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PROFESSOR: Ladies and gentlemen, welcome to lecture number two. In this lecture, I would like to discuss some basic concepts of finite element analysis with regard to the analysis of continuous systems. We discussed in the first lecture already some basic concepts of analysis of discrete systems. However, in actuality, in the analysis of a complex system, we are dealing with a continuous system, and there are some additional basic concepts that are used in analysis of continuous systems, using finite element methods. And those additional concept that are used, I want to discuss in this lecture.

Well, when we talk about the analysis of a continuous system, we can analyze that system via a differential formulation or a variational formulation. If we use a differential formulation or variational formulation, of course we obtain continuous variables, and we have an infinite number of state variables, or rather, if we talk about displacements, a U displacement, for example, of a rod, as I will be discussing just now, we will have infinite values of that displacement along the rod. In the differential formulation or the variational formulation, we would have to solve for that continuous variable along the rod. Well, we will also notice that in the analysis of a complex system, we cannot solve the differential equations that we are arriving at directly, and we have to resort to numerical methods.

Now, some concepts that have been used for a long time are the weighted residual methods. These have been used by Galerkin least squares approaches to solve the differential equations that govern the equilibrium motion of the system approximately. In the variational formulation, the Ritz method has been used for quite a long time. These are classical techniques, therefore, weighted residual method and the Ritz method. And what I want to show to you in this lecture is how the finite element method is really an extension off these methods, or how this

method, the finite element method, is related to these classical techniques.

Well, when we talk about the differential formulation, we are looking at the differential equilibrium, or the equilibrium of a differential element of the system. Now, I want to show to you the basic ideas of a differential formulation by looking at, or analyzing, this rod. Here we have a rod that is fixed at the left-hand side, on rollers. x is a variable along the rod. u is the displacement of that rod into this direction. The rod is subjected, as shown here, to a load R_0 at its right end.

Notice that in this analysis, we assume that plane sections remain plane. In other words, a section that was originally there at time t greater than 0 has moved to here. And this movement here is the u displacement. But notice that the vertical section here remains vertical during the motion.

So at every section, we have only 1 degree of freedom. There's no rotation of that section. However, this one degree of freedom, u , varies continuously along the rod. Therefore, the rod itself has really an infinite number of degrees of freedom.

For this very simple system, we could obtain an exact solution. However, I want to show you how we proceed in analyzing this rod via differential formulation, a variational formulation, and so on, simply as an example. Therefore, the basic ideas, really, that I will be putting forth to you, that I will be discussing with you, are really the important things that I want to expose to you. It's not the analysis of this very specific problem. It's really the basic idea that I want to clarify to you by looking at this one problem.

Well, for this problem here, the governing differential equation of motion is shown here. Notice u once again is the displacement of a section. That is, this displacement here at that coordinate x . c is given here as square root E over ρ , where E is Young's modulus of the material, ρ is the mass density of the material. t , of course, is the time variable. Notice the cross-sectional area a here, I have also written down here. This cross-sectional area cancels out on both sides, as you will see just now.

Now this question here is obtained in the differential formulation by looking at the equilibrium of an element. And we might consider this to be the element that I will now focus our attention on. Here we have that element drawn again. This element here of length dx is subjected at its left side, because x comes from here as a variable and brings us up to this station. At the left side here, the element is subjected to σ , the stress σ . At the right side, we have the stress σ plus partial $d\sigma/dx$. This here really means nothing else than a $d\sigma$, an increment in the σ .

Well, the equilibrium requirement for this element here is now that the force on this side here, that is, σ times a on this side, which is actually this one here, I should have pointed to this one-- and this force here, which is the force on the right side. If we subtract these two forces, that must be equal to the force applied. Or rather, the d'Alembert force.

Notice that if we look at this element, there's a force on the right, there's a force on the left. And this being the force on the right, let's call that, say, R_1 , and let's call that R_2 . So I put here R_1 , I put here R_2 . And R_1 minus R_2 must be equal to the d'Alembert force, which is due to the inertia of the material. This is the basic Newton's law applied to this differential element.

Now, if we referred back to how we proceeded in the analysis of a discrete system, we really proceeded in exactly the same way. But our element then was a discrete element, a discrete spring element. We now use the same concept, but apply those concepts to a differential element.

