Solutions Manual for Electromechanical Dynamics

PROBLEM 10.1

Part a

At $x = 0$, the net force on an incremental length of the string has to be zero.

$$-2B \frac{\partial \xi}{\partial t} - f \frac{\partial \xi}{\partial x} = 0$$

This is the required boundary condition at $x = 0$.

Part b

The power absorbed by the dashpots is the product of force $2B\ \partial\xi/\partial t$ and the velocity $\partial\xi/\partial t$. Thus

$$P = 2B \left(\frac{\partial \xi}{\partial t}\right)^2$$

If we solve Eq. 10.1.6 for

$$\xi(x,t) = \mathrm{Re}[\hat{\xi}\, e^{j(\omega t - kx)}]$$

and assume that $\omega < \omega_c$ we get

$$\xi(x,t) = \mathrm{Re}\left\{[A_1 \sinh|k|x + A_2 \cosh|k|x]e^{j\omega t}\right\}$$

where

$$k = \left[\frac{\omega^2 - \omega_c^2}{v_s^2}\right]^{1/2}$$

We can calculate A_1 and A_2 using the boundary condition of part (a) and the boundary condition at $x = \ell$

$$\xi(-\ell,t) = \mathrm{Re}\ \xi_o\, e^{j\omega t}$$

We then get

$$A_1 = -\frac{j\xi_o 2B\omega}{[f|k|\cosh|k|\ell + j\omega 2B \sinh|k|\ell]}$$

$$A_2 = \frac{\xi_o f|k|}{[f|k|\cosh|k|\ell + j\omega 2B \sinh|k|\ell]}$$

If we plug these values into the expression for power, and then time average, we have

$$\langle P\rangle = \frac{B(f|k|\xi_o\omega)^2}{[(f|k|\cosh|k|\ell)^2 + (2B\omega \sinh|k|\ell)^2]}$$

where it is convenient to use the identity

$$\langle \mathrm{Re}\, Ae^{j\omega t}\ \mathrm{Re} Be^{j\omega t}\rangle = \frac{1}{2} AB^*$$

PROBLEM 10.2

Part a

We use Eq. (10.1.6)

$$\frac{\partial^2\xi}{\partial t^2} = v_s^2 \frac{\partial^2\xi}{\partial x^2} - \omega_c^2\xi \ , \ \omega_c^2 = \frac{Ib}{m}$$

Assume solutions $\xi = \mathrm{Re}[(Ae^{-jkx} + Be^{jkx})e^{j\omega t}]$. The dispersion equation is:

$$k^2 = \frac{\omega_d^2 - \omega_c^2}{v_s^2}$$

Now use the boundary conditions, which require

$$A\ e^{jk\ell} + Be^{-jk\ell} = \xi_d$$

$$-jk[A - B] = 0$$

(i) $\omega_d < \omega_c$ (below cutoff)

$$\xi = \frac{\xi_d \cosh \alpha x}{\cosh \alpha\ell} \cos \omega_d t$$

$$\alpha = \sqrt{(\omega_c^2 - \omega_d^2)/v_s^2}$$

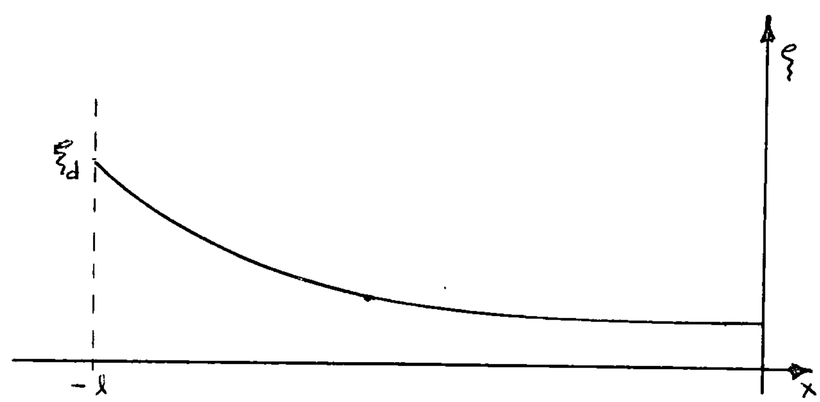
(ii) $\omega_d > \omega_c$ (above cutoff)

$$\xi(x,t) = \frac{\xi_d \cos \beta x}{\cos \beta\ell} \cos \omega_d t$$

$$\beta = \sqrt{(\omega_d^2 - \omega_c^2)/v_s^2}$$

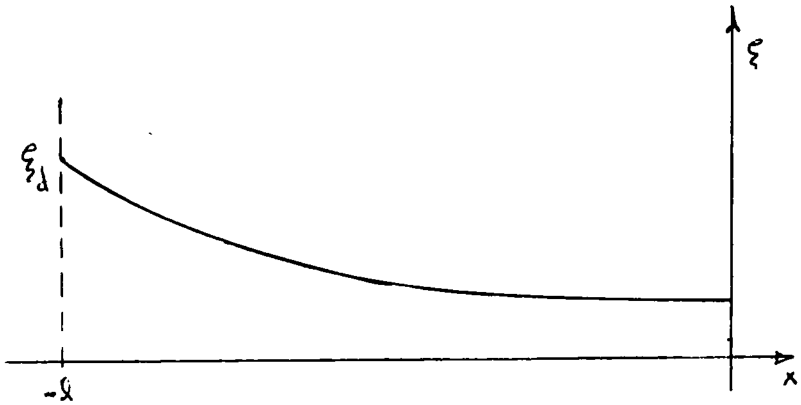
Part b

$\omega_d = 0$

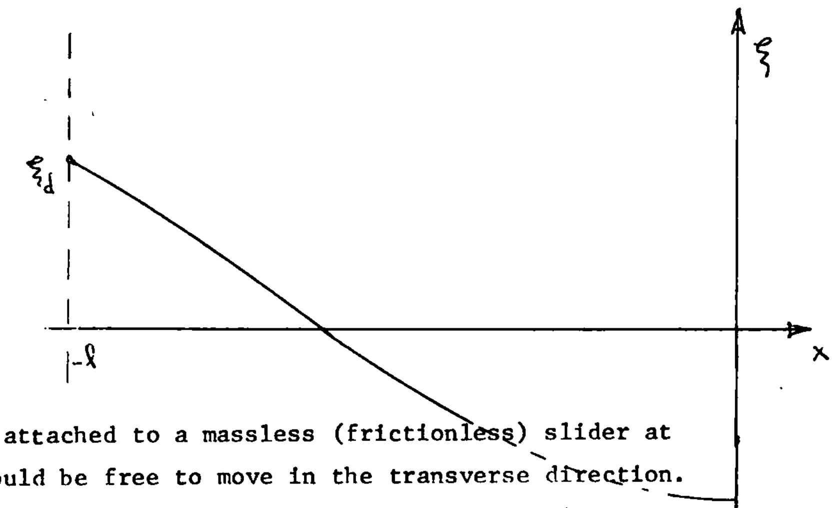


PROBLEM 10.2 (continued)

$\omega_d < \omega_c$



$\omega_d > \omega_c$



Part c

The string might be attached to a massless (frictionless) slider at $x = 0$, so that the end would be free to move in the transverse direction. Force equilibrium for the increment of length at $x = 0$ then requires $\partial\xi/\partial x = 0$ at $x = 0$.

PROBLEM 10.3

Part a

From Eq. 10.1.10 we have

$$k = \left[\frac{\omega^2 - \omega_c^2}{v_s^2}\right]^{1/2}$$

with our solution of the form

$$\xi(x,t) = \mathrm{Re}\left(A_1\, e^{j(\omega t - kx)} + A_2\, e^{j(\omega t + kx)}\right)$$

We have the boundary conditions

PROBLEM 10.3(continued)

$$\frac{\partial \xi}{\partial x}(0,t) = 0$$

and

$$\frac{\partial \xi}{\partial x}(-\ell,t) = 0.$$

From the first boundary condition, we obtain

$$A_1 = A_2 \ ; \ \xi(x,t) = \text{Re } A_3 \cos kx \, e^{j\omega t}$$

From the second boundary condition, we obtain

$$\sin k\ell = 0$$

This implies that

$$k = \frac{n\pi}{\ell} \ ; \ n = 0,1,2,3\ldots$$

Note that by contrast with the case where the ends are fixed, $n = 0$ is a valid (nontrivial) and crucial solution. It corresponds to an eigenmode which is simply a rigid body translation.

From Eq. 10.1.7

$$\omega^2 = k^2 v_s^2 + \omega_c^2$$

Therefore, the eigenfrequencies are

$$\omega = \pm \left[\omega_c^2 + \left(\frac{n\pi}{\ell} v_s\right)^2\right]^{1/2}$$

For the $n = 0$ mode, $\omega = \pm \omega_c$.

Part b

With I as in Fig. 10.1.9, we have the same equations as in part (a) if we replace ω_c^2 by $-\omega_c^2$. Therefore, for this case, the eigenfrequencies are

$$\omega = \left[\left(\frac{n\pi}{\ell} v_s\right)^2 - \omega_c^2\right]^{1/2}$$

Part c

With I as in Fig. 10.1.9, the $\bar{I}\times\bar{B}$ force is destabilizing, as a small perturbation from $x = 0$ tends to increase this force. If ω in part (b) became imaginary, the equilibrium $\xi = 0$ would become unstable as the solutions are unbounded in time. This will happen as

$$\left(\frac{n\pi}{\ell} v_s\right)^2 - \omega_c^2 < 0$$

PROBLEM 10.3 (continued)

or in terms of the current

$$I > \frac{n}{b}\left(\frac{\pi}{\ell} v_s\right)^2$$

Note that any finite current makes the n = 0 mode unstable, since for this mode there is no elastic restoring force.

PROBLEM 10.4

Multiply the system equation by $\frac{\partial\xi}{\partial t}$,

$$m\,\frac{\partial\xi}{\partial t}\cdot\frac{\partial^2\xi}{\partial t^2} = f\,\frac{\partial\xi}{\partial t}\cdot\frac{\partial^2\xi}{\partial x^2} - Ib\,\frac{\partial\xi}{\partial t}\cdot\xi + \frac{\partial\xi}{\partial t}\cdot F(x,t)$$

Proper substitution of partial differential identities yields:

$$\frac{\partial}{\partial t}\left[\frac{m}{2}\left(\frac{\partial\xi}{\partial t}\right)^2 + \frac{f}{2}\left(\frac{\partial\xi}{\partial x}\right)^2 + \frac{Ib}{2}\,\xi^2\right] - \frac{\partial}{\partial x}\left[f\,\frac{\partial\xi}{\partial x}\cdot\frac{\partial\xi}{\partial t}\right] = \frac{\partial\xi}{\partial t}\cdot F(x,t)$$

PROBLEM 10.5

We have that

$$\xi(x,t) = \mathrm{Re}\left[\hat{\xi}_+\,e^{j(\omega t-kx)} + \hat{\xi}_-\,e^{j(\omega t+kx)}\right]$$

Part a

For k real, we might write this in the form

$$\xi(x,t) = \frac{1}{2}\left[\hat{\xi}_+\,e^{j(\omega t-kx)} + \hat{\xi}_+^*\,e^{-j(\omega t-kx)} + \hat{\xi}_-\,e^{j(\omega t+kx)} + \hat{\xi}_-^*\,e^{-j(\omega t+kx)}\right]$$

From Prob. 10.4 we have that the power carried by the string is

$$P = -\,f\,\frac{\partial\xi}{\partial x}\,\frac{\partial\xi}{\partial t}\,.$$

If we do the indicated differentiations, then substitute into this expression, and then time average, we will obtain

$$\langle P\rangle = \frac{f\omega k}{2}\,[\hat{\xi}_+\,\hat{\xi}_+^* - \hat{\xi}_-\,\hat{\xi}_-^*]$$

Part b

For k purely imaginary

$$k = j\beta$$

with β real, we can write $\xi(x,t)$ in the form

PROBLEM 10.5 (continued)

$$\xi(x,t) = \frac{1}{2}\left[\hat{\xi}_+ e^{j\omega t+\beta x} + \hat{\xi}_+^* e^{-j\omega t+\beta x} + \hat{\xi}_- e^{j\omega t-\beta x} + \hat{\xi}_-^* e^{-j\omega t-\beta x}\right]$$

If we again substitute into our expression for power and average over time we obtain

$$\langle P\rangle = -\frac{j\omega\beta f}{2}\,[\hat{\xi}_+^*\,\hat{\xi}_- - \hat{\xi}_+\,\hat{\xi}_-^*]$$

From (b), we see that it is possible to have a net power flow from two evanescent waves, but not from a single evanescent wave. Suppose that a single evanescent wave did carry power away from the driving source. This would correspond physically to a string driven at the left and infinite to the right. With $\omega_d<\omega_c$, the response as $x \to \infty$ becomes vanishingly small; clearly there can be no power flow at $x \to \infty$. Yet, there is no mechanism for power absorption by the string and so there can be no power flow into the string from the drive. With a dissipative load, a second evanescent wave is established, decaying to the left, and the conditions for power flow are met.

PROBLEM 10.6

From the dispersion relation, we calculate:

$$v_g \equiv \frac{\partial\omega}{\partial k} = v_s\left[1 - \frac{\omega_c^2}{\omega^2}\right]^{1/2}$$

Now, assuming a single forward traveling wave:

$$\xi = \xi_+ \cos[\omega t - k(\omega)x]$$

Then:

$$\langle W\rangle = \left(\frac{m\omega^2}{4} + \frac{fk^2}{4} + \frac{Ib}{4}\right)\xi_+^2$$

$$\langle P\rangle = \left(\frac{fk\omega}{2}\right)\xi_+^2$$

Thus, substitution gives

$$\frac{\langle P\rangle}{\langle W\rangle} = \frac{fk\omega/2}{\left(\frac{m\omega^2}{4} + \frac{fk^2}{4} + \frac{Ib}{4}\right)}$$

$$= v_s\left[1 - \frac{\omega_c^2}{\omega^2}\right]^{1/2} = v_g$$

which is the desired relation. This result is of some general significance, but has been shown here for a particular case.

PROBLEM 10.7

Part a

The equations of motion for the membranes are

$$\sigma_m \frac{\partial^2 \xi_1}{\partial t^2} = S \frac{\partial^2 \xi_1}{\partial x^2} + T_1$$

$$\sigma_m \frac{\partial^2 \xi_2}{\partial t^2} = S \frac{\partial^2 \xi_2}{\partial x^2} + T_2$$

where T_1 and T_2 are the transverse magnetic forces/area. If the membranes extend a distance w into the paper, and if we define regions 1, 2, and 3 as the top, middle, and bottom regions respectively in Fig. 10P.7, the flux in each region is

$$\lambda_1 = \mu_o H_1 \, w(d-\xi_1)$$

$$\lambda_2 = \mu_o H_2 \, w(d+\xi_1-\xi_2)$$

$$\lambda_3 = \mu_o H_3 \, w(d+\xi_2)$$

where H_1, H_2, and H_3 are the magnetic field intensities within each region. Since the flux is conserved, when $\xi_1 = \xi_2 = 0$ we have

$$\lambda_1 = \lambda_3 = -\mu_o H_o wd \qquad \lambda_2 = +\mu_o H_o wd$$

Therefore,

$$\bar{H}_1 = -\frac{H_o d}{d-\xi_1} \bar{i}_x$$

and

$$\bar{H}_2 = +\frac{H_o d}{d+\xi_1-\xi_2} \bar{i}_x$$

and

$$\bar{H}_3 = -\frac{H_o d}{d+\xi_2} \bar{i}_x .$$

We will use the Maxwell stress tensor to calculate T_1 and T_2, using a pill-box volume enclosing a section of surface on each membrane.

We then obtain

$$T_1 \cong -\frac{\mu_o}{2} [H_1^2 - H_2^2]$$

and

$$T_2 \cong -\frac{\mu_o}{2} [H_2^2 - H_3^2]$$

Substituting the expression for the $\bar{H}$ fields, and realizing that $\xi_1 \ll d$ and $\xi_2 \ll d$, we finally obtain for the forces

PROBLEM 10.7 (continued)

$$T_1 = -\frac{\mu_o H_o^2\ (2\xi_1 - \xi_2)}{d}$$

and

$$T_2 = -\frac{\mu_o H_o^2\ (2\xi_2 - \xi_1)}{d}$$

Our equations of motion are then

$$\sigma_m \frac{\partial^2 \xi_1}{\partial t^2} = S\frac{\partial^2 \xi_1}{\partial x^2} - \frac{\mu_o H_o^2}{d}(2\xi_1 - \xi_2)$$

and

$$\sigma_m \frac{\partial^2 \xi_2}{\partial t^2} = S\frac{\partial^2 \xi_2}{\partial x^2} - \frac{\mu_o H_o^2}{d}(2\xi_2 - \xi_1)$$

Part b

We assume that

$$\xi_1 = \mathrm{Re}\ \hat{\xi}_1\ e^{j(\omega t - kx)}$$

and

$$\xi_2 = \mathrm{Re}\ \hat{\xi}_2\ e^{j(\omega t - kx)}$$

We can substitute these functions into the equations of motion from part (a), and solve for the relation between ω and k such that the 2 equations of motion are consistent. This dispersion relation is

$$-\sigma_m \omega^2 + Sk^2 + \frac{2\mu_o H_o^2}{d} = \pm \frac{\mu_o H_o^2}{d}$$

We see that the dispersion equation factors into two dispersion relations. If we substitute this relation back into the equations of motion from part (a), we see that we obtain even and odd solutions.

The dispersion relation

$$\omega^2 = \frac{Sk^2}{\sigma_m} + \frac{\mu_o H_o^2}{\sigma_m d}$$

yields

$$\xi_1 = \xi_2 .$$

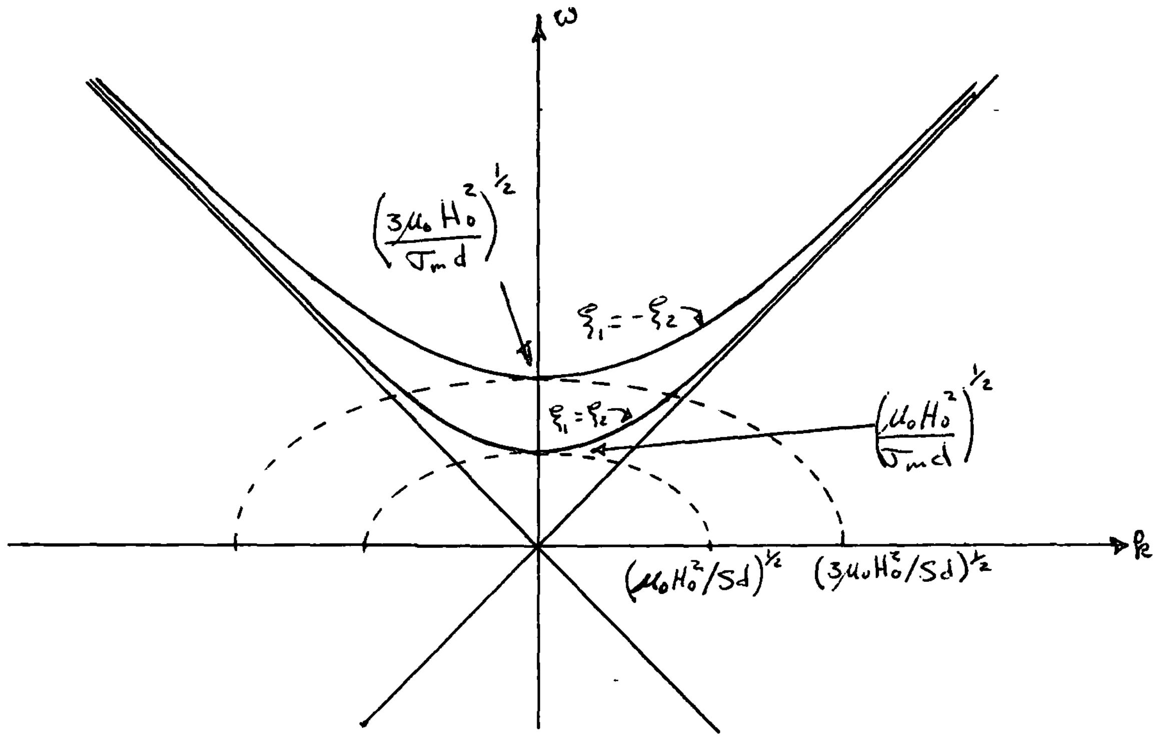
The dispersion relation

$$\omega^2 = \frac{Sk^2}{\sigma_m} + \frac{3\mu_o H_o^2}{\sigma_m d}$$

yields $\xi_1 = -\xi_2$.

