Solutions

Exercise 1.1: First note if $\varphi \in L^2(\lambda_{[0,1)};\mathbb{C})$, then $\sum_{n\in\mathbb{Z}}(\varphi,\mathfrak{e}_n)^2_{L^2(\lambda_{[0,1)};\mathbb{C})} < \infty$ and therefore $(\varphi,\mathfrak{e}_n)_{L^2(\lambda_{[0,1)};\mathbb{C})} \longrightarrow 0$. Next observe that

$$|(\varphi,\mathfrak{e}_n)_{L^2(\lambda_{[0,1)};\mathbb{C})} - (\psi,\mathfrak{e}_n)_{L^2(\lambda_{[0,1)};\mathbb{C})}| \le \|\varphi-\psi\|_{L^1([0,1];\mathbb{C})}$$

and therefore, if $\varphi \in L^1([0,1];\mathbb{C})$ and $\epsilon > 0$, there is an $R < \infty$ and an $m \ge 0$ such that

$$\sup_{n\in\mathbb{Z}} |(\varphi,\mathfrak{e}_n)_{L^2(\lambda_{[0,1)};\mathbb{C})} - (\varphi\wedge R,\mathfrak{e}_n)_{L^2(\lambda_{[0,1)};\mathbb{C})}| \le \|f-\varphi\wedge R\|_{L^1([0,1];\mathbb{C})} < \epsilon$$

and $\sup_{n\ge m}|(\varphi\wedge R,\mathfrak{e}_n)_{L^2(\lambda_{[0,1)};\mathbb{C})}| < \epsilon$. Thus, $|(\varphi,\mathfrak{e}_n)_{L^2(\lambda_{[0,1)};\mathbb{C})}| \le 2\epsilon$ for $n \ge m$.

Exercise 1.2: Set $\varphi_k = p_{\frac{1}{k}} * \varphi$, and check that $\varphi_k \in C^1([0,1];\mathbb{C})$ and $\|\varphi_k'\|_{\mathrm{u}} \le \|\varphi\|_{\mathrm{Lip}}$. Hence

$$|(\varphi,\mathfrak{e}_n)_{L^2(\lambda_{[0,1)};\mathbb{C})}| = \lim_{k\to\infty} |(\varphi_k,\mathfrak{e}_n)_{L^2(\lambda_{[0,1)};\mathbb{C})}| \le \frac{\|\varphi\|_{\mathrm{Lip}}}{2\pi|n|},$$

and so the required estimate follows by the same argument as was used when $\varphi \in C^1([0,1];\mathbb{C})$.

Exercise 2.1: Using the initial formula for $S_n(x)$, show that

$$S_n\left(\tfrac14\right) = \frac12 - \frac1\pi\left(\sum_{m=1}^n \frac{1}{4m+1} + \sum_{m=1}^n \frac{1}{4m+3}\right) = \frac12 + \frac2\pi \sum_{m=1}^\infty \frac{1}{(4m+1)(4m+3)}.$$

Hence, after $n \to \infty$, one has that

$$\frac14 = \frac12 - \frac2\pi \sum_{m=1}^\infty \frac{1}{(4m+1)(4m+3)},$$

from which

$$\pi = 8\sum_{m=1}^\infty \frac{1}{(4m+1)(4m+3)}$$

follows.

Exercise 2.2: Set $\eta(x) = \big(\varphi(1)-\varphi(0)\big)x$ and $\psi = \varphi - \eta$. Then $\psi \in C^1\big([0,1];\mathbb{C}\big)$ and $\psi(1) = \psi(0)$, and so

$$\|S_n\psi - \psi\|_{\mathrm{u}} \le \frac{\|\psi'\|_{L^1(\lambda_{\mathbb{R}};\mathbb{C})}}{\pi n^{\frac12}} \le \frac{2\|\varphi'\|_{L^1(\lambda_{[0,1)};\mathbb{C})}}{\pi n^{\frac12}},$$

and

$$|S_n\eta(x) - \eta(x)| \le \frac{6}{\pi n}|\varphi(1)-\varphi(0)|\left(\tfrac1x \vee \tfrac{1}{1-x}\right)| \le \frac{6\|\varphi'\|_{L^1(\lambda_{[0,1)};\mathbb{C})}}{\pi n}|\left(\tfrac1x \vee \tfrac{1}{1-x}\right)|.$$

Exercise 3.1: By Lemma 1.4,

$$\big|(\varphi,\mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})}\big| = \frac{|(\varphi^{(\ell)},\mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})}|}{(2\pi m)^\ell} \le \left(\frac{n}{m}\right)^\ell \frac{\|\varphi^{(\ell)}\|_{L^2(\lambda_{[0,1)};\mathbb{C})}}{(2\pi n)^\ell}.$$

Thus, if $\lim_{\ell\to\infty} \frac{\|\varphi^{(\ell)}\|_{L^2(\lambda_{[0,1)};\mathbb{C})}}{(2\pi n)^\ell} = 0$, then $(\varphi,\mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})} = 0$ for $|m| \ge n$. Conversely, if $(\varphi,\mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})} = 0$ for $|m| \ge n$, then

$$\|\varphi^{(\ell)}\|^2_{L^2(\lambda_{[0,1)};\mathbb{C})} = \sum_{|m|<n} (2\pi m)^{-2\ell}|(\varphi,\mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})}|^2 \le (2\pi)^{-2\ell}\|\varphi\|^2_{L^2(\lambda_{[0,1)};\mathbb{C})} \sum_{|m|<n} m^{-2\ell},$$

and so $\lim_{\ell\to\infty} \frac{\|\varphi^{(\ell)}\|_{L^2(\lambda_{[0,1)};\mathbb{C})}}{(2\pi n)^\ell} = 0$.