So this is the equilibrium equation of the element. The constitutive relation is given here-- that the stress is equal to E , the Young's Modulus, times the strain. This is the strain. And notice once again, since we are considering sections to be remaining plane, and simply move horizontally, the only strain that we're talking about is this one. If we combine these two equations here, we directly obtain that equation here. Notice as I mentioned earlier, a cancels out. If a is constant, that's why a does not enter into this equation, and this substitution for σ into here gives us the second

derivative here. Of course, this part here cancels out that part there, and that second derivative is this one here. The E brought over to this side gives us a 1 over c squared, where c , once again, is defined as shown here.

The important point, really, is that we are looking here in the differential formulation at a differential element of length dx at a particular station x . That we're looking at this element and we establish the equilibrium requirement of that element. R_1 minus R_2 shall be equal to the mass of the element times the acceleration.

We also introduced a constitutive relation. So far, clearly, we have used two conditions for the solution of the problem. The first one is the equilibrium condition. The second is the constitutive condition, or constitutive requirement. We have to ask ourselves, where do we satisfy the compatibility condition? Because there are always these three conditions that we have to satisfy.

Well, the compatibility condition is satisfied by solving this differential equation for this rod, and obtaining a u that is continuous. In other words, a u that tells that all the sections have remained together. We did not cut that bar apart.

In the discrete system analysis of the spring system of lecture 1, if you were to think back to it, we had to satisfy the compatibility condition explicitly, in establishing the equilibrium equations, because we had to make sure that all the springs remain attached to the carts. Here we satisfy the compatibility condition by solving this equation for a continuous u .

Well, the boundary conditions, of course, also have to be stated. And here we have a boundary condition on the left end of the rod. Remember, please, that the rod is fixed at its left end, so we have this condition here, and clearly u must be 0 for all times t at x equals 0 .

At the right end, we apply a load R_0 . And there we have, this being here the area times the stress at x equals L , E times a du/dx . Notice that $E du/dx$ is, of course, the stress, and so we have here this total force at the right end being equal to R_0 .

We also have initial conditions for the solution of this equation that I showed to you,

this equation here. We have to have two spatial conditions, two boundary conditions, one at the left and one at the right end. The ones that I just showed to you. We also have to have two initial conditions, one on the displacement and one on the velocity. Well, their initial conditions, in this particular example, might be as shown here. At time 0, all of the displacement are 0. And at time 0, all the velocities along the rod are 0.

So the basic differential equation given here, once again, plus these boundary conditions, plus these initial conditions, define the complete problem. I also like to point out here that this boundary condition here, which does not involve any derivative, is called an essential, or displacement boundary condition. An essential boundary condition because it does not involve any derivative when, and this is important, the highest derivative in this differential equation is 2. The right-hand side boundary condition is called a natural force boundary condition. It's really involving forces. And it involves, as a highest derivative, a derivative of order 1 when the differential equation here involves as the highest derivative a derivative of order 2.

So in general-- and this is a very important point-- we can say the following. If the highest order of the spatial derivative in the problem governing differential equation is $2m$, in our case, m is equal to 1 for our problem. The highest order of the spacial derivative in the essential boundary condition is m minus 1. In other words, in this case, of order 0. The highest order the spatial derivative in the natural boundary conditions that I just discussed is $2m$ minus 1, which is 1 in our particular case.

Then if we talk about this problem, we talk about a C^{m-1} variational problem. It will become apparent to you later on why we call it this way. C^{m-1} means continuity of order $m-1$. In fact, in the Ritz analysis that we will be performing later on, that I will discuss with you later on, we find that we need, in the solution of that kind of problem, only continuity in the Ritz functions of order $m-1$.

Well, let us now look at the variational formulation. I mentioned earlier that we have two different approaches. The first approach is a differential formulation, the second approach is the variational formulation. The variational formulation operates in

much the same way as I introduced it to you for the analysis of discrete systems. We talk about π a functional, being equal to the strain energy minus the potential of the loads.