PROBLEM 10.7 (continued)

Plotting ω versus k, we obtain



_______ k real

\- - - - k imaginery

From the plot we see that the lowest frequency for which we have propagation (k real) for the even mode is

$$\omega_{ce} = \left(\frac{\mu_o H_o^2}{\sigma_m d}\right)^{1/2}$$

For the odd mode, the cut off frequency is

$$\omega_{co} = \left(\frac{3\mu_o H_o^2}{\sigma_m d}\right)^{1/2}$$

Part d

We are given the boundary conditions that at x = 0

$$\xi_1 = 0 \quad \xi_2 = 0$$

and at x = - ℓ

$$\xi_1 = -\xi_2 = \text{Re}\ \xi_o\ e^{j\omega t}.$$

PROBLEM 10.7 (continued)

From the boundary condition, we see that our solution is purely odd. Therefore

$$k = \left[\frac{\omega^2\sigma_m}{S} - \frac{3\mu_o H_o^2}{Sd}\right]^{1/2}$$

We assume a solution of the form

$$\xi_1(x,t) = -\xi_2(x,t) = \text{Re}\{A_1\, e^{j(\omega t-kx)} + A_2\, e^{j(\omega t+kx)}\}$$

Evaluating A_1 and A_2 through the boundary conditions, we obtain

$$A_1 = -A_2 = \frac{\xi_o}{e^{jk\ell} - e^{-jk\ell}}$$

Therefore

$$\xi_1(x,t) = -\xi_2(x,t) = \text{Re}\,\frac{\xi_o[e^{-jkx} - e^{+jkx}]e^{j\omega t}}{[e^{jk\ell} - e^{-jk\ell}]}$$

For $\omega = 0$, k is pure imaginary. We define $k = j\beta$, with β real with value

$$\beta = \left(\frac{3\mu_o H_o^2}{Sd}\right)^{1/2}$$

Therefore

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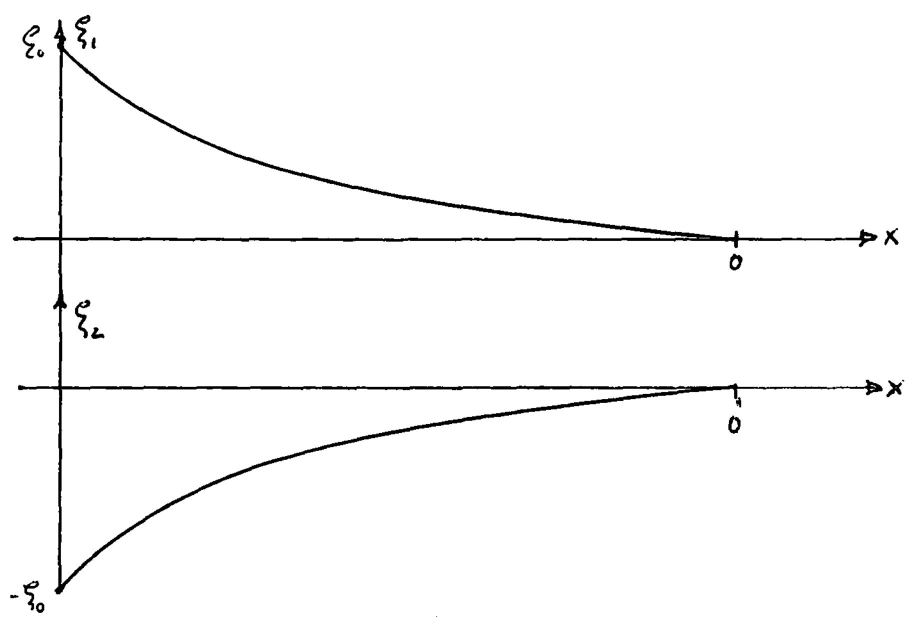
$$\xi_1(x,t) = -\frac{\xi_o \sinh \beta x}{\sinh \beta\ell} \cos\omega t$$

A sketch appears below.



PROBLEM 10.8

Part a

The given equations follow by writing out Maxwell's equations and assuming $\bar{E}$ and $\bar{H}$ have the given directions and dependences.

Part b

The force equation for an incremental volume element is

$$\bar{F} = \bar{i}_x \, mn_e \frac{\partial v_x}{\partial t} \tag{a}$$

where $\bar{F}$ is the force density due to electrical forces on the electrons

$$\bar{F} = - \bar{i}_x \, en_e E_x \tag{b}$$

Thus,

$$-en_e E_x = mn_e \frac{\partial v_x}{\partial t} \tag{c}$$

Part c

As the electrons move, they give rise to the current density

$$J_x \cong - en_e \, v_x \text{ (linearized)} \tag{d}$$

Part d

Assume $e^{j(\omega t-kx)}$ dependence and (c) and (d) require

$$\hat{J}_x = - j \frac{e^2 n_e}{\omega m} \hat{E}_x \tag{e}$$

$$= - j\omega\varepsilon_o \left[\frac{\omega_\rho^2}{\omega^2}\right] \hat{E}_x \tag{f}$$

where $\omega_\rho = \sqrt{e^2 n_e / m\varepsilon_o}$ is called the plasma frequency. (See page 600)
Combining this with Maxwell's equations:

$$k^2 = \frac{\omega^2}{c^2}\left[1 - \frac{\omega_\rho^2}{\omega^2}\right] \; ; \; c = \frac{1}{\sqrt{\varepsilon_o \mu_o}} \tag{g}$$

Part e

We have a dispersion which yields evanescent waves below the plasma (cutoff) frequency. Below this frequency, the electrons respond to the electric field associated with the wave in such a way as to reflect rather than transmit an incident electromagnetic wave.

Part f

Waves impinging upon a boundary between free space and plasma will be totally reflected if the wave frequency $\omega < \omega_p$. The plasma frequency for the ionosphere

PROBLEM 10.8 (continued)

is typically

$$f_p \sim 10 \text{ MH}_z$$

This result explains why AM broadcasts (500 KH_z < f < 1500 KH_z) can commonly be monitored all over the world, whereas FM (88 MH_z < f < 108 MH_z) has a range limited to "line-of-sight".

PROBLEM 10.9

In the regions

$$x < -\ell \text{ and } x > 0$$

the equation of motion for the string is

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2}$$

In the region $-\ell \leq x \leq 0$, this equation is modified due to the magnetic force to

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2} - \omega_c^2 \xi$$

If we assume

$$\xi(x,t) = \text{Re}\{\hat{\xi}\, e^{j(\omega t - kx)}\}$$

and substitute back into the equations of motion we obtain the dispersion relations

$$k = \pm \left[\frac{\omega^2 - \omega_c^2}{v_s^2}\right]^{1/2} \qquad -\ell \leq x \leq 0$$

$$k = \pm \frac{\omega}{v_s} \qquad x < -\ell, x > 0$$

The boundary conditions are

at $x = -\ell$ $\qquad \hat{\xi} = \hat{\xi}_o$

at $x = 0$ $\qquad \xi$ and $\frac{\partial \xi}{\partial x}$ must be continuous.

We assume that

$$\xi(x,t) = \text{Re}\left\{\left[A\, e^{-\beta x} + B\, e^{+\beta x}\right] e^{j\omega t}\right\} \qquad \text{for } -\ell < x < 0$$

PROBLEM 10.9 (continued)

where

$$\beta = \left[\frac{\omega_c^2 - \omega^2}{v_s^2}\right]^{1/2} \qquad \text{for } \omega < \omega_c$$

and

$$\xi(x,t) = \text{Re}\,\{\hat{\xi}_b\, e^{-jk_b x}\, e^{j\omega t}\} \quad \text{for } x > 0$$

where

$$k_b = \frac{\omega}{v_s}$$

Using the above boundary conditions, we obtain

$$A = \frac{\xi_o(\beta + jk_b)}{2(\beta \cosh \beta\ell + jk_b \sinh \beta\ell)}$$

$$B = \frac{\xi_o(\beta - jk_b)}{2(\beta \cosh \beta\ell + jk_b \sinh \beta\ell)}$$

But $\hat{\xi}_b = A + B$

Therefore

$$\frac{\hat{\xi}_b}{\xi_o} = \frac{1}{[\cosh \beta\ell + \frac{jk_b}{\beta} \sinh \beta\ell]}$$

Part b

As $\ell \to 0$

$$\frac{\hat{\xi}_b}{\hat{\xi}_o} \to 1$$

As $\ell \to \infty$

$$\frac{\hat{\xi}_b}{\hat{\xi}_o} \to 0$$

PROBLEM 10.10

Part a

The equation of motion for the string is

$$m\frac{\partial^2\xi}{\partial t^2} = f\frac{\partial^2\xi}{\partial x^2} + S - mg \tag{a}$$

PROBLEM 10.10 (continued)

where, for small deflections ξ in the "1/r" field from Q,

$$S \cong \frac{qQ}{2\pi\varepsilon_o d}\left[1 + \frac{\xi}{d}\right]$$

In static equilibrium, $\xi = 0$ and from (a)

$$qQ = 2\pi d\varepsilon_o \cdot mg \tag{b}$$

Part b

The perturbation equation of motion remains;

$$m\frac{\partial^2\xi}{\partial t^2} = f\frac{\partial^2\xi}{\partial x^2} + \left(\frac{qQ}{2\pi d^2\varepsilon_o}\right)\xi \tag{c}$$

Assume $e^{j(\omega t - kx)}$ dependence and (c) requires ($v_s = \sqrt{f/m}$)

$$\omega^2 = v_s^2 k^2 - \frac{qQ}{2\pi d^2\varepsilon_o m} \tag{d}$$

or from (b),

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$$\omega^2 = v_s^2 k^2 - \frac{g}{d}$$

The boundary conditions require $k = n\pi/\ell$, and for stability the most critical mode is $n = 1$; thus

$$v_s^2\left(\frac{\pi}{\ell}\right)^2 > \frac{g}{d} \tag{e}$$

$$m < \frac{fd}{g}\left(\frac{\pi}{\ell}\right)^2 \tag{f}$$

Part c

Increase f, d, or decrease ℓ.

PROBLEM 10.11

$$m\frac{\partial^2\xi}{\partial t^2} = f\frac{\partial^2\xi}{\partial x^2} + S - mg$$

where $S = (\bar{I}\times\bar{B})_{r=\xi_o}$ and $|B| = \frac{\mu_o I_o}{2 r}$, r the radial distance from the fixed wire.

Therefore $S = \frac{\mu_o I_o I^o}{2\pi r}$.

For static equilibrium

PROBLEM 10.11 (continued)

$$S = mg = \frac{\mu_o I_o I}{2\pi\xi_o}$$

Therefore

$$I = \frac{2\pi mg\xi_o}{\mu_o I_o}$$

Note that $I_o I > 0$ for the required equilibrium.

Part b

The force per unit length is linearized to obtain the perturbation equation.

$$S = \frac{\mu_o I_o I}{2\pi(\xi_o+\xi)} \approx \frac{\mu_o I_o I}{2\pi}\left[\frac{1}{\xi_o} - \frac{\xi}{\xi_o^2}\right]$$

Therefore

$$m\frac{\partial^2\xi}{\partial t^2} = f\frac{\partial^2\xi}{\partial x^2} - \frac{\mu_o I_o I}{2\pi\xi_o^2}\xi$$

Part c

Assuming $e^{j(\omega t-kx)}$ solutions, the dispersion relation is

$$-m\omega^2 = -f k^2 - \frac{\mu_o I_o I}{2\pi\xi_o^2}$$

Solving for ω, we obtain

$$\omega = \left[k^2\frac{f}{m} + \frac{\mu_o I_o I}{2\pi m\xi_o^2}\right]^{1/2}$$ × 3/29/99

As long as $I_o I > 0$ the equilibrium will always be stable as ω will always be real. Note that this condition is required for the desired static equilibrium to exist.

PROBLEM 10.12

The equation of motion is given as

$$m\frac{\partial^2\xi}{\partial t^2} = f\frac{\partial^2\xi}{\partial x^2} + P\xi \qquad (a)$$

Part a

Boundary conditions follow from force equilibrium for the ends of the wire

PROBLEM 10.12 (continued)

$$\text{(i)} \quad -2K\xi(0,t) + f\,\frac{\partial \xi(0,t)}{\partial x} = 0 \tag{b}$$

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$$\text{(ii)} \quad 2K\xi(\ell,t) + f\,\frac{\partial \xi(\ell,t)}{\partial x} = 0 \tag{c}$$

Part b

The dispersion relation follows from (a) as

$$\omega^2 = v_s^2\, k^2 - \frac{P}{m}\ ; \quad v_s = \sqrt{f/m} \tag{d}$$

where solutions have been assumed of the following form:

$$\xi = \mathrm{Re}[(A \sin kx + B \cos\ kx)e^{j\omega t}] \tag{e}$$

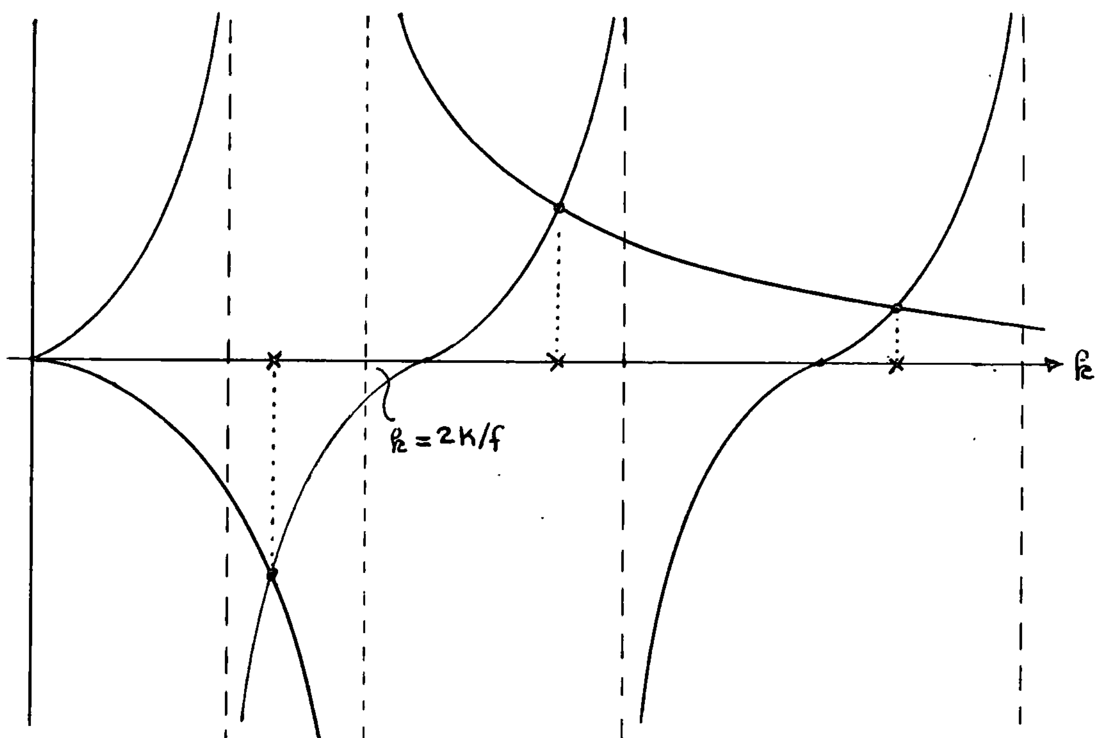
Application of the boundary conditions yields a transcendental equation for k:

$$\tan k\ell = \frac{4Kf\ k}{f^2k^2-4K^2} \tag{f}$$

where, from (d),

$$k = \frac{1}{v_s}\sqrt{\omega^2 + P/m} \tag{g}$$

Thus, (f) is the desired equation for the natural frequencies.



PROBLEM 10.12 (continued)

Part c

As $K \to 0$, the lowest root of graphic solution goes to $k \to 0$, for which stability criterion is:

$$0 > \frac{P}{m}$$

PROBLEM 10.13

Part a

This problem is very similar to that of problem 10.7. Using the same reasoning as in that problem, we obtain

$$\sigma_m \frac{\partial^2 \xi_1}{\partial t^2} = S \frac{\partial^2 \xi_1}{\partial x^2} + \frac{\varepsilon_o V_o^2}{d^3} (2\xi_1 - \xi_2)$$

$$\sigma_m \frac{\partial^2 \xi_2}{\partial t^2} = S \frac{\partial^2 \xi_2}{\partial x^2} + \frac{\varepsilon_o V_o^2}{d^3} (2\xi_2 - \xi_1)$$

Part b

Assuming sinusoidal solutions in time and space, the dispersion relation is

$$-\sigma_m \omega^2 + Sk^2 - \frac{2\varepsilon_o V_o^2}{d^3} = \pm \frac{\varepsilon_o V_o^2}{d^3}$$

We have a dispersion relation that factors into two parts. The odd mode, $\xi_1 = -\xi_2$ has the dispersion relation

$$\omega = \left[\frac{Sk^2}{\sigma_m} - \frac{3\varepsilon_o V_o^2}{\sigma_m d^3}\right]^{1/2}$$

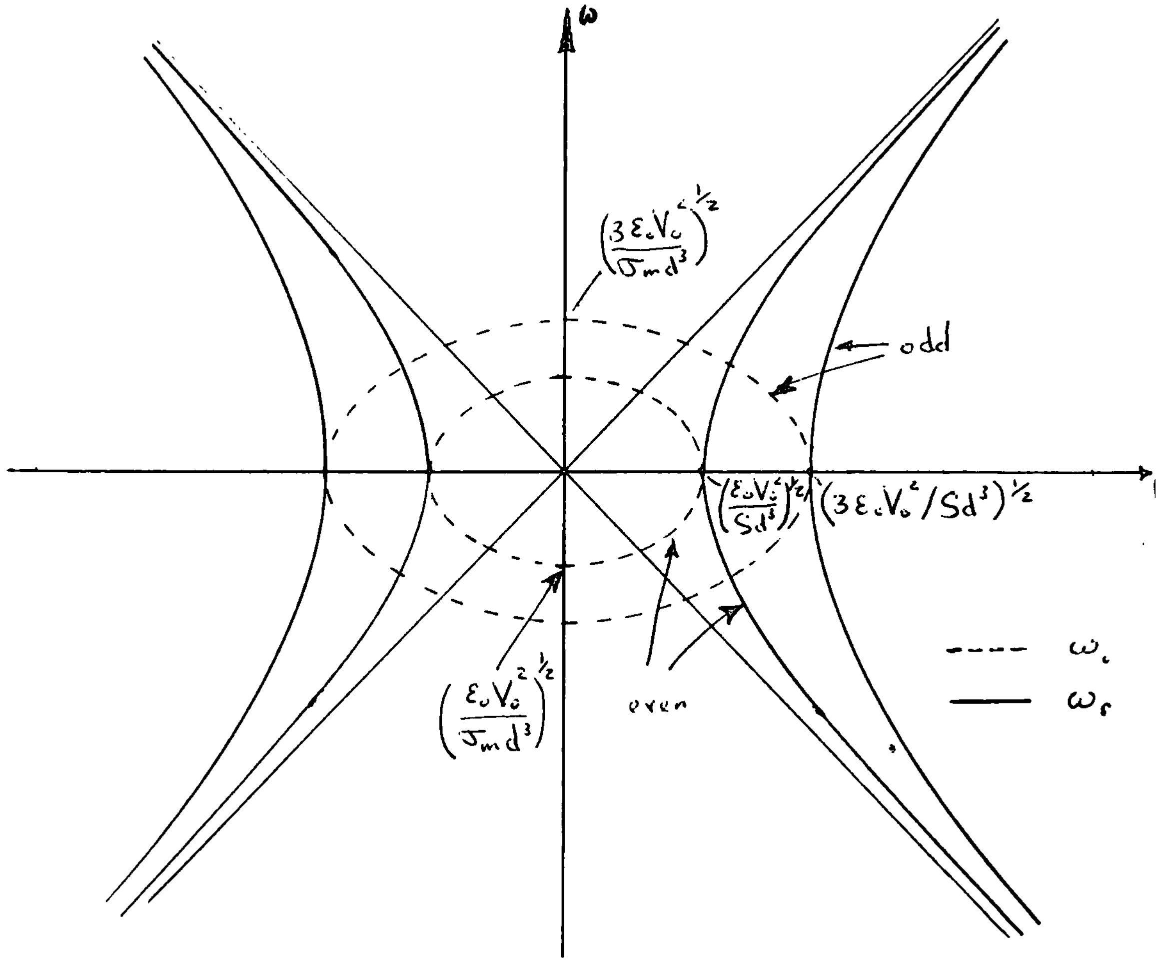
The even mode, $\xi_1 = \xi_2$ has the dispersion relation

$$\omega = \left[\frac{Sk^2}{\sigma_m} - \frac{\varepsilon_o V_o^2}{\sigma_m d^3}\right]^{1/2}$$

Part c

A plot of the dispersion relation appears below.