It is obvious that

$$(\varphi, \mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})} = 0 \text{ for } m \geq n \iff \varphi = \sum_{|m|<n} (\varphi, \mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})} \mathfrak{e}_m.$$

Finally, if $1 \leq |m| < n$, then

$$\sum_{j=1}^{n} \mathfrak{e}_m\left(\tfrac{j}{n}\right) = e^{\frac{\imath 2\pi}{n}} \frac{1 - e^{\imath 2\pi m}}{1 - e^{\frac{\imath 2\pi m}{n}}} = 0,$$

and so, if $(\varphi, \mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})} = 0$ for $|m| \geq n$, then

$$\begin{aligned} \frac{1}{n}\sum_{j=1} \varphi\left(\tfrac{j}{n}\right) &= \frac{1}{n}\sum_{j=1}^{n} \sum_{|m|<n} (\varphi, \mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})} \mathfrak{e}_m\left(\tfrac{j}{n}\right) \\ &= \int_0^1 \varphi(x)\, dx + \frac{1}{n} \sum_{1\leq|m|<n} (\varphi, \mathfrak{e}_m)_{L^2(\lambda_{[0,1)};\mathbb{C})} \left(\sum_{j=1}^{n} \mathfrak{e}_m\left(\tfrac{j}{n}\right) \right) = \int_0^1 \varphi(x)\, dx. \end{aligned}$$

Exercise 4.1:

(**i**) Clearly

$$S_n \equiv \sum_{m=1}^{n} (-1)^{m-1} = \begin{cases} 1 & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even,} \end{cases}$$

and so $\{S_n : n \geq 1\}$ doesn't converge and

$$\frac{1}{n}\sum_{m=1}^{n} S_m = \begin{cases} \frac{1}{2} & \text{if } n \text{ is even} \\ \frac{1}{2} - \frac{1}{2n} & \text{if } n \text{ is odd,} \end{cases}$$

which means that the series is Césaro summable to $\frac{1}{2}$.

(**ii**) Since $\frac{a_m}{m} \not\longrightarrow 0$, the series can't be Césaro summable. In fact, using induction on $n \geq 1$, one sees that $S_{2n+1} = n + 1 = -S_{2(n+1)}$, and therefore that $A_{2n} = 0$ and $A_{2n+1} = \frac{n+1}{2n+1}$ for $n \geq 1$. To see that it is Abel summable, observe that, for $r \in (0,1)$,

$$\sum_{m=1}^{\infty} (-1)^m m r^m = -r\partial \sum_{m=0}^{\infty} (-r)^m = \frac{r}{(1+r)^2} \longrightarrow \frac{1}{4}$$

as $r \nearrow 1$.

Exercise 5.1: Because $|\sin \pi x| \leq \pi|x|$ for all x and $|\sin \pi x| \geq 2^{-\frac{1}{2}}$ if $\frac{1}{4} \leq |x| \leq \frac{1}{2}$,

$$\left(\frac{\sin \pi n x}{\sin \pi x}\right) \geq \frac{1}{2\pi^2 x^2} \text{ if } \frac{1}{4n} \leq x \leq \frac{1}{2n}.$$

Thus,

$$n^\alpha \int_{-\frac{1}{2}}^{\frac{1}{2}} F_n(x)|x|^\alpha\, dx \geq \pi^{-2} n^{-1+\alpha} \int_{\frac{1}{4n}}^{\frac{1}{2n}} x^{-2+\alpha}\, dx \geq \frac{1}{4n\pi^2} n^{-2+\alpha}(2n)^{2-\alpha} = \frac{1}{2^\alpha \pi^2}.$$

Exercise 7.1: This is an elementary change of variables.

Exercise 7.2: Because $f', f'' \in L^1(\lambda_{\mathbb{R}};\mathbb{C})$ $\hat{f}'(\xi) = -\imath 2\pi\xi\hat{f}$ and $\widehat{f''} = -(2\pi\xi)^2\hat{f}$. Thus,

$$\int |f(x)|\,dx = \int_{|\xi|\le 1} |\hat{f}(\xi)|d\xi + \int_{|\xi|>1} \frac{|\widehat{f''}(\xi)|}{(2\pi\xi)^2}\,d\xi \le \|f\|_{L^1(\lambda_{\mathbb{R}};\mathbb{C})} + \frac{\|f''\|_{L^1(\lambda_{\mathbb{R}};\mathbb{C})}}{2\pi^2}.$$

Exercise 7.3: We know that

$$\sum_{n\in\mathbb{Z}} \frac{1}{y^2+n^2} = \frac{\pi\coth\pi y}{y}$$

and therefore

$$2\sum_{n\ge 1}\frac{1}{y^2+n^2} = \frac{\pi y\cosh\pi - \sinh\pi y}{y^2\sinh\pi y} = \frac{\pi y + \frac{\pi^3y^3}{2} - \pi y - \frac{\pi^3y^3}{6} + \mathcal{O}(y^5)}{\pi y + \mathcal{O}(y^3)} \longrightarrow \frac{\pi^2}{3}$$

as $y \searrow 0$.