Now for the rod, the strain energy is given here. Notice this is $1/2$ times the stress times the strain, and integrated over the total volume of the element, or of the rod, I should rather say. The total potential of the loads is given here. I could have written it this way, with a minus out there and a plus in there-- same thing. This, then, is really nothing other than the loads multiplied by the total displacement. And of course, there's an integration involved here, because the body forces, the body loads that I introduced here, f_B , are varying along the length of the rod. I introduce these f_B body forces because I want to use, later on, the d'Alembert principle, put these, in other words, equal to minus the acceleration forces, and can directly apply what I discussed now, also to the dynamic analysis of this rod, which we considered in the differential formulations.

Together with stating π as shown here, we also have to state the left-hand boundary condition, which is an essential boundary condition. We have to list all the essential boundary conditions here, or the displacement boundary conditions. Essential and displacement mean the same thing, in that sense.

Well, then we invoke the stationarity of π . We are saying that $\delta \pi$ shall be equal to 0 for any arbitrary variations of u that satisfy, however, the essential boundary conditions. This boundary condition here. So this has to hold, this statement shall hold, for any arbitrary variations in u . However, $\delta u = 0$ shall be 0, that satisfy, in other words, the essential boundary condition.

Well, if we apply this variation on π , we obtain directly this result here. Notice that this part here is obtained by applying the variation on this part here. The variation operator operates, δ operates, much in the same way as a differential operator. So this tool cancels this at $1/2$, and we are left with $EA \delta u \delta x$ times the variation on $\delta u \delta x$. And that is given right there.

The variation on this part here gives us simply a δu times f_B , and a δu_L times

R. Of course, u_L is equal to is the displacement at x equal to L . And this is, therefore, the final result.

Now if we look at this, we recognize, really, that this is the principle of virtual displacement. It's a principle of virtual displacement governing the problem. In general, we can write this principle as follows. Notice that here, we have variations in strains, which is that part there. The real stresses are given there, which are those. And here we have variations in displacement, those operating on the body forces. There, there.

Notice, I sum here over all body forces in general. We have three components in f_B - the x , y , and z component. So I list these components in a vector that I call f_B . Similarly, of course, we have three displacement components that appear here in this vector U . τ in general has 6 components. ϵ also has 6 components. Putting their transpose on the ϵ vector means that we're summing the product of the strains times these stresses. Similarly, we are summing here the product of the displacement components times the force components.

We also have here a contribution due to surface forces. f_S are the surface forces. 3 components, again, these are the variations in the surface displacement. So this part here, δu_{LR} , really corresponds to this part here, involving surface forces. So the surface forces read again, and the variations in the surface displacements, are those here.

Now notice that this principle here, or this equation, I should rather say, once again has to be satisfied for any arbitrary variations in displacements that satisfy the essential boundary conditions. For the problem of the rod, these δu 's have to satisfy the condition that the variation at the left-hand boundary on u is 0, because that is the essential boundary condition.

Also, of course, notice that this variation in the u 's corresponds to these variations in the strains. In other words, these strains here are obtained from the variations in the displacement. That's important. They are linked together. So if we impose certain variations in the displacement, we have to impose here the corresponding variations

in strains. Of course, these variations in the displacements give us also variations in surface displacement. So these here are again linked up with that.

Later on in our formulation of the finite element method, we will write the variations in the strains here as virtual strains, arbitrary virtual strains. And we are talking about the strain vector with a bar on top of it. Similarly, a bar here, a bar here, instead of the variation sign. However, the meaning is quite identical-- what we are saying here, and this is the principle of virtual displacement that we will be talking about later on, when we formulate the finite element equations, what we are saying here really is that this equation has to be satisfied for any arbitrary virtual displacements and corresponding virtual strains. However, the virtual displacements have to satisfy the displacement boundary conditions, the essential boundary conditions.

Well, from this, or rather that, which of course these equations are completely equivalent, we can go one step further. And if we go one step further, by applying integration by parts and recognizing that this part here is completely equal to that, by that I mean taking the variation on the derivative of x is the same as taking the sorry... by taking variation on the derivative of u with respect to x , that is completely identical to first taking the variation on u , and then taking the differentiation of that variation on u with respect to x .

Well, if we recognize that these two things are identical, then using integration by parts on this equation here, which means, really, on that equation here, for the special case of the rod that we're now considering, we directly obtain this equation here. The integration by parts is performed by integrating this relation here first, and notice that if we do integrate this, we want to lower the differentiation of the virtual part here and increase the order of differentiation on this part here. And that then directly gives us these two terms and that term here.

