Appendix B

REVIEW OF ELECTROMAGNETIC THEORY

B.1 BASIC LAWS AND DEFINITIONS

The laws of electricity and magnetism are empirical. Fortunately they can be traced to a few fundamental experiments and definitions, which are reviewed in the following sections. The rationalized MKS system of units is used.

B.1.1 Coulomb's Law, Electric Fields and Forces

Coulomb found that when a charge q (coulombs) is brought into the vicinity of a distribution of *charge density* $\rho_e(\mathbf{r}')$ (coulombs per cubic meter), as shown in Fig. B.1.1, a force of repulsion $\mathbf{f}$ (newtons) is given by

$$\mathbf{f} = q\mathbf{E}, \tag{B.1.1}$$

where the *electric field intensity* $\mathbf{E}$ (volts per meter) is evaluated at the position

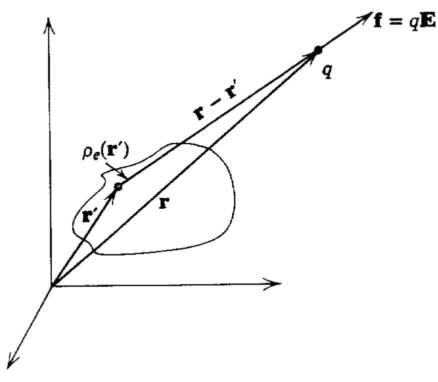

Fig. B.1.1 The force $\mathbf{f}$ on the point charge q in the vicinity of charges with density $\rho_e(\mathbf{r}')$ is represented by the electric field intensity $\mathbf{E}$ times q, where $\mathbf{E}$ is found from (B.1.2).

r of the charge q and determined from the distribution of charge density by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{V'} \rho_e(\mathbf{r}') \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\, dV'. \tag{B.1.2}$$

In the rationalized MKS system of units the permittivity ϵ_0 of free space is

$$\epsilon_0 = 8.854 \times 10^{-12} \approx \frac{1}{36\pi} \times 10^{-9}\, F/\mathrm{m}. \tag{B.1.3}$$

Note that the integration of (B.1.2) is carried out over all the charge distribution (excluding q), hence represents a superposition (at the location **r** of q) of the electric field intensities due to elements of charge density at the positions **r**′.

As an example, suppose that the charge distribution $\rho_e(\mathbf{r}')$ is simply a point charge Q (coulombs) at the origin (Fig. B.1.2); that is,

$$\rho_e = Q\, \delta(\mathbf{r}'), \tag{B.1.4}$$

where $\delta(\mathbf{r}')$ is the *delta function* defined by

$$\delta(\mathbf{r}') = 0, \qquad \mathbf{r}' \neq 0,$$
$$\int_{V'} \delta(\mathbf{r}')\, dV' = 1. \tag{B.1.5}$$

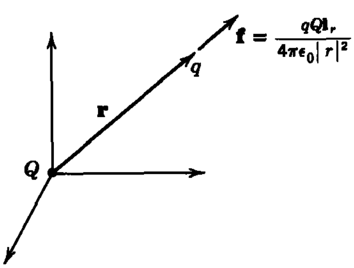

Fig. B.1.2 Coulomb's law for point charges Q (at the origin) and q (at the position **r**).

For the charge distribution of (B.1.4) integration of (B.1.2) gives

$$\mathbf{E}(\mathbf{r}) = \frac{Q\mathbf{r}}{4\pi\epsilon_0\, |r|^3}. \tag{B.1.6}$$

Hence the force on the point charge q, due to the point charge Q, is from (B.1.1)

$$\mathbf{f} = \frac{qQ\mathbf{r}}{4\pi\epsilon_0\, |r|^3}. \tag{B.1.7}$$

This expression takes the familiar form of *Coulomb's law* for the force of repulsion between point charges of like sign.

We know that electric charge occurs in integral multiples of the electronic charge (1.60×10^{-19} C). The charge density ρ_e, introduced with (B.1.2), is defined as

$$\rho_e(\mathbf{r}) = \lim_{\delta V \to 0} \frac{1}{\delta V} \sum_i q_i, \tag{B.1.8}$$

where δV is a small volume enclosing the point $\mathbf{r}$ and $\sum_i q_i$ is the algebraic sum of charges within δV. The charge density is an example of a continuum model. To be valid the limit $\delta V \to 0$ must represent a volume large enough to contain a large number of charges q_i, yet small enough to appear infinitesimal when compared with the significant dimensions of the system being analyzed. This condition is met in most electromechanical systems.

For example, in copper at a temperature of 20°C the number density of free electrons available for carrying current is approximately 10^{23} electrons/cm^3. If we consider a typical device dimension to be on the order of 1 cm, a reasonable size for δV would be a cube with 1-mm sides. The number of electrons in δV would be 10^{20}, which certainly justifies the continuum model.

The force, as expressed by (B.1.1), gives the total force on a single test charge in vacuum and, as such, is not appropriate for use in a continuum model of electromechanical systems. It is necessary to use an *electric force density* $\mathbf{F}$ (newtons per cubic meter) that can be found by averaging (B.1.1) over a small volume.

$$\mathbf{F} = \lim_{\delta V \to 0} \frac{\sum_i \mathbf{f}_i}{\delta V} = \lim_{\delta V \to 0} \frac{\sum q_i \mathbf{E}_i}{\delta V}. \tag{B.1.9}$$

Here q_i represents all of the charges in δV, $\mathbf{E}_i$ is the electric field intensity acting on the ith charge, and $\mathbf{f}_i$ is the force on the ith charge. As in the charge density defined by (B.1.8), the limit of (B.1.9) leads to a continuum model if the volume δV can be defined so that it is small compared with macroscopic dimensions of significance, yet large enough to contain many electronic charges. Further, there must be a sufficient amount of charge external to the volume δV that the electric field experienced by each of the test charges is essentially determined by the sources of field outside the volume. Fortunately these requirements are met in almost all physical situations that lead to useful electromechanical interactions. Because all charges in the volume δV experience essentially the same electric field $\mathbf{E}$, we use the definition of free charge density given by (B.1.8) to write (B.1.9) as

$$\mathbf{F} = \rho_e \mathbf{E}. \tag{B.1.10}$$

Although the static electric field intensity $\mathbf{E}$ can be computed from (B.1.2), it is often more convenient to state the relation between charge density and field intensity in the form of *Gauss's law:*

$$\oint_S \epsilon_0 \mathbf{E} \cdot \mathbf{n}\, da = \int_V \rho_e\, dV. \tag{B.1.11}$$

In this integral law $\mathbf{n}$ is the outward-directed unit vector normal to the surface S, which encloses the volume V. It is not our purpose in this brief review to show that (B.1.11) is implied by (B.1.2). It is helpful, however, to note that

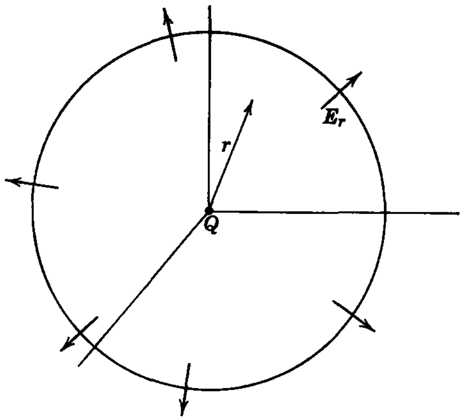

Fig. B.1.3 A hypothetical sphere of radius r encloses a charge Q at the origin. The integral of $\epsilon_0 \mathbf{E}_r$ over the surface of the sphere is equal to the charge Q enclosed.

in the case of a point charge Q at the origin it predicts the same electric field intensity (B.1.6) as found by using (B.1.2). For this purpose the surface S is taken as the sphere of radius r centered at the origin, as shown in Fig. B.1.3. By symmetry the only component of $\mathbf{E}$ is radial (E_r), and this is constant at a given radius r. Hence (B.1.11) becomes

$$4\pi r^2 E_r \epsilon_0 = Q. \tag{B.1.12}$$

Here the integration of the charge density over the volume V enclosed by S is the total charge enclosed Q but can be formally taken by using (B.1.4) with the definition provided by (B.1.5). It follows from (B.1.12) that

$$E_r = \frac{Q}{4\pi\epsilon_0 r^2}, \tag{B.1.13}$$

a result that is in agreement with (B.1.6).

Because the volume and surface of integration in (B.1.11) are arbitrary, the integral equation implies a differential law. This is found by making use of the *divergence theorem**

$$\oint_S \mathbf{A} \cdot \mathbf{n}\, da = \int_V \nabla \cdot \mathbf{A}\, dV \tag{B.1.14}$$

to write (B.1.11) as

$$\int_V (\nabla \cdot \epsilon_0 \mathbf{E} - \rho_e)\, dV = 0. \tag{B.1.15}$$

* For a discussion of the divergence theorem see F. B. Hildebrand, *Advanced Calculus for Engineers*, Prentice-Hall, New York, 1949, p. 312.

Since the volume of integration is arbitrary, it follows that

$$\nabla \cdot \epsilon_0 \mathbf{E} = \rho_e. \tag{B.1.16}$$

From this discussion it should be apparent that this *differential* form of *Gauss's law* is implied by Coulomb's law, with the electric field intensity defined as a force per unit charge.

B.1.2 Conservation of Charge

Experimental evidence supports the postulate that electric charge is conserved. When a negative charge appears (e.g., when an electron is removed from a previously neutral atom), an equal positive charge also appears (e.g., the positive ion remaining when the electron is removed from the atom).

We can make a mathematical statement of this postulate in the following way. Consider a volume V enclosed by a surface S. If charge is conserved, the net rate of flow of electric charge out through the surface S must equal the rate at which the total charge in the volume V decreases. The current density $\mathbf{J}$ (coulombs per square meter-second) is defined as having the direction of flow of positive charge and a magnitude proportional to the net rate of flow of charge per unit area. Then the statement of conservation of charge is

$$\oint_S \mathbf{J} \cdot \mathbf{n}\, da = -\frac{d}{dt}\int_V \rho_e\, dV. \tag{B.1.17}$$

Once again it follows from the arbitrary nature of S (which is fixed in space) and the divergence theorem (B.1.14) that

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho_e}{\partial t} = 0. \tag{B.1.18}$$

It is this equation that is used as a *differential* statement of *conservation of charge.*

To express conservation of charge it has been necessary to introduce a new continuum variable, the current density $\mathbf{J}$. Further insight into the relation between this quantity and the charge density ρ_e is obtained by considering a situation in which two types of charge contribute to the current, charges q_+ with velocity $\mathbf{v}_+$ and charges q_- with velocity $\mathbf{v}_-$. The current density $\mathbf{J}_+$ that results from the flow of positive charge is

$$\mathbf{J}_+ = \lim_{\delta V \to 0} \frac{1}{\delta V} \sum_i q_{+i} \mathbf{v}_{+i}. \tag{B.1.19}$$

If we define a *charge-average velocity* $\mathbf{v}_+$ for the positive charges as

$$\mathbf{v}_+ = \frac{\sum_i q_{+i}\mathbf{v}_{+i}}{\sum_i q_{+i}} \tag{B.1.20}$$

and the density ρ_+ of positive charges from (B.1.8) as

$$\rho_+ = \lim_{\delta V \to 0} \frac{1}{\delta V} \sum_i q_{+i}, \tag{B.1.21}$$

we can write the current density of (B.1.19) as

$$\mathbf{J}_+ = \rho_+ \mathbf{v}_+. \tag{B.1.22}$$

Similar definitions for the charge-average velocity $\mathbf{v}_-$ and charge density ρ_- of negative charges yields the component of current density

$$\mathbf{J}_- = \rho_- \mathbf{v}_-. \tag{B.1.23}$$

The total current density $\mathbf{J}$ is the vector sum of the two components

$$\mathbf{J} = \mathbf{J}_+ + \mathbf{J}_-. \tag{B.1.24}$$

Now consider the situation of a material that contains charge densities ρ_+ and ρ_- which have charge-average velocities $\mathbf{v}_+$ and $\mathbf{v}_-$ with respect to the material. Assume further that the material is moving with a velocity $\mathbf{v}$ with respect to an observer who is to measure the current. The net average velocities of positive and negative charges as seen by the observer are $\mathbf{v}_+ + \mathbf{v}$ and $\mathbf{v}_- + \mathbf{v}$, respectively. The current density measured by the observer is then from (B.1.24)

$$\mathbf{J} = (\rho_+ \mathbf{v}_+ + \rho_- \mathbf{v}_-) + \rho_e \mathbf{v}, \tag{B.1.25}$$

where the net charge density ρ_e is given by

$$\rho_e = \rho_+ + \rho_-. \tag{B.1.26}$$

The first term of (B.1.25) is a net flow of charge with respect to the material and is normally called a *conduction current*. (It is often described by Ohm's law.) The last term represents the transport of net charge and is conventionally called a *convection current*. It is crucial that *net flow of charge* be distinguished from *flow of net charge*. The net charge may be zero but a current can still be accounted for by the conduction term. This is the case in metallic conductors.

