



Electromechanical Dynamics

by Prof. Herbert H. Woodson and James R. Melcher

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ELECTROMECHANICAL DYNAMICS

Part I: Discrete Systems

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To our parents

PREFACE

Part I: Discrete Systems

In the early 1950's the option structure was abandoned and a common core curriculum was instituted for all electrical engineering students at M.I.T. The objective of the core curriculum was then, and is now, to provide a foundation in mathematics and science on which a student can build in his professional growth, regardless of the many opportunities in electrical engineering from which he may choose. In meeting this objective, core curriculum subjects cannot serve the needs of any professional area with respect to nomenclature, techniques, and problems unique to that area. Specialization comes in elective subjects, graduate study, and professional activities.

To be effective a core curriculum subject must be broad enough to be germane to the many directions an electrical engineer may go professionally, yet it must have adequate depth to be of lasting value. At the same time, the subject must be related to the real world by examples of application. This is true because students learn by seeing material in a familiar context, and engineering students are motivated largely by the relevance of the material to the realities of the world around them.

In the organization of the core curriculum in electrical engineering at M.I.T. electromechanics is one major component. As our core curriculum has evolved, there have been changes in emphasis and a broadening of the topic. The basic text in electromechanics until 1954, when a new departure was made, was *Electric Machinery* by Fitzgerald and Kingsley. This change produced *Electromechanical Energy Conversion* by White and Woodson, which was used until 1961. At that time we started the revision that resulted in the present book. During this period we went through many versions of notes while teaching the material three semesters a year.

Our objective has always been to teach a subject that combines classical mechanics with the fundamentals of electricity and magnetism. Thus the subject offers the opportunity to teach both mechanics and electromagnetic theory in a context vital to much of the electrical engineering community.

Our choice of material was to some extent determined by a desire to give the student a breadth of background sufficient for further study of almost any type of electromechanical interaction, whether in rotating machinery,

plasma dynamics, the electromechanics of biological systems, or magneto-elasticity. It was also chosen to achieve adequate depth while maintaining suitable unity, but, most important, examples were chosen that could be enlivened for the engineering student interested in the interplay of physical reality and the analytical model. There were many examples from which to choose, but only a few satisfied the requirement of being both mathematically lucid *and* physically demonstrable, so that the student could “push it or see it” and directly associate his observations with symbolic models. Among the areas of electrical engineering, electromechanics excels in offering the opportunity to establish that all-important “feel” for a physical phenomenon. Properly selected electromechanical examples can be the basis for discerning phenomena that are remote from human abilities to observe.

Before discussing how the material can be used to achieve these ends, a review of the contents is in order. The student who uses this book is assumed to have a background in electrostatics and magnetostatics. Consequently, Chapter 1 and Appendix B are essentially a review to define our starting point.

Chapter 2 is a generalization of the concepts of inductance and capacitance that are necessary to the treatment of electromechanical systems; it also provides a brief introduction to rigid-body mechanics. This treatment is included because many curricula no longer cover mechanics, other than particle mechanics in freshman physics. The basic ideas of Chapter 2 are repeated in Chapter 3 to establish some properties of electromechanical coupling in lumped-parameter systems and to obtain differential equations that describe the dynamics of lumped-parameter systems.

Next, the techniques of Chapters 2 and 3 are used to study rotating machines in Chapter 4. Physical models are defined, differential equations are written, machine types are classified, and steady-state characteristics are obtained and discussed. A separate chapter on rotating machines has been included not only because of the technological importance of machines but also because rotating machines are rich in examples of the kinds of phenomena that can be found in lumped-parameter electromechanical systems.

Chapter 5 is devoted to the study, with examples, of the dynamic behavior of lumped-parameter systems. Virtually all electromechanical systems are mathematically nonlinear; nonetheless, linear incremental models are useful for studying the stability of equilibria and the nature of the dynamical behavior in the vicinity of an equilibrium. The second half of this chapter develops the classic potential-well motions and loss-dominated dynamics in the context of electromechanics. These studies of nonlinear dynamics afford an opportunity to place linear models in perspective while forming further insights on the physical significance of, for example, flux conservation and state functions.