If we then also list this part here together with what we obtain from that part, we directly obtain this equation here, and we proceed similarly for the coefficients of δu_L , and we obtain this part here. So this equation here is obtained by simply using

integration by parts on, I repeat, this equation here. We have not used any assumption. All we did is a mathematical manipulation of this equation in a different form. And this is the new form that we obtained. Δu_0 , of course, is the variation of u at x equals 0.

Now when we look at this relation, we can extract now the differential equation of equilibrium and the natural or force boundary conditions. How do we extract them? Well, the first point is that I mentioned already earlier, this part here is 0, because Δu_0 is imposed to be 0. So that part is 0. We can strike it out directly.

Now when we look at this part here and that part here and recognize that Δu is now arbitrary, we can directly extract the relation that this must be 0 and that must be 0. How this is done? Well, first of all, we recognize that this integration here really goes from 0 plus to L minus, if you want to be really exact. Because at the boundary, we have the boundary conditions. Of course, this 0 plus means it is infinitesimally close to 0, and L minus, we are integrating up to a distance infinitesimally small to actually L . So putting the 0 plus and L minus here is just for conceptual understanding really necessary.

Well, if we then impose the following variations, say that Δu_L is 0, is exactly 0, then this part is out. And then we only have to look at this part. Now we can apply any arbitrary variation on u from 0 to L minus, and this, then, this total integration, must be equal to 0. It must satisfy the 0 condition. And that can only be true when this part here, what is in the brackets, is 0. Because if this is not 0, I can always select a certain Δu which, when multiplied by that and integrated as shown here-- remember, this part is not there-- will not give us 0. So therefore, this part must be 0. And that is our first condition, which is the equilibrium condition of a differential element. It's the equilibrium condition of a differential element.

Now, if we say, let us look at the right-hand side boundary, and put Δu_0 everywhere along the length of the rod, except at x equals exactly L . Then this part would be 0, and this part here is non-zero, provided-- or rather, this part would be present. And the only way that this part can be 0, as it must be, is that this part

here, the coefficient on δu_L , is actually 0.

So this way, we have extracted two points, two conditions. This part must be 0 and that part must be 0. We extract these conditions by looking at specific variations on u . First we look at the variation on u where δu_L is 0, and otherwise arbitrary from x equals 0 to L minus. And then we can directly conclude this must be 0, and second time around, we say, let δu equal 0 from x equals 0 to L minus, which makes all of this integral 0, and we can focus all our attention on this part here, and we directly can conclude that this part now must also be 0.

So this way, then, we extracted the differential equation of equilibrium and the natural boundary conditions. Very important. Differential equation of equilibrium and the natural boundary condition. And now we really recognize. of course, that if we put f_B equal to the d'Alembert force, or minus the d'Alembert force, equal to this right-hand side, substitute from here back into there, cancel out A , we directly obtain this differential equation. Notice that c is E divided by ρ square root E divided by ρ .

Now, the important point really is that by having started off with this π functional and that condition, this condition also, and that the variations on u at x equals 0 shall be identically 0, we directly can extract from this π functional the differential equation of equilibrium and the natural boundary conditions. The natural boundary conditions, in fact, are contained right in there. That's where the natural boundary conditions appear. The essential boundary conditions are there, and have to be satisfied by the variations.

So in general, then, we find the following points. The important points are that by invoking $\delta \pi$ equals 0 and using the essential boundary conditions only, we generate the principle of virtual displacement, an extremely important fact. And this will be the starting equation that we will be using to generate our finite element equations later on.

We also can extract the problem governing differential equation. Therefore, the problem governing differential equation is contained in the principle of virtual

displacement, and this one is contained in $\delta \pi = 0$, in the $\delta \pi = 0$ condition. Of course also satisfying the essential boundary condition.

We also can extract the natural boundary conditions. So these are also contained, in essence, in π , and as I showed to you, they're contained really in w . That's where they appear.

Now in the derivation of the problem governing differential equation, we used integration by parts. And the highest spatial derivative in π is of order m . We used integration by parts m times, and what we're finding is that the highest spatial derivative in the problem governing differential equation is then $2m$. And this then puts together the complete structure of the equations that we're talking about.

