Continuum Electromechanics

To Janet Damman Melcher

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James R. Melcher

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Preface

The three stages in which this text came into being give some insight as to how the material has matured. As "notes" written in the early 1960's, it was intended to serve as an introduction to the subject of electrohydrodynamics. Thus, it reflected the author's early research interests. During this period, the author had the privilege of collaborating with Herbert H. Woodson (now University of Texas) on the development of an undergraduate subject, "Fields, Forces and Motion". That effort resulted in the text Electromechanical Dynamics (Wiley, 1968). There has also been a strong influence from Hermann A. Haus, with whom the author has collaborated for a number of years in the development and teaching of an undergraduate electromagnetic field theory subject. Both Woodson, with his interests in rotating machinery and magnetohydrodynamics, and Haus, who then worked in areas ranging from electron beam engineering and plasmas to the electrodynamics of continuous media, stimulated the notion that there was a set of fundamental ideas that permeated many different "specialty areas". To be taught were widely applicable basic laws, approaches to modeling and mathematical techniques for disclosing what the models had to say.

The text took its second form in 1972-1973, when the objective was to achieve this broader and more enduring aspect of the material. Much of the writing was done while the author was on a Guggenheim Fellowship and a Fellow of Churchill College, Cambridge University, England. During that year, as a guest of George Batchelor's Department of Applied Mathematics and Theoretical Physics, and with the privilege of working with Sir Geoffrey Taylor, there was the opportunity to further broaden the perspective. Here, the influences were toward the disciplines of continuum mechanics.

Unfortunately, the manuscript resulting from this second writing was more in the nature of two books than one. More integration and culling of material was required if the self-imposed objective was to be achieved of helping to define a discipline rather than simply covering a number of interrelated topics.

The third version, this text, would probably not have come into being had it not been for the active encouragement of Aina Sils. Her editorial help and typewriter artistry provided teaching material that was immediately sufficiently attractive to serve as an incentive to commit nights and weekends to yet another rewrite.

As a close colleague who has been instrumental in establishing as an area the continuum electromechanics of biological systems, Alan J. Grodzinsky has been both a source of technical insight and an inspiration to complete the publication of material that for so many years had been referenced in theses as "notes."

Research carried out by still other colleagues at MIT will be seen to have influenced the scope and content. The Electric Power Systems Engineering Laboratory, directed by Gerald L. Wilson,is an example with its activities in superconducting machinery (James L. Kirtley, Jr.) and its model power system (Steven D. Umans). Others are the High Voltage Laboratory (John G. Trump and Chathan M. Cooke), the National Magnet Laboratory (Ronald R. Parker and Richard D. Thornton), the Research Laboratory of Electronics (Paul Penfield, Jr. and David H. Staelin), the Materials Processing Center (Merton C. Flemings), the Energy Laboratory (Janos M. Beer and Jean F. Louis), the Polymer Processing Program (Nam P. Suh), and the Laboratory for Insulation Research,(Arthur R. Von Hippel and William B. Westphal).

A great satisfaction and motivation has come from seeing the ideas promolgated here serve the needs of industry. The author's consulting activities, for more than 30 different companies, provided many useful examples. In the face of an increasing awareness of the importance of energy to our societal institutions and our way of life, it has been satisfying to see the concepts presented here applied not only to the development of new energy systems, but to the conflicting problem of environmental control as well.

Where possible, examples have intentionally been chosen that can be illustrated with generally available films. Referenced in Appendix C, these are in two series. The series from the National Committee on Fluid Mechanics Films was being developed at the Education Development Center while the author was active in making three films in the series from the National Committee on Electrical Engineering Films. Interaction with such individuals as Ascher H. Shapiro and J. A. Shercliff fostered an interest in using films to enliven and undergird classroom education.

While graduate students involved with the subject or carrying out their PhD theses, a number of people have made substantial contributions. Some of these are James F. Hoburg (Secs. 8.17 and 8.18), Jose Ignacio Perez Arriaga (Secs. 4.5 and 4.8), Peter W. Dietz (Sec. 5.17), Richard S. Withers (Secs. 5.8 and 5.9), Kent R. Davey (Sec. 8.5), and Richard M. Ehrlich (Sec. 5.9).

Problems at the ends of chapters were typed by Eleanor J. Nicholson. Figures were drawn by the author.

Solutions to the problems have been prepared in the form of a manual. Intended as an aid to those either presenting this material in the classroom or using it for self-study, this manual is available for the cost of reproduction from the author. Requests should be over the signature of either a member of a university faculty or the industrial equivalent.