Exercise 7.4: Take $f(x) = t^{-\frac{1}{2}}e^{-\frac{\pi x^2}{t}}$. Then $\hat{f}(\xi) = e^{-\frac{t\xi^2}{4\pi}}$, and therefore, by Theorem 7.2,

$$t^{-\frac{1}{2}}\sum_{n\in\mathbb{Z}} e^{-\frac{\pi n^2}{t}} = \sum_{n\in\mathbb{Z}} e^{-\pi t n^2}.$$

Exercise 8.1: By the result in Exercise 7.4, we know that $\hat{f} \in L^1(\lambda_{\mathbb{R}};\mathbb{C})$. Thus, by Lebesgue's dominated convergence theorem,

$$f(x) = \lim_{t\searrow 0}\int e^{-\frac{t\xi^2}{2}} e^{-\imath\xi x}\hat{f}(\xi)\,d\xi = \int e^{-\imath\xi x}\hat{f}(\xi)\,d\xi.$$

Exercise 8.2: Since $e^{-t|\xi|} = \widehat{p_t}(\xi)$,

$$2\pi p_t(x) = \int_0^\infty e^{-\imath\xi x - t\xi}\,d\xi + \int_0^\infty e^{\imath\xi x - t\xi}\,d\xi = \frac{1}{\imath x + t} + \frac{1}{-\imath x + t} = \frac{2t}{t^2+x^2}.$$

Exercise 8.3: For the cited facts about convolution, see § 6.3.3 in my text *Essentials of Integration Theory for Analysis, 2nd ed.* published by Springer in their GTM series. Given those facts, the asserted results follow easily from: $\widehat{\rho_t * f}(\xi) = \hat{\rho}(t\xi)\hat{f}(\xi) \in L^1(\lambda_{\mathbb{R}};\mathbb{C})$ and

$$2\pi\rho_t * f(x) = \int e^{-\imath\xi x}\hat{\rho}(t\xi)\hat{f}(\xi)\,d\xi.$$

Exercise 9.1: First show that it suffices to treat the case when $(\varphi, \mathbf{1})_{L^2(\gamma;\mathbb{C})} = 0$ and therefore $(P_t\varphi, \mathbf{1})_{L^2(\gamma;\mathbb{C})} = 0$ for all $t \ge 0$.

Let $\varphi \in C_{\mathrm{b}}(\mathbb{R};\mathbb{C})$, and check that

$\lim_{t\searrow 0} P_t\varphi = \varphi$ and $\lim_{t\nearrow\infty} P_t\varphi = (\varphi, \mathbf{1})_{L^2(\gamma;\mathbb{C})}\rangle$ boundedly and uniformly on compact subsets.

Next, suppose that $\varphi \in C^2_{\mathrm{b}}(\mathbb{R};\mathbb{C})$ with $(\varphi, \mathbf{1})_{L^2(\gamma;\mathbb{C})} = 0$. Then

$$\partial_x P_t\varphi(x) = \partial_x \int \varphi\big(y + e^{-\frac{t}{2}}x\big)p(t,0,y)\,dy = e^{-\frac{t}{2}}\int \varphi'(y)p(t,x,y)\,dy,$$

and

$$\begin{aligned}\partial_t\|P_t\varphi\|^2_{L^2(\gamma;\mathbb{C})} &= 2\big(P_t\varphi, \mathcal{L}P_t\varphi\big)_{L^2(\gamma;\mathbb{C})} = -\|(P_t\varphi)'\|^2_{L^2(\gamma;\mathbb{C})} \\ &= -e^{-t}\|P_t\varphi'\|^2_{L^2(\gamma;\mathbb{C})} \ge -e^{-t}\|\varphi'\|^2_{L^2(\gamma;\mathbb{C})}.\end{aligned} \qquad (*)$$

After integrating $(*)$ in t from 0 to ∞, one has

$$-\|\varphi\|^2_{L^2(\gamma;\mathbb{C})} \ge -\|\varphi'\|^2_{L^2(\gamma;\mathbb{C})}$$

which means that

$$\|\varphi - (\varphi, \mathbf{1})_{L^2(\gamma;\mathbb{C})}\|^2_{L^2(\gamma;\mathbb{C})} = \|\varphi\|^2_{L^2(\gamma;\mathbb{C})} - (\varphi, \mathbf{1})^2_{L^2(\gamma;\mathbb{C})} \le \|\varphi'\|^2_{L^2(\gamma;\mathbb{C})},$$

first for $\varphi \in C^2_{\mathrm{b}}(\mathbb{R};\mathbb{C})$ and then, by an easy limit argument, for $\varphi \in C^1_{\mathrm{b}}(\mathbb{R};\mathbb{C})$. Finally, because $(P_t\varphi, \mathbf{1})_{L^2(\gamma;\mathbb{C})} = 0$ for all $t \ge 0$ and therefore, by the preceding,

$$\partial_t \|P_t\varphi\|^2_{L^2(\gamma;\mathbb{C})} = -\|(P_t\varphi)'\|^2_{L^2(\gamma;\mathbb{C})} \le -\|P_t\varphi\|^2_{L^2(\gamma;\mathbb{C})}.$$

Thus $e^t\|P_t\varphi\|^2_{L^2(\gamma;\mathbb{C})}$ is non-increasing. Again an easy limit argument shows that the assumption that $\varphi \in C^2(\mathbb{R};\mathbb{C})$ can be replaced by $\varphi \in L^2(\gamma;\mathbb{C})$.