Here I have another view graph that summarizes the process once more. We are starting with the total potential π of the system being equal to the strain energy minus the total potential of the loads. We're using this condition and the essential boundary condition-- very important-- to generate the principle of virtual displacement. At this stage, we can solve the problem, and we will do so by using finite element methods.

We can, however, also go on from the principle of virtual displacement using integration by parts, and then we would derive the differential equation of equilibrium and the natural boundary conditions. We can solve the problem. Well, we can solve the problem at this level really only when we can solve the differential equation of equilibrium, subject to the natural boundary conditions. And that we can really only do for very simple systems.

Therefore, this process here, from here onward, can really be followed only for relatively simple systems. For complex systems, shell structures, complicated shell beam structures, plane stress systems, real engineering structural analysis systems, this is not possible, and we stop right there, and we solve our problem this way, using finite element methods.

Now, the one important point, however, I want to make once more clear-- when we

proceed this way, we are deriving the differential equation of equilibrium for each differential element. And this means that when we go this route for the simple systems that we can go this route, we satisfy the equilibrium condition on each differential element. However, when we go this route, we will find that we only satisfy the equilibrium conditions in a global sense, in an integrated sense, using finite element methods, and that to actually satisfy the equilibrium conditions in a local sense, meaning for each differential element, we have to use many elements, and only then, of course, our finite element solution will converge to the solution that we would have obtained solving the differential equation of equilibrium.

Therefore, when we start from the principle of virtual displacement, if we, as I will show you, use many elements, we really obtain the same solution as here. However, if we do not use many elements, if we use a coarse mesh-- we will talk about a coarse mesh later on-- then we will see that we only satisfy the equilibrium conditions in a global sense for the complete structure, in an integrated sense for the complete structure. For each finite element, we will satisfy the equilibrium conditions, but we will not satisfy the equilibrium conditions accurately for each differential element-- dx , dy , dz being arbitrarily small-- in the continuous body. That will become clearer, then, when we actually go through an example.

Now I mentioned earlier that a whole class of classical methods are the weighted residual methods. In the weighted residual methods, we proceed in the following way. We consider the steady state problem, which is given here, with these boundary conditions. Now this operator $L^2m \phi$ equals R -- this L^2m is the operator that governs the problem. Like in our particular rod problem, L^2m would be equal to $\Delta u \Delta x^2$. So the highest derivative in this spacial operator being $2m$, in this case, 2, for our example.

The boundary conditions can be written this way. And the basic step, then, in the weighted residual and the Ritz analysis is to assume a solution all of this form, where $\bar{\phi}$ gives us the assumed solution a_i are parameters that are unknown, and f_i are bases functions. These are functions that have to be assumed.

Well, there is, of course, considerable concern on what kind of functions to choose. In the weighted residual method, these functions here, if you directly operate on this equation here, should satisfy all boundary conditions. Notice that in the weighted residual method, here I'm really talking about going on this view graph through the differential equations of equilibrium and natural boundary conditions, deriving them, if you want, via this route, and then trying to solve the problem here numerically. We will actually, as I said earlier, in the finite element process, go this route, but we can also go that route with the weighted residual methods. In fact, there is a close relationship between using weighted residual methods at this level and the Ritz method at that level, the way I will be describing it later on.

But in the weighted residual method, this is the assumption. And then if we look at this equation here, we can construct an error R , substituting from here into there. And that capital R error is given by this equation. Notice that this is, of course, our trial function that we have. And if the right-hand side is 0, everywhere over the domain, then of course our error would be 0, and we would have solved our equation that we're looking at here.

That is the equation we want to solve. Since if all of these functions f_i satisfy all of the boundary conditions, then we are satisfying these equations, and all we have to worry about further is to satisfy this equation. If R is 0, we would also satisfy the differential equation of equilibrium, and we would have, in fact, the solution.

However, that, of course, would be a very lucky choice on the f_i functions. In general, R will not be identically 0 all over the domain. In fact, how much R , or how close R will be to 0 will of course depend on the a_i . And this is then basically our objective, namely, to calculate a_i values that are making this R , this left-hand side capital R , as close as possible to 0.

