Lecture 24: The Riesz Kernels

In a sense which can be made very precise, the basic C-Z kernels for $\mathbb{R}^N$ are the *Riesz kernels* $r_i(\mathbf{x}) = c_N \frac{x_i}{|\mathbf{x}|^{N+1}}$, $1 \le i \le N$, where

$$c_N \equiv \left(\frac{\pi}{2} \int_{\mathbb{S}^{N-1}} |\omega_1| \, \lambda_{\mathbb{S}^{N-1}}(d\omega) \right)^{-1}.$$

Obviously, the preceding applies to each of these. To get a feeling for how convolution with respect to r_i acts, apply (23.1) to see that

$$\widehat{r_i}(\xi) = \frac{\imath \pi c_N}{2} \int_{\mathbb{S}^{N-1}} \omega_i \mathrm{sgn}\big((\xi,\omega)\big) \, \lambda_{\mathbb{S}^{N-1}}(d\omega).$$

Obviously, $\widehat{r_i}$ is homogeneous of degree 0, and so we need only worry about $\xi \in \mathbb{S}^{N-1}$. Given $\xi \in \mathbb{S}^{N-1}$, write $\omega = (\omega,\xi)\xi + \omega^{\perp \xi}$. Then

$$\begin{aligned}\int_{\mathbb{S}^{N-1}} & \omega_i \mathrm{sgn}\big((\xi,\omega)\big) \, \lambda_{\mathbb{S}^{N-1}}(d\omega) \\ &= \xi_i \int_{\mathbb{S}^{N-1}} |(\omega,\xi)| \, d\omega + \int_{\mathbb{S}^{N-1}} \big(\omega^{\perp \xi}\big)_i \mathrm{sgn}\big((\xi,\omega)\big) \, \lambda_{\mathbb{S}^{N-1}}(d\omega).\end{aligned}$$

Because the integrand in the second term is an odd function of $\omega \rightsquigarrow (\xi,\omega)$, the second term vanishes. Hence,

$$\widehat{r_i}(\xi) = \frac{\imath \xi_i}{|\xi|}, \quad \xi \in \mathbb{R}^N \setminus \{0\}. \tag{24.1}$$

To evaluate c_N, observe that $c_1 = \frac{1}{\pi}$ is trivial. When $N \ge 2$, use

$$\begin{aligned}\int_{\mathbb{S}^{N-1}} |\omega_1| \, \lambda_{\mathbb{S}^{N-1}}(d\omega) &= \omega_{N-2} \int_{(-1,1)} |\rho| \big(1-\rho^2\big)^{\frac{N-3}{2}} \, d\rho \\ &= \omega_{N-2} \int_{(0,1)} (1-t)^{\frac{N-3}{2}} \, dt = \frac{2\omega_{N-2}}{N-1} = 2\Omega_{N-1},\end{aligned}$$

where Ω_{N-1} is the volume to the unit ball in $\mathbb{R}^{N-1}$.

From the Riesz transforms one can build other kernels. For instance, recall the kernels in (20.1). Because $\partial_{x_i}\partial_{x_j}\varphi = -(\Delta\varphi) * G^{(N)}_{i,j}$, $-\xi_i\xi_j\hat{\varphi} = |\xi|^2 \widehat{G^{(N)}_{i,j}} \hat{\varphi}$, and so

$$\widehat{G^{(N)}_{i,j}}(\xi) = -\frac{\xi_i\xi_j}{|\xi|^2} = -\widehat{r_i}(\xi)\widehat{r_j}(\xi).$$

Hence, $\varphi * G^{(N)}_{i,j} = -(\varphi * r_i) * r_j$, and so

$$\|\varphi * G^{(N)}_{i,j}\|_{L^p(\lambda_{\mathbb{R}^N};\mathbb{C})} \le C_p^2 \|\varphi\|_{L^p(\lambda_{\mathbb{R}^N};\mathbb{C})} \text{ for } p \in (1,\infty). \tag{24.2}$$

Equivalently, we now know that

$$\|\partial_{x_i}\partial_{x_j}\varphi\|_{L^p(\lambda_{\mathbb{R}^N};\mathbb{C})} \le C_p^2 \|\Delta\varphi\|_{L^p(\lambda_{\mathbb{R}^N};\mathbb{C})} \text{ for } p \in (1,\infty).$$

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