PROBLEM 10.13 (continued)



Part d

The lowest allowed value of k is $k = \frac{\pi}{L}$ since the membranes are fixed at x = 0 and x = L. Therefore the first mode to go unstable is the even mode.

This happens as

ODD

$$\left(\frac{3\varepsilon_o V_o^2}{Sd^3}\right) = \frac{\pi^2}{L^2}$$

or

$$V_o = \left|\frac{\pi^2}{L^2}\,\frac{Sd^3}{\varepsilon_o 3}\right|^{1/2}$$

PROBLEM 10.14

The equation of motion is

$$\frac{\partial^2\xi}{\partial t^2} = v_s^2 \frac{\partial^2\xi}{\partial x^2} - \nu \frac{\partial\xi}{\partial t} \tag{a}$$

Part a

The dispersion for this system is:

$$\omega^2 - j\nu\omega - v_s^2k^2 = 0 \tag{b}$$

We may solve for ω,

$$\omega = j\left[\frac{\nu}{2} \pm \sqrt{\left(\frac{\nu}{2}\right)^2 - v_s^2k^2}\right] \tag{c}$$

$$\equiv j[\alpha \pm \gamma]$$

We assume solutions of the form:

$$\xi(x,t)\ \mathrm{Re}\{\sum_{n\ \mathrm{odd}} [A_n e^{-(\alpha+\gamma_n)t} + B_n e^{-(\alpha-\gamma_n)t}]\sin\frac{n\pi x}{\ell}\} \tag{d}$$

Now, we may use the initial condition on $\frac{\partial\xi}{\partial t}$ to relate A_n and B_n. Thus we obtain:

$$\xi(x,t) = \mathrm{Re}\{\sum_{n\ \mathrm{odd}} A_n[e^{-\gamma_n t} - (\frac{\alpha+\gamma_n}{\alpha-\gamma_n})e^{\gamma_n t}]e^{-\alpha t}\sin\frac{n\pi x}{\ell}\} \tag{e}$$

Now, we apply the initial condition on $\xi(x,t = 0)$ to determine A_n.

$$\xi(x,0) = \sum_{n\ \mathrm{odd}} A_n\left[\frac{2\gamma_n}{\gamma_n-\alpha}\right]\sin\frac{n\pi x}{\ell} \tag{f}$$

$$\equiv \sum_{n\ \mathrm{odd}} A_n'\sin\frac{n\pi x}{\ell}$$

The coefficient A_n' is determined from a Fourier analysis of the displacement:

$$A_n' = \frac{4\xi_o}{n\pi}, \tag{g}$$

So that:

$$A_n = (\frac{\gamma_n-\alpha}{2\gamma_n})(\frac{4\xi_o}{n\pi}) \tag{h}$$

Part b

There is one important difference between this problem and the magnetic diffusion problems of Chap. VII. While magnetic diffusion is "true diffusion" and satisfies the normal diffusion equation, the string equation is basically a wave equation modified by viscosity. Hence, we note (c) that especially the

PROBLEM 10.14 (continued)

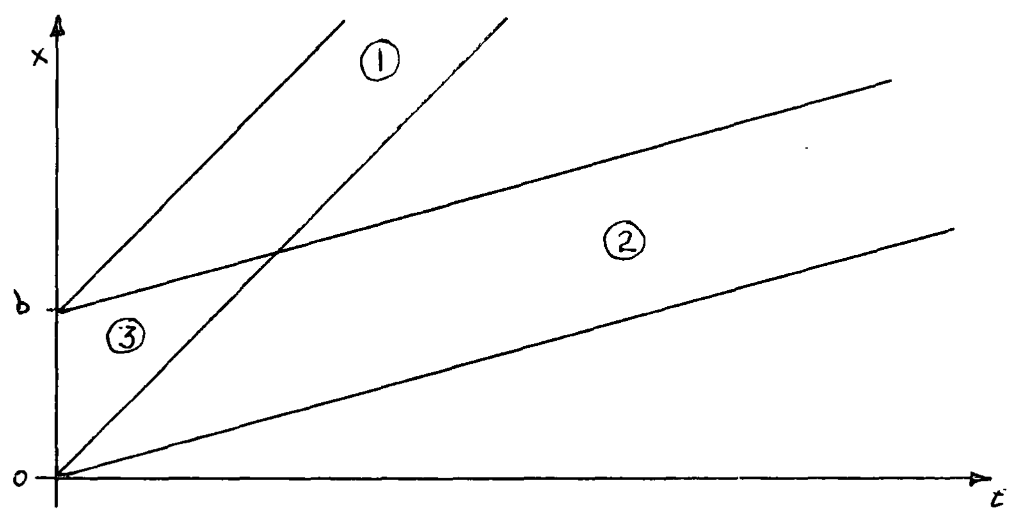
higher modes in the solution to this problem have sinusoidal time dependence as well as decay. Magnetic diffusion as discussed in Chap. 7 exhibits no such oscillation, because there is no mathematical analog to the inertia of the string. If we had included the effects of electromagnetic wave propagation (displacement current) the analogy would be more complete.

PROBLEM 10.15

From Chap. 10, page 588, Eqs. (e) and (f) we have

$$\frac{d\xi_+}{d\alpha} = \frac{(v_s - U)}{2v_s}\frac{\partial\xi}{\partial x} - \frac{1}{2v_s}\frac{\partial\xi}{\partial t}$$

$$\frac{d\xi_-}{d\beta} = \frac{(v_s + U)}{2v_s}\frac{\partial\xi}{\partial x} + \frac{1}{2v_s}\frac{\partial\xi}{\partial t}$$



Since $\frac{\partial\xi}{\partial x}(x = 0) = 0$, we have the following relations in the three regions.

Region 1

$$\frac{d\xi_+}{d\alpha} = -\frac{V_o}{2v_s}\ ;\ \frac{d\xi_-}{d\beta} = 0$$

Region 2

$$\frac{d\xi_+}{d\alpha} = 0;\ \frac{d\xi_-}{d\beta} = \frac{V_o}{2v_s}$$

Region 3

$$\frac{d\xi_+}{d\alpha} = -\frac{V_o}{2v_s}\ ;\ \frac{d\xi_-}{d\beta} = \frac{V_o}{2v_s}$$

PROBLEM 10.15 (continued)

In the other regions, the derivatives are zero. From Eq. 10.2.10 on page 586,

$$\frac{\partial\xi}{\partial x} = \frac{d\xi_+}{d\alpha} + \frac{d\xi_-}{d\beta}$$

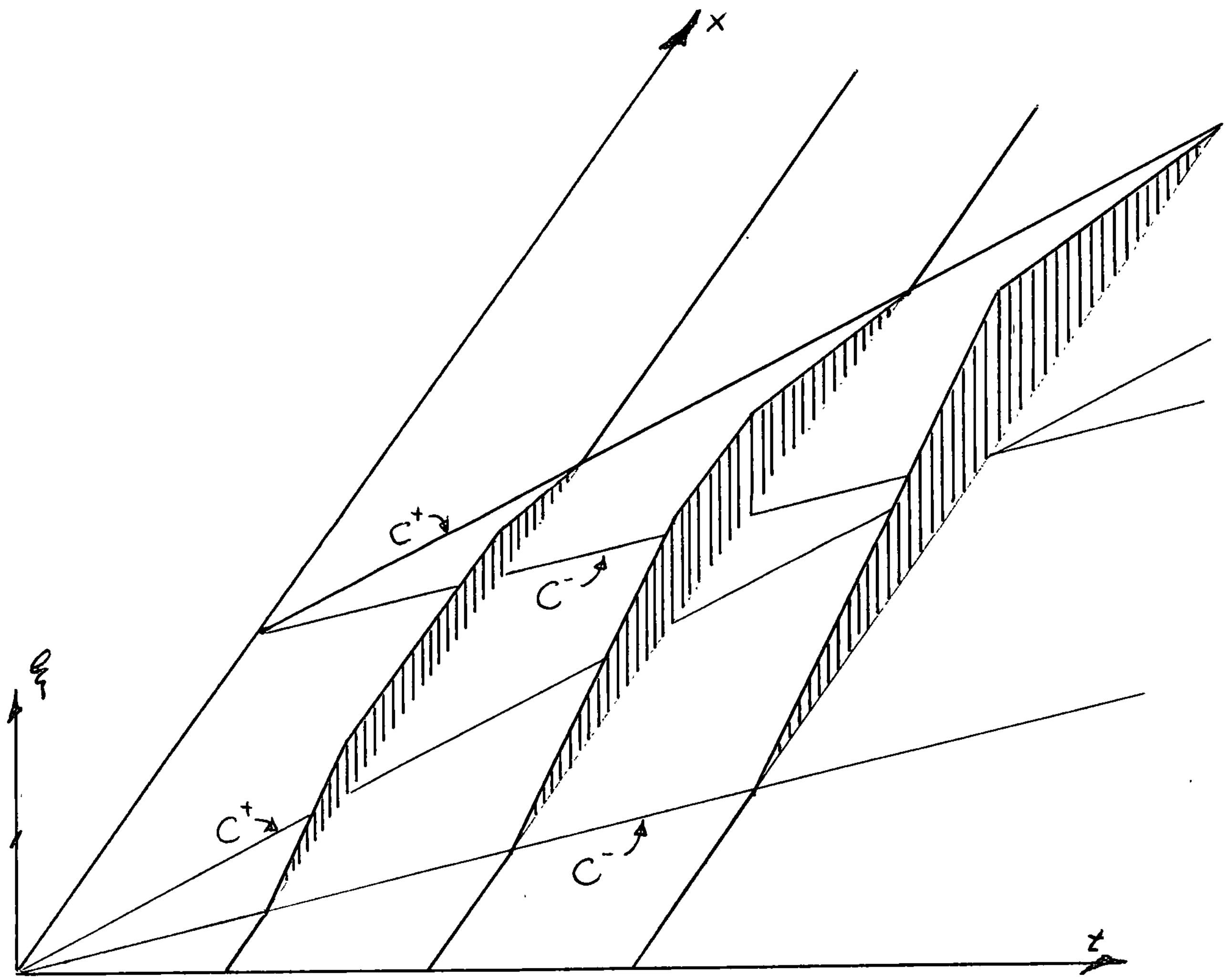
we have

$$\frac{\partial\xi}{\partial x} = \frac{V_o}{2v_s}\,[u_{-1}(\beta)-u_{-1}(\beta-b) - u_{-1}(\alpha) + u_{-1}(\alpha-b)]$$

Integrating with respect to x, we obtain

$$\xi(x,t) = \frac{V_o}{2v_s}\,[u_{-2}(\beta) - u_{-2}(\beta-b) - u_{-2}(\alpha) + u_{-2}(\alpha-b)]$$

A sketch of this deflection is shown in the figure.



PROBLEM 10.16

Part a

The equation of motion is simply

$$m \frac{\partial^2 \xi}{\partial t^2} = f \frac{\partial^2 \xi}{\partial x^2}$$

The dispersion equation follows as:

$$(\omega - kU)^2 = v_s^2 k^2$$

or

$$k_{\pm} = \frac{\omega}{U \pm v_s} = \frac{\omega(U \mp v_s)}{U^2 - v_s^2} \equiv \alpha \mp \beta$$

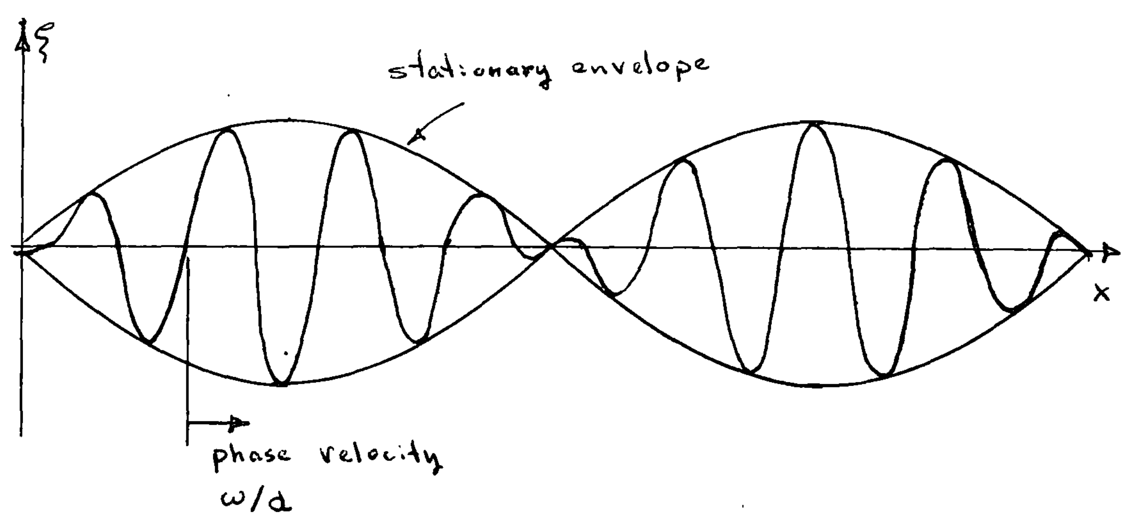
Where solutions are assumed of the form:

$$\xi(x,t) = \mathrm{Re}\{(\xi_+ e^{j\beta x} + \xi_- e^{-j\beta x})e^{j(\omega t - \alpha x)}\}$$

The boundary conditions are both applied at x = 0, because string is moving at a "supersonic" velocity.

$$\xi(x,t) = \xi_o\{\cos \beta x \cos[\omega t - \alpha x] - \frac{U}{v_s} \sin \beta x \sin[\omega t - \alpha x]\}$$

Part b



PROBLEM 10.17

We use Eq. 10.2.9

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \xi = v_s^2 \frac{\partial^2 \xi}{\partial x^2}$$

Assuming sinusoidal solutions in time and space we obtain the dispersion relation

$$(\omega - kU)^2 = k^2 v_s^2$$

Thus

$$k = \frac{\omega}{U \pm v_s} = \frac{\omega(U \mp v_s)}{U^2 - v_s^2}$$

We let

$$\alpha = \frac{\omega U}{U^2 - v_s^2}$$

$$\beta = \frac{\omega v_s}{U^2 - v_s^2}$$

Therefore, $k = \alpha \mp \beta$ and

$$\xi(x,t) = \text{Re}[A\, e^{-j(\alpha-\beta)x} + B\, e^{-j(\alpha+\beta)x}]e^{j\omega t}$$

The boundary conditions are

$\xi(x = 0) = 0$ which implies $A = -B$

$\xi(x = -\ell) = \xi_o$

Therefore

$$\xi(x,t) = \text{Re}\, A[e^{-j(\alpha-\beta)x} - e^{-j(\alpha+\beta)x}]e^{j\omega t}$$

$$= \text{Re}\, A2j \sin\beta x\, e^{j(\omega t - \alpha x)}$$

However,

$$\xi(-\ell,t) = \text{Re}\, \xi_o\, e^{j\omega t}$$

Therefore

$$\xi(x,t) = -\frac{\xi_o}{\sin\beta\ell}\sin\beta x \cos[\omega t - \alpha(x+\ell)]$$

Part b

For $\xi = 0$ at $x = 0$ and at $x = -\ell$ we must have $\beta = n\pi/\ell$

$$\frac{\omega v_s}{U^2 - v_s^2} = \frac{n\pi}{\ell}$$

or

PROBLEM 10.17 (continued)

$$\omega = \frac{n\pi}{\ell}\,\frac{(U^2 - v_s^2)}{v_s}$$

These are the natural frequencies of the wire.

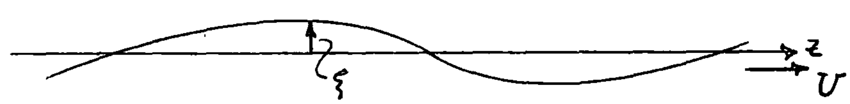
Part c

The results are meaningful only for $|U| < |v_s|$. If this inequality were not true, we would not be able to use a downstream boundary condition to determine upstream behavior and arrive at a result that would be obtained by "turning the driv on". That is, if $U > v_s$ the predictions are not consistent with causality.

