\big|^{p_0}\mu(\Gamma_m)=\sum_{m=1}^n|a_m|^p\mu(\Gamma_m)=1$$

Similarly

$$\|f_{1+\imath y}\|^{p_1}_{L^{p_1}(\mu;\mathbb{C})}=1,\ \|g_{\imath y}\|^{q_0'}_{L^{q_0'}(\mu;\mathbb{C})}=1,\ \text{and } \|g_{1+\imath y}\|^{q_1'}_{L^{q_1'}(\mu;\mathbb{C})}=1.$$

□

By combining (21.1) and Theorem 22.1, we know that there is a $C_p<\infty$ such $\sup_{y>0}\|\varphi*h_y\|_{L^p(\lambda_{\mathbb{R}};\mathbb{C})}\le C_p\|\varphi\|_{L^p(\lambda_{\mathbb{R}};\mathbb{C})}$ for each $p\in[2,\infty)$. To extend this result to $p\in(1,2)$, observe that if $p\in(1,2)$, then $p'\in(2,\infty)$. Hence, since

$$(\psi,\varphi*h_y)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}=-(\psi*h_y,\varphi)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})},$$

we have that

$$|(\psi, \varphi * h_y)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}| \leq C_{p'} \|\psi\|_{L^{p'}(\lambda_{\mathbb{R}};\mathbb{C})} \|\varphi\|_{L^p(\lambda_{\mathbb{R}};\mathbb{C})}$$

and therefore that, for all $p \in (1, \infty)$,

$$\sup_{y>0} \|\varphi * h_y\|_{L^p(\lambda_{\mathbb{R}};\mathbb{C})} \leq C_p \|\varphi\|_{L^p(\lambda_{\mathbb{R}};\mathbb{C})}, \tag{22.1}$$

where $C_p = C_{p'}$ when $p \in (1, 2)$.

Exercise 22.3. Note that $\|\hat{\varphi}\|_{L^2(\lambda_{\mathbb{R}^N};\mathbb{C})} = (2\pi)^{\frac{N}{2}} \|\varphi\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}$ and $\|\hat{\varphi}\|_{L^\infty(\lambda_{\mathbb{R}^N};\mathbb{C})} \leq \|\varphi\|_{L^1(\lambda_{\mathbb{R}^N};\mathbb{C})}$, and use Theorem 22.1 to prove that $\|\hat{\varphi}\|_{L^{p'}(\lambda_{\mathbb{R}^N};\mathbb{C})} \leq (2\pi)^{\frac{N}{p'}} \|\varphi\|_{L^p(\lambda_{\mathbb{R}^N};\mathbb{C})}$ for $p \in [1, 2]$. Next, let $\psi \in L^p(\lambda_{\mathbb{R}^N};\mathbb{C})$ for some $p \in [1, \infty)$, and define $T\varphi = \varphi * \psi$. Remember that $\|T\varphi\|_{L^p(\lambda_{\mathbb{R}^N};\mathbb{C})} \leq \|\varphi\|_{L^p(\lambda_{\mathbb{R}^N};\mathbb{C})} \|\psi\|_{L^1(\lambda_{\mathbb{R}^N};\mathbb{C})}$ and $\|T\varphi\|_{L^\infty(\lambda_{\mathbb{R}^N};\mathbb{C})} \leq \|\varphi\|_{L^p(\lambda_{\mathbb{R}^N};\mathbb{C})} \|\psi\|_{L^{p'}(\lambda_{\mathbb{R}^N};\mathbb{C})}$, and use Theorem 22.1 to prove *Young's inequality*

$$\|\psi * \psi\|_{L^r(\lambda_{\mathbb{R}^N};\mathbb{C})} \leq \|\varphi\|_{L^p(\lambda_{\mathbb{R}^N};\mathbb{C})} \|\psi\|_{L^q(\lambda_{\mathbb{R}^N};\mathbb{C})} \text{ if } \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0.$$