B.1.3 Ampère's Law, Magnetic Fields and Forces

The *magnetic flux density* $\mathbf{B}$ is defined to express the force on a current element $i\,d\mathbf{l}$ placed in the vicinity of other currents. This element is shown in Fig. B.1.4 at the position $\mathbf{r}$. Then, according to Ampère's experiments, the force is given by

$$\mathbf{f} = i\,d\mathbf{l} \times \mathbf{B}, \tag{B.1.27}$$

where

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\, dV'. \tag{B.1.28}$$

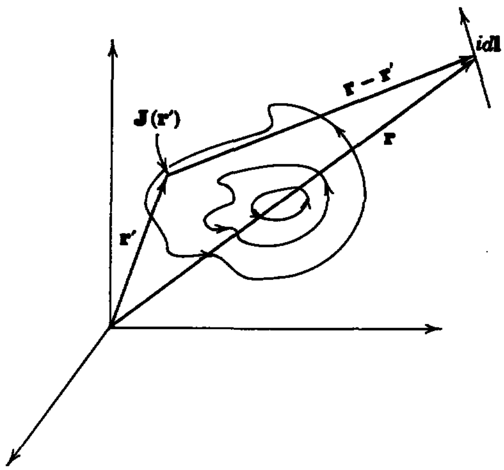

Fig. B.1.4 A distribution of current density $\mathbf{J}(\mathbf{r}')$ produces a force on the current element $i d\mathbf{l}$ which is represented in terms of the magnetic flux density $\mathbf{B}$ by (B.1.27) and (B.1.28).

Hence the flux density at the position $\mathbf{r}$ of the current element $i\, d\mathbf{l}$ is the superposition of fields produced by currents at the positions $\mathbf{r}'$. In this expression the permeability of free space μ_0 is

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m}. \tag{B.1.29}$$

As an example, suppose that the distribution of current density $\mathbf{J}$ is composed of a current I (amperes) in the z direction and along the z-axis, as shown in Fig. B.1.5. The magnetic flux density at the position $\mathbf{r}$ can be computed

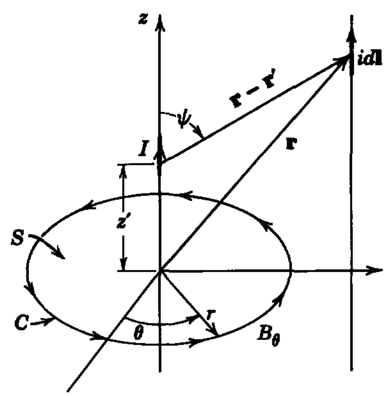

Fig. B.1.5 A current I (amperes) along the z-axis produces a magnetic field at the position $\mathbf{r}$ of the current element $i d\mathbf{l}$.

from (B.1.28), which for this case reduces to*

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{+\infty}\frac{\mathbf{i}_z \times (\mathbf{r} - z'\mathbf{i}_z)}{|\mathbf{r} - z'\mathbf{i}_z|^3}\,dz'. \tag{B.1.30}$$

Here the coordinate of the source current I is z', as shown in Fig. B.1.5, whereas the coordinate $\mathbf{r}$ that designates the position at which $\mathbf{B}$ is evaluated can be written in terms of the cylindrical coordinates (r, θ, z). Hence (B.1.30) becomes

$$\mathbf{B} = \frac{\mu_0 I \mathbf{i}_\theta}{4\pi}\int_{-\infty}^{+\infty}\frac{\sin\psi\sqrt{(z - z')^2 + r^2}}{[(z - z')^2 + r^2]^{3/2}}\,dz', \tag{B.1.31}$$

where, from Fig. B.1.5, $\sin\psi = r/\sqrt{(z - z')^2 + r^2}$. Integration on z' gives the magnetic flux density

$$\mathbf{B} = \frac{\mu_0 I \mathbf{i}_\theta}{2\pi r}. \tag{B.1.32}$$

It is often more convenient to relate the magnetic flux density to the current density $\mathbf{J}$ by the integral of *Ampère's law* for static fields, which takes the form

$$\oint_C \mathbf{B}\cdot d\mathbf{l} = \mu_0\int_S \mathbf{J}\cdot\mathbf{n}\,da. \tag{B.1.33}$$

Here C is a closed contour of line integration and S is a surface enclosed by C. We wish to present a review of electromagnetic theory and therefore we shall not embark on a proof that (B.1.33) is implied by (B.1.28). Our purpose is served by recognizing that (B.1.33) can also be used to predict the flux density in the situation in Fig. B.1.5. By symmetry we recognize that $\mathbf{B}$ is azimuthally directed and independent of θ and z. Then, if we select the contour C in a plane z equals constant and at a radius r, as shown in Fig. B.1.5, (B.1.33) becomes

$$2\pi r B_\theta = \mu_0 I. \tag{B.1.34}$$

Solution of this expression for B_θ gives the same result as predicted by (B.1.28). [See (B.1.32).]

The contour C and surface S in (B.1.33) are arbitrary and therefore the equation can be cast in a differential form. This is done by using Stokes' theorem†,

$$\oint_C \mathbf{A}\cdot d\mathbf{l} = \int_S \mathbf{n}\cdot(\nabla\times\mathbf{A})\,da, \tag{B.1.35}$$

* Unit vectors in the coordinate directions are designated by $\mathbf{i}$. Thus $\mathbf{i}_z$ is a unit vector in the z-direction.

† See F. B. Hildebrand, *Advanced Calculus for Engineers*, Prentice-Hall, New York, 1949, p. 318.

to write (B.1.33) as

$$\int_S (\nabla \times \mathbf{B} - \mu_0 \mathbf{J}) \cdot \mathbf{n}\, da = 0, \tag{B.1.36}$$

from which the differential form of Ampère's law follows as

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{B.1.37}$$

So far the assumption has been made that the current $\mathbf{J}$ is constant in time. Maxwell's contribution consisted in recognizing that if the sources ρ_e and $\mathbf{J}$ (hence the fields $\mathbf{E}$ and $\mathbf{B}$) are time varying the displacement current $\epsilon_0 \partial \mathbf{E}/\partial t$ must be included on the right-hand side of (B.1.37). Thus for dynamic fields Ampère's law takes the form

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \frac{\partial \epsilon_0 \mathbf{E}}{\partial t}. \tag{B.1.38}$$

This alteration of (B.1.37) is necessary if conservation of charge expressed by (B.1.18) is to be satisfied. Because the divergence of any vector having the form $\nabla \times \mathbf{A}$ is zero, the divergence of (B.1.38) becomes

$$\nabla \cdot \mathbf{J} + \frac{\partial(\nabla \cdot \epsilon_0 \mathbf{E})}{\partial t} = 0. \tag{B.1.39}$$

Then, if we recall that ρ_e is related to $\mathbf{E}$ by Gauss's law (B.1.16), the conservation of charge equation (B.1.18) follows. The displacement current in (B.1.38) accounts for the rate of change of ρ_e in (B.1.18).

We shall make considerable use of Ampère's law, as expressed by (B.1.38), with Maxwell's displacement current included. From our discussion it is clear that the static form of this law results from the force law of interaction between currents. The magnetic flux density is defined in terms of the force produced on a current element. Here we are interested primarily in a continuum description of the force, hence require (B.1.27) expressed as a force density. With the same continuum restrictions implied in writing (B.1.10), we write the magnetic force density (newtons per cubic meter) as

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}. \tag{B.1.40}$$

In view of our remarks it should be clear that this force density is not something that we have derived but rather arises from the definition of the flux density $\mathbf{B}$. Further remarks on this subject are found in Section 8.1.

B.1.4 Faraday's Law of Induction and the Potential Difference

Two extensions of static field theory are required to describe dynamic fields. One of these, the introduction of the displacement current in Ampère's law, was discussed in the preceding section. Much of the significance of this generalization stems from the apparent fact that an electric field can lead to the generation of a magnetic field. As a second extension of static field theory, Faraday discovered that, conversely, time-varying magnetic fields can lead to the generation of electric fields.

Faraday's law of induction can be written in the integral form

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot \mathbf{n}\, da, \tag{B.1.41}$$

where again C is a contour that encloses the surface S. The contour and surface are arbitrary; hence it follows from Stokes' theorem (B.1.35) that Faraday's law has the differential form

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{B.1.42}$$

Note that in the static case this expression reduces to $\nabla \times \mathbf{E} = 0$, which is, in addition to Gauss's law, a condition on the static electric field. That this further equation is consistent with the electric field, as given by (B.1.2), is not shown in this review. Clearly the one differential equation represented by Gauss's law could not alone determine the three components of **E**.

In regions in which the magnetic field is either static or negligible the electric field intensity can be derived as the gradient of a scalar potential ϕ:

$$\mathbf{E} = -\nabla\phi. \tag{B.1.43}$$

This is true because the curl of the gradient is zero and (B.1.42) is satisfied. The difference in potential between two points, say a and b, is a measure of the line integral of **E**, for

$$\int_a^b \mathbf{E} \cdot d\mathbf{l} = -\int_a^b \nabla\phi \cdot d\mathbf{l} = \phi_a - \phi_b. \tag{B.1.44}$$

The potential difference $\phi_a - \phi_b$ is referred to as the voltage of point a with respect to b. If there is no magnetic field **B** in the region of interest, the integral of (B.1.44) is independent of path. In the presence of a time-varying magnetic field the integral of **E** around a closed path is not in general zero, and if a potential is defined in some region by (B.1.43) the path of integration will in part determine the measured potential difference.