PROBLEM 10.18

Part a

In the limit of wavelength short compared to the radius, we may "unwrap" the system:



$$m\left(\frac{\partial}{\partial t} + U\,\frac{\partial}{\partial z}\right)^2 \xi = f\,\frac{\partial^2 \xi}{\partial z^2} \tag{a}$$

Now let $z \rightarrow R\theta$, $U \rightarrow R\Omega$. Then, it follows that

$$m\left(\frac{\partial}{\partial t} + \Omega\,\frac{\partial}{\partial \theta}\right)^2 \xi = \frac{f}{R^2}\,\frac{\partial^2 \xi}{\partial \theta^2} \tag{b}$$

where $\Omega_s = \sqrt{f/(m\,R^2)}$

Part b

The initial conditions are

$$\partial\xi/\partial t(\theta, t = 0) = 0 \tag{c}$$

$$\xi(\theta, t = 0) = \begin{cases} \xi_o & 0 \leq \theta \leq \pi/4 \\ 0, & \text{elsewhere} \end{cases} \tag{d}$$

Solutions take the form

$$\xi = \xi_+(\alpha) + \xi_-(\beta) \tag{e}$$

where

PROBLEM 10.18 (continued)

$$\alpha = \theta - \Omega_s t$$

$$\beta = \theta - 3\Omega_s t$$

Because $\partial\xi/\partial t\ (t = 0) = 0$,

$$= -\Omega_s \frac{d\xi_+}{d\alpha} - 3\Omega_s \frac{d\xi_-}{d\beta} \qquad \text{(f)}$$

Also,

$$\frac{\partial\xi}{\partial\theta}\ (t=0) = \frac{d\xi_+}{d\alpha} + \frac{d\xi_-}{d\beta} = \xi_o[u_o(0) - u_o(\pi/4)] \qquad \text{(g)}$$

Thus, from (f) and (g)

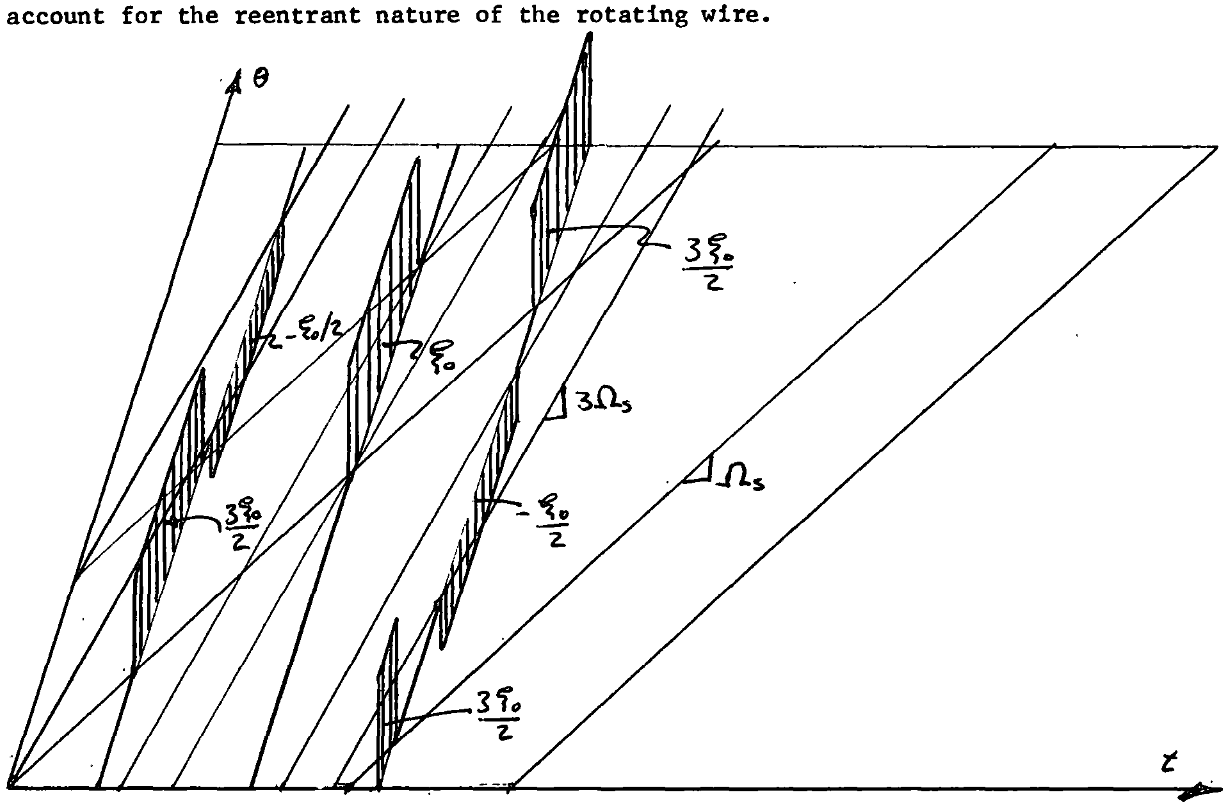
$$\frac{d\xi_+}{d\alpha} = \frac{3}{2}\,\xi_o[u_o(0) - u_o(\pi/4)];\ \text{on } \alpha \qquad \text{(h)}$$

$$\frac{d\xi_-}{d\beta} = -\frac{1}{2}\,\xi_o[u_o(0) - u_o(\pi/4)];\ \text{on } \beta$$

The solution in the θ-t plane follows from

$$\frac{\partial\xi}{\partial\theta} = \frac{d\xi_+}{d\alpha} + \frac{d\xi_-}{d\beta} \qquad \text{(j)}$$

and an integration at constant t on θ. The result is shown in the figure. Note that the characteristics that leave the interval $0 < \theta < 2\pi$ at $\theta = 2\pi$ reappear at $\theta = 0$ to account for the reentrant nature of the rotating wire.



PROBLEM 10.19

In the moving frame we can write

$$m \frac{\partial^2\xi}{\partial t'^2} = f \frac{\partial^2\xi}{\partial x'^2} + F(x',t') \tag{a}$$

and so from Prob. 10.4, we can write

$$P'_{in} = \frac{\partial W'}{\partial t'} + \frac{\partial P'}{\partial x'} \tag{b}$$

where

$$P'_{in} = F \frac{\partial \xi}{\partial t'} \tag{c}$$

$$W' = \frac{1}{2} m \left(\frac{\partial \xi}{\partial t'}\right)^2 + \frac{1}{2} f \left(\frac{\partial \xi}{\partial x'}\right)^2 \tag{d}$$

$$P' = - f \frac{\partial \xi}{\partial x'} \frac{\partial \xi}{\partial t'} \tag{e}$$

But $\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + U \frac{\partial}{\partial x}$

Therefore (c)-(e) become

$$P'_{in} = F\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right)\xi \tag{f}$$

$$W' = \frac{1}{2} m\left(\frac{\partial \xi}{\partial t} + U \frac{\partial \xi}{\partial x}\right)^2 + \frac{1}{2} f\left(\frac{\partial \xi}{\partial x}\right)^2 \tag{g}$$

$$P' = - f \frac{\partial \xi}{\partial x} \left(\frac{\partial \xi}{\partial t} + U \frac{\partial \xi}{\partial x}\right) \tag{h}$$

The conservation of energy equation, in terms of fixed frame coordinates, becomes

$$P'_{in} = \frac{\partial W'}{\partial t} + U \frac{\partial W'}{\partial x} + \frac{\partial P'}{\partial x} \tag{i}$$

$$= \frac{\partial W'}{\partial t} + \frac{\partial}{\partial x} (P' + W'U) \tag{j}$$

If we let

$$P_{in} = P'_{in}$$

$$W = W' \tag{k}$$

$$P = P' + W'U$$

we can write

$$P_{in} = \frac{\partial W}{\partial t} + \frac{\partial P}{\partial x} \tag{ℓ}$$

which is the required form.

PROBLEM 10.20

The equation of motion is given by Eq. 10.2.33, and hence the dispersion equation is 10.2.36;

$$k = \eta \pm j\gamma \tag{a}$$

where

$$\eta = \omega_d U/(U^2-v_s^2)$$

$$\gamma = v_s \sqrt{(U^2-v_s^2)k_c^2-\omega_d^2} \; /(U^2-v_s^2)$$

Solutions are assumed of the form

$$\xi = \text{Re}[A \sinh \gamma x + B \cosh \gamma x]e^{j(\omega t-\eta x)} \tag{b}$$

Boundary conditions require;

$$B = \xi_o \tag{c}$$

$$A = j\eta\xi_o/\gamma \tag{d}$$

Thus

$$\xi = \text{Re}\ \xi_o[\frac{j\eta}{\gamma} \sinh \gamma x + \cosh \gamma x]e^{j(\omega t-\eta x)} \tag{e}$$

The deflection has an envelope with an essentially exponentially increasing dependence on x, with the instantaneous deflection traveling in the + x direction.

PROBLEM 10.21 "m" SHOULD BE "σ_m" THROUGHOUT PROBLEM

Part a

The equation of motion is

$$\sigma_m \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right)^2 \xi = S \frac{\partial^2\xi}{\partial x^2} - mg + T \tag{a}$$

with $T = \frac{\varepsilon_o}{2} \frac{V_o^2}{(d-\xi)^2} \simeq \frac{\varepsilon_o}{2} V_o^2 \left[\frac{1}{d^2} + \frac{2\xi}{d^3}\right]$

For equilibrium, $\xi = 0$ and from (a)

$$\frac{\varepsilon_o V_o^2}{2d^2} = mg \tag{b}$$

or

$$V_o = \left[\frac{2mg\ d^2}{\varepsilon_o}\right]^{1/2} \tag{c}$$

PROBLEM 10.21 (continued)

Part b

With solutions of the form $e^{j(\omega t-kx)}$ the dispersion relation is

$$(\omega-kU)^2 = \frac{S}{\sigma_m}k^2 - \frac{\varepsilon_o V_o^2}{\sigma_m d^3} \tag{d}$$

Solving for k, we obtain

$$k = \frac{\omega U \pm \sqrt{\frac{S}{\sigma_m}\omega^2 - \left(U^2 - \frac{S}{\sigma_m}\right)\left(\frac{\varepsilon_o V_o^2}{\sigma_m d^3}\right)}}{(U^2 - \frac{S}{\sigma_m})} \tag{e}$$

For $U > \sqrt{S/\sigma_m}$, and not to have spatially growing waves

$$\frac{S}{\sigma_m}\omega^2 - (U^2 - \frac{S}{\sigma_m})\left(\frac{\varepsilon_o V_o^2}{\sigma_m d^3}\right) > 0 \tag{f}$$

or

$$\omega^2 > \left[(U^2 - \frac{S}{\sigma_m})\frac{\varepsilon_o V_o^2}{Sd^3}\right] \tag{g}$$

POOBLEM 10.22

Part a

Neglecting the curvature of the system, as in Prob. 10.18, we write:

$$\sigma_m\left(\frac{\partial}{\partial t} + \frac{\Omega R}{R}\frac{\partial}{\partial\theta}\right)^2\xi = \frac{S}{R^2}\cdot\frac{\partial^2\xi}{\partial\theta^2} + T_r \tag{a}$$

where the linearized perturbation force/unit area is

$$T_r = \left(\frac{2\varepsilon_o V_o^2}{a^3}\right)\xi \tag{b}$$

Therefore, the equation of motion is

$$\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\theta}\right)^2\xi = \Omega_s^2\left(\frac{\partial^2\xi}{\partial\theta^2} + m_c^2\,\xi\right) \tag{c}$$

$$\Omega_s^2 = \frac{S}{\sigma_m R^2}$$

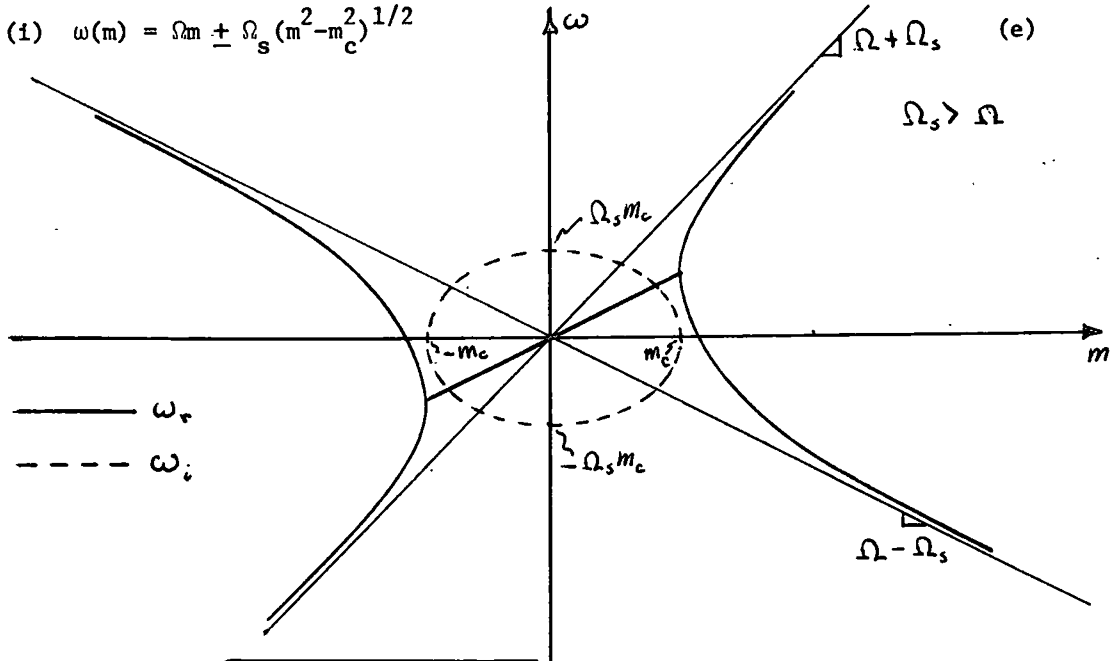
$$m_c^2 = \frac{2\varepsilon_o V_o^2}{a^3}\frac{R^2}{S}$$

PROBLEM 10.22

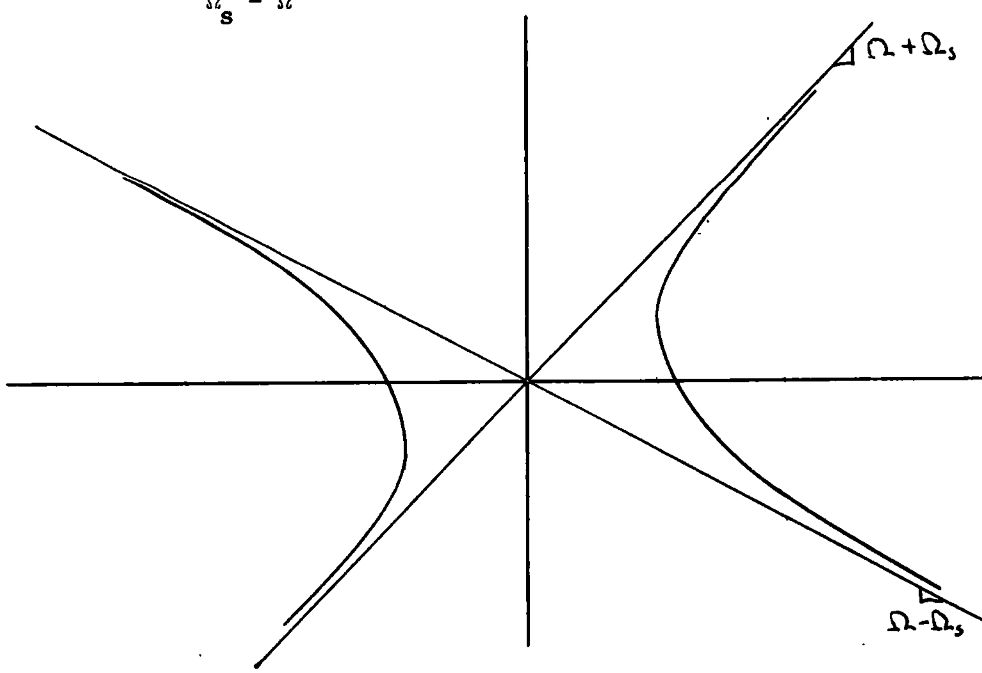
Part b

$$(\omega - m\Omega)^2 = \Omega_s^2(m^2 - m_c^2) \tag{d}$$

(i) $$\omega(m) = \Omega m \pm \Omega_s(m^2 - m_c^2)^{1/2} \tag{e}$$



(ii) $$m = \frac{\Omega\omega \pm \sqrt{\Omega_s^2\omega^2 + (\Omega_s^2 - \Omega^2)(\Omega_s^2 m_c^2)}}{\Omega_s^2 - \Omega^2} \tag{f}$$



PROBLEM 10.22

Part c

Because the membrane closes on itself it can be absolutely unstable regardless of Ω relative to Ω_s. Allowed values of m are determined by the requirement that the deflections be periodic in θ; m = 0, 1,2,3,... Thus, from (e) any finite m_c will lead to instability in the m = 0 mode. Note however that this mode does not meet the requirement that wavelengths be short compared to R.

PROBLEM 10.23

We may take the results of Prob. 10.13, replacing $\frac{\partial}{\partial t}$ by $\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}$ and replacing ω by ω-kU.

Part a

The equations of motion are

$$\sigma_m\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \xi_1 = S\frac{\partial^2\xi_1}{\partial x^2} + \frac{\varepsilon_o V_o^2}{d^3}(2\xi_1 - \xi_2) \qquad \text{(a)}$$

and

$$\sigma_m\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\xi_2 = S\frac{\partial^2\xi_2}{\partial x^2} + \frac{\varepsilon_o V_o^2}{d^3}(2\xi_2 - \xi_1) \qquad \text{(b)}$$

Part b

The dispersion relation is biquadratic, and factors into

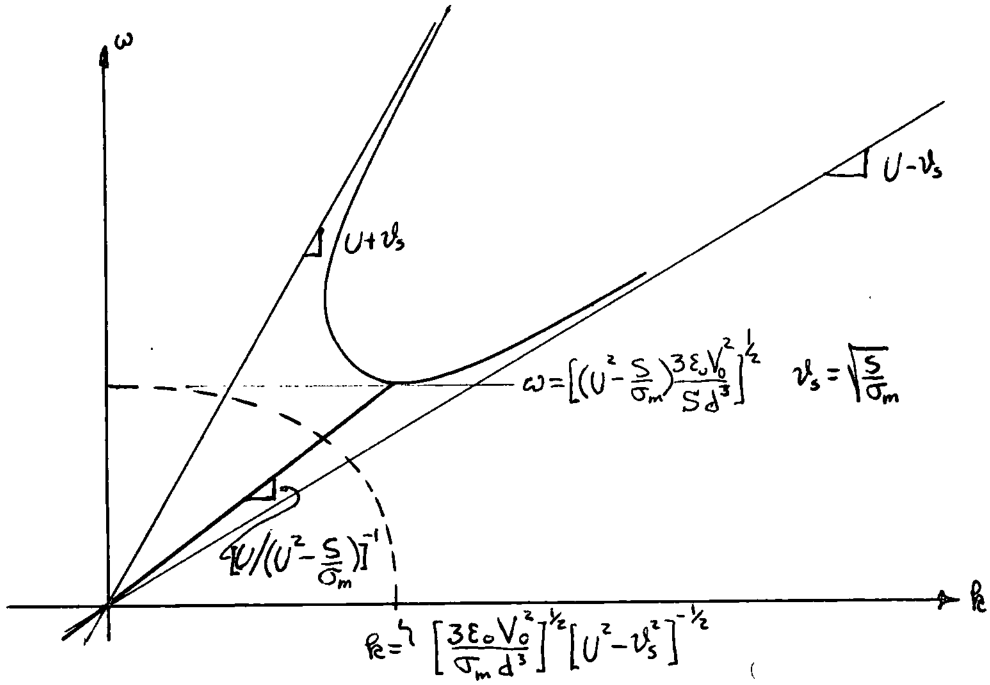
$$-\sigma_m(\omega-kU)^2 + Sk^2 - \frac{2\varepsilon_o V_o^2}{d^3} = \pm\frac{\varepsilon_o V_o^2}{d^3} \qquad \text{(c)}$$

The ($\pm$) signs correspond to the cases $\xi_1 = -\xi_2$ and $\xi_1 = \xi_2$ respectively, as will be seen in part (d).

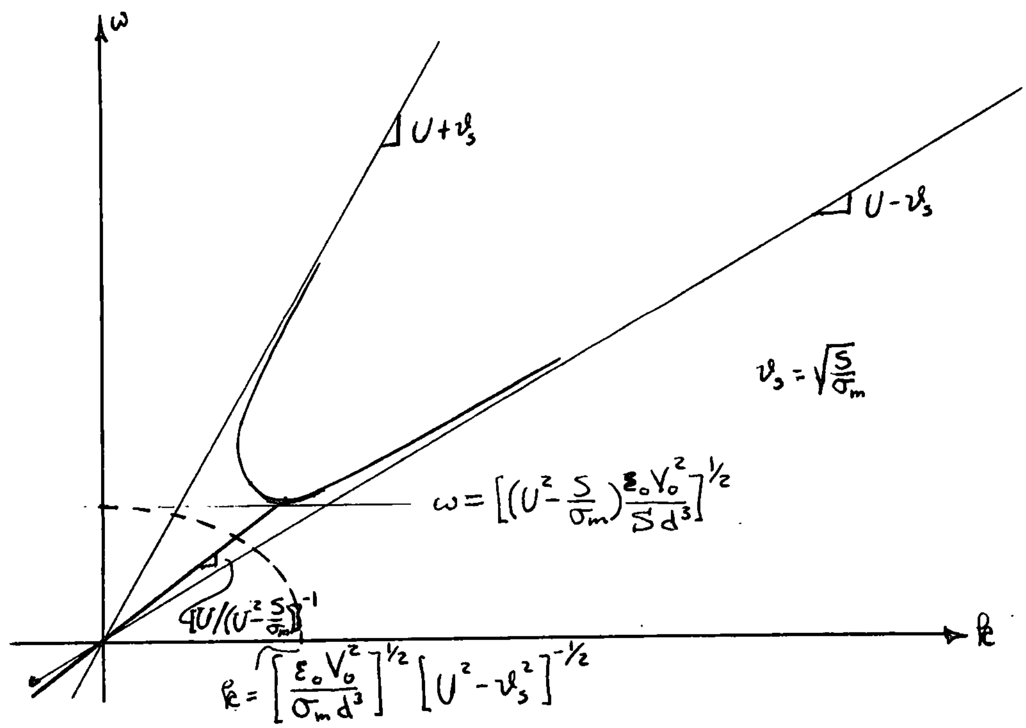
Part c

The dispersion relations are plotted in the figure for $U > \sqrt{S/\sigma_m}$.