And that can be achieved via the Galerkin method, for example. This is the basic process. Here we're substituting the R . These are the trial functions, and we're integrating the product of these over the total domain. The domain here, in the case of our rod, would simply be the volume of the rod. This is the mechanism that

generates to us n equations in the trial parameters a_i .

In another approach, the least squares method, we would operate on the square of the error, and minimize the square of the error when integrated over the total domain with respect to the trial parameters a_i . That again gives us n equations, and we set up these n equations just as we're doing here, to solve for the a_i . Knowing, then, the a_i , we can back substitute into this assumption here, and now we have our approximate solution, $\bar{\phi}$.

If the trial functions have been selected to satisfy the boundary conditions, then of course $\bar{\phi}$ will satisfy the boundary conditions. However, what $\bar{\phi}$ will not satisfy exactly is this equation here. However, we have minimized the error in the satisfaction of this equation in some sense using the Galerkin method or the least squares method.

These methods can also be extended when the trial functions do not satisfy all of the boundary conditions, namely, not the natural boundary conditions. They can be extended to their places, but classically, they have been used with trial functions that satisfy all boundary conditions. When we were to extend the Galerkin method for the case where the trial functions do not satisfy the natural boundary conditions, then we really talk basically about already a Ritz analysis, and that is the next procedure that I want to introduce to you.

Now of course, if we wanted to really start deriving the Ritz method from the Galerkin approach-- in other words, if we wanted to derive this Ritz analysis method from the Galerkin approach, we would have to extend, first of all, this Galerkin approach to include the natural boundary conditions, and then we would have to perform integrations on this equation, and we would obtain the Ritz analysis method. The actual way of starting, of introducing the Ritz analysis method, is to introduce it as a separate tool.

And it is introduced in the following way. Let π be the functional of the C^{m-1} variational problem that is equivalent to the differential formulation that we talked about earlier. Now, the differential formulation that I talked about here is this one.

That's a differential formulation. And as an example once again, the operator L_2^m is $\frac{d^2 u}{dx^2}$. There's a constant here. We could also put an EA in front here, but that only is a constant. It doesn't change the character of the operator. But this is basically the operator for the problem that we considered, and of course, our boundary conditions are also there.

So if we have a π functional that is equivalent to the differential formulation given in those two equations that I just pointed out to you again, then in the Ritz method, we substitute the trial functions, $\bar{\phi}$ -- let us look at them again, these are the trial functions-- into π , and we generate n simultaneous equations for the parameters that appear in this assumption here by invoking the stationarity of π . Notice that by invoking that $\delta \pi = 0$, we really say that $\delta \pi_{a_i} = 0$ for all i . And that gives us the condition which we used to set up the individual equations that we need to set up to solve for the trial parameters a_i .

Well, let us look at some of the properties. The trial functions used in the Ritz analysis need only satisfy the essential boundary conditions, an extremely important fact. In the classical weighted residual method, as I pointed out, the trial function should satisfy all boundary conditions. Therefore, they can be very difficult to choose. In the Ritz analysis method, we only need to satisfy the essential boundary conditions.

The application of $\delta \pi = 0$ generates the principle of virtual displacement. I mentioned that to you earlier already. And therefore, in effect, we use in the Ritz analysis this principle of virtual displacement. By invoking $\delta \pi = 0$, we minimize basically the violation of the internal equilibrium requirements and the violation of the natural boundary conditions.

Well, remember that invoking $\delta \pi = 0$ and then using integration by parts, we actually generate, we could generate the differential equations of equilibrium and the natural boundary conditions the way I've shown it to you, for the simple bar structure. Now, what I'm saying here is that we do not want to generate these differential equations of equilibrium and natural boundary conditions. However,

please recognize that they are contained in the equation $\delta \Pi = 0$. Therefore, by substituting our trial functions, we violate the internal equilibrium requirements and the natural boundary conditions, but we will see that we are minimizing that violation in these conditions here. Also, we will see that we generate a symmetric coefficient matrix, and K and the governing equations then, our $KU = R$, that we want to solve.