Exercise 10.1: Reduce to the case when $(\varphi, \mathbf{1})_{L^2(\gamma;\mathbb{C})} = 0$. Then, from (10.3), one has

$$P_t\varphi = \sum_{m=1}^{\infty} e^{-\frac{mt}{2}} (\varphi, H_m)_{L^2(\gamma;\mathbb{C})} H_m,$$

and so

$$\begin{aligned}\|P_t\varphi\|^2_{L^2(\gamma;\mathbb{C})} &= \sum_{m=1}^{\infty} e^{-mt} |(\varphi, H_m)_{L^2(\gamma;\mathbb{C})}|^2 \\ &\le e^{-t} \sum_{m=0}^{\infty} |(\varphi, H_{m+1})_{L^2(\gamma;\mathbb{C})}|^2 = e^{-t}\|\varphi\|_{L^2(\gamma;\mathbb{C})},\end{aligned}$$

and

$$\begin{aligned}\|P_t\varphi\|^2_{L^2(\gamma;\mathbb{C})} &= e^{-t} \sum_{m=0}^{\infty} e^{-mt} |(\varphi, H_{m+1})_{L^2(\gamma;\mathbb{C})}|^2 = e^{-t} \sum_{m=0}^{\infty} e^{-mt} |(\varphi, A_+ H_m)_{L^2(\gamma;\mathbb{C})}|^2 \\ &= e^{-t} \sum_{m=0}^{\infty} e^{-mt} |(\varphi', H_m)_{L^2(\gamma;\mathbb{C})}|^2 \le e^{-t}\|\varphi'\|^2_{L^2(\gamma;\mathbb{C})}.\end{aligned}$$

Exercise 11.1: Using the estimates in Corollary 11.2, one knows that the series $\sum_{m=0}^{\infty} e^{-\frac{mt}{2}} H_m(x)H_m(y)$ is absolutely and uniformly convergent on compact subsets of $(0,\infty) \times \mathbb{R} \times \mathbb{R}$. Thus, if $\varphi \in C_{\mathrm{c}}(\mathbb{R};\mathbb{C})$,

$$P_t\varphi = \lim_{n\to\infty} \sum_{m=0}^{n} (\varphi, H_m)_{L^2(\gamma;\mathbb{C})} P_t H_m = \sum_{m=0}^{\infty} e^{\frac{mt}{2}} (\varphi, H_m)_{L^2(\gamma;\mathbb{C})} H_m,$$

and so

$$\int \varphi(y) p(t,x,y)\, dy = (2\pi)^{-\frac{1}{2}} \int \left(\sum_{m=0}^{\infty} e^{-\frac{mt}{2}} H_m(x) H_m(y) \right) e^{-\frac{y^2}{2}} \varphi(y)\, dy,$$

from which it follows that

$$(2\pi)^{\frac{1}{2}} p(t,x,y) e^{\frac{y^2}{2}} = \sum_{m=0}^{\infty} e^{-\frac{mt}{2}} H_m(x) H_m(y).$$

Finally, set $e^{-\frac{t}{2}} = \theta$, and check that the preceding is Mehler's formula.

Exercise 12.1: Since $f \rightsquigarrow \hat{f}$ is an isomorphism on $L^2(\lambda_{\mathbb{R}};\mathbb{C})$ and

$$\|\mathcal{F}f\|^2_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} = \int |\hat{f}(2\pi\xi)|^2\, d\xi = (2\pi)^{-1}\|\hat{f}\|^2_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} = \|f\|^2_{L^2(\lambda_{\mathbb{R}};\mathbb{C})},$$

$\mathcal{F}$ is an orthogonal operator on $L^2(\lambda_{\mathbb{R}};\mathbb{C})$. Thus $\mathcal{F}^{-1} = \mathcal{F}^*$. Finally, by Fubini's theorem,

$$\big(f, \mathcal{F}g\big)_{L^2(\lambda_{[0,1)};\mathbb{C})} = \int f(\xi)\left(\int e^{-2\pi\imath\xi x}\overline{g(x)}\,dx\right)d\xi = \int \overline{g(x)}\left(\int e^{-2\pi\xi x} f(\xi)\,d\xi\right)dx,$$

and so $\mathcal{F}^* f = (\mathcal{F}f)^{\cup} = \mathcal{F}\check{f}$.