PROBLEM 10.23 (continued)



and



PROBLEM 10.23 (continued)

Part d

Let $\xi_1 = \xi_2$. Then (a) and (b) become

$$\sigma_m\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \xi_1 = S\frac{\partial^2 \xi_1}{\partial x^2} + \frac{\varepsilon_o V_o^2}{d^3}\xi_1 \tag{d}$$

and

$$\sigma_m\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \xi_2 = S\frac{\partial^2 \xi_2}{\partial x^2} + \frac{\varepsilon_o V_o^2}{d^3}\xi_2 \tag{e}$$

These equations are identical for $\xi_1 = \xi_2$; the dispersion equation is (c) with the minus sign. Now let $\xi_1 = -\xi_2$ and (a) and (b) require

$$\sigma_m\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \xi_1 = S\frac{\partial^2 \xi_1}{\partial x^2} + \frac{3\varepsilon_o V_o^2}{d^3}\xi_1 \tag{f}$$

and

$$\sigma_m\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \xi_2 = S\frac{\partial^2 \xi_2}{\partial x^2} + \frac{3\varepsilon_o V_o^2}{d^3}\xi_2 \tag{g}$$

These equations are identical for $\xi_1 = -\xi_2$; the dispersion equation is (c) with the + sign.

Part e

$$\xi_1(0,t) = \text{Re}\,\hat{\xi}\,e^{j\omega t} = -\xi_2(0,t) \tag{h}$$

$$\frac{\partial \xi_1}{\partial x} = \frac{\partial \xi_2}{\partial x} = 0 \text{ at } x = 0 \tag{i}$$

The odd mode is excited. Hence, we use the + sign in (c)

$$-\sigma_m(\omega - kU)^2 + Sk^2 - \frac{3\varepsilon_o V_o^2}{d^3} = 0 \tag{j}$$

$$k^2(S - \sigma_m U^2) + 2\sigma_m \omega k U - \sigma_m \omega^2 - \frac{3\varepsilon_o V_o^2}{d^3} = 0 \tag{k}$$

Solving for k, we obtain

$$k_\pm = \alpha \pm \beta \tag{ℓ}$$

where $\alpha = \dfrac{\omega U}{U^2 - v_s^2}$ (m)

$$\beta = \frac{\left[\omega^2 v_s^2 - \dfrac{3\varepsilon_o V_o^2 (U^2 - v_s^2)}{\sigma_m d^3}\right]^{1/2}}{U^2 - v_s^2} \tag{n}$$

PROBLEM 10.23 (continued)

with $v_s^2 = S/\sigma_m$.

Therefore

$$\xi_1 = \mathrm{Re}\,\{[A\,e^{-j(\alpha+\beta)x} + B\,e^{-j(\alpha-\beta)x}]e^{j\omega t}\} \tag{o}$$

Applying the boundary conditions, we obtain

$$A = \hat{\xi}\,\frac{(\beta-\alpha)}{2\beta} \tag{p}$$

$$B = \frac{(\alpha+\beta)\hat{\xi}}{2\beta} \tag{q}$$

Therefore, if $\hat{\xi}$ is real

$$\xi_1(x,t) = -\,\xi_2(x,t) = \hat{\xi}\cos\beta x\cos(\omega t-\alpha x) - \frac{\alpha}{\beta}\,\hat{\xi}\sin\beta x\sin(\omega t-\alpha x) \tag{r}$$

Part f

We can see that β can be imaginary, for which we will have spatially growing curves. This can happen when

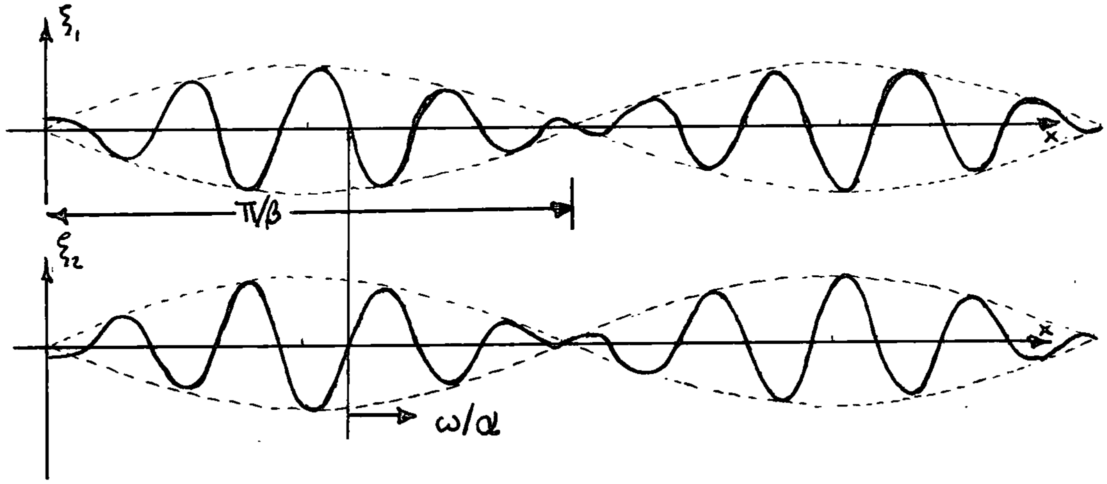
$$\omega^2 v_s^2 - \frac{3\varepsilon_o V_o^2}{\sigma_m d^3}(U^2 - v_s^2) < 0 \tag{s}$$

or

$$V_o^2 > \frac{\sigma_m d^3\,\omega^2\,v_s^2}{3\varepsilon_o(U^2 - v_s^2)} \tag{t}$$

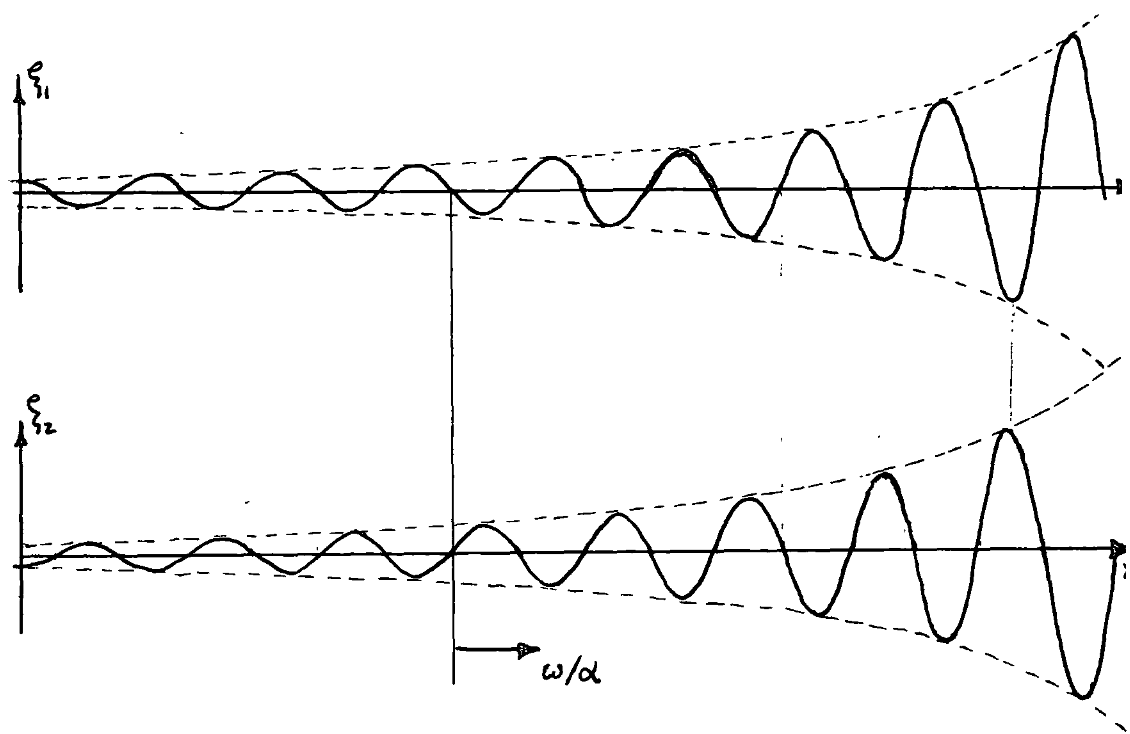
Part g

With $V_o = 0$ and $v > v_s$;



PROBLEM 10.23 (continued)

Amplifying waves are obtained as (t) is satisfied;



PROBLEM 10.24

Part a

The equation of motion for the membrane is:

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S\left[\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2}\right] + T_z \tag{a}$$

where

$$T_z = T_{zz} = 2\varepsilon_o V_o^2\, \xi/s^3 \tag{b}$$

The equation may be rewritten as follows:

PROBLEM 10.24 (continued)

$$\frac{\partial^2\xi}{\partial t^2} = v_s^2\left[\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2} + k_c^2\ \xi\right] \tag{c}$$

where

$$k_c^2 \equiv \frac{2\varepsilon_o V_o^2}{S\ s^3}$$

Assume solutions of the following form:

$$\xi(x,y,t) = \text{Re}[\hat{\xi}\ e^{j(\omega t - k_x x - k_y y)}] \tag{d}$$

The dispersion is:

$$\omega^2 = v_s^2[k_x^2 + k_y^2 - k_c^2] \tag{e}$$

The mode which goes unstable first is the lowest spatial mode:

$$k_x = \frac{\pi}{a}, \quad k_y = \frac{\pi}{b} \tag{f}$$

Instability occurs at

$$k_c^2 = (\frac{\pi}{a})^2 + (\frac{\pi}{b})^2 \tag{g}$$

or,

$$V_o = \{\frac{S\ s^3}{2\varepsilon_o}\ [(\frac{\pi}{a})^2 + (\frac{\pi}{b})^2]\}^{1/2} \tag{h}$$

Part b

The natural frequencies follow from Eq. (e) as

$$\omega_{mn} = v_s[(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 - k_c^2]^{1/2} \tag{i}$$

Part c

We superimpose eigensolutions to obtain the membrane motion for $t > 0$. The solution that already satisfies the initial condition on velocity is

$$\xi(x,y,t) = \sum_m \sum_n \xi_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \cos\omega_{mn}t \tag{j}$$

where m and n are odd only, since the initial condition on $\xi(x,y,t=0)$ requires no even modes. Now use the principle of orthogonality of modes. Multiply (j) by $\sin(p\pi x/a)\ \sin(q\pi y/b)$ and integrate over the area of the membrane. The left hand side becomes

PROBLEM 10.24 (continued

$$\int_0^b \int_0^a \left[\xi(x,y,t=0)\sin \frac{p\pi x}{a} \sin \frac{q\pi y}{b}\right] dx\, dy \tag{k}$$

$$= \int_0^b \int_0^a J_o u_o\,(x-\tfrac{a}{2})u_o(y-\tfrac{b}{2})\sin \frac{p\pi x}{a} \sin \frac{q\pi y}{b}\, dx\, dy$$

Thus, (j) reducesto

$$J_o = \xi_{pq}\,(\tfrac{a}{2})(\tfrac{b}{2}) \tag{ℓ}$$

which makes it possible to evaluate the Fourier amplitudes

$$\xi_{mn} = \frac{4J_o}{ab} \tag{m}$$

The desired response is (j) with ξ_{mn} given by (m).

$$\xi(x,y,t) = \sum_{\substack{m\\(odd)}}\sum_{n} (\frac{4J_o}{ab}) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos \omega_{mn} t$$

Note that the analysis is valid even if the lowest mode(s) is (are) unstable, for which case:

$$\cos \omega_{pq} t \rightarrow \cosh \alpha_{pq} t$$

PROBLEM 10.25

The equation of motion is (see Table 9.2, page 535):

$$\sigma_m \frac{\partial^2\xi}{\partial t^2} = S\left(\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2}\right) \tag{a}$$

With solutions of the form $\xi = \text{Re}\,\hat{\xi}\, e^{j(\omega t - k_x x - k_y y)}$, the dispersion equation is

$$\omega = \pm v_s \sqrt{k_x^2 + k_y^2} \tag{b}$$

A particular superposition of these solutions that satisfies the boundary conditions along three of the four edges is

$$\xi = A \sin \frac{n\pi y}{a} \sin k_x(x-b) \cos \omega_o t \tag{c}$$

where in view of (b),

$$\omega_o^2 = v_s^2\,[k_x^2 + (\frac{n\pi}{a})^2] \tag{d}$$

Thus, there is a solution for each value of n, and

PROBLEM 10.25 (continued)

$$\xi = \sum_{n=1}^{\infty} A_n \sin k_n(x-b)\cos \omega_o t \sin \frac{n\pi y}{a} \qquad \text{(e)}$$

where, from (d)

$$k_n = \left[\left(\frac{\omega_o}{v_s}\right)^2 - \left(\frac{n\pi}{a}\right)^2\right]^{1/2} \qquad \text{(f)}$$

At x = 0, (e) takes the form of a Fourier series

$$\xi(y=0) = \sum_{n=1}^{\infty} -A_n \sin k_n b \cos \omega_o t \sin \frac{n\pi y}{a} \qquad \text{(g)}$$

This function of (y,t) has the correct dependence on t. The dependence on y is made that of Fig. 10P.25 by adjusting the coefficients A_n as is usual in a Fourier series. Note that because of the symmetry of the excitation about y = a/2, only odd values of n give finite A_n. Thus

$$\int_o^{a/2} \frac{2\xi_o}{a} y \sin \frac{m\pi}{a} y\, dy + \int_{a/2}^{a} \frac{2\xi_o}{a} (a-y)\sin \frac{m\pi}{a} y\, dy \qquad \text{(h)}$$

$$= -\int_o^a A_n \sin k_n b \sin \frac{n\pi y}{a} \sin \frac{m\pi y}{a}\, dy$$

Evaluation of the integrals gives

$$\frac{4\xi_o a}{(m\pi)^2} \sin\left(\frac{m\pi}{2}\right) = -\frac{A_n a \sin k_n b}{2} \qquad \text{(i)}$$

Hence, the required function is (e) with k_n given by (f) and A_n given by solving (i)

$$A_n = -\frac{8\xi_o}{(m\pi)^2} \sin\left(\frac{m\pi}{2}\right)/\sin k_n b \qquad \text{(j)}$$

PROBLEM 10.26

The force per unit length is $\mu_o \bar{I} \times \bar{H}$, where $\bar{H}$ is the magnetic field intensity evaluated at the position of the wire. That is,

$$\bar{S} = \mu_o I[H_x \bar{i}_y - H_y \bar{i}_x] \qquad \text{(a)}$$

To evaluate H_x and H_y at $u\bar{i}_x + v\bar{i}_y$ note that $\bar{H}(0,0) = 0$. By symmetry $H_y(0,y) = 0$ and therefore $\partial H_y/\partial y\ (0,0) = 0$. Then, $\nabla\cdot\bar{B} = 0$ requires that $\partial H_x/\partial x(0,0) = 0$. Thus, an expansion of (a) about the origin gives

PROBLEM 10.26 (continued)

$$\bar{S} \cong \mu_o I \left[\frac{\partial H_x}{\partial y} v \bar{i}_y - \frac{\partial H_y}{\partial x} u \bar{i}_x\right] \tag{b}$$

Note that because $\nabla x \bar{H} = 0$ at the origin, $\partial H_x/\partial y = \partial H_y/\partial x$. Thus, (b) becomes

$$\bar{S} = \mu_o I \frac{\partial H_y}{\partial x} [-u \bar{i}_x + v \bar{i}_y] \tag{c}$$

and $\partial H_y/\partial x$ (0,0) is easily computed because

$$H_y(x,0) = \frac{I_o}{2\pi} [\frac{1}{a-x} - \frac{1}{a+x}] \cong \frac{I_o}{2\pi} [\frac{2x}{a^2}] \tag{d}$$

Thus,

$$\bar{S} = \frac{\mu_o I I_o}{\pi a^2} [-u \bar{i}_x + v \bar{i}_y] \tag{e}$$

It is the fact that $\nabla \times \bar{H} = 0$ in the neighborhood of the origin that requires that the contributions to (e) be negatives.

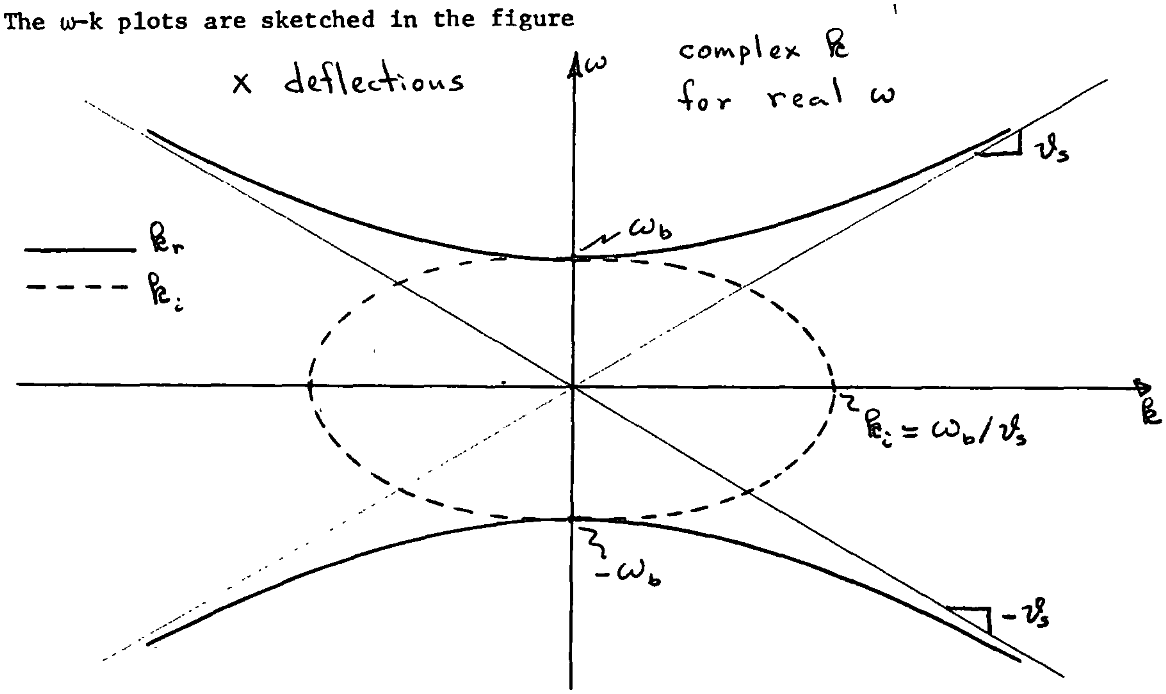
Part b

(i) Assume $u = \mathrm{Re}[\hat{u} e^{j(\omega t-kz)}]$ (f)