And that is really the basis of the finite element method for the analysis of continuous systems. Let me now go in detail through an example. Here we have a simple bar structure which has an area 1 square centimeter from A to B, and from B to C, B being this point here where the area changes and C being that point there. From B to C, we have a varying area. This variation in the area is shown here. It's $1 + y/40$ squared is the area at any station y , y being measured from point B, as you can see here. The length here is 100 centimeters. This length is 80 centimeters. The structure is subjected to a load of 100 newtons here. Notice that this arrow really lies on top of that dashed line we just separated out for you to understand that there is this arrow.

So this is the load applied at the mid-line of this structure. We assume once again for the structure also that there is only the following displacement mechanism. If a section was originally there, and it is a vertical section to the midline, then it has moved over, and I grossly exaggerate now, to this position, where this is the displacement that we're talking about u . Grossly exaggerated, of course.

So we're having a bar structure subjected to a concentrated load fixed at the left end. And our objective now is to solve this structure, to solve for the unknown displacement u , being 0 here, of course as a function of x , when this structure is subjected to that load. Well, in the calculation of this example, I want to display to you as many of the concepts that we just discussed. Here we have Π being equal to this value here. The strain energy is given here. Notice this is $1/2$ times the stress times the strain integrated over the volume of the structure. The integration goes from 0 to 180, because that is the length of that structure. The total potential of the external load is 100, which is the intensity of the load times the displacement at x

equal to 180... at x equal to 180. Well, the essential boundary condition is that u is 0 at x equals 0.

I'd like to now consider two different cases for the Ritz analysis. In the case one I want to use a function that spans u , which spans continuously-- and let me draw it out here-- from u , from x equals 0, to x equals 180. That is the endpoint. So here we have this function, this part here and that part there, these are the two trial parameters that we want to solve for. And we will select them, we will calculate them, rather, using the Ritz analysis.

Case two, I also use trial functions, but notice now that I'm performing the following. We have a domain AB-- let me go back once more-- a domain AB, and a domain BC. And I want to now use one function for AB and one function for BC. The AB function is simply this one here, a linear variation up to this point. Now notice that u_B is our trial parameter-- that's the one we don't know, our trial function parameter. $x/100$ is simply is the function that I'm talking about. And notice that this function only is applicable for this domain where, let me put down here the length that is x equal to 100, and this is here x equal to 180.

Now for this part here, I use this function here. Now notice what this function does. Well, if we look at this part here in front, it involves u_B , which is also there, and it involves u_C . u_B , by the way, is the physical displacement right here into this direction. Of course, I'm plotting u upwards here to be able to show it to you. But the displacement, u_B , is the displacement of this point B to the right. u_C is the displacement of this point C to the right.

Then we recognize that this part here corresponds really to a variation such as that. Notice when x is equal to 100, which is that point, this function here is 1. When x is equal to 180, which is that point there, this part is equal to 0, because $180 - 100$ is 80, divided by 80 is 1, and $1 - 1$ is 0. So this dashed line corresponds to this function here. Let me put a dashed line underneath there.

Well, if we now look at this part here, we notice that this part is 0, or this trial function here is 0 at this point B. And it varies linearly like that across, where this

part here, of course, denotes u_C . That is this one here, solid black line, and here also, solid black line.

Now the superposition of both these functions, the dashed blue and the solid black line, give us this function here. So the actual function that I'm talking about is a linear variation along here, and a linear variation along here, where I plot it vertically up here-- u_B and u_C here.

Now, this is a specific case that I want to draw your attention on, because this really corresponds, as we shall see, to a true finite element analysis. And the reason for it is that we're talking about one domain here and another domain there. And both of these domains are identified as finite elements. Well, the first step now is to use π , invoke the stationarity condition as we did earlier, and this gives us the principle of virtual displacement. I mentioned it earlier already. Our virtual strains are here. The stresses are here. The virtual work is on this side. I discussed it earlier already.

We do not want to go now via this route. We first of all want to now obtain the exact solution. The exact solution is obtained by using integration by parts on $\delta \pi$, being, of course, equal to 0, and extracting the differential equation of equilibrium for each differential element in this structure. This means that if we are talking here about the differential element equilibrium of each differential element dx long anywhere along the structure, in other words, the equilibrium of typically an element like that. That is a differential equation of equilibrium. And we also, of course, have the natural boundary conditions. We can also derive the natural boundary conditions. The solution to this is obtained by integration, and this is the solution given.