Chapter 6 represents our first departure from lumped-parameter systems into continuum systems with a discussion of how observers in relative motion will define and measure field quantities and the related effects of material motion on electromagnetic fields. It is our belief that dc rotating machines are most easily understood in this context. Certainly they are a good demonstration of field transformations at work.

As part of any continuum electromechanics problem, one must know how the electric and magnetic fields are influenced by excitations and motion. In quasi-static systems the distribution of charge and current are controlled by magnetic diffusion and charge relaxation, the subjects of Chapter 7. In Chapter 7 simple examples isolate significant cases of magnetic diffusion or charge relaxation, so that the physical processes involved can be better understood.

Chapters 6 and 7 describe the electrical side of a continuum electromechanical system with the material motion predetermined. The mechanical side of the subject is undertaken in Chapter 8 in a study of force densities of electric and magnetic origin. Because it is a useful concept in the analysis of many systems, we introduce the Maxwell stress tensor. The study of useful properties of tensors sets the stage for later use of mechanical stress tensors in elastic and fluid media.

At this point the additional ingredient necessary to the study of continuum electromechanics is the mechanical medium. In Chapter 9 we introduce simple elastic continua—longitudinal motion of a thin rod and transverse motion of wires and membranes. These models are used to study simple continuum mechanical motions (nondispersive waves) as excited electromechanically at boundaries.

Next, in Chapter 10 a string or membrane is coupled on a continuum basis to electric and magnetic fields and the variety of resulting dynamic behavior is studied. The unifying thread of this treatment is the dispersion equation that relates complex frequency ω with complex wavenumber k . Without material convection there can be simple nondispersive waves, cut off or evanescent waves, absolute instabilities, and diffusion waves. The effect of material convection on evanescent waves and oscillations and on wave amplification are topics that make a strong connection with electron beam and plasma dynamics. The method of characteristics is introduced as a convenient tool in the study of wave propagation.

In Chapter 11 the concepts and techniques of Chapters 9 and 10 are extended to three-dimensional systems. Strain displacement and stress-strain relations are introduced, with tensor concepts, and simple electromechanical examples of three-dimensional elasticity are given.

In Chapter 12 we turn to a different mechanical medium, a fluid. We first study electromechanical interactions with inviscid, incompressible

fluids to establish essential phenomena in the simplest context. It is here that we introduce the basic notions of MHD energy conversion that can result when a conducting fluid flows through a transverse magnetic field. We also bring in electric-field interactions with fluids, in which ion drag phenomena are used as an example. In addition to these basically conducting processes, we treat the electromechanical consequences of polarization and magnetization in fluids. We demonstrate how highly conducting fluids immersed in magnetic fields can propagate Alfvén waves.

In Chapter 13 we introduce compressibility to the fluid model. This can have a marked effect on electromechanical behavior, as demonstrated with the MHD conduction machine. With compressibility, a fluid will propagate longitudinal disturbances (acoustic waves). A transverse magnetic field and high electrical conductivity modify these disturbances to magnetoacoustic waves.

Finally, in Chapter 14 we add viscosity to the fluid model and study the consequences in electromechanical interactions with steady flow. Hartmann flow demonstrates the effect of viscosity on the dc magnetohydrodynamic machine.

To be successful a text must have a theme; the material must be inter-related. Our philosophy has been to get into the subject where the student is most comfortable, with lumped-parameter (circuit) concepts. Thus many of the subtle approximations associated with quasi-statics are made naturally, and the student is faced with the implications of what he has assumed only after having become thoroughly familiar with the physical significance and usefulness of his approximations. By the time he reaches Chapter 4 he will have drawn a circle around at least a class of problems in which electromagnetic fields interact usefully with media in motion.