The physical situation shown in Fig. B.1.6 serves as an illustration of the implications of Faraday's law. A magnetic circuit is excited by a current source $I(t)$ as shown. Because the magnetic material is highly permeable, the induced flux density $B(t)$ is confined to the cross section A which links a circuit formed by resistances R_a and R_b in series. A cross-sectional view of the

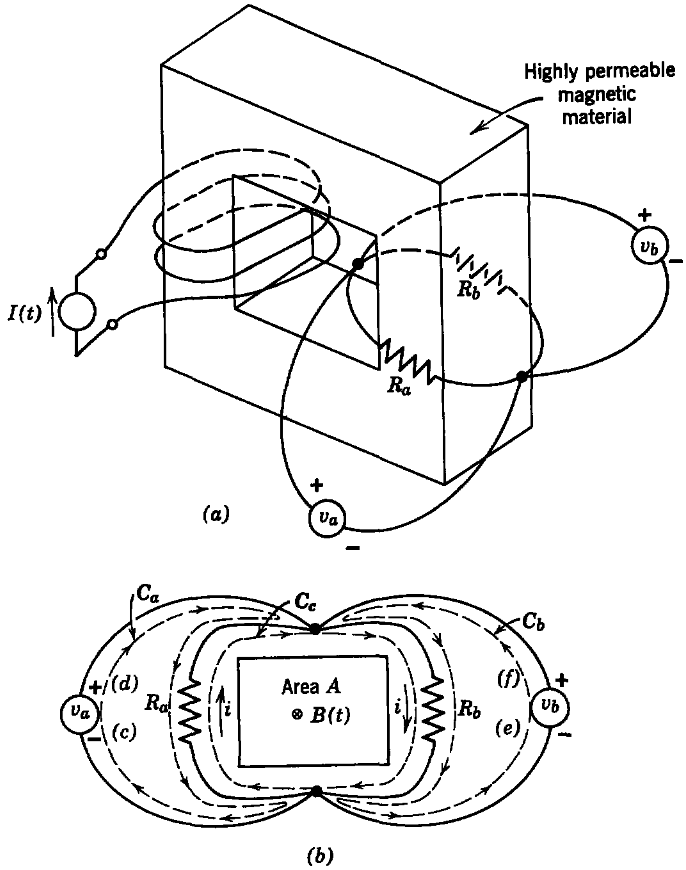

Fig. B.1.6 (*a*) A magnetic circuit excited by $I(t)$ so that flux $AB(t)$ links the resistive loop (*b*) a cross-sectional view of the loop showing connection of the voltmeters.

circuit is shown in Fig. B.1.6*b*, in which high impedance voltmeters v_a and v_b are shown connected to the same nodes. Under the assumption that no current is drawn by the voltmeters, and given the flux density $B(t)$, we wish to compute the voltages that would be indicated by v_a and v_b.

Three contours of integration C are defined in Fig. B.1.6*b* and are used with Faraday's integral law (B.1.41). The integral of $\mathbf{E}$ around the contour C_c is equal to the drop in potential across both of the resistances, which carry the same current i. Hence, since this path encloses a total flux $AB(t)$, we have

$$i(R_a + R_b) = -\frac{d}{dt}[AB(t)]. \tag{B.1.45}$$

The paths of integration C_a and C_b do not enclose a magnetic flux; hence for these paths (B.1.41) gives

$$v_a = -iR_a = \frac{R_a}{R_a + R_b}\frac{d}{dt}[AB(t)] \quad \text{for} \quad C_a, \tag{B.1.46}$$

$$v_b = iR_b = \frac{-R_b}{R_a + R_b}\frac{d}{dt}[AB(t)] \quad \text{for} \quad C_b, \tag{B.1.47}$$

where the current i is evaluated by using (B.1.45). The most obvious attribute of this result is that although the voltmeters are connected to the same nodes they do not indicate the same values. In the presence of the magnetic induction the contour of the voltmeter leads plays a role in determining the voltage indicated.

The situation shown in Fig. B.1.6 can be thought of as a transformer with a single turn secondary. With this in mind, it is clear that Faraday's law plays an essential role in electrical technology.

The divergence of an arbitrary vector $\nabla \times \mathbf{A}$ is zero. Hence the divergence of (B.1.42) shows that the divergence of $\mathbf{B}$ is constant. This fact also follows from (B.1.28), from which it can be shown that this constant is zero. Hence an additional differential equation for $\mathbf{B}$ is

$$\nabla \cdot \mathbf{B} = 0. \tag{B.1.48}$$

Integration of this expression over an arbitrary volume V and use of the divergence theorem (B.1.14) gives

$$\oint_S \mathbf{B} \cdot \mathbf{n}\, da = 0. \tag{B.1.49}$$

This integral law makes more apparent the fact that there can be no net magnetic flux emanating from a given region of space.

B.2 MAXWELL'S EQUATIONS

The generality and far-reaching applications of the laws of electricity and magnetism are not immediately obvious; for example, the law of induction given by (B.1.42) was recognized by Faraday as true when applied to a conducting circuit. The fact that (B.1.42) has significance even in regions of space unoccupied by matter is a generalization that is crucial to the theory of electricity and magnetism. We can summarize the differential laws introduced in Section B.1 as

$$\nabla \cdot \epsilon_0 \mathbf{E} = \rho_e, \tag{B.2.1}$$

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho_e}{\partial t} = 0, \tag{B.2.2}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \frac{\partial \epsilon_0 \mathbf{E}}{\partial t}, \tag{B.2.3}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{B.2.4}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{B.2.5}$$

Taken together, these laws are called *Maxwell's equations* in honor of the man who was instrumental in recognizing that they have a more general significance than any one of the experiments from which they originate. For example, we can think of a time-varying magnetic flux that induces an electric field according to (B.2.4) even in the absence of a material circuit. Similarly, (B.2.3) is taken to mean that even in regions of space in which there is no circuit, hence $\mathbf{J} = 0$, a time-varying electric field leads to an induced magnetic flux density $\mathbf{B}$.

The coupling between time-varying electric and magnetic fields, as predicted by (B.2.1 to B.2.5), accounts for the existence of electromagnetic waves, whether they be radio or light waves or even gamma rays. As we might guess from the electromechanical origins of electromagnetic theory, the propagation of electromagnetic waves is of secondary importance in the study of most electromechanical phenomena. This does not mean that electromechanical interactions are confined to frequencies that are low compared with radio frequencies. Indeed, electromechanical interactions of practical significance extend into the gigahertz range of frequencies.

To take a mature approach to the study of electromechanics it is necessary that we discriminate at the outset between essential and nonessential aspects of interactions between fields and media. This makes it possible to embark immediately on a study of nontrivial interactions. An essential purpose of this section is the motivation of approximations used in this book.

Although electromagnetic waves usually represent an unimportant consideration in electromechanics and are not discussed here in depth, they are important to an understanding of the quasi-static approximations that are introduced in Section B.2.2. Hence we begin with a brief simplified discussion of electromagnetic waves.

B.2.1 Electromagnetic Waves

Consider fields predicted by (B.2.3) and (B.2.4) in a region of free space in which $\mathbf{J} = 0$. In particular, we confine our interest to situations in which the fields depend only on (x, t) (the fields are one-dimensional) and write the y-component of (B.2.3) and the z-component of (B.2.4)

$$-\frac{\partial B_z}{\partial x} = \mu_0 \epsilon_0 \frac{\partial E_y}{\partial t}, \tag{B.2.6}$$

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}. \tag{B.2.7}$$

This pair of equations, which make evident the coupling between the dynamic electric and magnetic fields, is sufficient to determine the field components B_z and E_y. In fact, if we take the time derivative of (B.2.6) and use the resulting expression to eliminate B_z from the derivative with respect to x of (B.2.7), we obtain

$$\frac{\partial^2 E_y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2}, \tag{B.2.8}$$

where

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \quad \text{(m/sec)}.$$

This equation for E_y is called the *wave equation* because it has solutions in the form of

$$E_y(x, t) = E_+(x - ct) + E_-(x + ct). \tag{B.2.9}$$

That this is true may be verified by substituting (B.2.9) into (B.2.8). Hence solutions for E_y can be analyzed into components E_+ and E_- that represent waves traveling, respectively, in the $+x$- and $-x$-directions with the *velocity of light* c, given by (B.2.8). The prediction of electromagnetic wave propagation is a salient feature of Maxwell's equations. It results, as is evident from the derivation, because time-varying magnetic fields can induce electric fields [Faraday's law, (B.2.7)] while at the same time dynamic electric fields induce magnetic fields [Ampère's law with the displacement current included (B.2.6)]. It is also evident from the derivation that if we break this two-way coupling by leaving out the displacement current *or* omitting the magnetic induction term electromagnetic waves are not predicted.

Electromechanical interactions are usually not appreciably affected by the propagational character of electromagnetic fields because the velocity of propagation c is very large. Suppose that we are concerned with a system whose largest dimension is l. The time l/c required for the propagation of a wave between extremes of the system is usually short compared with characteristic dynamical times of interest; for example, in a device in which $l = 0.3$ m the time l/c equals 10^{-9} sec. If we were concerned with electromechanical motions with a time constant of a microsecond (which is extremely short for a device characterized by 30 cm), it would be reasonable to ignore the wave propagation. In the absence of other dynamic effects this could be done by assuming that the fields were established everywhere within the device instantaneously.

Even though it is clear that the propagation of electromagnetic waves has nothing to do with the dynamics of interest, it is not obvious how to go about simplifying Maxwell's equations to remove this feature of the dynamics. A pair of particular examples will help to clarify approximations made in the next section. These examples, which are considered simultaneously so that they can be placed in contrast, are shown in Fig. B.2.1.

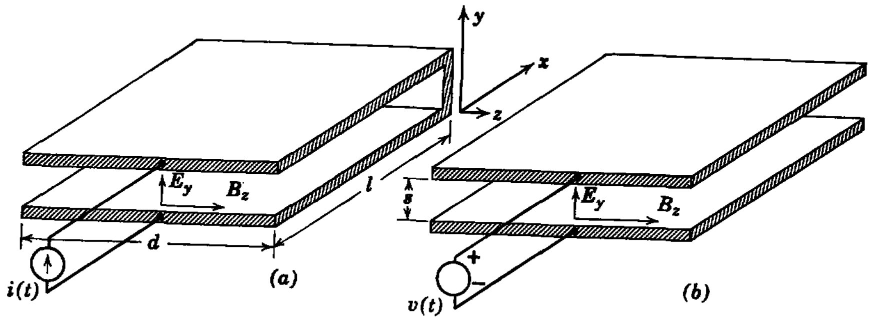

Fig. B.2.1 Perfectly conducting plane-parallel electrodes driven at $x = -l$: (*a*) $i(t) = i_0 \cos \omega t$; (*b*) $v(t) = v_0 \cos \omega t$.

A pair of perfectly conducting parallel plates has the spacing s which is much smaller than the x-z dimensions l and d. The plates are excited at $x = -l$ by

a current source	a voltage source
$i(t) = i_0 \cos \omega t$ (amperes). (B.2.10*a*)	$v(t) = v_0 \cos \omega t$ (volts). (B.2.10*b*)

At $x = 0$, the plates are terminated in

a perfectly conducting short circuit plate.	an open circuit.