Exercise 13.1: The lower bound is an easy application of Lemma 13.1. To prove the upper bound, first check that $a_-^n h_{k+n} = \frac{(k+n)!}{k!} h_k$ and therefore that $a_-^n \tilde{h}_{k+n} = \left(\frac{(k+n)!}{k!}\right)^{\frac{1}{2}} \tilde{h}_k$. Hence

$$\big|\big(\varphi, \tilde{h}_k\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\big|^2 = \frac{k!}{(k+n)!}\big|\big(\varphi, a_-^n\tilde{h}_{k+n}\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\big|^2 = \frac{k!}{(k+n)!}\big|\big(a_+^n\varphi, \tilde{h}_{k+n}\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\big|^2,$$

and so

$$\|\varphi\|^2_{\mathscr{S}^{(m+n)}(\mathbb{R};\mathbb{C})} = \sum_{k=0}^{\infty} \mu_k^m\left(\frac{k!\mu_k^n}{(k+n)!}\right)\big|\big(a_+^n\varphi, \tilde{h}_{k+n}\big)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}\big|^2.$$

Using Stirling's formula, show that

$$C = \sup_{k\geq 0}\left(\frac{k!\mu_k^n}{(k+n)!}\right) < \infty,$$

and therefore that $\|\varphi\|_{\mathscr{S}^{(m+n)}(\mathbb{R};\mathbb{C})} \leq C^{\frac{1}{2}}\|a_+^n\varphi\|_{\mathscr{S}^{(m)}(\mathbb{R};\mathbb{C})}$. Finally, write $a_+^n\varphi$ as a linear combination of terms of the form $x^k\partial^\ell\varphi$ with $k+\ell \leq n$, and apply the lower bound to each of these terms.

Exercise 13.2: Because, by Theorem 13.5, the sequence is relatively compact, and, by assumption, it is pointwise convergent, it can have at most one limit. Thus it must be convergent.

Exercise 13.3: Choose $\eta \in C^\infty\big(\mathbb{R};[0,1]\big)$ so that $\eta(x) = 1$ if $|x| \leq 1$ and $\eta(x) = 0$ if $|x| \geq 2$. For $R > 0$ define $\eta_R(x) = \eta\big(R^{-1}x\big)$.

(**i**) Given $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{C})$, set $\varphi_R = \eta_R\varphi$ for $R > 0$. Clearly $\varphi_{\mathbb{R}} \in C_c^\infty(\mathbb{R};\mathbb{C})$ and $\varphi_R = \varphi$ on $[-R,R]$. In addition,

$$\begin{aligned}\|x^k\partial^\ell(\varphi - \varphi_R)\|_{\mathrm{u}} &\leq \sup_{|x|\geq R}\|x^k\partial^\ell\varphi(x)| + \sup_{|x|\geq R}\|x^k\partial^\ell\varphi_R(x)| \\ &\leq \frac{1}{R}\big(\|x^{k+1}\partial^\ell\varphi\|_{\mathrm{u}} + \|x^{k+1}\partial^\ell\varphi_R^\ell\varphi\|_{\mathrm{u}}\big) \leq \frac{1}{R}\big(\|\varphi\|_{\mathrm{u}}^{(k+\ell+1)} + \|\varphi_R\|_{\mathrm{u}}^{(k+\ell+1)}\big).\end{aligned}$$

Finally, because $\partial^\ell\varphi_R(x)$ is a linear combination of terms of the form

$$R^{-k}\eta^{(k)}\big(R^{-1}x\big)\varphi^{\ell-k}(x),$$

$\sup_{R\geq 1}\|\varphi_R\|_{\mathrm{u}}^{(k+\ell+1)} < \infty$.

(**ii**) Because $C_0(\mathbb{R};\mathbb{C})$ is a closed subset of $C_{\mathrm{b}}(\mathbb{R};\mathbb{C})$ with respect of the uniform topology, it is a Banach space. Now choose $\rho \in C^\infty\big(\mathbb{R};[0,\infty)\big)$ so that $\rho = 0$ off of $(-1,1)$ and $\int\rho(x)\,dx = 1$, and define $\rho_\epsilon(x) = \epsilon^{-1}\rho\big(\epsilon^{-1}x\big)$ for $\epsilon > 0$. Given $\varphi \in C_c(\mathbb{R};\mathbb{C})$, $\rho_\epsilon * \varphi \in C_c^\infty(\mathbb{R};\mathbb{C})$ and $\|\rho_\epsilon * \varphi - \varphi\|_{\mathrm{u}} \longrightarrow 0$ as $\epsilon \searrow 0$. Thus, we will know that $C_c^\infty(\mathbb{R};\mathbb{C})$, and therefore also $\mathscr{S}(\mathbb{R};\mathbb{C})$, is dense in $C_0(\mathbb{R};\mathbb{C})$ once we show that $C_c(\mathbb{R};\mathbb{C})$ is dense in $C_0(\mathbb{R};\mathbb{C})$. But if $\varphi \in C_0(\mathbb{R};\mathbb{C})$, then $\varphi_R \in C_c(\mathbb{R};\mathbb{C})$ and $\|\varphi_R - \varphi\|_{\mathrm{u}} \leq 2\sup_{|x|\geq R}|\varphi(x)|$ as $R \to \infty$.

Exercise 13.4: Begin by checking that

$$|y| \le \begin{cases} 2|x| & \text{if } |y| \le 2|x| \\ 2|x+y| & \text{if } |y| \ge 2|x|. \end{cases}$$

Thus,

$$|y^k \partial^\ell \varphi(x+y)| \le \begin{cases} 2^k |x|^k |\partial^\ell \varphi(x+y)| & \text{if } |y| \le 2|x| \\ 2^k |x+y|^k |\partial^\ell \varphi(x+y)| & \text{if } |y| \ge 2|x|, \end{cases}$$

and so, if $k+\ell \le m$, then

$$\left\| y^k \partial^\ell \tau_x \varphi \right\|_{\mathrm{u}} \le 2^m (|x| \vee 1)^m \|\varphi\|_{\mathrm{u}}^{(m)}.$$