James R. Melcher

Cambridge, Massachusetts
January, 1981

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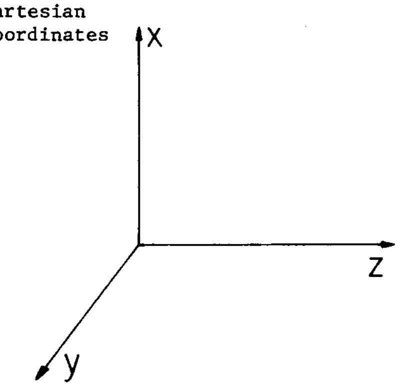
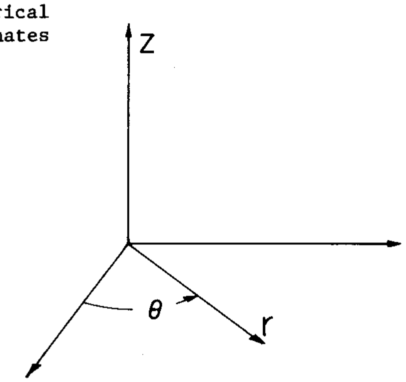
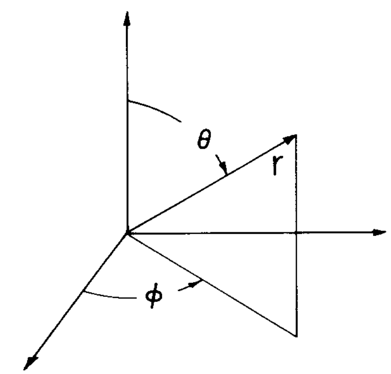
Appendix A

Differential Operators in Cartesian, Cylindrical and Spherical Coordinates

APPENDIX A. Differential Operators in Cartesian, Cylindrical and Spherical Coordinates

Operator	Cartesian coordinates	Cylindrical coordinates	Spherical coordinates
			
$(\nabla\cdot\vec{A})$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$
$\nabla\Phi$	$\frac{\partial\Phi}{\partial x}\vec{i}_x + \frac{\partial\Phi}{\partial y}\vec{i}_y + \frac{\partial\Phi}{\partial z}\vec{i}_z$	$\frac{\partial\Phi}{\partial r}\vec{i}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\vec{i}_\theta + \frac{\partial\Phi}{\partial z}\vec{i}_z$	$\frac{\partial\Phi}{\partial r}\vec{i}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\vec{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\vec{i}_\phi$
$(\nabla^2\phi)$	$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$	$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\Phi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\Phi}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Phi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$
$(\nabla\times\vec{A})$	$\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{i}_x$ $+\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{i}_y$ $+\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{i}_z$	$\left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right)\vec{i}_r$ $+\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{i}_\theta$ $+\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)\vec{i}_z$	$\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}\right)\vec{i}_r$ $+\left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right)\vec{i}_\theta$ $+\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)\vec{i}_\phi$