If we assume that the spacing s is small enough to warrant ignoring the effects of fringing and that the driving sources at $x = -l$ are distributed along the z-axis, the one-dimensional fields B_z and E_y predicted by (B.2.6) and (B.2.7) represent the fields between the plates. Hence we can think of the current and voltage sources as exciting electromagnetic waves that propagate along the x-axis between the plates. The driving sources impose conditions on the fields at $x = -l$. They are obtained by

integrating (B.1.33) around the contour C (Fig. B.2.2*a*) which encloses the upper plate adjacent to the current source. (The surface S enclosed by C is very thin so that negligible displacement current links the loop).	integrating the electric field between (*a*) and (*b*) in Fig. B.2.2*b* to relate the potential difference of the voltage source to the electric field intensity $E_y(-l, t)$.
$$B_z(-l, t) = -\mu_0 K = -\frac{\mu_0 i(t)}{d} \quad \text{(B.2.11}a\text{)}$$	$$\int_s^0 E_y\, dy = -sE_y(-l, t) = v(t). \quad \text{(B.2.11}b\text{)}$$

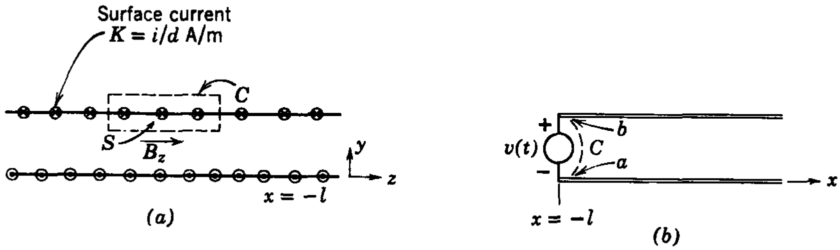

Fig. B.2.2 Boundary conditions for the systems in Fig. B.2.1

Similar conditions used at $x = 0$ give the boundary conditions

$$E_y(0, t) = 0 \qquad \text{(B.2.12}a\text{)}$$

$$B_z(0, t) = 0 \qquad \text{(B.2.12}b\text{)}$$

It is not our purpose in this chapter to become involved with the formalism of solving the wave equation [or (B.2.6) and (B.2.7)] subject to the boundary conditions given by (B.2.11) and (B.2.12). There is ample opportunity to solve boundary value problems for electromechanical systems in the text, and the particular problem at hand forms a topic within the context of transmission lines and waveguides. For our present purposes, it suffices to guess solutions to these equations that will satisfy the appropriate boundary conditions. Then direct substitution into the differential equations will show that we have made the right choice.

$$E_y = -i_o \frac{\sin \omega t \sin (\omega x/c)}{d\epsilon_0 c \cos (\omega l/c)}, \qquad \text{(B.2.13}a\text{)}$$

$$B_z = -\frac{\mu_0 i_o \cos \omega t \cos(\omega x/c)}{d \cos (\omega l/c)}, \qquad \text{(B.2.14}a\text{)}$$

$$E_y = -\frac{v_o \cos \omega t \cos (\omega x/c)}{s \cos (\omega l/c)}, \qquad \text{(B.2.13}b\text{)}$$

$$B_z = -\frac{v_o \sin \omega t \sin (\omega x/c)}{cs \cos (\omega l/c)} \qquad \text{(B.2.14}b\text{)}$$

Note that at $x = -l$ the boundary conditions B.2.11 are satisfied, whereas at $x = 0$ the conditions of (B.2.12) are met. One way to show that Maxwell's equations are satisfied also (aside from direct substitution) is to use trigometric identities* to rewrite these standing wave solutions as the superposition of two traveling waves in the form of (B.2.9). Our solutions are sinusoidal, steady-state solutions, so that with the understanding that the amplitude of the field at any point along the x-axis is varying sinusoidally with time we can obtain an impression of the dynamics by plotting the instantaneous amplitudes, as shown in Fig. B.2.3. In general, the fields have the sinusoidal distribution along the x-axis of a standing wave. From (B.2.13 to B.2.14) it

* For example in (B.2.13a) $\sin \omega t \sin (\omega x/c) \equiv \frac{1}{2}\{\cos [\omega(t - x/c)] - \cos [\omega(t + x/c)]\}$.

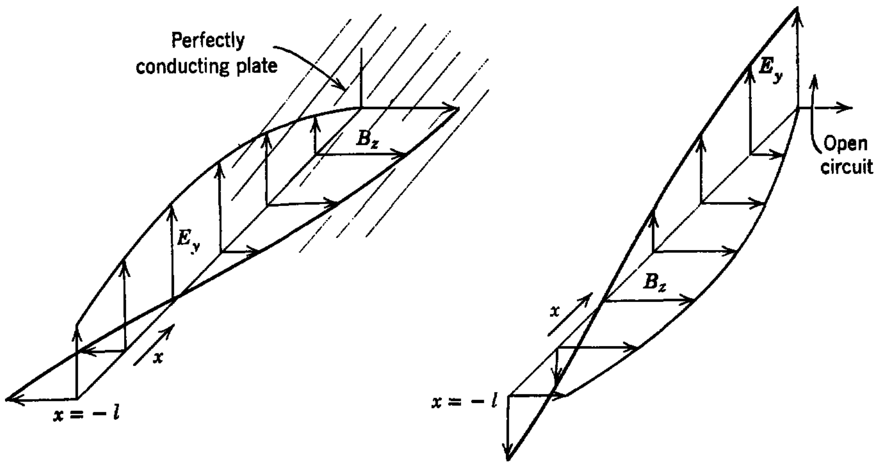

Fig. B.2.3 Amplitude of the electric field intensity and magnetic flux density along the x-axis of the parallel-plate structures shown in Fig. B.2.1 For these plots $\omega l/c = 3\pi/4$.

is clear that as a function of time the electric field reaches its maximum amplitude when $B_z = 0$ and vice versa. Hence the amplitudes of E_y and B_z shown in Fig. B.2.3 are for different instants of time. The fields near $x = 0$ do not in general have the same phase as those excited at $x = -l$. If, however, we can make the approximation that times of interest (which in this case are $1/\omega$) are much longer than the propagation time l/c,

$$\frac{l/c}{1/\omega} = \frac{\omega l}{c} \ll 1. \tag{B.2.15}$$

The sine functions can then be approximated by their arguments (which are small compared with unity) and the cosine functions are essentially equal to unity. Hence, when (B.2.15) is satisfied, the field distributions (B.2.13) and (B.2.14) become

$$E_y \simeq -\frac{i_o \sin \omega t}{d\epsilon_0 c}\left(\frac{\omega x}{c}\right), \tag{B.2.16a}$$

$$B_z \simeq -\frac{\mu_0 i_o \cos \omega t}{d}, \tag{B.2.17a}$$

$$E_y \simeq -\frac{v_o}{s} \cos \omega t, \tag{B.2.16b}$$

$$B_z \simeq -\frac{v_o}{cs} \sin \omega t\left(\frac{\omega x}{c}\right). \tag{B.2.17b}$$

The distribution of field amplitudes in this limit is shown in Fig. B.2.4. The most significant feature of the limiting solutions is that

the magnetic field between the short-circuited plates has the same distribution as if the excitation current were static.

the electric field between the open-circuited plates has the same distribution as if the excitation voltage were constant.

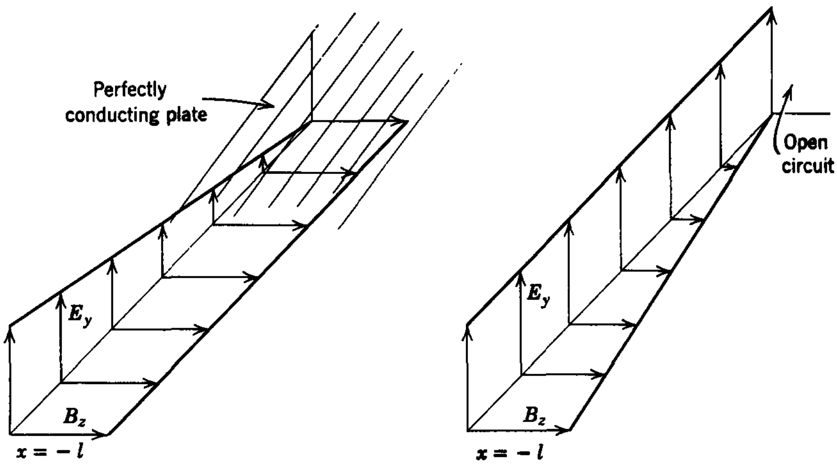

Fig. B.2.4 The distribution of field amplitudes between the parallel plates of Fig. B.2.1 in the limit in which $(\omega l/c) \ll 1$.

Note that the fields as they are excited at $x = -l$ retain the same phase everywhere between the plates. This simply reflects the fact that according to the approximate equations there is no time lag between an excitation at $x = -l$ and the field response elsewhere along the x-axis. It is in this limit that the ideas of circuit theory are applicable, for if we now compute

the voltage $v(t)$ at $x = -l$

$$v(t) = -sE_y(-l, t) \qquad \text{(B.2.18}a\text{)}$$

we obtain the terminal equation for an inductance

$$v = L\frac{d}{dt}(i_o \cos \omega t), \qquad \text{(B.2.19}a\text{)}$$

where the inductance L is

$$L = \frac{sl\mu_0}{d}.$$

the current $i(t)$ at $x = -l$

$$i(t) = -B_z(-l, t)\frac{d}{\mu_0} \qquad \text{(B.2.18}b\text{)}$$

we obtain the terminal equation for a capacitance

$$i(t) = C\frac{d}{dt}(v_o \cos \omega t), \qquad \text{(B.2.19}b\text{)}$$

where the capacitance C is

$$C = \frac{\epsilon_0 dl}{s}.$$

A comparison of the examples will be useful for motivating many of the somewhat subtle ideas introduced in the main body of the book. One of the most important points that we can make here is that even though we have solved the same pair of Maxwell's equations (B.2.6) and (B.2.7) for both examples, subject to the same approximation that $\omega l/c \ll 1$ (B.2.15), we have been led to very different physical results. The difference between these two examples arises from the boundary condition at $x = 0$. In the case of

a short circuit a static excitation leads to a uniform magnetic field but no electric field. The electric field is generated by Faraday's law because the magnetic field is in fact *only quasi-static* and varies slowly with time.

an open circuit a static excitation results in a uniform electric field but no magnetic field. The magnetic field is induced by the displacement current in Ampère's law because the electric field is, in fact, *only quasi-static* and varies slowly with time.

B.2.2 Quasi-Static Electromagnetic Field Equations

As long as we are not interested in phenomena related to the propagation of electromagnetic waves, it is helpful to recognize that most electromechanical situations are in one of two classes, exemplified by the two cases shown in Fig. B.2.1. In the situation in which the plates are short-circuited together (Fig. B.2.1*a*) the limit $\omega l/c \ll 1$ means that the *displacement current* is of *negligible* importance. A characteristic of this system is that with a static excitation a large current results; hence there is a large static magnetic field. For this reason it exemplifies a *magnetic field system*. By contrast, in the case in which the plates are open-circuited, as shown in Fig. B.2.1*b*, a static excitation gives rise to a static electric field but no magnetic field. This example exemplifies an *electric field system*, in which the *magnetic induction* of Faraday's law is of *negligible* importance. To emphasize these points consider how we can use these approximations at the outset to obtain the approximate solutions of (B.2.19). Suppose that the excitations in Fig. B.2.1 were static. The fields between the plates are then independent of x and given by

$$E_y = 0, \tag{B.2.20a}$$

$$B_z = -\frac{\mu_0 i}{d}, \tag{B.2.21a}$$

$$E_y = -\frac{v}{s}, \tag{B.2.20b}$$

$$B_z = 0. \tag{B.2.21b}$$

Now suppose that the fields vary slowly with time [the systems are quasi-static in the sense of a condition like (B.2.15)]. Then i and v in these equations are time-varying, hence

B_z is a function of time.
From Faraday's law of induction as expressed by (B.2.7)

$$\frac{\partial E_y}{\partial x} = \frac{\mu_0}{d}\frac{di}{dt}. \tag{B.2.22a}$$

E_y is a function of time.
From Ampère's law, as expressed by (B.2.6)

$$\frac{\partial B_z}{\partial x} = \frac{\mu_0 \epsilon_0}{s}\frac{dv}{dt}. \tag{B.2.22b}$$

Now the right-hand side of each of these equations is independent of x; hence they can be integrated on x. At the same time, we recognize that

$$E_y(0, t) = 0, \qquad \text{(B.2.23a)} \qquad\qquad B_z(0, t) = 0, \qquad \text{(B.2.23b)}$$

so that integration gives

$$E_y = \frac{\mu_0 x}{d}\frac{di}{dt}. \qquad \text{(B.2.24a)} \qquad\qquad B_z = \frac{\mu_0 \epsilon_0 x}{s}\frac{dv}{dt}. \qquad \text{(B.2.24b)}$$

Recall how the terminal voltage and current are related to these field quantities (B.2.18) and these equations become

$$v(t) = L\frac{di}{dt}, \qquad \text{(B.2.25a)} \qquad\qquad i(t) = C\frac{dv}{dt}, \qquad \text{(B.2.25b)}$$

where again the inductance L and capacitance C are defined as following (B.2.19). Hence making these approximations at the outset has led to the same approximate results as those found in the preceding section by computing the exact solution and taking the limits appropriate to $\omega l/c \ll 1$.

