Lecture 23: The Method of Rotations

Calderòn and Zygmund noticed that the Hilbert transform, and especially (22.1), can be used to prove the L^p boundedness of their kernels when $\Omega \in L^1(\lambda_{\mathbb{S}^{N-1}};\mathbb{C})$ is odd (i.e., $\Omega(-\omega) = -\Omega(\omega)$ for $\omega \in \mathbb{S}^{N-1}$). For example, set $k_y(x) = \mathbf{1}_{(y,\infty)}(|x|)k(x)$. Then because

$$\begin{aligned}\widehat{k_y}(\xi) &= \lim_{R\to\infty}\int_{y<|x|\le R} e^{\imath(\xi,x)}k(x)\,dx\\ &= \lim_{R\to\infty}\int_{\mathbb{S}^{N-1}}\Omega(\omega)\left(\int_{(y,R]} e^{\imath r(\xi,\omega)}\frac{1}{r}\,dr\right)d\omega,\end{aligned}$$

if Ω is odd, one has that

$$\begin{aligned}\widehat{k_y}(\xi) &= \lim_{R\to\infty}\frac{1}{2}\int_{\mathbb{S}^{N-1}}\Omega(\omega)\left(\int_{y<|r|\le R} e^{\imath r(\xi,\omega)}\frac{1}{r}\,dr\right)d\omega\\ &= \frac{\pi}{2}\int_{\mathbb{S}^{N-1}}\Omega(\omega)\widehat{h_y}\big((\xi,\omega)\big)\,d\omega.\end{aligned}$$

Hence,

$$\widehat{k_y}(\xi) = \frac{\imath\pi}{2}\int_{\mathbb{S}^{N-1}}\Omega(\omega)\widehat{h_y}\big((\xi,\omega)\big)\,d\omega, \tag{23.1}$$

and so

$$\|\widehat{k_y}\|_{\mathrm{u}} \le \frac{\pi\|\Omega\|_{L^1(\lambda_{\mathbb{S}^{N-1}};\mathbb{C})}\|\widehat{h_y}\|_{\mathrm{u}}}{2}.$$

In particular, we already know that

$$\|\varphi * k\|_{L^2(\lambda_{\mathbb{R}^N};\mathbb{C})} \le \frac{\pi\|\Omega\|_{L^1(\lambda_{\mathbb{S}^{N-1}};\mathbb{C})}}{2}\|\varphi\|_{L^2(\lambda_{\mathbb{R}^N};\mathbb{C})}.$$

The same trick as we just used allows us to prove estimates for general $p \in (1,\infty)$. Namely, again using the oddness of k, one can first write

$$\varphi * k_y(x) = \frac{1}{2}\int_{\mathbb{S}^{N-1}}\Omega(\omega)\left(\int_{|r|>y}\varphi(x-r\omega)\frac{dr}{r}\right)\lambda_{\mathbb{S}^{N-1}}(d\omega),$$

and then, after applying Minkowski's inequality,

$$\|\varphi * k_\epsilon\|_{L^p(\lambda_{\mathbb{R}^N};\mathbb{C})} \le \frac{1}{2}\int_{\mathbb{S}^{N-1}}|\Omega(\omega)|\left(\int_{\mathbb{R}^N}\left|\int_{|r|>y}\varphi(x-r\omega)\,\frac{dr}{r}\right|^p\,dx\right)^{\frac{1}{p}}\lambda_{\mathbb{S}^{N-1}}(d\omega).$$

Finally, for fixed $\omega \in \mathbb{S}^{N-1}$, choose Euclidean coordinates for $\mathbb{R}^N$ so that ω points in the direction of the first coordinate. Then

$$\begin{aligned}&\int_{\mathbb{R}^N}\left|\int_{|r|>y}\varphi(x-r\omega)\,\frac{dr}{r}\right|^p\,dx\\ &\quad = \pi^p\underset{\mathbb{R}^{N-1}}{\int\cdots\int}\left(\int_{\mathbb{R}}\big|[\varphi * h_y(\,\cdot\,,x_2,\dots,x_N)](x_1)\big|^p\,dx_1\right)dx_2\cdots dx_N\\ &\quad \le (\pi C_p)^p\underset{\mathbb{R}^{N-1}}{\int\cdots\int}\|\varphi(\,\cdot\,,x_2,\dots,x_N)\|^p_{L^p(\lambda_{\mathbb{R}};\mathbb{C})}\,dx_2\cdots dx_N = (\pi C_p)^p\|\varphi\|^p_{L^p(\lambda_{\mathbb{R}^N};\mathbb{C})},\end{aligned}$$

which, together with the preceding, leads immediately to the existence of $C_p < \infty$ such that

$$\|\varphi * k\|_{L^p(\lambda_{\mathbb{R}^N};\mathbb{C})} \le C_p \|\varphi\|_{L^p(\lambda_{\mathbb{R}^N};\mathbb{C})} \text{ for } p \in (1,\infty). \tag{23.2}$$

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