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Continuum Electromechanics

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Problems for Chapter 10

For Section 10.2:

Prob. 10.2.1 The region between two planes, at $x = \Delta$ and $x = 0$, is filled with a material having uniform thermal properties that sustains fully developed flow with velocity $\vec{v} = v(x)\hat{i}_y$. The surfaces are at the respective constant temperatures (T^α, T^β). In the volume, there is an arbitrary dissipation $\phi_d(x)$.

- (a) Determine the temperature distribution $T(x)$.
- (b) What is the thermal flux at the boundaries? Note that this is one of a group of "fully developed" heat conduction configurations, playing a role in heat transfer analogous to the fluid mechanics relations of Table 9.3.1.

For Section 10.3:

Prob. 10.3.1 The magnetically excited layer considered in this section is embedded in a system in which the surroundings are relatively thermally insulating. The temperature of the layer rises to a sufficient extent that the steady dissipation is accommodated by the steady heat flux. However, insofar as the time-varying part of the heat flux is concerned, the layer surfaces are bounded by thermal insulators. What are the temperatures at the layer surfaces?

Prob. 10.3.2 The moving slab of Fig. 10.3.1 is now a semi-insulating dielectric having uniform electrical conductivity σ and permittivity ϵ . Potential distributions at the α and β surfaces are respectively $\text{Re}\hat{\Phi}^\alpha \exp j(\omega t - ky)$ and $\text{Re}\hat{\Phi}^\beta \exp j(\omega t - ky)$.

- (a) Write the electrical dissipation density in the form of Eq. 10.3.6.
- (b) Find the temperature distribution throughout the slab and the heat fluxes at its surfaces. Assume that at the α and β surfaces the respective temperatures are

$$T_0^\alpha + \text{Re}\hat{T}^\alpha \exp j(\omega_2 t - k_2 y) \text{ and } T_0^\beta + \text{Re}\hat{T}^\beta \exp j(\omega_2 t - k_2 y).$$

For Section 10.4:

Prob. 10.4.1 A ferrofluid has a permeability that has the temperature dependence $\mu = \mu_a [1 - \alpha_\mu (T - T_a)]$, where μ_a and α_μ are constant parameters. In the channel of Fig. 10.4.1, the fluid is subjected to a uniform transverse magnetic field intensity H_0 . The object is to pump the fluid by imposing the temperatures T_a and T_b on the grids, and hence producing a variation in the permeability in the direction of the heat flux. Assume that the boundary layer thickness is small compared to the channel cross section, so that the velocity is uniform across the channel. Determine the pressure-velocity relation that is analogous to Eq. 10.4.7 and the temperature distribution and heat flux.

For Section 10.5:

Prob. 10.5.1 The rotor described by Eqs. 10.5.16 - 10.5.18 is in the state of steady rotation described by Eqs. 10.5.23.

- (a) Show that this stationary state is overstable if R_a exceeds

$$R_a = P_T \frac{(1+f)}{f} \frac{P_T + 4(1+f)}{P_T - 2(1+f)}$$

- (b) Show that the frequency of oscillation at the onset of this instability is

$$\omega = \frac{1}{\tau_T} \sqrt{2P_T(1+f) \frac{[P_T + (1+f)]}{[P_T - 2(1+f)]}}$$

Prob. 10.5.2 The rotor of Fig. 10.5.1 is heated from the side rather than from below. Thus the external temperature distribution is given by Eq. 10.5.1 with $\sin\theta \rightarrow \cos\theta$.

- (a) Deduce the equations of motion, similar to Eqs. 10.5.16 - 10.5.18.
- (b) Use a graphical solution similar to that pictured by Fig. 10.5.3 to determine the steady angular velocity. Explain qualitatively the direction of rotation.

For Section 10.6:

Prob. 10.6.1 Implicit to Eq. 10.6.17 is the principle of exchange of stabilities. That is, as R_{am} is raised, each temporal mode becomes unstable with $s_n = 0$. If it is only the condition for onset of instability that is of interest, it can be assumed at the outset that $s_n = 0$ and R_{am} can be treated as an eigenvalue. Thus $(R_{am})_n$ is the value of R_{am} that reduces the frequency of the n^{th} mode to zero.

- (a) Use Eqs. 10.6.8 and 10.6.9 with the boundary conditions that $\underline{T} = 0$ and $\underline{v}_x = 0$ on the boundaries $\underline{x} = 0, \underline{x} = 1$ to show that, provided $R_{am} > 0$, the principle of exchange of stabilities holds. (See the Temporal Modes subsection of Sec. 8.18.)
- (b) Set $\underline{\omega} = 0$ in Eqs. 8 and 9 and solve the eigenvalue problem for R_{am} . The result should be Eq. 10.6.18 and hence 10.6.19.