The simple example in Fig. B.2.1 makes it plausible that Maxwell's equations can be written in two quasi-static limits appropriate to the analysis of two major classes of electromechanical interaction:

Magnetic Field Systems		Electric Field Systems	
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J},$	(B.2.26a)	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t},$	(B.2.26b)
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t},$	(B.2.27a)	$\nabla \times \mathbf{E} = 0,$	(B.2.27b)
$\nabla \cdot \mathbf{B} = 0,$	(B.2.28a)	$\nabla \cdot \epsilon_0 \mathbf{E} = \rho_e,$	(B.2.28b)
$\nabla \cdot \mathbf{J} = 0,$	(B.2.29a)	$\nabla \cdot \mathbf{J} + \dfrac{\partial \rho_e}{\partial t} = 0.$	(B.2.29b)

Here the displacement current has been omitted from Ampère's law in the magnetic field system, whereas the magnetic induction has been dropped from Faraday's law in the electric field system. Note that if the displacement current is dropped from (B.2.26*a*) the charge density must be omitted from the conservation of charge equation (B.2.29*a*) because the latter expression is the divergence of (B.2.26*a*).

We have not included Guass's law for the charge density in the magnetic field system or the divergence equation for **B** in the electric field system because in the respective situations these expressions are of no interest. In fact, only the divergence of (B.2.26*b*) is of interest in determining the dynamics of most electric field systems and that is (B.2.29*b*).

It must be emphasized that the examples of Fig. B.2.1 serve only to motivate the approximations introduced by (B.2.26 to B.2.29). The two systems of equations have a wide range of application. The recognition that a given physical situation can be described as a magnetic field system, as opposed to an electric field system, requires judgment based on experience. A major intent of this book is to establish that kind of experience.

In the cases of Fig. B.2.1 we could establish the accuracy of the approximate equations by calculating effects induced by the omitted terms; for example, in the magnetic field system of Fig. B.2.1*a* we ignored the displacement current to obtain the quasi-static solution of (B.2.21*a*) and (B.2.24*a*). We could now compute the correction B_z^c to the quasi-static magnetic field induced by the displacement current by using (B.2.6), with **E** given by (B.2.24*a*). This produces

$$\frac{\partial B_z^c}{\partial x} = -\frac{\mu_0^2\epsilon_0 x}{d}\frac{d^2 i}{dt^2}. \tag{B.2.30}$$

Because the right-hand side of this expression is a known function of x, it can be integrated. The constant of integration is evaluated by recognizing that the quasi-static solution satisfies the driving condition at $x = -l$; hence the correction field B_z^c must be zero there and

$$B_z^c = -\frac{\mu_0^2\epsilon_0(x^2 - l^2)}{2d}\frac{d^2 i}{dt^2}. \tag{B.2.31}$$

Now, to determine the error incurred in ignoring this field we take the ratio of its largest value (at $x = 0$) to the quasi-static field of (B.2.21*a*):

$$\frac{|B_z^c|}{|B_z|} = \frac{l^2}{2c^2}\frac{|d^2 i/dt^2|}{|i|}. \tag{B.2.32}$$

If this ratio is small compared with 1, the quasi-static solution is adequate. It is evident that in this case the ratio depends on the time rate of change of the excitation. In Section B.2.1, in which $i = i_0 \cos \omega t$, (B.2.32) becomes

$$\frac{|B_z^c|}{|B_z|} = \frac{1}{2}\left(\frac{\omega l}{c}\right)^2 \ll 1, \tag{B.2.33}$$

which is essentially the same condition given by (B.2.15).

Once the fields have been determined by using either the magnetic field or the electric field representation it is possible to calculate the effects of the omitted terms. This procedure results in a condition characterized by (B.2.33). For this example, if the device were 30 cm long and driven at 1 MHz (this is an extremely high frequency for anything 30 cm long to respond to electromechanically) (B.2.33) becomes

$$\frac{1}{2}\left(\frac{\omega l}{c}\right)^2 = \frac{1}{2}\left(\frac{2 \cdot \pi \cdot 10^6 \cdot 0.3}{3 \times 10^8}\right)^2 = 2\pi^2 \times 10^{-6} \ll 1, \qquad \text{(B.2.34)}$$

and the quasi-static approximation is extremely good.

It is significant that the magnetic and electric field systems can be thought of in terms of their respective modes of electromagnetic energy storage. In the quasi-static systems the energy that can be attributed to the electromagnetic fields is stored either in the magnetic or electric field. This can be seen by using (B.2.26 to B.2.27) to derive Poynting's theorem for the conservation of electromagnetic energy. If the equations in (B.2.27) are multiplied by $\mathbf{B}/\mu_0$ and subtracted from the equations in (B.2.26) multiplied by $\mathbf{E}/\mu_0$, it follows that

$$\frac{\mathbf{E}}{\mu_0} \cdot \nabla \times \mathbf{B} - \frac{\mathbf{B}}{\mu_0} \cdot \nabla \times \mathbf{E} = \mathbf{E} \cdot \mathbf{J} + \frac{\mathbf{B}}{\mu_0} \cdot \frac{\partial \mathbf{B}}{\partial t}. \qquad \text{(B.2.35a)}$$

$$\frac{\mathbf{E}}{\mu_0} \cdot \nabla \times \mathbf{B} - \frac{\mathbf{B}}{\mu_0} \cdot \nabla \times \mathbf{E} = \mathbf{E} \cdot \mathbf{J} + \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}. \qquad \text{(B.2.35b)}$$

Then, because of a vector identity,* these equations take the form

$$-\nabla \cdot \left(\mathbf{E} \times \frac{\mathbf{B}}{\mu_0}\right) = \mathbf{E} \cdot \mathbf{J} + \frac{\partial}{\partial t}\left(\frac{1}{2}\frac{\mathbf{B} \cdot \mathbf{B}}{\mu_0}\right). \qquad \text{(B.2.36a)}$$

$$-\nabla \cdot \left(\mathbf{E} \times \frac{\mathbf{B}}{\mu_0}\right) = \mathbf{E} \cdot \mathbf{J} + \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0 \mathbf{E} \cdot \mathbf{E}\right). \qquad \text{(B.2.36b)}$$

Now, if we integrate these equations over a volume V enclosed by a surface S, the divergence theorem (B.1.14) gives

$$-\oint_S \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \cdot \mathbf{n}\, da = \int_V \mathbf{E} \cdot \mathbf{J}\, dV + \frac{\partial}{\partial t}\int_V w\, dV, \qquad \text{(B.2.37)}$$

where

$$w = \frac{1}{2}\frac{\mathbf{B} \cdot \mathbf{B}}{\mu_0}. \qquad \text{(B.2.38a)}$$

$$w = \frac{1}{2}\epsilon_0 \mathbf{E} \cdot \mathbf{E}. \qquad \text{(B.2.38b)}$$

The term on the left in (B.2.37) (including the minus sign) can be interpreted as the flux of energy into the volume V through the surface S. This energy is either dissipated within the volume V, as expressed by the first term on the right, or stored in the volume V, as expressed by the second term. Hence

* $\nabla \cdot (\mathbf{A} \times \mathbf{C}) = \mathbf{C} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{C}$.

(w) can be interpreted as an electromagnetic energy density. The electromagnetic energy of the magnetic field system is stored in the magnetic field alone. Similarly, an electric field system is one in which the electromagnetic energy is stored in the electric field.

The familiar elements of electrical circuit theory illustrate the division of interactions into those defined as magnetic field systems and those defined as electric field systems. From the discussion in this and the preceding section it is evident that the short-circuited plates in Fig. B.2.1 constitute an inductor, whereas the open-circuited plates can be represented as a capacitor. This fact is the basis for the development of electromechanical interactions undertaken in Chapter 2. From this specific example it is evident that the magnetic field system includes interactions in which we can define lumped-parameter variables like the inductance, but it is not so evident that this model also describes the magnetohydrodynamic interactions of a fluid and some plasmas with a magnetic field and the magnetoelastic interactions of solids in a magnetic field, even including electromechanical aspects of microwave magnetics.

Similarly, the electric field system includes not only the electromechanics of systems that can be modeled in terms of circuit concepts like the capacitance but ferroelectric interactions between solids and electric fields, the electrohydrodynamics of a variety of liquids and slightly ionized gases in an electric field, and even the most important oscillations of an electron beam. Of course, if we are interested in the propagation of an electromagnetic wave through an ionospheric plasma or through the slightly ionized wake of a space vehicle, the full set of Maxwell's equations must be used.

There are situations in which the propagational aspects of the electromagnetic fields are not of interest, yet neither of the quasi-static systems is appropriate. This is illustrated by short-circuiting the parallel plates of Fig. B.2.1 at $x = 0$ by a resistive sheet. A static current or voltage applied to the plates at $x = -l$ then leads to both electric and magnetic fields between the plates. If the resistance of the sheet is small, the electric field between the plates is also small, and use of the exact field equations would show that we are still justified in ignoring the displacement current. In this case the inductance of Fig. B.2.1*a* is in series with a resistance. In the opposite extreme, if the resistance of the resistive sheet were very high, we would still be justified in ignoring the magnetic induction of Faraday's law. The situation shown in Fig. B.2.1*b* would then be modeled by a capacitance shunted by a resistance. The obvious questions are, when do we make a transition from the first case to the second and why is not this intermediate case of more interest in electromechanics?