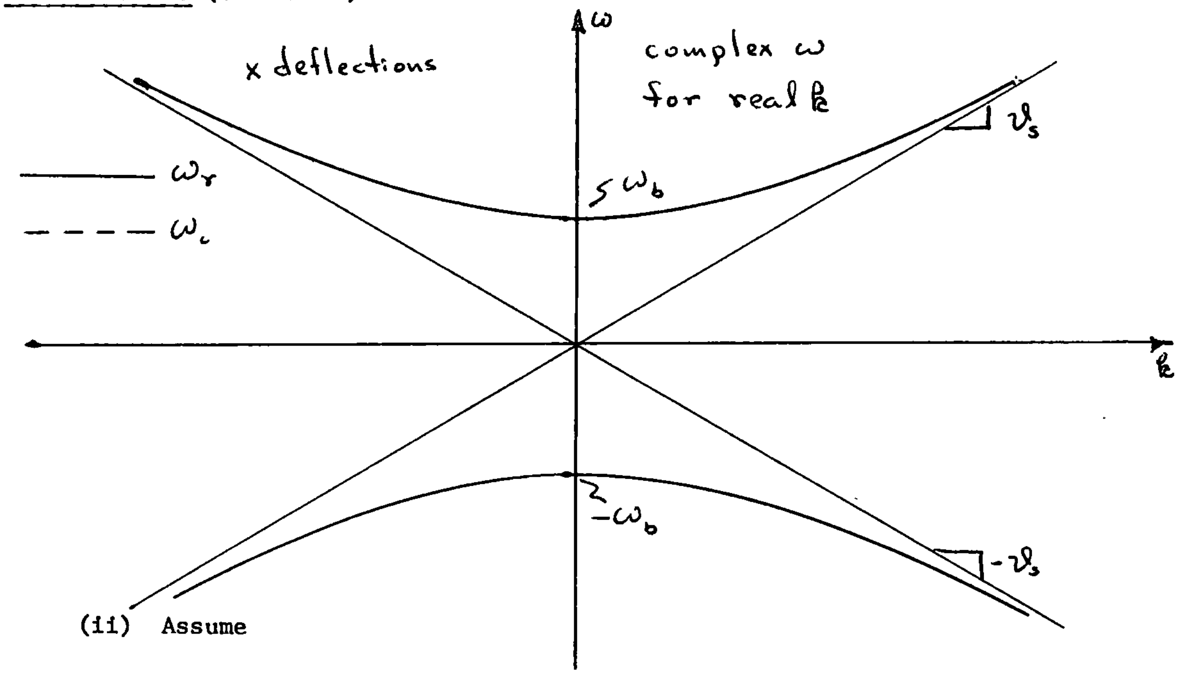
Then

$$\omega^2 = v_s^2 k^2 + \omega_b^2, \; v_s^2 = \frac{f}{m} ; \omega_b^2 = \frac{Ib}{m} \tag{g}$$

The ω-k plots are sketched in the figure



PROBLEM 10.26 (continued)



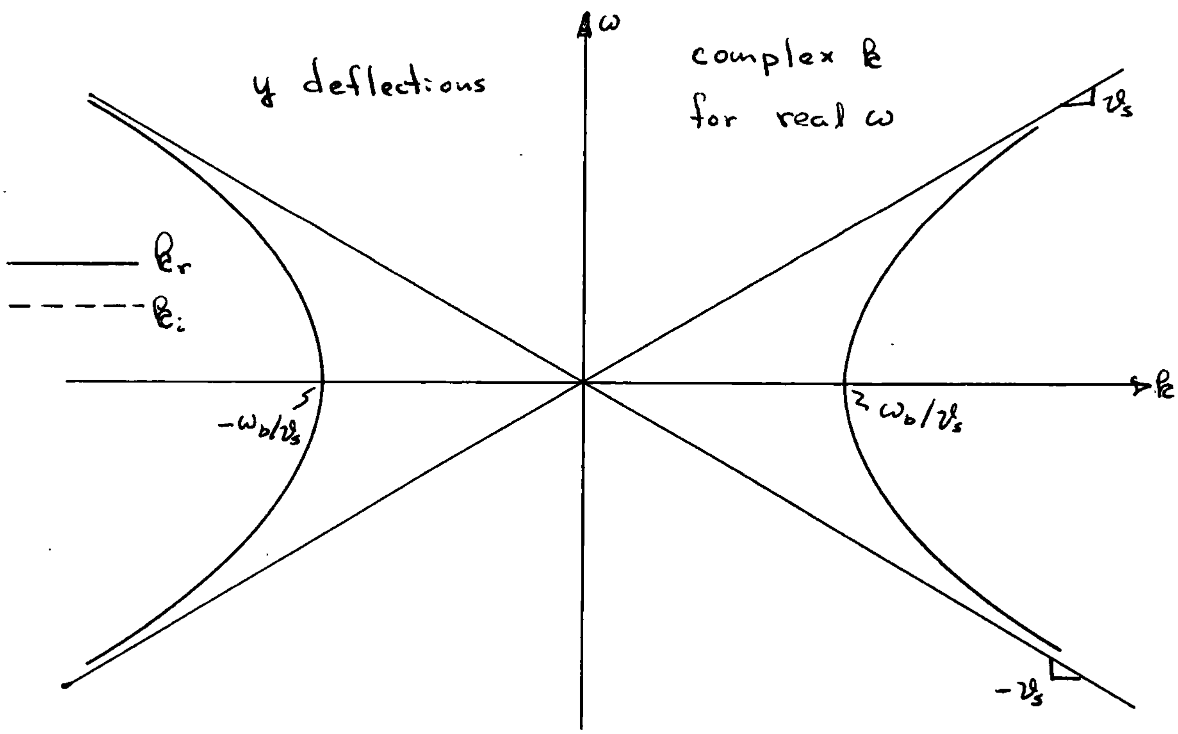
(ii) Assume

$$v = \mathrm{Re}[\hat{v}\, e^{j(\omega t - kz)}] \tag{h}$$

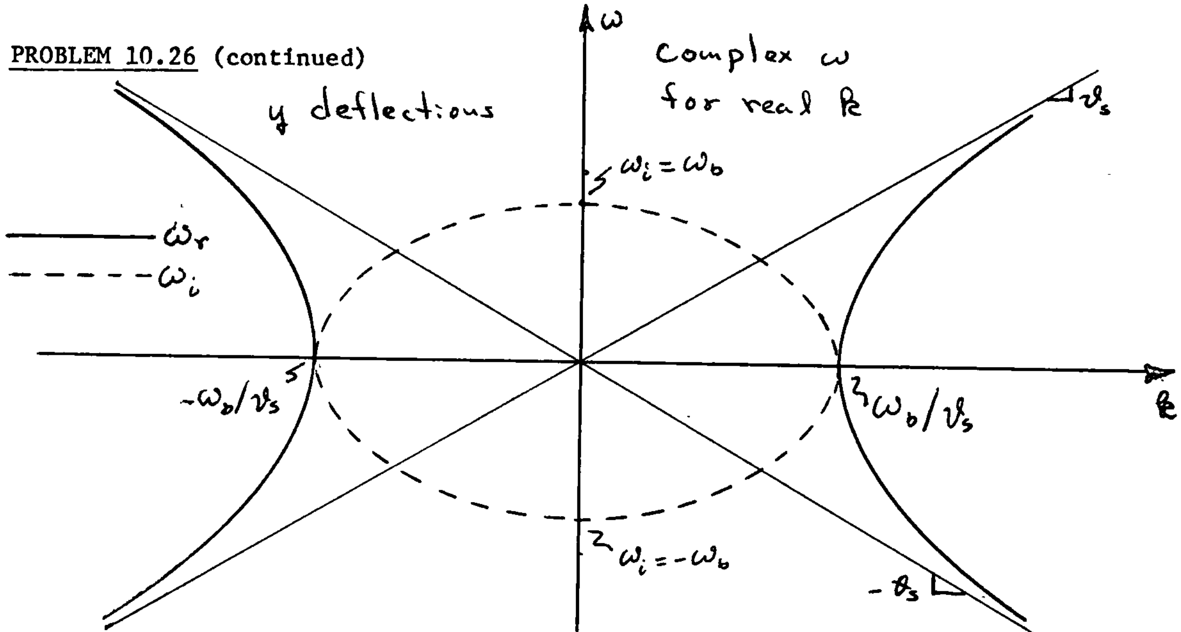
Then

$$\omega^2 = v_s^2 k^2 - \omega_b^2 \tag{i}$$

and the ω-k plots are as shown



PROBLEM 10.26 (continued)



Note that if the effect of the field is to stabilize motions in one direction, it must destabilize motions in the other direction.

Part c

Driven response is found in a manner similar to that for Prob. 10.2. Thus for

$\omega < \omega_b$ (cutoff)

$$u(z,t) = -\frac{u_o \sinh \alpha_u x}{\sinh \alpha \ell} \cos \omega_o t \tag{j}$$

$$v(z,t) = -\frac{v_o \sin k_r x}{\sin k_v \ell} \sin \omega_o t \tag{k}$$

$\omega > \omega_b$

$$u(z,t) = -\frac{u_o \sin k_u x}{\sin k_u \ell} \cos \omega_o t \tag{ℓ}$$

$$v(z,t) = -\frac{v_o \sin k_v x}{\sin k_v \ell} \sin \omega_o t \tag{m}$$

where

$$\alpha_u = \left[\frac{\omega_b^2 - \omega_o^2}{v_s^2}\right]^{1/2}$$

$$k_u = \left[\frac{\omega_o^2 - \omega_b^2}{v_s^2}\right]^{1/2}$$

PROBLEM 10.26 (continued)

$$k_v = \left[\frac{\omega_o^2 + \omega_b^2}{v_s^2}\right]^{1/2}$$

Part d

We must suppress instability of lowest natural mode in v.

$$v_s^2 \left(\frac{\pi}{\ell}\right)^2 > \omega_b^2 \tag{n}$$

or

$$I\ I_o < \frac{f\pi a^2}{\mu_o}\left(\frac{\pi}{\ell}\right)^2 \tag{o}$$

for evanescent waves

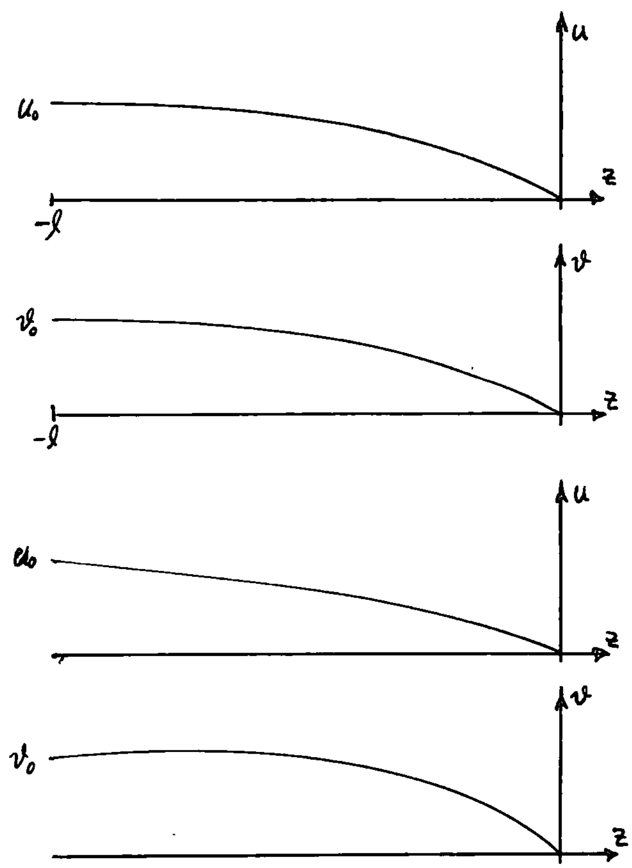
$$\omega_o^2 < \omega_b^2 \tag{p}$$

Thus, from (n) and (p), $\omega_o^2 < v_s^2\ (\pi/\ell)^2$.

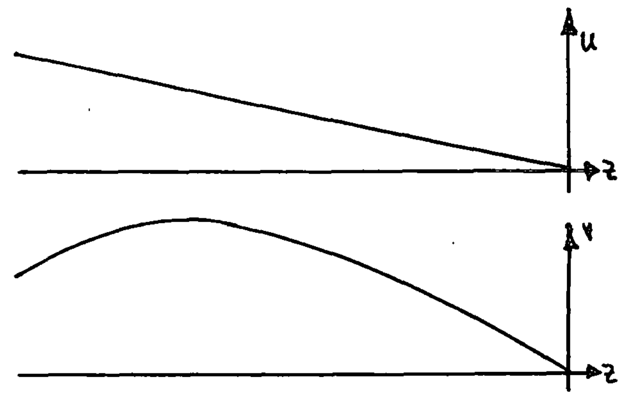
Part e

$I = 0$

$I < \omega_o^2/(b/m)$

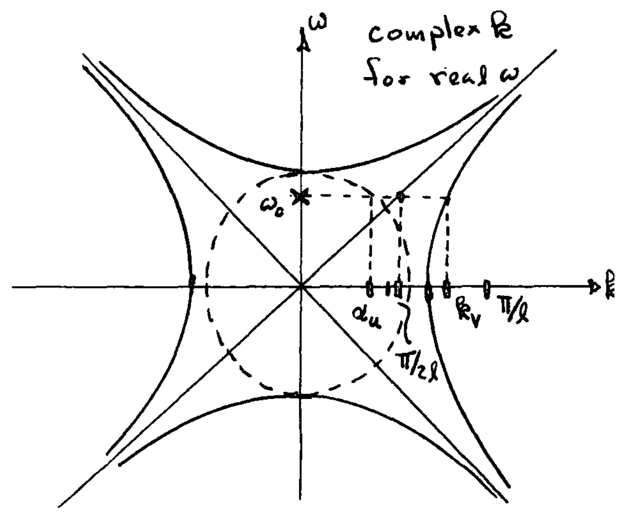


PROBLEM 10.26 (continued)



$$I = \frac{\omega_o^2}{b/m}$$

The effect of raising the current is summarized by the ω-k plot, with complex k plotted for real ω. As I is raised the hyperbola moves outward. Thus, k_v increases and k_u decreases to zero and becomes imaginary. Thus, wavelengths for the v deflection shorten while those for u lengthen to infinity and then deflections decay. Note that v waves shorter than $\lambda=2\ell$ will not be observed because of instability.



PROBLEM 10.27

Part a

We may take the results of problem 10.26 and replace $\frac{\partial}{\partial t}$ by $\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}$ in the differential equations, and ω by ω-kU in the dispersion equations. Therefore, the equations of motion are

$$m\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}\right)^2 u = f\frac{\partial^2 u}{\partial z^2} - Ibu \qquad \text{(a)}$$

$$m\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}\right)^2 v = f\frac{\partial^2 v}{\partial z^2} + Ibv \qquad \text{(b)}$$

Part b

For the x motions, the dispersion relation is

$$-m(\omega-kU)^2 = -fk^2 - Ib \qquad \text{(c)}$$

We let

PROBLEM 10.27 (continued)

$$\frac{Ib}{m} = \omega_b^2$$

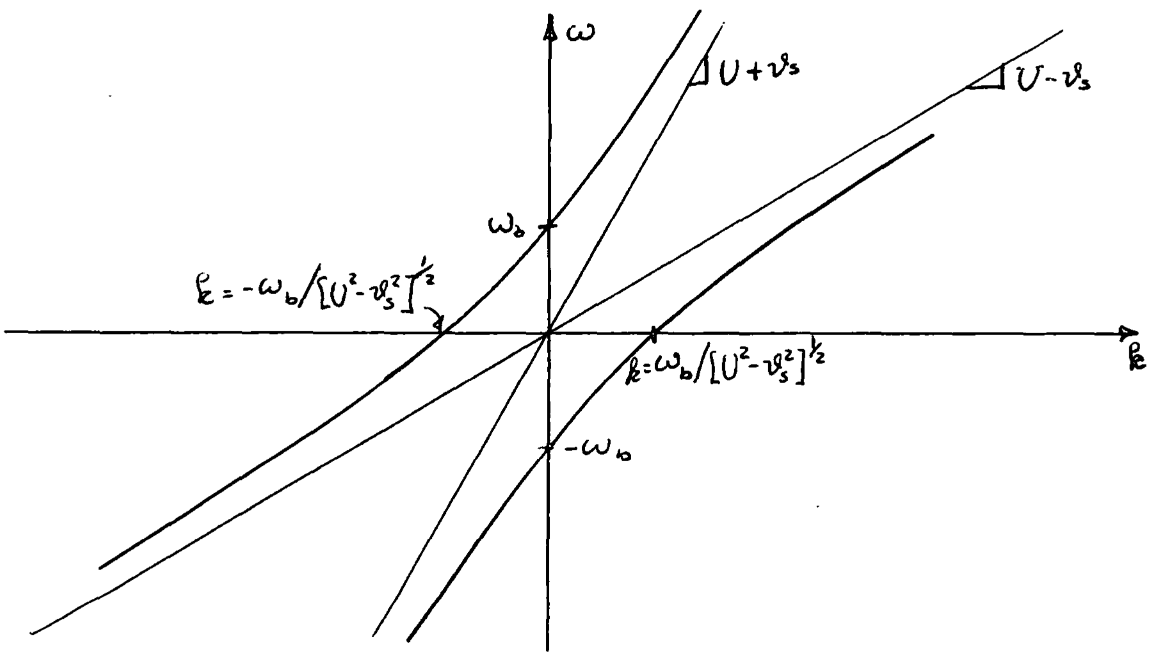
$$\frac{f}{m} = v_s^2$$

Therefore $\omega = kU \pm \sqrt{k^2 v_s^2 + \omega_b^2}$ (d)

or solving for k

$$k = \frac{\omega U \pm \sqrt{\omega^2 v_s^2 + \omega_b^2 (U^2 - v_s^2)}}{(U^2 - v_s^2)} \tag{e}$$

The ω-k plot for x motions is sketched as



For real ω, we have only k real. For real k, we have only real ω.

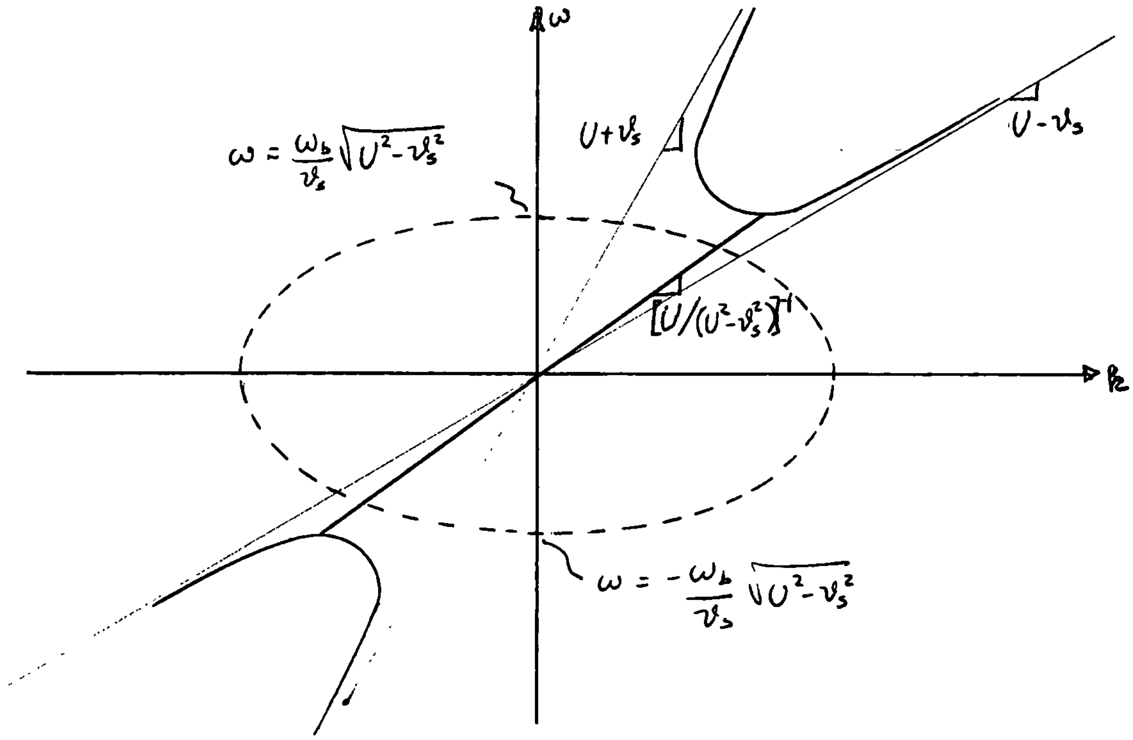
For the y motions, we obtain

$$\omega = kU \pm \sqrt{v_s^2 k^2 - \omega_b^2} \tag{f}$$

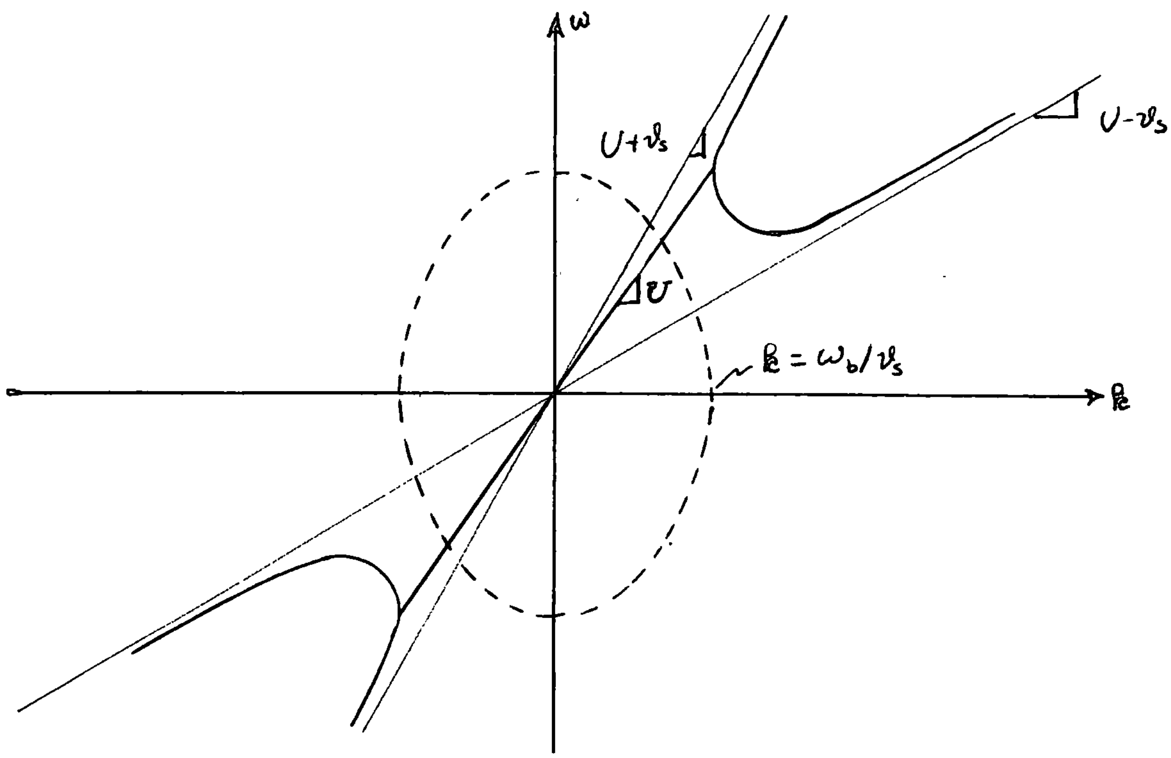
$$k = \frac{\omega U \pm \sqrt{\omega^2 v_s^2 - \omega_b^2 (U^2 - v_s^2)}}{(U^2 - v_s^2)} \tag{g}$$

Thus for real ω, the sketch is

PROBLEM 10.27 (continued)



while for real k, the ω-k plot is



PROBLEM 10.27 (continued)

Part c

Since the wire is traveling at a "supersonic" velocity, we cannot impose a downstream boundary condition to determine upstream behavior.