Prob. 10.6.2 For the thermal-hyromagnetic layer between the planes α and β as treated in this section, determine the transfer relations

$$\begin{bmatrix} \hat{\Gamma}^\alpha \\ \hat{\Gamma}^\beta \\ \hat{p} \\ \hat{p} \end{bmatrix} = [C_{ij}] \begin{bmatrix} \hat{T}^\alpha \\ \hat{T}^\beta \\ \hat{v}_x^\alpha \\ \hat{v}_x^\beta \end{bmatrix}$$

Prob. 10.6.3 Consider the layer of Fig. 10.6.2, but with viscosity.

- (a) Show that the normalized equations replacing Eqs. 10.6.8 and 10.6.9 are

$$\begin{aligned} [(D^2 - k^2) - \frac{j\omega}{P_T} (D^2 - k^2) - H_m^2 D^2] \hat{v}_x &= -R_a \hat{T} \\ [j\omega - (D^2 - k^2)] \hat{T} &= -\hat{v}_x \end{aligned}$$

where

$$\begin{aligned} \omega &= \underline{\omega} K_T / \Delta^2 & \hat{T} &= \underline{T} \Delta D T_s \\ x &= \underline{x} \Delta & \hat{v}_x &= \underline{v}_x K_T / \Delta \\ k &= \underline{k} / \Delta & p &= p \Delta^2 / K_T^2 \rho_o \end{aligned}$$

and the conventional Rayleigh, Prandtl and Hartmann numbers are

$$R_a \equiv \frac{\alpha \rho_o g \Delta^4 D T_s}{K_T \eta} ; \quad P_T \equiv \frac{\eta}{\rho K_T} = \frac{\tau_T}{\tau_v} ; \quad H_m^2 \equiv \frac{\mu \sigma \Delta^2 \mu_o H_o^2}{\eta}$$

- (b) Outline a scheme to determine the transfer relations expressing the surface stresses and heat flux ($\hat{s}_x^\alpha, \hat{s}_x^\beta, \hat{s}_y^\alpha, \hat{s}_y^\beta, \hat{T}_x^\alpha, \hat{T}_x^\beta$) in terms of the surface velocities and temperatures ($\hat{v}_x^\alpha, \hat{v}_x^\beta, \hat{v}_y^\alpha, \hat{v}_y^\beta, \hat{T}^\alpha, \hat{T}^\beta$). The motions may be assumed to be independent of z , so $k_z = 0$.

For Section 10.7:

Prob. 10.7.1 A thin metal cylinder having radius R is charged by unipolar ions having the density ρ_o at the radius a from the cylinder's center. Assume that at a given instant the charge per unit length on the cylinder is λ and that the self fields of the ions in the volume are negligible compared to those due to the charge on the cylinder.

- (a) Determine the ion charge density as a function of radial distance r .
- (b) What is the current per unit length collected by the cylinder as a function of the voltage of the cylinder relative to that at $r = a$?
- (c) If the cylinder is allowed to charge up, what is $\lambda(t)$ given that when $t = 0, \lambda = 0$?

Prob. 10.7.2 A pair of horizontal capacitor plates are used to impose a perpendicularly directed electric field on a homogeneous layer of liquid bounded from above (at $x = 0$) by air. Model the liquid as devoid of all but one positive species of electrical carriers with charge density ρ_+ . Assume that charge in the neighborhood of the interface shields the field from the liquid bulk so that $\vec{E} = E_0 \vec{i}_x$ at $x = 0$ and $\vec{E} \rightarrow 0$ as $x \rightarrow -\infty$. Hence, self fields of the ions are included.

- (a) With negligible net current through the air, and hence in the liquid, show that the electric field and charge density comprising the monolayer of surface charge for $x < 0$ are

$$E_x = E_0 / (1 - x/\ell_d) ; \quad \rho_+ = (\epsilon E_0 / \ell_d) / (1 - x/\ell_d)^2 ; \quad \ell_d = 2K_+ / bE_0$$

- (b) For $E_0 = 10^4$ v/m, what is a typical value of ℓ_d ?

For Section 10.8:

Prob. 10.8.1 An electrolyte is bounded by plane parallel boundaries, each having the potential $-\zeta$. They are positioned at $x = 0$ and $x = \Delta$.