In dealing with physical and mathematical subjects, as we are here, in which the job is incomplete unless the student sees the physical laws put to work in some kind of physical embodiment, it is necessary for the thread of continuity to be woven into the material in diverse and subtle ways. A number of attempts have been made, to which we can add our early versions of notes, to write texts with one obvious, pedagogically logical basis for evolving the material; for example, it can be recognized that classes of physical phenomena could be grouped according to the differential equation that describes the pertinent dynamics. Thus we could treat magnetic diffusion, diffusion waves on elastic continua, and viscous diffusion waves in one chapter, even though the physical embodiments are entirely different. Alternatively, we could devise a subject limited to certain technological applications or cover superficially a wide range of basically unrelated topics such as "energy conversion" under one heading. This was the prevalent approach in engineering education a decade or so ago, even at the

undergraduate level. It seems clear to us that organizing material in a teachable and meaningful fashion is far more demanding than this. To confess our own mistakes, our material went originally from the general to the specific; it began with the relativistic form of Maxwell's equations, including the effects of motion, and ended with lumped-parameter devices as special cases. Even if this were a pedagogically tenable approach, which we found it was not, what a bad example to set for students who should be learning to distinguish between the essential and the superfluous! Ideas connected with the propagation of electromagnetic waves (relativistic ideas) must be included in the curriculum, but their connection with media in motion should be made after the student is aware of the first-order issues.

A meaningful presentation to *engineers* must interweave and interrelate mathematical concepts, physical characteristics, the modeling process, and the establishment of a physical "feel" for the world of reality. Our approach is to come to grips with each of these goals as quickly as possible (let the student "get wet" within the first two weeks) and then, while reinforcing what he has learned, continually add something new. Thus, if one looks, he will see the same ideas coming into the flow of material over and over again.

For the organization of this book one should look for many threads of different types. We can list here only a few, in the hope that the subtle reinforcing interplay of mathematical and physical threads will be made evident. Probably the essential theme is Maxwell's equations and the ideas of quasi-statics. The material introduced in Chapter 1 is completely abstract, but it is reinforced in the first few chapters with material that is close to home for the student. By the time he reaches Chapter 10 he will have learned that waves exist which intimately involve electric and magnetic fields that are altogether quasistatic. (This is something that comes as a surprise to many late in life.) Lumped-parameter ideas are based on the integral forms of Maxwell's equations, so that the dynamical effects found with lumped-parameter time constants L/R and RC in Chapter 5 are easily associated with the subjects of magnetic diffusion and charge relaxation. A close tie is made between the "speed voltage" of Chapter 5 and the effects of motion on magnetic fields, as described by field transformations in Chapters 6 to 14. Constant flux dynamics of a lumped coil in Chapter 5 are strongly associated with the dynamics of perfectly conducting continuous media; for example, Alfvén waves in Chapter 12.

Consider another thread of continuity. The book begins with the mathematics of circuit theory. The machines of Chapter 4 are essentially circuits in the sinusoidal steady state. In Chapter 5 we linearize to pursue lumped-parameter ideas of stability and other transient responses and then proceed to nonlinear dynamics, potential-well theory, and other approaches that should form a part of any engineer's mathematical background. By the time

the end of Chapter 10 is reached these ideas will have been carried into the continuum with the addition of tensor concepts, simple cases of the method of characteristics, and eigenvalue theory. The ω - k plot and its implication for all sorts of subjects in modern electrical engineering can be considered as a mathematical or a physical objective. The ideas of stability introduced with ordinary differential equations ($\exp st$) in Chapter 5 evolve into the continuum stability studies of Chapter 10 [$\exp j(\omega t - kx)$] and can be regarded as a mathematical or a physical thread in our treatment. We could list many other threads: witness the evolution of energy and thermodynamic notions from Chapters 3 to 5, 5 to 8, and 8 to 13.

We hope that this book is not just one more in the mathematics of electrical engineering or the technical aspects of rotating machines, transducers, delay lines, MHD converters, and so on, but rather that it is the mathematics, the physics, and, most of all, the engineering combined into one.