Next suppose that $x_1 < x_2$. Then

$$y^k \big(\tau_{x_2} \varphi^{(\ell)}(y) - \tau_{x_1} \varphi^{(\ell)}(y) \big) = \int_{x_1}^{x_2} y^k \varphi^{(\ell+1)}(t+y)\, dt,$$

and so, if $k+\ell \le m$, then

$$\begin{aligned} \left| y^k \big(\tau_{x_2} \varphi^{(\ell)}(y) - \tau_{x_1} \varphi^{(\ell)}(y) \big) \right| &\le |x_2 - x_1| \max_{x_1 \le t \le x_2} \left| y^k \varphi^{(\ell+1)}(t+y) \right| \\ &\le 2^k (|x_2| \vee 1)^k \|\varphi\|_{\mathrm{u}}^{(m+1)} |x_2 - x_1|. \end{aligned}$$

Exercise 14.1: Set $u = f(|x|)$. Then

$$\begin{aligned} \langle \varphi, u' \rangle &= -\int_0^\infty \varphi'(x) \bar{f}(x)\, dx - \int_{-\infty}^0 \varphi'(x) \bar{f}(-x)\, dx \\ &= \varphi(0)\bar{f}(0) + \int_0^\infty \varphi(x) \bar{f}'(x)\, dx - \varphi(0)\bar{f}(0) - \int_{-\infty}^0 \varphi(x) \bar{f}'(-x)\, dx \\ &= \int \varphi(x) \mathrm{sgn}(x) \bar{f}'(|x|)\, dx, \end{aligned}$$

and so $u' = \mathrm{sgn}(x)\bar{f}'(|x|)$. Next

$$\begin{aligned} \langle \varphi, u'' \rangle &= -\int_0^\infty \varphi'(x) \bar{f}'(x)\, dx + \int_{-\infty}^0 \varphi'(x) \bar{f}'(-x)\, dx \\ &= 2\varphi(0)\bar{f}'(0) + \int_0^\infty \varphi(x) \overline{f''(x)}\, dx - \int_{-\infty}^0 \varphi(x) \overline{f''(-x)}\, dx \\ &= 2\bar{f}(0)\varphi(0) + \int \varphi(x) \mathrm{sgn}(x) \overline{f''(|x|)}\, dx, \end{aligned}$$

which means that $u'' = 2f(0)\delta_0 + \mathrm{sgn}(x) f''(|x|)$.

Exercise 15.1:

(**i**) By Fubini's theorem,

$$\langle \varphi, \psi * \mu \rangle = \int \left(\int \varphi(x+y) \bar{\psi}(x)\, dx \right) \mu(dy) = \int \varphi(x) \left(\overline{\int \psi(x-y)\, \mu(dy)} \right) dx,$$

and so $\psi * \mu = \int \psi(x-y)\, \mu(dy)$.

(**ii**) Again by Fubini's theorem,

$$\langle \varphi, \hat{\mu} \rangle = \langle \check{\varphi}, \mu \rangle = \int \left(\int e^{-\imath \xi x} \varphi(\xi)\, d\xi \right) \bar{\mu}(dx)$$
$$= \int \varphi(\xi) \left(\overline{\int e^{\imath \xi x} \mu(dx)} \right) d\xi,$$

and so $\hat{\mu} = \int e^{\imath \xi x}\, \mu(dx)$.

(**iii**) Using Lebesgue's dominated convergence theorem, one can check that

$$\int |x|^{k+1}\, \mu(dx) \implies \partial_\xi \int x^k e^{\imath \xi x}\, dx = \int x^{k+1} e^{\imath \xi x}\, \mu(dx).$$

Thus, $\hat{\mu} \in C^\infty_{\mathrm{b}}(\mathbb{R};\mathbb{C})$, and so, since $\hat{\psi} \in \mathscr{S}(\mathbb{R};\mathbb{C})$, $\hat{\psi}\hat{\mu} \in \mathscr{S}(\mathbb{R};\mathbb{C})$.

Exercise 16.1: Because the real and imaginary parts of f are harmonic elements of $\mathscr{S}(\mathbb{R};\mathbb{C})$, they, and therefore f, are polynomials. Thus, if f is an entire function that is not a polynomial, it can't be an element of $\mathscr{S}(\mathbb{R};\mathbb{C})$, which means that it must grow at infinity faster than a polynomial.

Exercise 17.1: Trivially, $\mu_n \xrightarrow{\mathrm{w}} \mu \implies \mu_n \longrightarrow \mu$ in $\mathscr{S}(\mathbb{R};\mathbb{C})^*$, and, by Theorem 17.3, $\mu \longrightarrow \mu$ in $\mathscr{S}(\mathbb{R};\mathbb{C})^* \implies \mu_n \xrightarrow{\mathrm{w}} \mu$.