The purpose of practical electromechanical systems is either the conversion of an electromagnetic excitation into a force that can perform work on a mechanical system or the reciprocal generation of electromagnetic energy from a force of mechanical origin. From (B.1.10) and (B.1.40) there are two fundamental types of electromagnetic force. Suppose that we are interested in producing a force of electrical origin on the upper of the two plates in Fig. B.2.1. We have the option of imposing a large current to interact with its induced magnetic field or of using a large potential to create an electric field that would interact with induced charges on the upper plate. Clearly, we are not going to impose a large potential on the plates if they are terminated in a small resistance or attempt to drive a large current through the plates with an essentially open circuit at $x = 0$. The electrical dissipation in both cases would be prohibitively large. More likely, if we intended to use the force $\mathbf{J} \times \mathbf{B}$, we would make the resistance as small as possible to minimize the dissipation of electric power and approach the case of Fig. B.2.1*a*. The essentially open circuit shown in Fig. B.2.1*b* would make it possible to use a large potential to create a significant force of the type $\rho_e\mathbf{E}$ without undue power dissipation. In the intermediate case the terminating resistance could be adjusted to make the electric and magnetic forces about equal. As a practical matter, however, the resulting device would probably melt before it served any useful electromechanical function. The power dissipated in the termination resistance would be a significant fraction of any electric power converted to mechanical form.*

The energy densities of (B.2.38) provide one means of determining when the problem shown in Fig. B.2.1 (but with a resistive sheet terminating the plates at $x = 0$) is intermediate between a magnetic and an electric field system. In the intermediate case the energy densities are equal

$$\frac{1}{2}\epsilon_0\mathbf{E}\cdot\mathbf{E} = \frac{1}{2}\frac{\mathbf{B}\cdot\mathbf{B}}{\mu_0}. \tag{B.2.39}$$

Now, if the resistive sheet has a total resistance of R, then from (B.2.18*a*) applied at $x = 0$

$$E_y s = -iR. \tag{B.2.40}$$

The current can be evaluated in terms of the magnetic field at $x = 0$ by using (B.2.18*b*):

$$E_y s = B_z\frac{dR}{\mu_0}. \tag{B.2.41}$$

Substitution of the electric field, as found from this expression into (B.2.39), gives

$$\frac{\epsilon_0}{2}\mathrm{B}_z{}^2\left(\frac{Rd}{s\mu_0}\right)^2 = \frac{1}{2}\frac{B_z{}^2}{\mu_0}. \tag{B.2.42}$$

* It is interesting that for this particular intermediate case the electric force tends to pull the plates together, whereas the magnetic force tends to push them apart. Hence, because the two forces are equal in magnitude, they just cancel.

Hence, if the energy densities are equal, we obtain the following relation among the physical parameters of the system:

$$\frac{dR}{s} = \left(\frac{\mu_0}{\epsilon_0}\right)^{1/2}. \tag{B.2.43}$$

It would be a digression to pursue this point here, but (B.2.43) is the condition that must be satisfied if an electromagnetic wave launched between the plates at $x = -l$ is to be absorbed, without reflection, by the resistive sheet*; that is, the intermediate case is one in which all the power fed into the system, regardless of the frequency or time constant, is dissipated by the resistive sheet.

B.3 MACROSCOPIC MODELS AND CONSTITUENT RELATIONS

When solids, liquids, and gases are placed in electromagnetic fields, they influence the field distribution. This is another way of saying that the force of interaction between charges or between currents is influenced by the presence of media. The effect is not surprising because the materials are comprised of charged particles.

Problems of physical significance can usually be decomposed into parts with widely differing scales. At the molecular or submolecular level we may be concerned with the dynamics of individual charges or of the atoms or molecules to which they are attached. These systems tend to have extremely small dimensions when compared with the size of a physical device. On the macroscopic scale we are not interested in the detailed behavior of the microscopic constituents of a material but rather only a knowledge of the average behavior of variables, since only these averages are observable on a macroscopic scale. The charge and current densities introduced in Section B.1 are examples of such variables, hence it is a macroscopic picture of fields and media that we require here.

There are three major ways in which media influence macroscopic electromagnetic fields. Hence the following sections undertake a review of magnetization, polarization, and conduction in common materials.

B.3.1 Magnetization

The macroscopic motions of electrons, even though associated with individual atoms or molecules, account for aggregates of charge and current

* The propagation of an electromagnetic wave on structures of this type is discussed in texts concerned with transmission lines or TEM wave guide modes. For a discussion of this matching problem see R. B. Adler, L. J. Chu, and R. M. Fano, *Electromagnetic Energy Transmission and Radiation*, Wiley, New York, 1960, p. 111, or S. Ramo, J. R. Whinnery, and T. Van Duzer, *Fields and Waves in Communication Electronics*, Wiley, New York, p. 27.

(when viewed at the macroscopic level) that induce electric and magnetic fields. These field sources are not directly accessible; for example, the equivalent currents within the material cannot be circulated through an external circuit. The most obvious sources of magnetic field that are inaccessible in this sense are those responsible for the field of a permanent magnet. The earliest observations on magnetic fields involved the lodestone, a primitive form of the permanent magnet. Early investigators such as Oersted found that magnetic fields produced by a permanent magnet are equivalent to those induced by a circulating current. In the formulation of electromagnetic theory we must distinguish between fields due to sources within the material and those from applied currents simply because it is only the latter sources that can be controlled directly. Hence we divide the source currents into *free currents* (with the density $\mathbf{J}_f$) and *magnetization currents* (with the density $\mathbf{J}_m$). Ampère's law then takes the form

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0}\right) = \mathbf{J}_m + \mathbf{J}_f. \tag{B.3.1}$$

By convention it is also helpful to attribute a fraction of the field induced by these currents to the magnetization currents in the material. Hence (B.3.1) is written as

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J}_f, \tag{B.3.2}$$

where the *magnetization density* $\mathbf{M}$ is defined by

$$\nabla \times \mathbf{M} = \mathbf{J}_m. \tag{B.3.3}$$

Up to this point in this chapter it has been necessary to introduce only two field quantities to account for interactions between charges and between currents. To account for the macroscopic properties of media we have now introduced a new field quantity, the magnetization density $\mathbf{M}$, and in the next section similar considerations concerning electric polarization of media lead to the introduction of the polarization density $\mathbf{P}$. It is therefore apparent that macroscopic field theory is formulated in terms of four field variables. In our discussion these variables have been $\mathbf{E}$, $\mathbf{B}$, $\mathbf{M}$, and $\mathbf{P}$. An alternative representation of the fields introduces the *magnetic field intensity* $\mathbf{H}$, in our development *defined* as

$$\mathbf{H} = \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right). \tag{B.3.4}$$

From our definition it is clear that we could just as well deal with $\mathbf{B}$ and $\mathbf{H}$ as the macroscopic magnetic field vectors rather than with $\mathbf{B}$ and $\mathbf{M}$. This is particularly appealing, for then (B.3.2) takes the simple form

$$\nabla \times \mathbf{H} = \mathbf{J}_f. \tag{B.3.5}$$

When the source quantities $\mathbf{J}_f$ and $\mathbf{M}$ are specified independently, the magnetic field intensity $\mathbf{H}$ (or magnetic flux density $\mathbf{B}$) can be found from the quasi-static magnetic field equations. A given constant magnetization density corresponds to the case of the permanent magnet. In most cases, however, the source quantities are functions of the field vectors, and these funtional relations, called *constituent relations*, must be known before the problems can be solved. The constituent relations represent the constraints placed on the fields by the internal physics of the media being considered. Hence it is these relations that make it possible to separate the microscopic problem from the macroscopic one of interest here.

The simplest form of constituent relation for a magnetic material arises when it can be considered *electrically linear* and *isotropic*. Then the *permeability* μ is constant in the relation

$$\mathbf{B} = \mu \mathbf{H}. \tag{B.3.6}$$

The material is isotropic because $\mathbf{B}$ is collinear with $\mathbf{H}$ and a particular constant (μ) times $\mathbf{H}$, regardless of the direction of $\mathbf{H}$. A material that is *homogeneous* and isotropic will in addition have a permeability μ that does not vary with position in the material. Another way of expressing (B.3.6) is to define a magnetic susceptibility χ_m (dimensionless) such that

$$\mathbf{M} = \chi_m \mathbf{H}, \tag{B.3.7}$$

where

$$\mu = \mu_0(1 + \chi_m). \tag{B.3.8}$$

Magnetic materials are commonly found with $\mathbf{B}$ not a linear function of $\mathbf{H}$ and the constitutive law takes the general form

$$\mathbf{B} = \mathbf{B}(\mathbf{H}). \tag{B.3.9}$$

We deal with some problems involving materials of this type, but with few exceptions confine our examples to situations in which $\mathbf{B}$ is a single-valued function of $\mathbf{H}$. In certain magnetic materials in some applications the $\mathbf{B}$-$\mathbf{H}$ curve must include hysteresis and (B.3.9) is not single-valued.*

The differential equations for a magnetic field system in the presence of moving magnetized media are summarized in Table 1.2.

B.3.2 Polarization

The force between a charge distribution and a test charge is observed to change if a dielectric material is brought near the region occupied by the test

* G. R. Slemon, *Magnetoelectric Devices*, Wiley, New York, 1966, p. 115.

charge. Like the test charge, the charged particles which compose the dielectric material experience forces due to the applied field. Although these charges remain identified with the molecules of the material, their positions can be distorted incrementally by the electric force and thus lead to a polarization of the molecules.

The basic sources of the electric field are charges. Hence it is natural to define a *polarization charge density* ρ_p as a source of a fraction of the electric field which can be attributed to the inaccessible sources within the media. Thus Gauss's law (B.1.16) is written

$$\nabla \cdot \epsilon_0 \mathbf{E} = \rho_f + \rho_p, \tag{B.3.10}$$

where the *free charge density* ρ_f resides on conducting electrodes and other parts of the system capable of supporting conduction currents. The free charges do not remain attached to individual molecules but rather can be conducted from one point to another in the system.

In view of the form taken by Gauss's law, it is convenient to identify a field induced by the polarization charges by writing (B.3.10) as

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f, \tag{B.3.11}$$

where the *polarization density* **P** is related to the polarization charge density by

$$\rho_p = -\nabla \cdot \mathbf{P}. \tag{B.3.12}$$

As in Section B.3.1, it is convenient to define a new vector field that serves as an alternative to **P** in formulating the electrodynamics of polarized media. This is the *electric displacement* **D**, defined as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{B.3.13}$$

In terms of this field, Gauss's law for electric fields (B.3.11) becomes

$$\nabla \cdot \mathbf{D} = \rho_f. \tag{B.3.14}$$

The simple form of this expression makes it desirable to use **D** rather than **P** in the formulation of problems.

If a polarization charge model is to be used to account for the effects of polarizable media on electric fields, we must recognize that the motion of these charges can lead to a current. In fact, now that two classes of charge density have been identified we must distinguish between two classes of current density. The free current density $\mathbf{J}_f$ accounts for the conservation of free charge so that (B.1.18) can be written as

$$\nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t} = 0. \tag{B.3.15}$$

In view of (B.3.11), this expression becomes

$$\nabla \cdot \mathbf{J}_f + \frac{\partial}{\partial t} \nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = 0. \tag{B.3.16}$$

Now, if we write Ampère's law (B.2.26*b*) as

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0}\right) = \mathbf{J}_f + \mathbf{J}_p + \frac{\partial}{\partial t} \epsilon_0 \mathbf{E}, \tag{B.3.17}$$

where $\mathbf{J}_p$ is a current density due to the motion of polarization charges, the divergence of (B.3.17) must give (B.3.16). Therefore

$$\nabla \cdot \mathbf{J}_p + \frac{\partial}{\partial t}(-\nabla \cdot \mathbf{P}) = 0. \tag{B.3.18}$$

which from (B.3.12) is an expression for the conservation of polarization charge. This expression does not fully determine the polarization current density $\mathbf{J}_p$, because in general we could write

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{A}, \tag{B.3.19}$$

where $\mathbf{A}$ is an arbitrary vector, and still satisfy (B.3.18). At this point we could derive the quantity $\mathbf{A}$ (which would turn out to be $\mathbf{P} \times \mathbf{v}$, where $\mathbf{v}$ is the velocity of the polarized medium). It is important, however, to recognize that this represents an unnecessary digression. In the electric field system the magnetic field appears in only one of the equations of motion—Ampère's law. It does not appear in (B.2.27*b*) to (B.2.29*b*), nor will it appear in any constitutive law used in this book. For this reason the magnetic field serves simply as a quantity to be calculated once the electromechanical problem has been solved. We might just as well lump the quantity $\mathbf{A}$ with the magnetic field in writing Ampère's law. In fact, if we are consistent, the magnetic field intensity $\mathbf{H}$ can be defined as given by

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}, \tag{B.3.20}$$

with no loss of physical significance. In an electric field system the magnetic field is an alternative representation of the current density $\mathbf{J}_f$. A review of the quasi-static solutions for the system in Fig. B.2.1*b* illustrates this point.