$\nabla^2\vec{A}$	$\left[\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_x}{\partial y^2}+\frac{\partial^2 A_x}{\partial z^2}\right]\vec{i}_x + \left[\frac{\partial^2 A_y}{\partial x^2}+\frac{\partial^2 A_y}{\partial y^2}+\frac{\partial^2 A_y}{\partial z^2}\right]\vec{i}_y + \left[\frac{\partial^2 A_z}{\partial x^2}+\frac{\partial^2 A_z}{\partial y^2}+\frac{\partial^2 A_z}{\partial z^2}\right]\vec{i}_z$	$\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_r)\right)+\frac{1}{r^2}\frac{\partial^2 A_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial A_\theta}{\partial \theta}+\frac{\partial^2 A_r}{\partial z^2}\right]\vec{i}_r + \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 A_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial A_r}{\partial \theta}+\frac{\partial^2 A_\theta}{\partial z^2}\right]\vec{i}_\theta + \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 A_z}{\partial \theta^2}+\frac{\partial^2 A_z}{\partial z^2}\right]\vec{i}_z$	$\left[\nabla^2 A_r-\frac{2A_r}{r^2}-\frac{2}{r^2}\frac{\partial A_\theta}{\partial \theta}-\frac{2A_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial \phi}\right]\vec{i}_r + \left[\nabla^2 A_\theta+\frac{2}{r^2}\frac{\partial A_r}{\partial \theta}-\frac{A_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\phi}{\partial \phi}\right]\vec{i}_\theta + \left[\nabla^2 A_\phi-\frac{A_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial \phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial \phi}\right]\vec{i}_\phi$
$\vec{C}\cdot\nabla\vec{A}$	$\left(C_x\frac{\partial A_x}{\partial x}+C_y\frac{\partial A_x}{\partial y}+C_z\frac{\partial A_x}{\partial z}\right)\vec{i}_x + \left(C_x\frac{\partial A_y}{\partial x}+C_y\frac{\partial A_y}{\partial y}+C_z\frac{\partial A_y}{\partial z}\right)\vec{i}_y + \left(C_x\frac{\partial A_z}{\partial x}+C_y\frac{\partial A_z}{\partial y}+C_z\frac{\partial A_z}{\partial z}\right)\vec{i}_z$	$\left(C_r\frac{\partial A_r}{\partial r}+\frac{C_\theta}{r}\frac{\partial A_r}{\partial \theta}+C_z\frac{\partial A_r}{\partial z}-\frac{C_\theta A_\theta}{r}\right)\vec{i}_r + \left(C_r\frac{\partial A_\theta}{\partial r}+\frac{C_\theta}{r}\frac{\partial A_\theta}{\partial \theta}+C_z\frac{\partial A_\theta}{\partial z}+\frac{C_\theta A_r}{r}\right)\vec{i}_\theta + \left(C_r\frac{\partial A_z}{\partial r}+\frac{C_\theta}{r}\frac{\partial A_z}{\partial \theta}+C_z\frac{\partial A_z}{\partial z}\right)\vec{i}_z$	$\left(C_r\frac{\partial A_r}{\partial r}+\frac{C_\theta}{r}\frac{\partial A_r}{\partial \theta}+\frac{C_\phi}{r\sin\theta}\frac{\partial A_r}{\partial \phi}-\frac{C_\theta A_\theta}{r}-\frac{C_\phi A_\phi}{r}\right)\vec{i}_r + \left(C_r\frac{\partial A_\theta}{\partial r}+\frac{C_\theta}{r}\frac{\partial A_\theta}{\partial \theta}+\frac{C_\phi}{r\sin\theta}\frac{\partial A_\theta}{\partial \phi}+\frac{C_\theta A_r}{r}-\frac{C_\phi A_\phi\cot\theta}{r}\right)\vec{i}_\theta + \left(C_r\frac{\partial A_\phi}{\partial r}+\frac{C_\theta}{r}\frac{\partial A_\phi}{\partial \theta}+\frac{C_\phi}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}+\frac{C_\phi A_r}{r}+\frac{C_\phi A_\theta\cot\theta}{r}\right)\vec{i}_\phi$
$(\overset{\Rightarrow}{T}:\nabla\vec{A})$	$T_{xx}\left(\frac{\partial A_x}{\partial x}\right)+T_{yy}\left(\frac{\partial A_y}{\partial y}\right)+T_{zz}\left(\frac{\partial A_z}{\partial z}\right) + T_{xy}\left(\frac{\partial A_x}{\partial y}+\frac{\partial A_y}{\partial x}\right) + T_{yz}\left(\frac{\partial A_y}{\partial z}+\frac{\partial A_z}{\partial y}\right)+T_{zx}\left(\frac{\partial A_z}{\partial x}+\frac{\partial A_x}{\partial z}\right)$	$T_{rr}\left(\frac{\partial A_r}{\partial r}\right)+T_{\theta\theta}\left(\frac{1}{r}\frac{\partial A_\theta}{\partial \theta}+\frac{A_r}{r}\right)+T_{zz}\left(\frac{\partial A_z}{\partial z}\right) + T_{r\theta}\left(r\frac{\partial}{\partial r}\left(\frac{A_\theta}{r}\right)+\frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)+T_{\theta z}\left(\frac{1}{r}\frac{\partial A_z}{\partial \theta}+\frac{\partial A_\theta}{\partial z}\right) + T_{rz}\left(\frac{\partial A_z}{\partial r}+\frac{\partial A_r}{\partial z}\right)$	$T_{rr}\left(\frac{\partial A_r}{\partial r}\right)+T_{\theta\theta}\left(\frac{1}{r}\frac{\partial A_\theta}{\partial \theta}+\frac{A_r}{r}\right)+T_{\phi\phi}\left(\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}+\frac{A_r}{r}+\frac{A_\theta\cot\theta}{r}\right) + T_{r\theta}\left(\frac{\partial A_\theta}{\partial r}+\frac{1}{r}\frac{\partial A_r}{\partial \theta}-\frac{A_\theta}{r}\right)+T_{r\phi}\left(\frac{\partial A_\phi}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \phi}-\frac{A_\phi}{r}\right) + T_{\theta\phi}\left(\frac{1}{r}\frac{\partial A_\phi}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial \phi}-\frac{\cot\theta}{r}A_\phi\right)$
$\nabla\cdot\overset{\Rightarrow}{T}$	$\left(\frac{\partial T_{xx}}{\partial x}+\frac{\partial T_{xy}}{\partial y}+\frac{\partial T_{xz}}{\partial z}\right)\vec{i}_x + \left(\frac{\partial T_{yx}}{\partial x}+\frac{\partial T_{yy}}{\partial y}+\frac{\partial T_{yz}}{\partial z}\right)\vec{i}_y + \left(\frac{\partial T_{zx}}{\partial x}+\frac{\partial T_{zy}}{\partial y}+\frac{\partial T_{zz}}{\partial z}\right)\vec{i}_z$	$\left(\frac{1}{r}\frac{\partial}{\partial r}(rT_{rr})+\frac{1}{r}\frac{\partial}{\partial \theta}T_{r\theta}-\frac{1}{r}T_{\theta\theta}+\frac{\partial T_{rz}}{\partial z}\right)\vec{i}_r + \left(\frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta}+\frac{\partial T_{r\theta}}{\partial r}+\frac{2}{r}T_{r\theta}+\frac{\partial T_{\theta z}}{\partial z}\right)\vec{i}_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}(rT_{zr})+\frac{1}{r}\frac{\partial T_{z\theta}}{\partial \theta}+\frac{\partial T_{zz}}{\partial z}\right)\vec{i}_z$	$\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{rr})+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(T_{r\theta}\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial T_{r\phi}}{\partial \phi}-\frac{T_{\theta\theta}+T_{\phi\phi}}{r}\right)\vec{i}_r + \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{r\theta})+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(T_{\theta\theta}\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial T_{r\phi}}{\partial \phi}+\frac{T_{r\theta}}{r}-\frac{\cot\theta}{r}T_{\phi\phi}\right)\vec{i}_\theta + \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{r\phi})+\frac{1}{r}\frac{\partial T_{\theta\phi}}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial \phi}+\frac{T_{r\phi}}{r}+\frac{2\cot\theta}{r}T_{\theta\phi}\right)\vec{i}_\phi$