- (a) Under the assumption that $\underline{\Phi} \ll 1$, what is the distribution of $\underline{\Phi}$? What is the potential $\underline{\Phi} \equiv \underline{\Phi}_c$ at the midplane?
- (b) For arbitrary magnitude of $\underline{\Phi}$, show that in terms of normalized variables the potential distribution is

$$\underline{x} = \int_{-\underline{\zeta}}^{\underline{\Phi}} \frac{d\underline{\Phi}}{\sqrt{2 \cosh \underline{\Phi} - \cosh \underline{\Phi}_c}}$$

where again $\underline{\Phi}_c$ is the potential at the midplane.

- (c) Given the normalized spacing $\underline{\Delta} \equiv \Delta/\delta_0$, describe a numerical procedure for finding $\underline{\Phi}_c$ and hence determining the potential distribution.
- (d) For $\underline{\Delta} = 2$ and $\underline{\zeta} = 3$, what is $\underline{\Phi}_c$? Plot the potential distribution.

For Section 10.9:

Prob. 10.9.1 The boundaries of a planar duct, such as pictured in Table 9.3.1, have a spacing Δ that is not necessarily large compared to δ_d .

- (a) Used Eq. a from Table 9.3.1 to express the velocity distribution in terms of the potential distribution.
- (b) Show that this expression reduces to Eq. 10.9.5 in the case where the Debye length is short compared to the channel width.
- (c) In Prob. 10.8.1, a procedure is developed for finding the potential distribution with arbitrary wall spacing. Show that the velocity distribution can be written in the normalized form

$$\underline{v} = \frac{\Delta^2 q}{2\epsilon E_y kT} \frac{\partial p'}{\partial y} \underline{x}^2 \left(\frac{\delta_D}{\Delta}\right)^2 - \underline{x} \frac{\delta_D}{\Delta} + \underline{\Phi}(\underline{x}) + \underline{\zeta}$$

where $v = v_y \epsilon E_y kT / \eta q$ and $x = x \delta_D$ and where $\underline{\Phi}(x)$ follows from Prob. 10.8.1.

Prob. 10.9.2 A two-dimensional channel having width Δ has walls with potentials $\Phi = -\zeta$. The current density in the y direction is "fully developed" and hence the total current through the channel is given by Eq. 10.9.13.

- (a) Show that the current is related to the imposed E_y and the pressure gradient $\partial p / \partial y$ by

$$i = \sigma \Delta + \frac{2\rho_o \zeta^2 \epsilon \delta_D}{\eta(kT/q)} E_y - \frac{\zeta \Delta \epsilon}{\eta} \frac{\partial p'}{\partial y}$$

- (b) For an "open-circuit" channel ($i = 0$) having a length ℓ and pressure difference $\Delta p = -\ell \partial p / \partial y$, what is the streaming potential $v \equiv -E_y \ell$?

For Section 10.10:

Prob. 10.10.1 Following Eq. 10.10.2, it is argued that the shear stress induced surface current is ignorable compared to that driven by the imposed field. Approximate the shear stress contribution using the velocity U that was determined and justify this approximation.

Prob. 10.10.2 The particle considered in this section is fixed on a "stinger" which does not distort the field or impede the flow but does constrain the particle to a fixed position relative to the fluid at infinity. What is the force imparted by the electric field to the stinger?

Prob. 10.10.3 The particle is fixed on a stinger, as in Prob. 10.10.2, but both a uniform electric field and a uniform flow velocity are imposed at infinity. Because the flow is now forced, the contributions of the shear stress to the surface current can be significant. In view of Eq. 10.9.12, represent the surface current as

$$K_{\theta} = \sigma_s E_{\theta} + \beta S_{\theta r}$$

where for $\zeta < kT/q$, $\beta = 2\rho_o \zeta \delta_D^2 / \eta(kT/q)$ and determine the potential distribution around the particle as a function of E_o and U . What is the potential if $E_o = 0$? What is f_z ?

For Section 10.11:

Prob. 10.11.1 A clean interface is modeled as having a surface tension γ_o at the voltage $v_d = \phi_d$, the tension being independent of the area A , and a Helmholtz double layer consisting of a plane parallel capacitor having spacing Δ , permittivity ϵ and zero double layer charge at $v_d = \phi_d$. Determine C_d , σ_d and W'_s , and compare to Fig. 10.11.1.