The material brought together here can be used in a variety of ways. It has been used by Professors C. N. Weygandt and F. D. Ketterer at the University of Pennsylvania for two subjects. The first restricts attention to Chapters 1 to 6 and Appendix B for a course in lumped-parameter electromechanics that both supplants the traditional one on rotating machines in the electrical engineering curriculum and gives the background required for further study in a second term (elective) covering Chapter 7 and beyond. Professors C. D. Hendricks and J. M. Crowley at the University of Illinois have used the material to follow a format that covers up through Chapter 10 in one term but omits much of the material in Chapter 7. Professor W. D. Getty at the University of Michigan has used the material to follow a one-term subject in lumped-parameter electromechanics taught from a different set of notes. Thus he has been able to use the early chapters as a review and to get well into the later chapters in a one-term subject.

At M.I.T. our curriculum seems always to be in a state of change. It is clear that much of the material, Chapters 1 to 10, will be part of our required (core) curriculum for the foreseeable future, but the manner in which it is packaged is continually changing. During the fall term, 1967, we covered Chapters 1 to 10 in a one-semester subject taught to juniors and seniors. The material from Chapters 4 and 6 on rotating machines was used selectively, so that students had "a foot solidly in the door" on this important subject but also that the coverage could retain an orientation toward the needs of all the diverse areas found in electrical engineering today. We have found the material useful as the basis for early graduate work and as a starting point in several courses related to electromechanics.

Finally, to those who open this book and then close it with the benediction, "good material but unteachable," we apologize because to them we have not made our point. Perhaps not as presented here, but certainly as it is

represented here, this material is rich in teaching possibilities. The demands on the teacher to see the subject in its total context, especially the related problems that lie between the lines, are significant. We have taught this subject many times to undergraduates, yet each term has been more enjoyable than the last. There are so many ways in which drama can be added to the material, and we do not need to ask the students (bless them) when we have been successful in doing so.

In developing this material we have found lecture demonstrations and demonstration films to be most helpful, both for motivation and for developing understanding. We have learned that when we want a student to see a particular phenomenon it is far better for us to do the experiment and let the student focus his attention on what he should see rather than on the wrong connections and blown fuses that result when he tries to do the experiment himself. The most successful experiments are often the simplest—those that give the student an opportunity to handle the apparatus himself. Every student should “chop up some magnetic field lines” with a copper “axe” or he will never really appreciate the subject. We have also found that some of the more complex demonstrations that are difficult and expensive to store and resurrect each semester come through very well in films. In addition to our own short films, three films have been produced professionally in connection with this material for the National Committee on Electrical Engineering Films, under a grant from the National Science Foundation, by the Education Development Center, Newton, Mass.

Synchronous Machines: Electromechanical Dynamics by H. H. Woodson
Complex Waves I: Propagation, Evanescence and Instability by J. R. Melcher

Complex Waves II: Instability, Convection and Amplification by J. R. Melcher

An additional film is in the early stages of production. Other films that are useful have been produced by the Education Development Center for the National Committee on Fluid Mechanics Films and for the College Physics Film Program. Of particular interest, from the former series, is *Magnetohydrodynamics* by Arthur Shercliff.

A book like this can be produced only with plenty of assistance. We gratefully acknowledge the help we received from many directions and hope we have forgotten no one after seven years of work. First of all we want to acknowledge our students with whom we worked as the material developed. They are the one most essential ingredient in an effort of this sort. Next we want to thank Dr. S. I. Freedman, Professor H. H. Richardson, and Dr. C. V. Smith, Jr., for their assistance in framing worthwhile approaches to several of our key topics. In seven years we have had the help of many able