$T_{\theta r}$

$T_{\phi\theta}$

Appendix B

Vector and Operator Identities

$$\vec{A} \times \vec{B} \cdot \vec{C} = \vec{A} \cdot \vec{B} \times \vec{C} \tag{1}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \tag{2}$$

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi \tag{3}$$

$$\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B} \tag{4}$$

$$\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B} \tag{5}$$

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi \tag{6}$$

$$\nabla \cdot (\psi\vec{A}) = \vec{A} \cdot \nabla\psi + \psi\nabla \cdot \vec{A} \tag{7}$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \nabla \times \vec{A} - \vec{A} \cdot \nabla \times \vec{B} \tag{8}$$

$$\nabla \cdot \nabla\phi = \nabla^2\phi \tag{9}$$

$$\nabla \cdot \nabla \times \vec{A} = 0 \tag{10}$$

$$\nabla \times \nabla\phi = 0 \tag{11}$$

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A} \tag{12}$$

$$(\nabla \times \vec{A}) \times \vec{A} = (\vec{A} \cdot \nabla)\vec{A} - 1/2\ \nabla(\vec{A} \cdot \vec{A}) \tag{13}$$

$$\nabla(\vec{A} \cdot \vec{B}) = (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} + \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) \tag{14}$$

$$\nabla \times (\phi\vec{A}) = \nabla\phi \times \vec{A} + \phi\nabla \times \vec{A} \tag{15}$$

$$\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\ \vec{A} - (\vec{A} \cdot \nabla)\vec{B} \tag{16}$$

Appendix C

Films

Developed for educational purposes with the support of the National Science Foundation at the Education Development Center, films cited fall in one of two series.

Produced by the National Committee for Fluid Mechanics Films and distributed by Encyclopedia Britannica Educational Corp., 425 N. Michigan Ave., Chicago, Illinois (60611) are:

(1) Channel Flow of a Compressible Fluid
(2) Current-induced Instability of a Mercury Jet
(3) Eulerian and Lagrangian Descriptions in Fluid Mechanics
(4) Flow Instabilities
(5) Fundamentals of Boundary Layers
(6) Low-Reynolds Number Flows
(7) Magnetohydrodynamics
(8) Pressure Fields and Fluid Acceleration
(9) Surface Tension and Fluid Mechanics
(10) Waves in Fluids

Produced by the National Committee for Electrical Engineering Films and distributed by Education Development Center, 39 Chapel Street, Newton, Mass. 02160 are:

(11) Complex Waves I and Complex Waves II
(12) Electric Fields and Moving Media

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