Well, the stresses, then, of course are obtained by differentiation of the u 's to get strains, and multiplying those by E , and these are the stresses in the bar. These are the exact stresses in the bar that satisfy the differential equations of equilibrium and the natural boundary conditions. This is the exact solution of this bar problem, the way I have formulated it.

Now will perform our Ritz analysis. In case one, we use π equal to this. Notice that I

have substituted now our trial functions corresponding to case one into the functional π . That gives us this term, that term here, and that term here. Notice that I have broken up the integration from 0 to 100, and 100 to 180, because the area changes from over this length here, and only for that reason, really, I have broken up the integrations.

Now this is π . And if we now invoke, we can integrate this out, and then invoke that $\delta \pi$ shall be 0, we obtain this set of equations. We solve for a_1 and a_2 , substitute back into our assumption that we had earlier, and we got this u . Notice that of course this u displacement does satisfy the essential boundary conditions. It does satisfy the essential boundary conditions at x equals 0. You can just substitute x equals 0, and you would see that u is 0.

It does not satisfy, however, the natural boundary condition at x equal to 180. σ is given here, obtained by calculating the strains from here and multiplying by E -- this is our approximate solution to the problem. We are satisfying the compatibility conditions, because the bar has remained together. No material has been cut away from it. Also, we are satisfying the constitutive relations, but we do not satisfy the internal equilibrium on a differential local elements sense. We do not satisfy the differential equilibrium, and we do not satisfy the natural boundary conditions. But we satisfy them in an approximate sense.

Case two. Here now we're talking about our two linear functions. And here we naturally integrate from 0 to 100 for the first linear function, and from 100 to 180 for the second linear function. Notice this is, again, the area, and notice that this is here the strain squared. It's strain squared here because our E is out there, which would give us the stress. And the area here, of course, is equal to 1, which we did not write down.

The important point is that this is now our π for these two functions. We again invoke $\delta \pi$ equal to 0. We obtain now this set of equations. We are solving from this set of equations u_B and u_C , given here. Having got u_B and u_C , of course we now have the complete displacements along the bar, because we only need to

substitute back into our original approximations that we looked at earlier. Let me just get them once more here. We had them here. If we now substitute from u_B and u_C into these two equations, we have the complete displacement solution. Of course, this is an approximate displacement solution. And similarly, our stresses are approximate.

Now on these last few graphs, I have plotted the solution. And notice that this is here the direction x . Here we have the point B, here we have the point C, here we have the point A. Our exact solution, which satisfies the constitutive relations, compatibility relations, and the differential equations of equilibrium, and all bounded conditions, is the solid line here. Our solution one, case one, Ritz analysis, is the dashed line here, and the solution two is this dashed dotted line, down there.

Notice that we are quite close in our Ritz analysis to the exact solution in the displacement. However, the strains and stresses are obtained by the differentiation of these displacement solutions, and here I show to you the calculated stresses. Again, point A here, point B here, point C there.

The important point is the following. In the exact solution, we have the stress of 100 in domain AB, and then we have this curve here, a very high slope there. And in our solution one, we had this variation in stress. Notice that it goes continuously over the complete domain, because our assumed displacement function was continuous also over this domain, and its first derivative was continuous over this complete domain. So that's why our solution one is continuous there, and in fact, we're seeing just the straight line there, because our displacement approximation was parabolic. Our solution two is exact here, 100, and very approximate here for the displacement. But notice that at the midpoint between B and C, we get very good results.

Now the important point really is shown here on the last view graph. We note that in this last analysis, we use trial functions that do not satisfy the natural boundary condition, and I'm talking now about the piecewise linear functions, in other words, from A to B and B to C each, just a straight line. We use trial functions that do not

satisfy the natural boundary conditions. The trial functions themselves are continuous, but the derivatives are discontinuous at point B. Notice our stresses here are discontinuous at point B.

For a C^{m-1} variational problem, the way I've defined it, we only need continuity in the $m-1$ first derivatives of the functions. In this problem, m is 1, and therefore we only need continuity in the functions themselves, and not in any derivatives, because we only need continuity in the $m-1$ first derivative. The domains A and B and B and C are finite elements, and in actuality, we've performed a finite element analysis.

This is all I wanted to say in this lecture. Thank you for your attention.