Exercise 18.1: In the proof of Theorem 18.3, it was shown that f is a characteristic function if $f(\mathbf{0}) = 1$ and

$$\iint f(\boldsymbol{\xi} - \boldsymbol{\eta}) \varphi(\boldsymbol{\xi}) \overline{\varphi(\boldsymbol{\eta})}\, d\boldsymbol{\xi} d\boldsymbol{\eta} \geq 0$$

for all $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{C})$. Thus, f is a characteristic function if and only if $f(\mathbf{0}) = 1$ and $(\varphi, \psi)_f$ is a non-negative quadric form. Conversely, if f is a Finally, if $f = \hat{\mu}$, then, by Parseval's idententy and Fourier inversion formula,

$$\int \hat{\varphi}(\mathbf{x}) \overline{\hat{\psi}(\mathbf{x})}\, \mu(d\mathbf{x}) = \int \hat{\varphi}(\mathbf{x}) \big(\breve{\bar{\psi}}\big)^{\wedge}(\mathbf{x})\, \mu(d\mathbf{x}) = (2\pi)^{-N} \int \Big(\hat{\varphi} \big(\breve{\bar{\psi}}\big)^{\wedge} \Big)^{\vee} (\boldsymbol{\xi}) \overline{\breve{\mu}(\boldsymbol{\xi})}\, d\boldsymbol{\xi}$$
$$= \int \big(\varphi * \breve{\bar{\psi}} \big)(\boldsymbol{\xi}) f(\boldsymbol{\xi})\, d\boldsymbol{\xi} = \iint f(\boldsymbol{\xi}) \varphi(\boldsymbol{\xi} - \boldsymbol{\eta}) \overline{\psi(-\boldsymbol{\eta})}\, d\boldsymbol{\xi}\, d\boldsymbol{\eta}$$
$$= \iint f(\boldsymbol{\xi} - \boldsymbol{\eta}) \varphi(\boldsymbol{\xi}) \overline{\psi(\boldsymbol{\eta})}\, d\boldsymbol{\xi} d\boldsymbol{\eta}.$$

Exercise 18.2:

(**i**) If A is an bounded operator on a complex Hilbert H space and $(Ah, h)_H \in \mathbb{R}$ for all $h \in H$, then A is self-adjoint. Thus, if $A = \begin{pmatrix} 1 & f(-\boldsymbol{\xi}) \\ f(\boldsymbol{\xi}) & 1 \end{pmatrix}$, then $f(-\boldsymbol{\xi}) = \overline{f(\boldsymbol{\xi})}$, and so A is non-negative definite Hermitian matrix and $0 \leq \det(A) = 1 - |f(\boldsymbol{\xi})|^2$.

Next take

$$A = \begin{pmatrix} 1 & f(-\boldsymbol{\xi}) & f(-\boldsymbol{\eta}) \\ f(\boldsymbol{\xi}) & 1 & f(\boldsymbol{\xi} - \boldsymbol{\eta}) \\ f(\boldsymbol{\eta}) & f(\boldsymbol{\eta} - \boldsymbol{\eta}) & 1 \end{pmatrix}$$

and $\alpha_1 = z,\ \alpha_2 = -1,\ \alpha_3 = 1$. Then

$$0 \le \sum_{k,\ell=1}^{2} A_{k,\ell}\alpha_k\overline{\alpha_\ell} = |z|^2 - 2\mathfrak{Re}\,\bar z f(\xi) + 2\mathfrak{Re}\,\bar z f(\boldsymbol{\eta}) + 2 - \mathfrak{Re}\,f(\boldsymbol{\eta}-\boldsymbol{\xi})$$
$$= |z|^2 - 2\mathfrak{Re}\,\bar z\big(f(\boldsymbol{\eta}) - f(\boldsymbol{\xi})\big) + 2\big(1 - \mathfrak{Re}\,f(\boldsymbol{\eta}-\boldsymbol{\xi})\big).$$

Now take $z = f(\boldsymbol{\eta}) - f(\boldsymbol{\xi})$.

(**ii**) Clearly $af_1 + bf_2$ is non-negative definite for all $a, b \ge 0$. To see that $f_1 f_2$ is non-negative definite, it suffices to show that if A and B are non-negative definite matrices, then so is $\big((A_{k,\ell}B_{k,\ell})\big)_{1\le k,\ell\le N}$. To this end, use the fact that $B = C\bar C$, where C is again a non-negative definite matrix. Hence

$$\sum_{k,\ell=1}^{N} A_{k,\ell}B_{k,\ell}\alpha_l\overline{\alpha_\ell} = \sum_{j=1}^{N}\sum_{k,\ell=1}^{N} A_{k,\ell}\alpha_k C_{k,j}\overline{C_{\ell,j}\alpha_\ell} \ge 0$$

(**iii**) Assume $\lim_{|\mathbf{x}|\searrow 0}\frac{1-f(\mathbf{x})}{|\mathbf{x}|^2} = 0$. Since, for each $\mathbf{e}\in\mathbb{S}^{N-1}$,

$$\frac{|f(\boldsymbol{\xi}+t\mathbf{e}) - f(\boldsymbol{\xi})|^2}{t^2} \le 2\frac{1-\mathfrak{Re}\,f(t\mathbf{e})}{t^2} \longrightarrow 0$$

as $t\to 0$, f' is 0 everwhere and therefore f is constant.