Prob. 10.11.2 A hemisphere of mercury submerged in an electrolyte is shown in cross section in Fig. P10.11.2. The interface between liquids forms a double layer of thickness Δ , pictured here as being a "Helmholtz" layer. (Prob. 10.11.1)

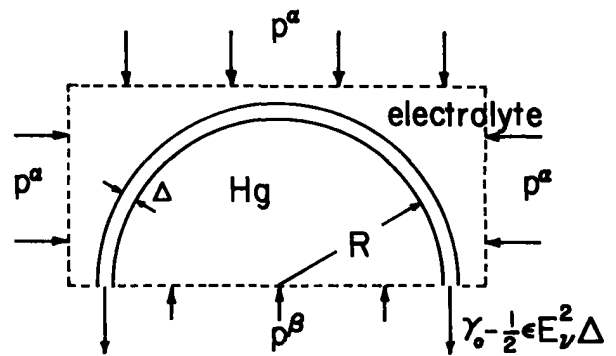


Fig. P10.11.2

- Write an expression for static equilibrium using the control volume shown to balance the pressure forces against those due to the combined surface tension and Maxwell stresses. Show that the resulting expression is as would be deduced from Eq. 10.11.1, where the electrocapillary surface tension is found in Prob. 10.11.1.
- Now suppose that, by means of an orifice at the center of the hemisphere, a small additional amount of mercury is introduced, so that the interface expands from R to $R + \delta\xi$. Use the result of (a) to compute the incremental change in pressure implied by the electrocapillary model.
- An alternative model might depict the double layer as composed of a film of insulating fluid. In that case, the equilibrium would take the same form as found in (a). But, suppose that with the addition of an increment of mercury the surface expands in such a way that the insulating layer of fluid preserves its volume. Find an expression for the change in pressure associated with an incremental change in radius $\delta\xi$. Compare the result to that found in (b) and explain the difference.

For Section 10.12:

Prob. 10.12.1 With the objective of determining the mobility $b = U/E_o$ of the mercury drop in an electrolyte, consider a drop that is highly conducting, with a surrounding electrolyte permeated by an electric field which is $E_o \hat{i}_z$ far from the drop. Following steps paralleling those in this section, show that the mobility is

$$b = \sigma_o R / \left(\frac{\sigma_o}{\sigma} \right) + (2\eta_a + 3\eta_o)$$

A mercury drop in an electrolyte is the configuration of a dropping mercury electrode, widely used to study electrochemical double layers because the surface is constantly renewed by continual

Prob. 10.12.1 (continued)

generation of drops.¹ The dropping mercury electrode is used in analytical chemistry as a sensitive means of measuring trace constituents of the electrolyte.²

Prob. 10.12.2 A linear volume rate of flow is secured in the configuration of Fig. P10.12.2 by exploiting the double layer shearing surface force density. An electrolyte is bounded from above by insulating walls and from below by alternate sections of insulator and pools of mercury, each having length $l \gg a$ or b .

Electrodes fixed adjacent to the pool edges are driven by an external current source and cause a "standing wave" of current with the distribution sketched. Hence, the ideally polarized double layer experiences a shearing surface force density tending to carry the liquid in one direction, while the insulating sections prevent backward motion where that force density would be reversed.

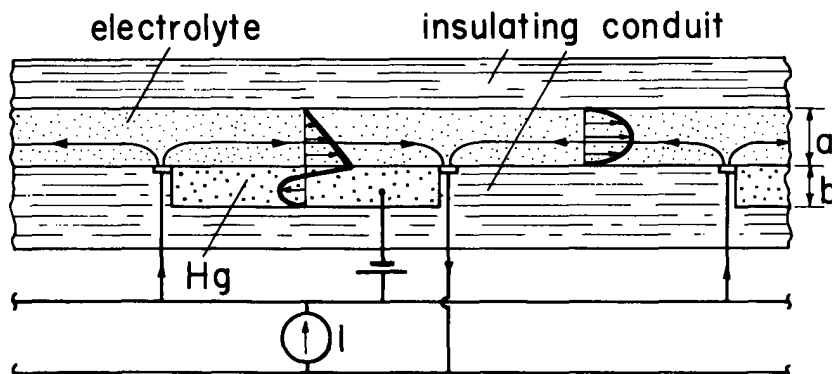


Fig. P10.12.2

- (a) Model the system as quasi-one-dimensional, assuming fully developed plane flow in each of the sections and using mass and momentum conservation to piece these flows together at the pool edges. Assume that gravity holds the interface flat and that the system is closed on itself. Assume that the electrolyte is sufficiently highly conducting that charge convection at the interface can be ignored and the interface can be regarded as essentially uniformly polarized (even with the driving current producing a voltage drop in the interfacial plane).
- (b) Find the volume rate of flow of the electrolyte as a function of the driving current.

1. An extensive treatment of the subject is given by V. G. Levich, Physicochemical Hydrodynamics, Prentice-Hall, Englewood Cliffs, N.J., 1965, pp. 493-551.

2. J. Heyrovský and K. Jaroslav, Principles of Polarography, Academic Press, New York, 1966.