We are given

$$u(0,t)\bar{i}_x + v(0,t)\bar{i}_y = u_o \cos\omega_o t\ \bar{i}_x + v_o \sin\omega_o t\ \bar{i}_y \tag{h}$$

and the boundary conditions

$$\frac{\partial u}{\partial z}(0,t) = 0; \frac{\partial v}{\partial z}(0,t) = 0 \tag{i}$$

We let

$$\alpha = \frac{\omega U}{U^2-v_s^2};\ \beta = \frac{\sqrt{\omega^2 v_s^2 + \omega_b^2(U^2-v_s^2)}}{(U^2-v_s^2)}$$

$$\gamma = \frac{\sqrt{\omega^2 v_s^2 - \omega_b^2(U^2 - v_s^2)}}{(U^2 - v_s^2)} \tag{j}$$

For the x motions, the allowed values of k are

$$k_1 = \alpha + \beta \qquad \text{with } \omega = \omega_o$$

$$k_2 = \alpha - \beta \tag{k}$$

Therefore

$$u = \mathrm{Re}\left\{\left[A_1 e^{-jk_1 z} + A_2 e^{-jk_2 z}\right] e^{j\omega_o t}\right\} \tag{ℓ}$$

Applying the boundary conditions and simplying, we obtain

$$u = u_o \mathrm{Re}\left[\left(\frac{j\alpha}{\beta}\sin\beta z + \cos\beta z\right) e^{j(\omega t-\alpha z)}\right] \tag{m}$$

For the y motions, the allowed values of k are

$$k_3 = \alpha + \gamma$$

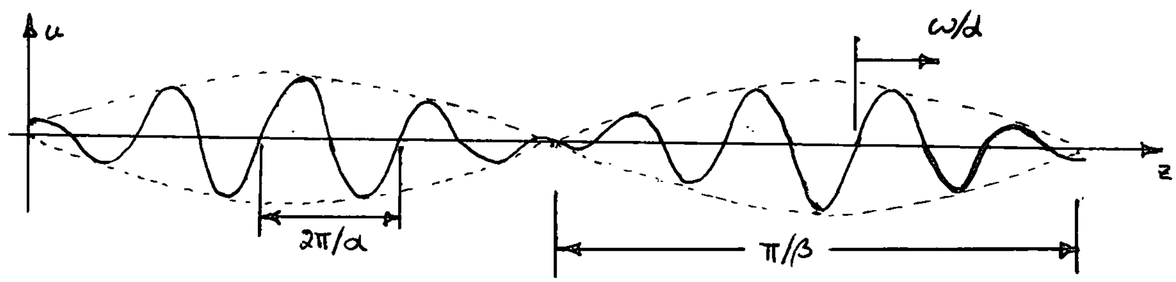
$$k_4 = \alpha - \gamma \tag{n}$$

Therefore

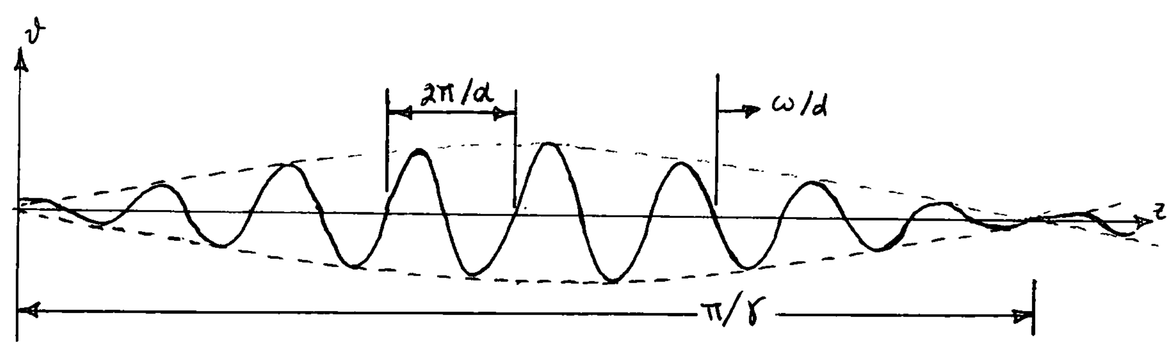
$$v = -v_o\ \mathrm{Re}\left[\left(-\frac{\alpha}{\gamma}\sin\gamma z + j\cos\gamma z\right)e^{j(\omega t-\alpha z)}\right] \tag{o}$$

PROBLEM 10.27 (continued)

Part d



As long as $U > v_s$ this is the form of u, no matter what the value of I (as long as $I > 0$). As the magnitude of I increases, β increases but α remains unchanged.

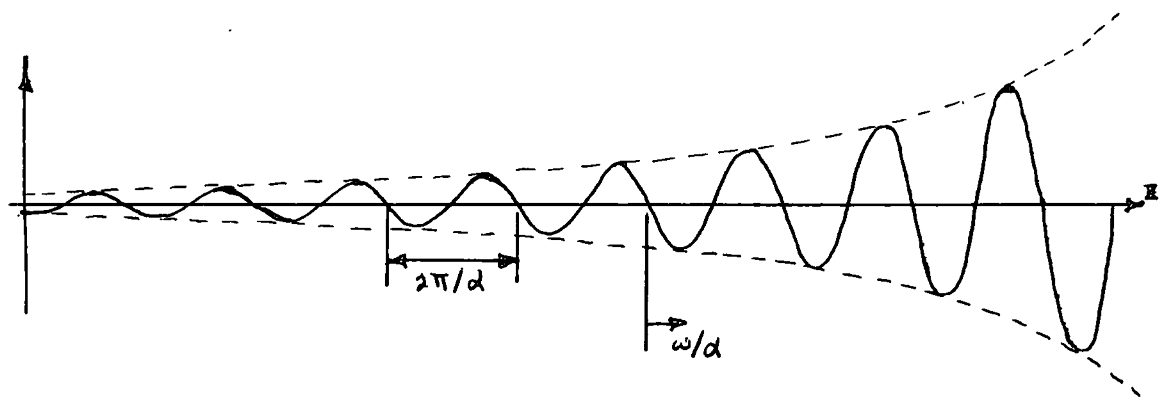


PROBLEM 10.27 (continued)

This is the form of v, as long as

$$\omega^2 v_s^2 - \omega_b^2(U^2 - v_s^2) > 0 \tag{p}$$

As I is increased, we reach a value whereby this inequality no longer holds. At this point γ becomes imaginary and we have spatial growth.



As I is increased beyond the critical value, v will begin to grow exponentially with z.

Part e

To simulate the moving wire, we could use a moving stream of a conducting liquid such as mercury. We would introduce current onto the stream at the nozzle and complete the circuit by having the stream strike a metal plate at some downstream postion.

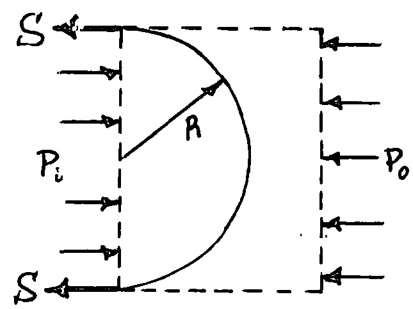
PROBLEM 10.28

Part a

A simple static argument establishes the required pressure difference. The pressure, as a mechanical stress that occurs in a fluid, always acts on a surface in the normal direction. The figure shows a section of length Δz from the membrane. Since the volume which encloses this section must be in force equilibrium, we can write

PROBLEM 10.28 (continued)

$$2R(\Delta z)[p_i - p_o] = 2S(\Delta z) \tag{a}$$



where we have summed the forces acting on the surfaces. It follows that the required pressure difference is

$$p_i - p_o = \frac{S}{R} \tag{b}$$

Part b

To answer this question, and other questions concerning the dynamics of the circular membrane, we must include in our description a perturbation displacement from the equilibrium at $r = R$. Hence, we define the membrane surface by the relation

$$r = R + \xi(\theta,z,t) \tag{c}$$

The pressure difference $p_i - p_o$ is a force per unit area acting on the membrane in the normal direction. It is the surface force density necessary to counteract a mechanical force per unit area T_m

$$T_m = -\frac{S}{R} \tag{d}$$

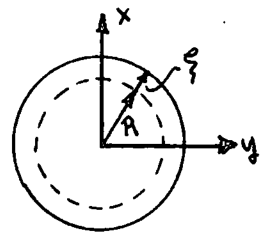
which acts on each section of the membrane in the radial direction. We wish now to determine the mechanical force acting on each section, when the surface is perturbed to the position given by (c). We can do this in steps. First, consider the case where ξ is independent of θ and z, as shown in the figure. Then from (d)

$$T_m = -\frac{S}{R+\xi} = -S[\frac{1}{R} - \frac{\xi}{R^2}] \tag{e}$$

where we have kept only the linear term in the expansion of T_m about $r = R$.

When the perturbation ξ depends on θ, the surface has a tilt, as shown. We can sum the components to S acting on the section in the radial direction as

PROBLEM 10.28 (continued)



$$\lim_{\Delta\theta\to 0} \frac{S}{R\Delta\theta}\left[\frac{1}{R}\frac{\partial\xi}{\partial\theta}\bigg|_{\theta+\frac{\Delta\theta}{2}} - \frac{1}{R}\frac{\partial\xi}{\partial\theta}\bigg|_{\theta-\frac{\Delta\theta}{2}}\right] = \frac{S}{R}\frac{\partial^2\xi}{\partial\theta^2} \tag{f}$$

Similarly, a dependence on z gives rise to a radial force on the section due to the mechanical tension S,

$$\lim_{\Delta z\to 0} \frac{S}{\Delta z}\left[\frac{\partial\xi}{\partial z}\bigg|_{z+\frac{\Delta z}{2}} - \frac{\partial\xi}{\partial z}\bigg|_{z-\frac{\Delta z}{2}}\right] = S\frac{\partial^2\xi}{\partial z^2} \tag{g}$$

In general, the force per unit area exerted on a small section of membrane under the constant tension S from the adjacent material is the sum of the forces given by (e), (f) and (g),

$$T_m = S\left(-\frac{1}{R} + \frac{\xi}{R^2} + \frac{1}{R}\frac{\partial^2\xi}{\partial\theta^2} + \frac{\partial^2\xi}{\partial z^2}\right) \tag{h}$$

It is now possible to write the dynamic force equation for radial motions. In addition to the pressure difference $p_i - p_o$ acting in the radial direction, we will include the inertial force density $\sigma_m/(\partial^2\xi/\partial t^2)$ and a surface force density T_r due to electric or magnetic fields. Hence,

$$\sigma_m\frac{\partial^2\xi}{\partial t^2} = S\left(-\frac{1}{R} + \frac{\xi}{R^2} + \frac{1}{R^2}\frac{\partial^2\xi}{\partial\theta^2} + \frac{\partial^2\xi}{\partial z^2}\right) + T_r + p_i - p_o \tag{i}$$

Consider now the case where there is no electromechanical interaction. Then $T_r = 0$, and static equilibrium requires that (b) hold. Hence, the constant terms in (i) cancel, leaving the perturbation equation

$$\sigma_m\frac{\partial^2\xi}{\partial t^2} = S\left(\frac{\xi}{R^2} + \frac{1}{R^2}\frac{\partial^2\xi}{\partial\theta^2} + \frac{\partial^2\xi}{\partial z^2}\right) \tag{j}$$

PROBLEM 10.28 (continued)
Parts c & d

This equation is formally the same as those that we have encountered previously (see Sec. 10.1.3). However, the cylindrical geometry imposes additional requirements on the solutions. That is, if we assume solutions having the form,

$$\xi = \text{Re}\ \hat{\xi}(z)e^{j(\omega t + m\theta)} \tag{k}$$

the assumed dependence on θ is a linear combination of $\sin m\theta$ and $\cos m\theta$. If the displacement is to be single valued, m must have integer values. Otherwise we would not have $\xi(\theta,z,t) = \xi(\theta + 2\pi,z,t)$.

With the assumed dependence on θ and t, (j) becomes,

$$\frac{d^2\hat{\xi}}{dz^2} + k^2\hat{\xi} = 0 \tag{ℓ}$$

where

$$k^2 = \frac{1}{R^2}(1-m^2) + \frac{\omega^2\sigma_m}{S}$$

The membrane is attached to the rigid tubes at $z = 0$ and $z = \ell$. The solution to (ℓ) which satisfies this condition is

$$\hat{\xi} = A \sin k_n x \tag{m}$$

where

$$k_n = \frac{n\pi}{\ell}\ ,\ n = 1,2,3,...$$

The eigenvalue k_n determines the eigenfrequency, because of (ℓ).

$$\omega_n^2 = [(\frac{n\pi}{\ell})^2 + \frac{(m^2 - 1)}{R^2}]\frac{S}{\sigma_m} \tag{n}$$

To obtain a picture of how these modes appear, consider the case where A is real, and (m) and (k) become

$$\xi(\theta,z,t) = A \sin \frac{n\pi}{\ell} x \cos m\theta \cos \omega t \tag{o}$$

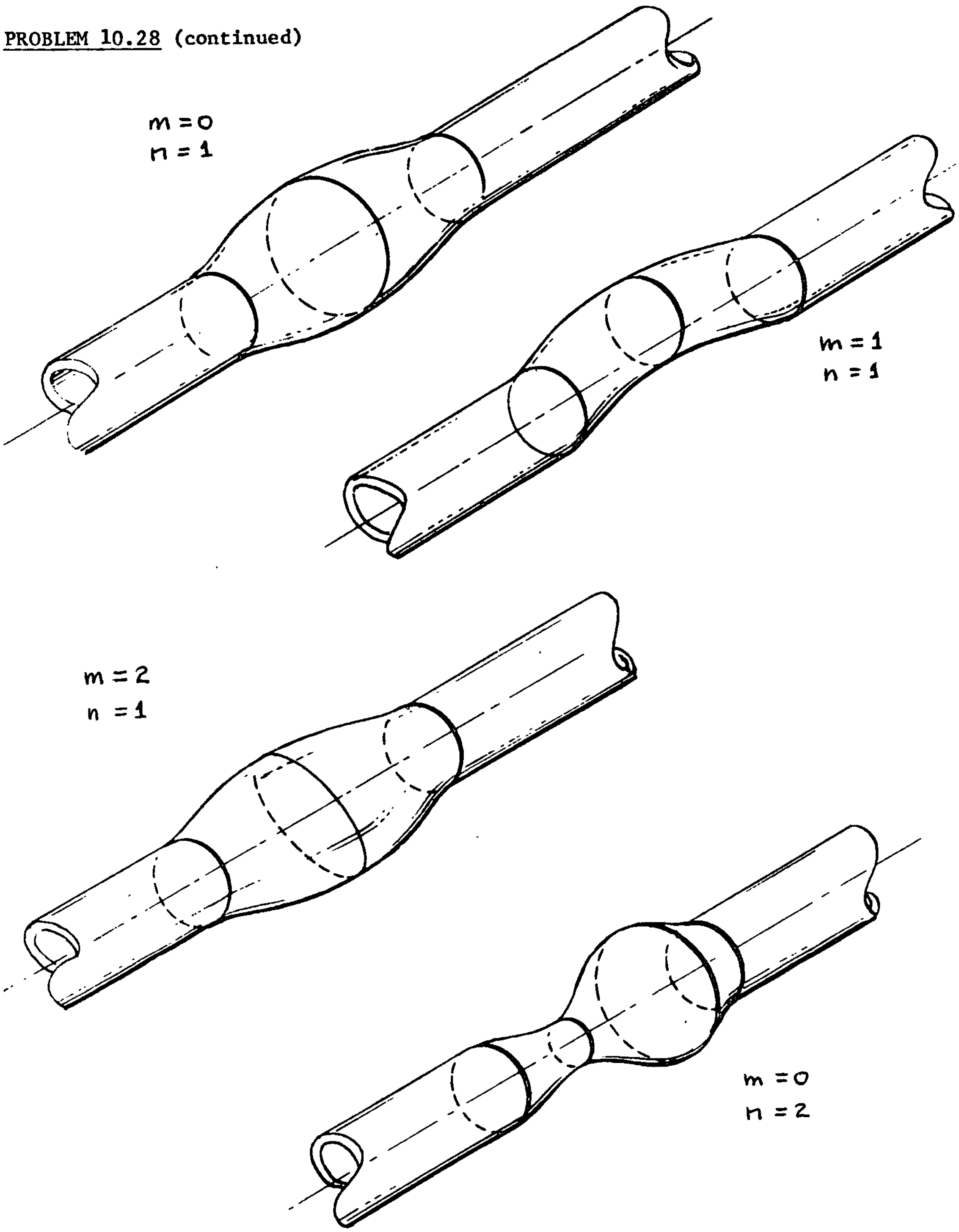
The instantaneous displacements for the first four modes are shown in the figure.

There is the possibility that the $m = 0$ mode is unstable, as can be seen from (n), where if

$$(\frac{n\pi}{\ell})^2 < \frac{1}{R^2} \tag{p}$$

we find that the time dependence has the form $\exp \pm |\omega|t$. The first mode to meet this condition for instability is the $n = 1$ mode. Hence, it is not possible

PROBLEM 10.28 (continued)



PROBLEM 10.28 (continued)

to maintain the uniform cylindrical shape of the static equilibrium if

$$R\pi/\ell < 1 \tag{q}$$

This condition for static instability is easily understood if we remember that in the m = 0, n = 1 mode, there are two perturbation surface forces on a small section of the membrane surface. One of these is the perturbation part of (e) and arises because of the curvature in static equilibrium. This force acts in the same direction as the displacement, and hence tends to produce static instability. It is counteracted by a restoring force proportional to the second derivative in the z direction, as given by (g). Condition (q) is satisfied when the effect of the initial curvature predominates the stiffness from the boundaries.