In some materials (ferroelectrics) the polarization density $\mathbf{P}$ is constant. In most common dielectrics, however, the polarization density is a function of $\mathbf{E}$. The simplest constituent relation for a dielectric is that of linear and isotropic material,

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \tag{B.3.21}$$

where χ_e is the *dielectric susceptibility* (dimensionless) that may be a function of space but not of $\mathbf{E}$. For such a material we define the *permittivity* ϵ as

$$\epsilon = \epsilon_0(1 + \chi_e). \tag{B.3.22}$$

and then write the relation between $\mathbf{D}$ and $\mathbf{E}$ as [see (B.3.13)]

$$\mathbf{D} = \epsilon\mathbf{E}. \tag{B.3.23}$$

This mathematical model of polarizable material is used extensively in this book.

The differential equations for the electric field system, in the presence of moving polarized media, are summarized in Table 1.2.

B.3.3 Electrical Conduction

In both magnetic and electric field systems the conduction process accounts for the free current density $\mathbf{J}_f$ in a fixed conductor. The most common model for this process is appropriate in the case of an isotropic, linear, conducting medium which, when stationary, has the constituent relation (often called *Ohm's law*)

$$\mathbf{J}_f = \sigma\mathbf{E}. \tag{B.3.24}$$

Although (B.3.24) is the most widely used mathematical model of the conduction process, there are important electromechanical systems for which it is not adequate. This becomes apparent if we attempt to derive (B.3.24), an exercise that will contribute to our physical understanding of Ohm's law.

In many materials the conduction process involves two types of charge carrier (say, ions and electrons). As discussed in Section B.1.2, a macroscopic model for this case would recognize the existence of free charge densities ρ_+ and ρ_- with charge average velocities $\mathbf{v}_+$ and $\mathbf{v}_-$, respectively. Then

$$\mathbf{J}_f = \rho_+\mathbf{v}_+ + \rho_-\mathbf{v}_-. \tag{B.3.25}$$

The problem of relating the free current density to the electric field intensity is thus a problem in electromechanics in which the velocities of the particles carrying the free charge must be related to the electric fields that apply forces to the charges.

The charge carriers have finite mass and thus accelerate when subjected to a force. In this case there are forces on the positive and negative carriers, respectively, given by (B.1.10) (here we assume that effects from a magnetic field are ignorable):

$$\mathbf{F}_+ = \rho_+\mathbf{E}, \tag{B.3.26}$$

$$\mathbf{F}_- = \rho_-\mathbf{E}. \tag{B.3.27}$$

As the charge carriers move, their motion is retarded by collisions with other particles. On a macroscopic basis the retarding force of collisions can be thought of as a viscous damping force that is proportional to velocity. Hence we can picture the conduction process in two extremes. With no collisions between particles the electric force densities of (B.3.26 and B.3.27) continually accelerate the charges, for the only retarding forces are due to acceleration expressed by Newton's law. In the opposite extreme a charge carrier suffers collisions with other particles so frequently that its average velocity quickly reaches a limiting value, which in view of (B.3.26 and B.3.27) is proportional to the applied electric field. It is in this second limiting case that Ohm's law assumes physical significance. By convention *mobilities* μ_+ and μ_- which relate these limiting velocities to the field $\mathbf{E}$ are defined

$$\mathbf{v}_+ = \mu_+\mathbf{E}, \tag{B.3.28}$$

$$\mathbf{v}_- = \mu_-\mathbf{E}. \tag{B.3.29}$$

In terms of these quantities, (B.3.25) becomes

$$\mathbf{J}_f = (\rho_+\mu_+ + \rho_-\mu_-)\mathbf{E}. \tag{B.3.30}$$

It is important to recognize that it is only when the collisions between carriers and other particles dominate the accelerating effect of the electric field that the conduction current takes on a form in which it is dependent on the instantaneous value of $\mathbf{E}$. Fortunately, (B.3.30) is valid in a wide range of physical situations. In fact, in a metallic conductor the number of charge carriers is extremely high and very nearly independent of the applied electric field. The current carriers in most metals are the electrons, which are detached from atoms held in the lattice structure of the solid. Therefore the negatively charged electrons move in a background field of positive charge and, to a good approximation, $\rho_+ = -\rho_-$. Then (B.3.30) becomes

$$\mathbf{J} = \sigma\mathbf{E}, \tag{B.3.31}$$

where the conductivity is defined as

$$\rho_+(\mu_+ - \mu_-). \tag{B.3.32}$$

The usefulness of the conductivity as a parameter stems from the fact that both the number of charges available for conduction and the net mobility (essentially that of the electrons) are constant. This makes the conductivity essentially independent of the electric field, as assumed in (B.3.24).*

* We assume here that the temperature remains constant. A worthwhile qualitative description of conduction processes in solids is given in J. M. Ham and G. R. Slemon, *Scientific Basis of Electrical Engineering*, Wiley, New York, 1961, p. 453.

In some types of material (notably slightly ionized gases) which behave like insulators, the conduction process cannot be described simply by Ohm's law. In such materials the densities of charge carriers and even the mobilities may vary strongly with electric field intensity.

B.4 INTEGRAL LAWS

The extensive use of circuit theory bears testimony to the usefulness of the laws of electricity and magnetism in integral form. Physical situations that would be far too involved to describe profitably in terms of field theory have a lucid and convenient representation in terms of circuits. Conventional circuit elements are deduced from the integral forms of the field equations. The description of lumped-parameter electromechanical systems, as undertaken in Chapter 2, requires that we generalize the integral laws to include time-varying surfaces and contours of integration. Hence it is natural that we conclude this appendix with a discussion of the integral laws.

B.4.1 Magnetic Field Systems

Faraday's law of induction, as given by (B.1.42), has the differential form

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{B.4.1}$$

This expression can be integrated over a surface S enclosed by the contour C. Then, according to Stokes's theorem,

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, da. \tag{B.4.2}$$

Now, if S and C are fixed in space, the time derivative on the right can be taken either before or after the surface integral of $\mathbf{B} \cdot \mathbf{n}$ is evaluated. Note that $\int_S \mathbf{B} \cdot \mathbf{n}\, da$ is only a function of time. For this reason (B.1.41) could be written with the total derivative outside the surface integral. It is implied in the integral equation (B.1.41) that S is fixed in space.

Figure B.4.1 shows an example in which it is desirable to be able to use (B.4.2), with S and C varying in position as a function of time. The contour C is a rectangular loop that encloses a surface S which makes an angle $\theta(t)$ with the horizontal. Although the induction law is not limited to this case, the loop could comprise a one-turn coil, in which case it is desirable to be able to use (B.4.2) with C fixed to the coil. The integral law of induction would be much more useful if it could be written in the form

$$\oint_C \mathbf{E}' \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot \mathbf{n}\, da. \tag{B.4.3}$$

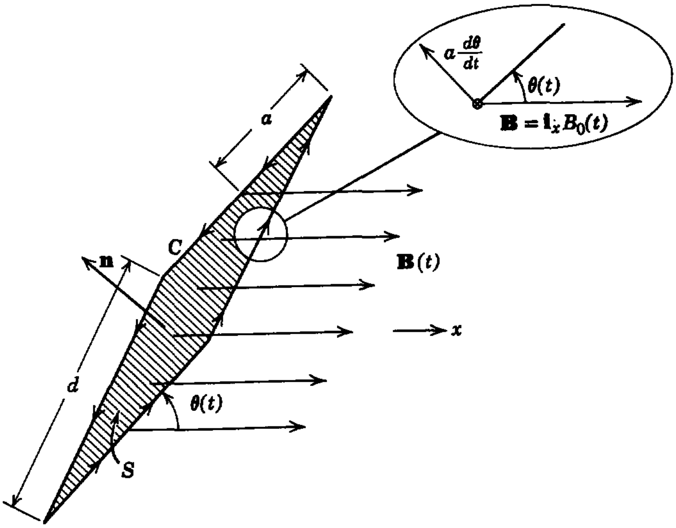

Fig. B.4.1 Contour C enclosing a surface S which varies as a function of time. The rectangular loop links no magnetic flux when $\theta = 0, \pi, \ldots$.

In this form the quantity on the right is the negative time rate of change of the flux linked by the contour C, whereas $\mathbf{E}'$ is interpreted as the electric field measured in the moving frame of the loop. An integral law of induction in the form of (B.4.3) is essential to the lumped-parameter description of magnetic field systems. At this point we have two choices. We can accept (B.4.3) as an empirical result of Faraday's original investigations or we can show mathematically that (B.4.2) takes the form of (B.4.3) if

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \tag{B.4.4}$$

where $\mathbf{v}$ is the velocity of $d\mathbf{l}$ at the point of integration. In any case this topic is pursued in Chapter 6 to clarify the significance of electric and magnetic fields measured in different frames of reference.

The mathematical connection between (B.4.2) and (B.4.3) is made by using the integral theorem

$$\frac{d}{dt}\int_S \mathbf{A}\cdot\mathbf{n}\,da = \int_S \left[\frac{\partial \mathbf{A}}{\partial t} + (\nabla\cdot\mathbf{A})\mathbf{v}\right]\cdot\mathbf{n}\,da + \oint_C (\mathbf{A}\times\mathbf{v})\cdot d\mathbf{l}, \tag{B.4.5}$$

where $\mathbf{v}$ is the velocity of S and C and in the case of (B.4.3), $\mathbf{A} \rightarrow \mathbf{B}$. Before we embark on a proof of this theorem, an example will clarify its significance.

Example B.4.1. The coil shown in Fig. B.4.1 rotates with the angular deflection $\theta(t)$ in a uniform magnetic flux density $\mathbf{B}(t)$, directed as shown. We wish to compute the rate of change of the flux linked by the coil in two ways: first by computing $\int_S \mathbf{B}\cdot\mathbf{n}\,da$ and taking its derivative [the left-hand side of (B.4.5)], then by using the surface and contour integrations indicated on the right-hand side of (B.4.5). This illustrates how the identity allows us to carry out the surface integration before rather than after the time derivative is taken. From Fig. B.4.1 we observe that

$$\int_S \mathbf{B} \cdot \mathbf{n}\, da = -B_0(t) 2ad \sin\theta, \tag{a}$$

so that the first calculation gives

$$\frac{d}{dt}\int_S \mathbf{B} \cdot \mathbf{n}\, da = -2ad \sin\theta \frac{dB_0}{dt} - B_0 2ad \cos\theta \frac{d\theta}{dt}. \tag{b}$$

To evaluate the right-hand side of (B.4.5) observe that $\nabla \cdot \mathbf{B} = 0$ and [from (a)]

$$\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, da = -2ad \sin\theta \frac{dB_0}{dt}. \tag{c}$$

The quantity $\mathbf{B} \times \mathbf{v}$ is collinear with the axis of rotation in Fig. B.4.1; hence there is no contribution to the line integral along the pivoted ends of the loop. Because both the velocity $\mathbf{v} = \mathbf{i}_\theta a\,(d\theta/dt)$ and line elements $d\mathbf{l}$ are reversed in going from the upper to the lower horizontal contours, the line integral reduces to twice the value from the upper contour.

$$\oint_C \mathbf{B} \times \mathbf{v} \cdot d\mathbf{l} = -2B_0 ad \cos\theta \frac{d\theta}{dt} \tag{d}$$

From (c) and (d) it follows that the right-hand side of (B.4.5) also gives (b). Thus, at least for this example, (B.4.5) provides alternative ways of evaluating the time rate of change of the flux linked by the contour C.