(**iv**) Using the Hahn decompostion theorem, write $\mu = \mu_+ - \mu_-$ where $\mu_+ \perp \mu_-$. Then, for $\varphi\in\mathscr{S}(\mathbb{R}^N;\mathbb{C})$,

$$(2\pi)^N\int\varphi(\mathbf{x})\,\mu(d\mathbf{x}) = \int\hat\varphi(\boldsymbol{\xi})\,\widehat{\mu_+}(\boldsymbol{\xi})\,d\boldsymbol{\xi} - \int\hat\varphi(\boldsymbol{\xi})\,\widehat{\mu_-}(-\boldsymbol{\xi})\,d\boldsymbol{\xi} = \int\hat\varphi(\boldsymbol{\xi})\hat\mu(-\boldsymbol{\xi})\,d\boldsymbol{\xi} = 0,$$

and so $\int\varphi\,d\mu_+ = \int\varphi(x)\,d\mu_-$ for all $\varphi\in\mathscr{S}(\mathbb{R}^N;\mathbb{C})$ and therefore $\mu_+ = \mu_-$.

(**v**) Choose $\mu\in\mathbf{M}_1(\mathbb{R})$ so that $f = \hat\mu$. Since f is not constant, $\mu\ne\delta_0$.

The first step is to show that $f''(\xi) = -\int x^2 e^{\imath\xi x}\,\mu(dx)$. To this end, observe that

$$-f''(0) = \lim_{t\to 0}\frac{2 - f(t) - f(-t)}{t^2} = \lim_{t\to 0} 2\int\frac{1-\cos tx}{t^2}\,\mu(dx) \longrightarrow \int x^2\,\mu(dx).$$

Knowing that $\int x^2\,\mu(dx) < \infty$, the same reasoning shows that $-f''(\xi) = \int x^2 e^{\imath\xi x}\,\mu(dx)$. Since $\mu\ne\delta_0$, $-f''(0) < 0$, and so $\frac{f''}{f''(0)} = \hat\nu$, where $\nu(dx) = \frac{x^2}{-f''(0)}\mu(dx)$.

Clearly $|f''(\xi)| \le \int x^2\,\mu(dx) = |f''(0)|$. Knowing that x is μ-square integrable, it is easy to check that $f'(\xi) = \imath\int x e^{\imath\xi x}\,\mu(dx)$, and therefore that $|f'(\xi)|^2 \le \int x^2\,\mu(dx) = |f''(0)|$. Finally, $|f(\eta) - f(\xi)| \le \|f'\|_{\mathrm{u}}|\eta-\xi| \le |f''(0)|^{\frac12}|\eta-\xi|$.

(**vi**) It is easy to check that f must be a non-negative definite function for which $f(\mathbf{0}) = 1$. By part (**i**), we know that f is continuous everywhere since it is continuous at $\mathbf{0}$. Hence, $f = \hat\mu$ for some $\mu\in\mathbf{M}_1(\mathbb{R})$, and so, by Theorem 18.1, $\mu_n\xrightarrow{\mathrm{w}}\mu$. Clearly, this proves that if $\{\hat\mu_n : n\ge 1\}$ is unformly convergent at $\mathbf{0}$, then $\mu_n\xrightarrow{\mathrm{w}}\mu$ for some

(**vii**) There is essentially nothing to do.

Exercise 19.1: Set $M_r(d\mathbf{y}) = \mathbf{1}_{[r,\infty)}(|\mathbf{y}|)\,M(d\mathbf{y})$ for $r > 0$. If M is symmetric, then

$$\begin{aligned}\int &\Big(e^{\imath(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 - \mathbf{1}_{B(\mathbf{0},1)}(\mathbf{y})\Big)\,M_r(d\mathbf{y})\\ &= \int \big(\cos(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N} - 1\big)M_r(d\mathbf{y}) + \int \big(\imath\sin(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N} - \mathbf{1}_{B(\mathbf{0},1)}(\mathbf{y})(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}\big)M_r(d\mathbf{y})\\ &= \int \big(\cos(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N} - 1\big)M_r(d\mathbf{y})\end{aligned}$$

for all $r > 0$. Since $|\cos(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N} - 1| \le |\boldsymbol{\xi}|^2|\mathbf{x}|^2$, Lebesgue's dominated convergence justifies

$$\lim_{r\searrow 0}\int \big(\cos(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N} - 1\big)M_r(d\mathbf{y}) = \int \big(\cos(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N} - 1\big)M(d\mathbf{y}).$$

Next assume that M is invariant under orthogonal transformations and choose $\mathbf{e} \in \mathbb{S}^{N-1}$. Then it is symmetric and

$$\int \big(\cos(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N} - 1\big)M(d\mathbf{y}) = \int \big(\cos(|\boldsymbol{\xi}|\mathbf{e},\mathbf{y}) - 1\big)\,M(d\mathbf{y}).$$

Finally, if $M(d\mathbf{y}) = |\mathbf{y}|^{-N-\alpha}\lambda_{\mathbb{R}^N}(d\mathbf{y})$ for some $\alpha \in (0,2)$ and $\mathbf{e} \in \mathbb{S}^{N-1}$, then

$$\begin{aligned}\int \big(\cos(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N} - 1\big)M(d\mathbf{y}) &= \int \big(\cos(\mathbf{e},|\boldsymbol{\xi}|\mathbf{y})_{\mathbb{R}^N} - 1\big)|\mathbf{y}|^{-1-\alpha}\,\lambda_{\mathbb{R}^N}(d\mathbf{y})\\ &= |\boldsymbol{\xi}|^{\alpha}\int \big(\cos(\mathbf{e},\mathbf{y})_{\mathbb{R}^N} - 1\big)|\mathbf{y}|^{-1-\alpha}\,\lambda_{\mathbb{R}^N}(d\mathbf{y}).\end{aligned}$$

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