Part e

With rotation, the dispersion becomes:

$$(\omega - m\Omega)^2 = \Omega_s^2[m^2 - 1 - m_c^2]$$

with

$$\Omega_s^2 = \frac{S}{\sigma_m R^2}$$

$$m_c^2 = \frac{2\varepsilon_o V_o^2}{a^3} \frac{R^2}{S}$$

Because there is no z dependence (no surface curvature in the z direction) the equilibrium is unstable in the m = 0 mode even in the absence of an applied voltage.

PROBLEM 10.29

The solution is of the form

$$\xi = \xi_+(\alpha) + \xi_-(\beta) \tag{a}$$

where

$$\alpha = x - y$$

$$\beta = x + y$$

We are given that at x = 0

$$\frac{\partial \xi}{\partial x} = \frac{d\xi_+}{d\alpha} + \frac{d\xi_-}{d\beta} = \Delta[u_{-1}(y) - u_{-1}(y-a)] \tag{b}$$

and that

PROBLEM 10.29 (continued)

$$\xi = 0$$

which implies that

$$\frac{\partial\xi}{\partial y} = 0 = -\frac{d\xi_+}{d\alpha} + \frac{d\xi_-}{d\beta} \qquad \text{(c)}$$

We therefore have

$$\frac{d\xi_+}{d\alpha} = \frac{\Delta}{2}\,[u_{-1}(-\alpha) - u_{-1}(-\alpha-a)]$$

and

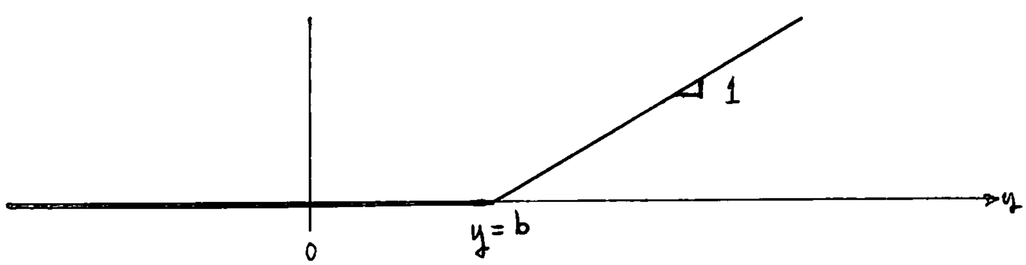
$$\frac{d\xi_-}{d\beta} = \frac{\Delta}{2}\,[u_{-1}(\beta) - u_{-1}(\beta-a)] \qquad \text{(e)}$$

Then $\frac{\partial\xi}{\partial y} = -\frac{d\xi_+}{d\alpha} + \frac{d\xi_-}{d\beta} = \frac{\Delta}{2}\left\{u_{-1}(y-x) - u_{-1}(y-x-a) + u_{-1}(x+y) - u_{-1}(x+y-a)\right\}$ (f)

Integrating with respect to y, we obtain

$$\xi = \frac{\Delta}{2}\,\{-u_{-2}(y-x) + u_{-2}(y-x-a) + u_{-2}(y+x) - u_{-2}(y+x-a) \qquad \text{(g)}$$

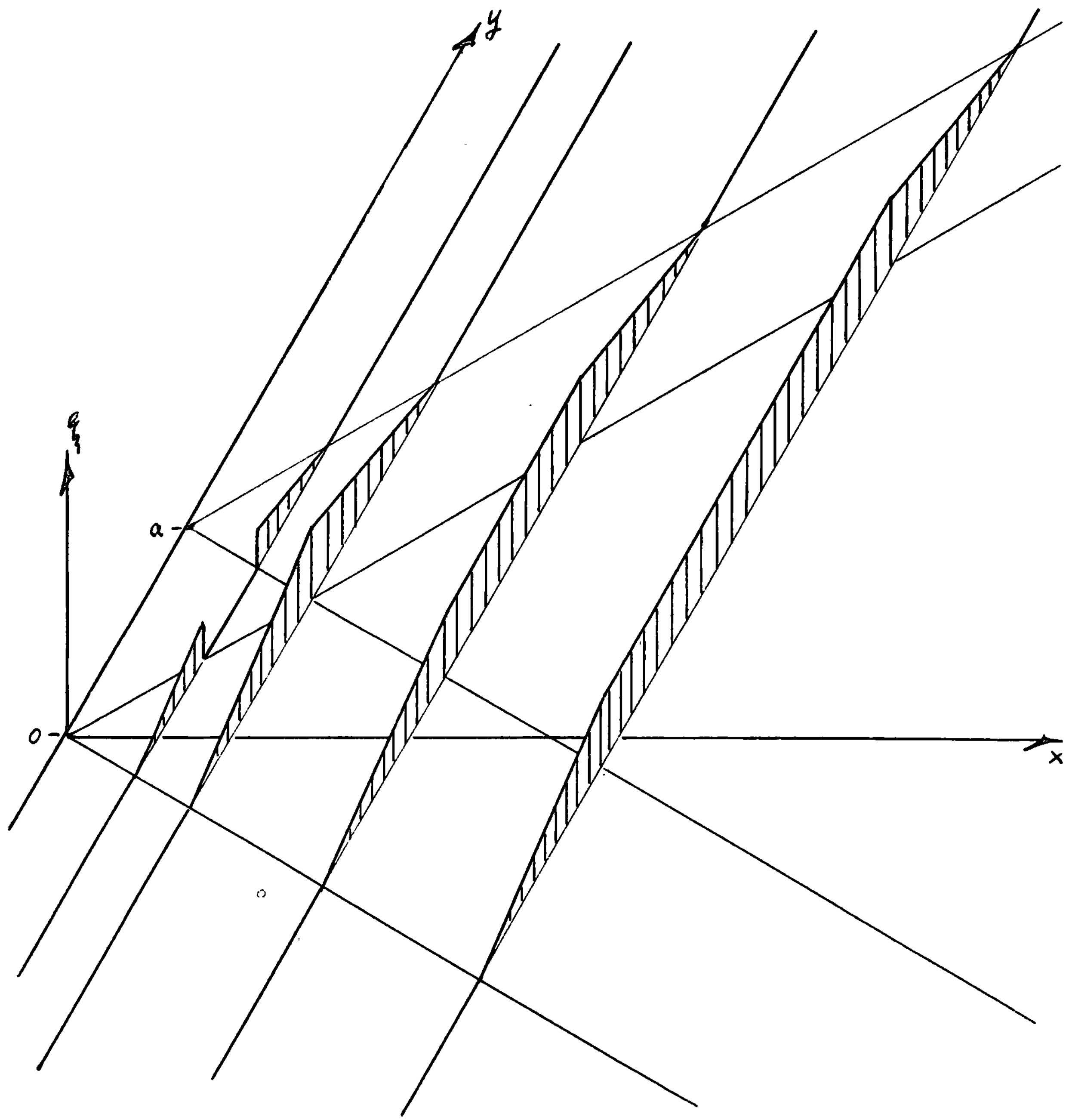
where u_{-2} is a ramp function; that is $u_{-2}(y-b)$ is defined as



Part b

The constraint represented by Fig. 10P.29 could be obtained by ejecting the membrane from a slit at x = 0 that is planar, but tilted over the range $0 < y < a$. Thus, the membrane would have no deflection ξ at $x = 0$, but would have the required constant slope Δ over the range $0 < y < a$, and zero slope elsewhere.

PROBLEM 10.29 (continued)



PROBLEM 10.30

For this situation, the governing equation is (10.4.15) of the text.

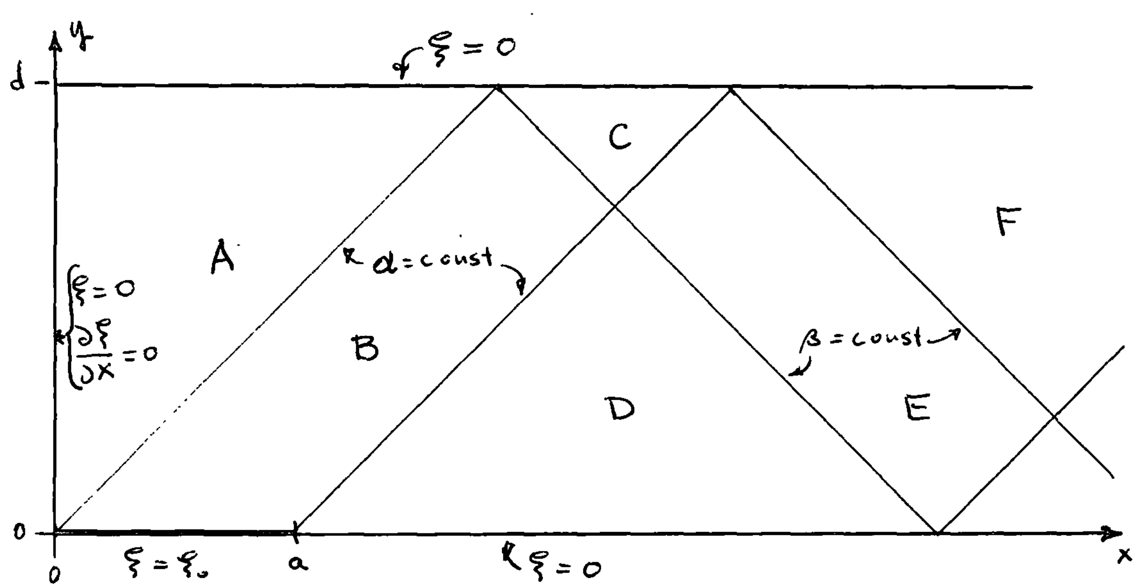
$$(M^2-1)\,\frac{\partial^2 \xi}{\partial x^2} = \frac{\partial^2 \xi}{\partial y^2} \tag{a}$$

Here $M^2 = 2$; so we have the equation:

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{\partial^2 \xi}{\partial y^2} \tag{b}$$

The characteristics are determined from equations (10.4.17) and (10.4.18) to be:

$$\begin{aligned} \alpha &= x - y \\ \beta &= x + y \end{aligned} \tag{c}$$



The x-y plane divides into regions A...F, as shown in the sketch. Tracing back on the characteristics from points in regions A, D, F... shows that in these regions $\xi = 0$; the characteristics originate on "zero" boundary conditions. At points in region B, only the C^+ characteristic originates on finite data; $\xi_+(\alpha) = \xi_o$, $\xi_-(\beta) = 0$ and hence

$$\xi = \xi_o \text{ in region B}$$

PROBLEM 10.30 (continued)

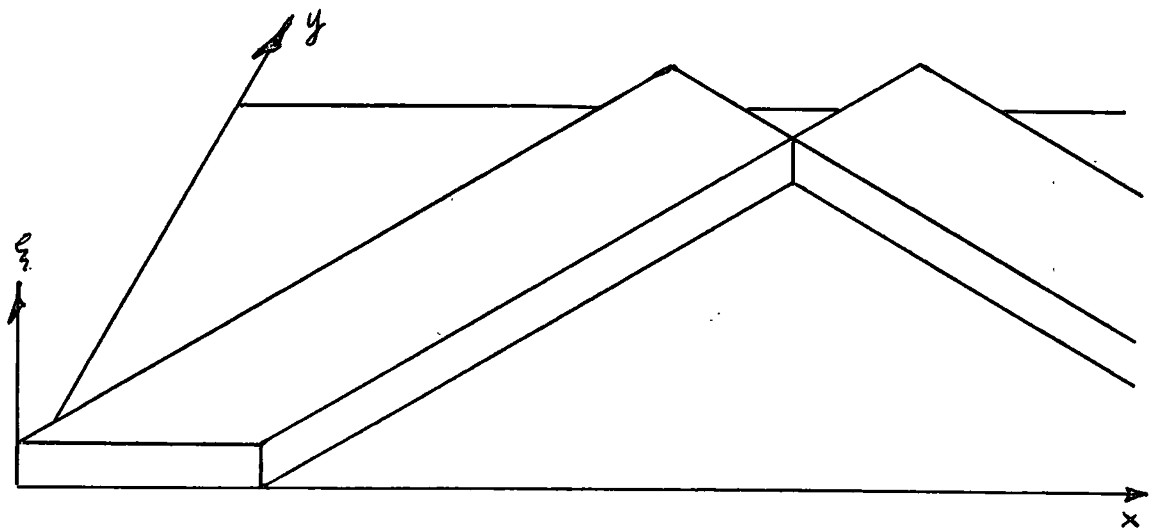
In C, deflections are determined by waves, both originating from the initial data. Hence $\xi_+(\alpha) = \xi_o$, but $\xi_-(\beta)$ is determined by the reflection of an incident wave on the boundary at y = d. Hence $\xi_-(\beta) = -\xi_o$ and

$$\xi = 0 \text{ in region C}$$

In region E only the $\xi_-(\beta)$ wave is finite because the $\xi_+(\alpha)$ wave originates on zero conditions and

$$\xi = -\xi_o \text{ in region E}$$

The deflection has the stationary appearance shown in the figure.



PROBLEM 10.31

From equation 10.4.30, we have

$$\omega^2 = k^2 v_s^2 \pm \frac{kB_o I}{m} \quad \text{(a)}$$

We define

$$\alpha = + \frac{IB_o}{2mv_s^2} \quad \text{(b)}$$

and

$$\beta = \sqrt{\left(\frac{B_o I}{2mv_s^2}\right)^2 + \frac{\omega^2}{v_s^2}} \quad \text{(c)}$$

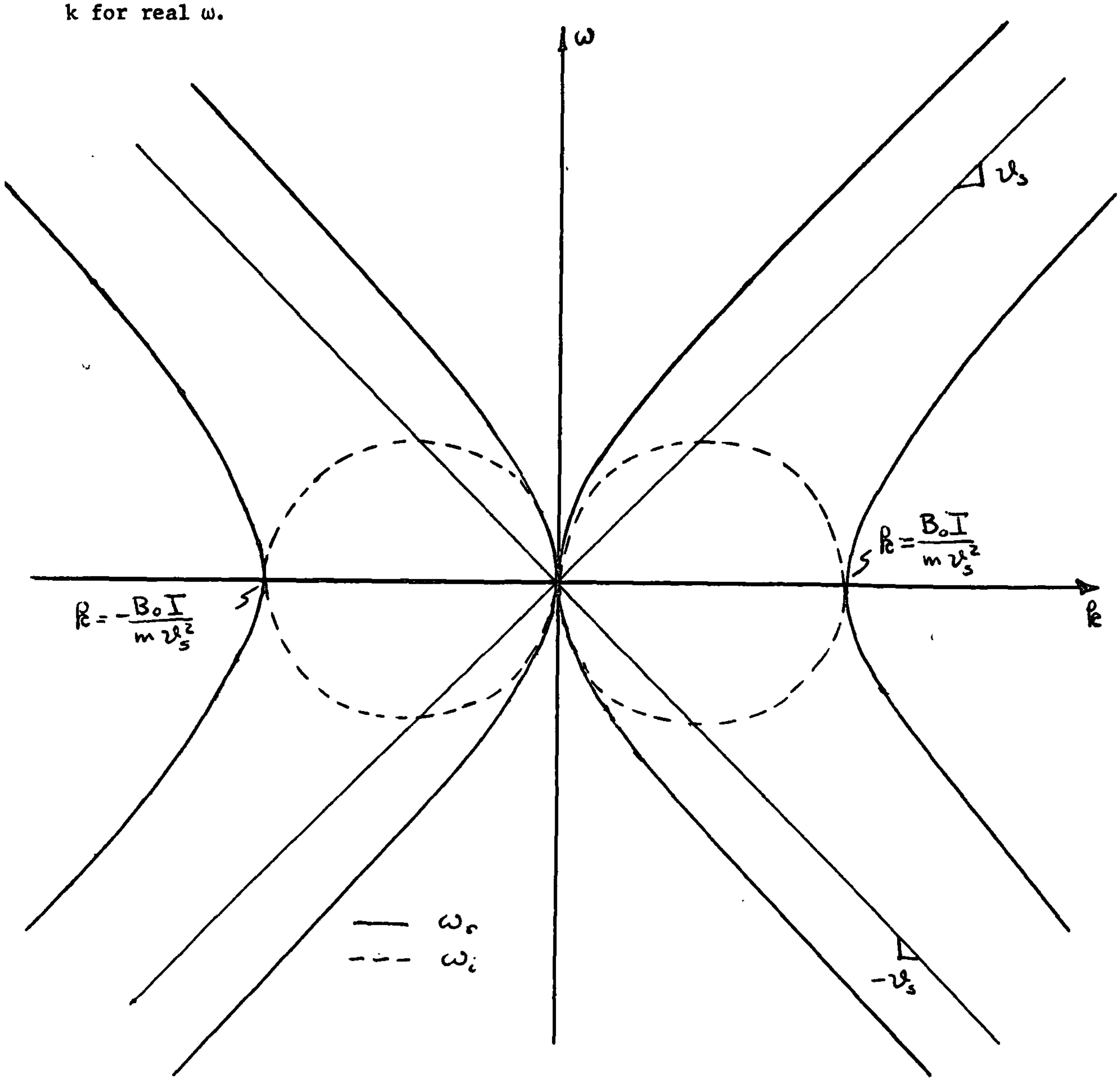
PROBLEM 10.31 (continued)

The four allowed branches of k as a function of ω are therefore $k = \pm k_1$, and $\pm k_2$, where

$$k_1 = \alpha + \beta \quad \text{(d)}$$

$$k_2 = -\alpha + \beta \quad \text{(e)}$$

The sketch shows complex ω for real k. Note however, that only real values of k are given if ω is real and hence the solid lines represent the plot of complex k for real ω.



PROBLEM 10.32

The effect of the longitudinal convection is accounted for by replacing ω in Eq. 10.4.3 by ω-kU (see for example page 594). Thus,

$$(\omega-kU)^2 = k^2v_s^2 \pm \frac{kB_oI}{m} \tag{a}$$

This expression can be solved to give

$$k = \frac{\left(2\omega U \pm \frac{B_oI}{m}\right) \pm \sqrt{4v_s^2\omega^2 + 4\omega U\left(\pm\frac{B_oI}{m}\right) + \left(\frac{B_oI}{m}\right)^2}}{2(U^2 - v_s^2)} \tag{b}$$

The sketch of complex k for real ω is made with the help of the following observations: Consider the modes that are represented by $-B_o$.

1) Asymptotes for branches are $k = \omega/(U \mp v_s)$ as $\omega \to \infty$.

2) As ω is lowered, the $(-B_o)$ branches become complex as

$$4v_s^2\omega^2 + 4\omega U\left(\frac{-B_oI}{m}\right) + \left(\frac{B_oI}{m}\right)^2 = 0$$

or at the frequencies

$$\omega = \frac{B_oI}{2v_s^2m} \quad \left(U \pm \sqrt{U^2 - v_s^2}\right)$$

Thus, for $U > v_s$ there is a lower as well as an upper positive frequency at which k switches from real to complex values.

In this range of complex k, real k is

$$k = \left(2\omega U - \frac{B_oI}{m}\right)/2(U^2 - v_s^2)$$

or a straight line intercepting the k = 0 axis at

$$\omega = \frac{B_oI}{2Um}$$

3) as $\omega \to 0$,

$$k \to 0 \text{ and } k \to \pm B_oI/m(U^2 - v_s^2)$$

where the - sign goes with the unstable branches.

4) As $\omega \to -\infty$ the values of k are real and approach the asymptotes $k = \omega/(U \mp v_s)$.

PROBLEM 10.32 (continued)

Similar reasoning gives the modes represented in (b) by $+B_o$. Note that these modes have a plot obtained by replacing $\omega \to -\omega$ and $k \to -k$ in the figure.