The integral theorem of (B.4.5) can be derived by considering the deforming surface S shown at two instants of time in Fig. B.4.2. In the incremental time interval Δt the surface S moves from S_1 to S_2, and therefore by

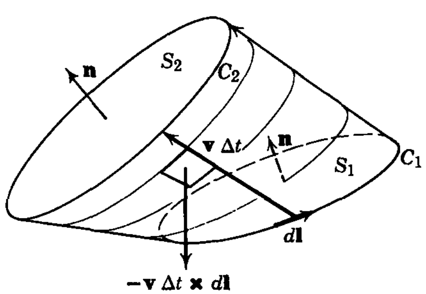

Fig. B.4.2 When $t = t$, the surface S enclosed by the contour C is as indicated by S_1 and C_1. By the time $t = t + \Delta t$ this surface has moved to S_2, where it is enclosed by the contour C_2.

definition

$$\frac{d}{dt}\int_S \mathbf{A}\cdot\mathbf{n}\,da = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\int_{S_2}\mathbf{A}\Big|_{t+\Delta t}\cdot\mathbf{n}\,da - \int_{S_1}\mathbf{A}\Big|_t\cdot\mathbf{n}\,da\right). \quad \text{(B.4.6)}$$

Here we have been careful to show that when the integral on S_2 is evaluated $t = t + \Delta t$, in contrast to the integration on S_1, which is carried out when $t = t$.

The expression on the right in (B.4.6) can be evaluated at a given instant in time by using the divergence theorem (B.1.14) to write

$$\int_V \nabla\cdot\mathbf{A}\,dV \cong \int_{S2}\mathbf{A}\Big|_t\cdot\mathbf{n}\,da - \int_{S_1}\mathbf{A}\Big|_t\cdot\mathbf{n}\,da - \Delta t\oint_{C_1}\mathbf{A}\cdot\mathbf{v}\times d\mathbf{l} \quad \text{(B.4.7)}$$

for the volume V traced out by the surface S in the time Δt. Here we have used the fact that $-\mathbf{v}\,\Delta t \times d\mathbf{l}$ is equivalent to a surface element $\mathbf{n}\,da$ on the surface traced out by the contour C in going from C_1 to C_2 in Fig. B.4.2. To use (B.4.7) we make three observations. First, as $\Delta t \to 0$,

$$\int_{S_2}\mathbf{A}\Big|_{t+\Delta t}\cdot\mathbf{n}\,da \cong \int_{S_2}\mathbf{A}\Big|_t\cdot\mathbf{n}\,da + \int_{S_1}\frac{\partial\mathbf{A}}{\partial t}\Big|_t\,\Delta t\cdot\mathbf{n}\,da + \cdots. \quad \text{(B.4.8)}$$

Second, it is a vector identity that

$$\mathbf{A}\cdot\mathbf{v}\times d\mathbf{l} = \mathbf{A}\times\mathbf{v}\cdot d\mathbf{l}. \quad \text{(B.4.9)}$$

Third, an incremental volume dV swept out by the surface da is essentially the base times the perpendicular height or

$$dV = \Delta t\mathbf{v}\cdot\mathbf{n}\,da. \quad \text{(B.4.10)}$$

From these observations (B.4.7) becomes

$$\begin{aligned}\Delta t\int_{S_1}(\nabla\cdot\mathbf{A})\mathbf{v}\cdot\mathbf{n}\,da \cong \int_{S_2}\mathbf{A}\Big|_{t+\Delta t}\cdot\mathbf{n}\,da - \int_{S_1}\Delta t\frac{\partial\mathbf{A}}{\partial t}\Big|_t\cdot\mathbf{n}\,da \\ - \int_{S_1}\mathbf{A}\Big|_t\cdot\mathbf{n}\,da - \Delta t\oint_{C_1}\mathbf{A}\times\mathbf{v}\cdot d\mathbf{l}. \quad \text{(B.4.11)}\end{aligned}$$

This expression can be solved for the quantity on the right in (B.4.6) to give

$$\frac{d}{dt}\int_S\mathbf{A}\cdot\mathbf{n}\,da = \lim_{\Delta t\to 0}\left\{\int_{S_1}\left[(\nabla\cdot\mathbf{A})\mathbf{v} + \frac{\partial\mathbf{A}}{\partial t}\right]\cdot\mathbf{n}\,da + \oint_{C_1}\mathbf{A}\times\mathbf{v}\cdot d\mathbf{l}\right\}. \quad \text{(B.4.12)}$$

The limit of this expression produces the required relation (B.4.5).

Use of (B.4.5) to express the right-hand side of (B.4.2) results in

$$\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, da = \frac{d}{dt}\int_S \mathbf{B} \cdot \mathbf{n}\, da - \int_S (\nabla \cdot \mathbf{B})\mathbf{v} \cdot \mathbf{n}\, da - \oint_C (\mathbf{B} \times \mathbf{v}) \cdot d\mathbf{l}. \tag{B.4.13}$$

Because $\nabla \cdot \mathbf{B} = 0$, (B.4.2) then reduces to (B.4.3), with $\mathbf{E}'$ given by (B.4.4).

The integral laws for the magnetic field system are summarized in Table 1.2 at the end of Chapter 1. In these equations surfaces and contours of integration can, in general, be time-varying.

B.4.2 Electric Field System

Although the integral form of Faraday's law can be taken as an empirical fact, we require (B.4.5) to write Ampère's law in integral form for an electric field system. If we integrate (B.3.20) over a surface S enclosed by a contour C, by Stokes's theorem it becomes

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot \mathbf{n}\, da + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{n}\, da. \tag{B.4.14}$$

As with the induction law for the magnetic field system, this expression can be generalized to include the possibility of a deforming surface S by using (B.4.13) with $\mathbf{B} \to \mathbf{D}$ to rewrite the last term. If, in addition, we use (B.3.14) to replace $\nabla \cdot \mathbf{D}$ with ρ_f, (B.4.14) becomes

$$\oint_C \mathbf{H}' \cdot d\mathbf{l} = \int_S \mathbf{J}_f' \cdot \mathbf{n}\, da + \frac{d}{dt}\int_S \mathbf{D} \cdot \mathbf{n}\, da, \tag{B.4.15}$$

where

$$\mathbf{H}' = \mathbf{H} - \mathbf{v} \times \mathbf{D}, \tag{B.4.16}$$

$$\mathbf{J}_f' = \mathbf{J}_f - \rho_f \mathbf{v}. \tag{B.4.17}$$

The fields $\mathbf{H}'$ and $\mathbf{J}_f'$ can be interpreted as the magnetic field intensity and free current density measured in the moving frame of the deforming contour. The significance of these field transformations is discussed in Chapter 6. Certainly the relationship between $\mathbf{J}_f'$ (the current density in a frame moving with a velocity $\mathbf{v}$) and the current density $\mathbf{J}_f$ (measured in a fixed frame), as given by (B.4.17), is physically reasonable. The free charge density appears as a current in the negative $\mathbf{v}$-direction when viewed from a frame moving at the velocity $\mathbf{v}$. If was reasoning of this kind that led to (B.1.25).

As we have emphasized, it is the divergence of Ampère's differential law that assumes the greatest importance in electric field systems, for it accounts for conservation of charge. The integral form of the conservation of charge

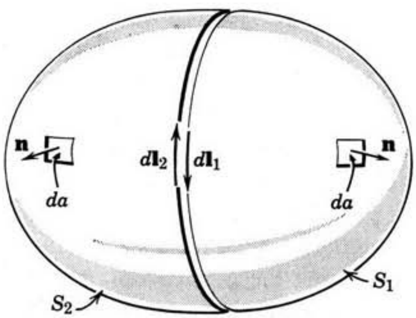

Fig. B.4.3 The sum of two surfaces S_1 and S_2 "spliced" together at the contour to enclose the volume V.

equation, including the possibility of a deforming surface of integration, is obtained by using (B.4.15). For this purpose integrations are considered over two deforming surfaces, S_1 and S_2, as shown in Fig. B.4.3. These surfaces are chosen so that they are enclosed by the same contour C. Hence, taken together, S_1 and S_2 enclose a volume V.

Integration of (B.4.15) over each surface gives

$$\oint_C \mathbf{H}' \cdot d\mathbf{l}_1 = \int_{S_1} \mathbf{J}_f' \cdot \mathbf{n}\, da + \frac{d}{dt}\int_{S_1} \mathbf{D} \cdot \mathbf{n}\, da. \tag{B.4.18}$$

$$\oint_C \mathbf{H}' \cdot d\mathbf{l}_2 = \int_{S_2} \mathbf{J}_f' \cdot \mathbf{n}\, da + \frac{d}{dt}\int_{S_2} \mathbf{D} \cdot \mathbf{n}\, da. \tag{B.4.19}$$

Now, if $\mathbf{n}$ is defined so that it is directed out of the volume V on each surface, the line integral enclosing S_1 will be the negative of that enclosing S_2. Then the sum of (B.4.18 and B.4.19) gives the desired integral form of the conservation of charge equation:

$$\oint_S \mathbf{J}_f' \cdot \mathbf{n}\, da + \frac{d}{dt}\int_V \rho_f \, dV = 0. \tag{B.4.20}$$

In writing this expression we have used Gauss's theorem and (B.3.14) to show the explicit dependence of the current density through the deforming surface on the enclosed charge density.

The integral laws for electric field systems are summarized in Table 1.2 at the end of Chapter 1.

B.5 RECOMMENDED READING

The following texts treat the subject of electrodynamics and provide a comprehensive development of the fundamental laws of electricity and magnetism.

R. M. Fano, L. J. Chu, and R. B. Adler, *ElectromagneticFields, Energy, and Forces*, Wiley, New York, 1960; J. D. Jackson, *Classical Electrodynamics*, Wiley, New York, 1962: S. Ramo, J. R. Whinnery, and T. Van Duzer, *Fields and Waves in Communication Electronics*, Wiley, New York, 1965; W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, Addison-Wesley, Reading, Mass., 1956; J. A. Stratton, *Electromagnetic Theory*, McGraw-Hill, New York, 1941.

Many questions arise in the study of the effects of moving media on electric and magnetic fields concerning the macroscopic representation of polarized and magnetized media; for example, in this appendix we introduced the fields **E** and **B** as the quantities defined by the force law. Then **P** and **M** (or **D** and **H**) were introduced to account for the effects of polarization and magnetization. Hence the effect of the medium was accounted for by equivalent polarization charges ρ_p and magnetization currents $\mathbf{J}_m$. Other representations can be used in which a different pair of fundamental vectors is taken, as defined by the force law (say, **E** and **H**), and in which the effects of media are accounted for by an equivalent magnetic charge instead of an equivalent current. If we are consistent in using the alternative formulations of the field equations, they predict the same physical results, including the force on magnetized and polarized media. For a complete discussion of these matters see P. Penfield, and H. Haus, *Electrodynamics of Moving Media*, M.I.T. Press, Cambridge, Mass., 1967.