teachers in presenting this material to students. Their discussions and advice have been most useful. In this category we want particularly to mention Professors H. A. Haus, P. L. Penfield, D. C. White, G. L. Wilson, R. Gallager, and E. Pierson and Doctors J. Reynolds, W. H. Heiser, and A. Kusko. Professor Ketterer, who has taught this material at M.I.T. and the University of Pennsylvania, Professors C. D. Hendricks and J. M. Crowley, who have taught it at M.I.T. and the University of Illinois, and Professor W. D. Getty, who has taught it at M.I.T. and the University of Michigan, have been most generous with their comments. Messrs. Edmund Devitt, John Dressler, and Dr. Kent Edwards have checked the correctness of many of the mathematical treatments. Such a task as typing a manuscript repeatedly is enough to try the patience of anyone. Our young ladies of the keyboard, Miss M. A. Daly, Mrs. D. S. Figgins, Mrs. B. S. Morton, Mrs. E. M. Holmes, and Mrs. M. Mazroff, have been gentle and kind with us.

A lengthy undertaking of this sort can be successful only when it has the backing of a sympathetic administration. This work was started with the helpful support of Professor P. Elias, who was then head of the Department of Electrical Engineering at M.I.T. It was finished with the active encouragement of Professor L. D. Smullin, who is presently head of the Department.

Finally, and most sincerely, we want to acknowledge the perseverance of our families during this effort. Our wives, Blanche S. Woodson and Janet D. Melcher, have been particularly tolerant of the demands of this work.

This book appears in three separately bound, consecutively paged parts that can be used individually or in any combination. Flexibility is ensured by including with each part a complete Table of Contents and Index. In addition, for convenient reference, Parts II and III are supplemented by brief appendices which summarize the relevant material from the preceding chapters. Part I includes Chapters 1 to 6, hence emphasizes lumped-parameter models while developing background in field concepts for further studies.

H. H. Woodson
J. R. Melcher

Cambridge, Massachusetts
January 1968

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Chapter 1

INTRODUCTION

1.0 INTRODUCTION

The human is first of all a mechanical entity who exists in a mechanical environment. The day-by-day habits of man are dictated largely by such considerations as how rapidly he can transport or feed himself. Communication with his environment is geared to such mechanical operations as the time required for his eye to scan a page or the speed with which he can speak or typewrite. Furthermore, his standard of living is very much a function of his ability to augment human muscle for better transportation and for the diverse industrial processes needed in an advanced society.

There are two major conclusions to be drawn from these thoughts. First, the unaided human faculties operate on limited time and size scales. Thus the mechanical effects of electric and magnetic forces on ponderable bodies were observed and recorded by the Greeks as early as 500 B.C., and electricity and magnetism were developed largely as classical sciences in the nineteenth century, on the basis of unaided human observations. Coulomb enunciated his inverse square law from measurements taken with an electrical torsion balance; magnetic forces, as they influenced ponderable objects such as magnetized needles, were the basis of experiments conducted by Oersted and Ampère. These electromechanical experiments constituted the origins of the modern theories of electricity and magnetism. Faraday and Maxwell unified the subjects of electrostatics and magnetostatics into a dynamical theory that predicted phenomena largely beyond the powers of direct human observation. Thus today we recognize that electromagnetic theory encompasses not only the electromechanical effects that first suggested the existence of electric and magnetic fields but also numerous radiation effects, whether they involve radio frequency waves or x-rays. Nonetheless, when man controls these phenomena, detects their existence, and puts them to good use, he most often does so by some type of electromechanical interaction—from the simple act of turning a switch to the remote operation of a computer with a teletypewriter.

The second major conclusion to be drawn from our opening remarks is that man's need for motive power for transportation and industrial processes is satisfied largely by conversion of electric energy to mechanical energy. Energy in electric form is virtually useless, yet the largest and fastest growing segment of our economy is the electric utility industry, whose source of income is the sale of electric energy. This is eloquent testimony to the fact that electric energy can be converted easily into a variety of forms to aid man in his mechanical environment. It is remarkable that the same 60-Hz power line can supply the energy requirements of a rolling mill, a television station, a digital computer, a subway train, and many other systems and devices that provide a fuller and more comfortable life. In the vast majority of these examples electromechanical energy conversion is required because of man's basic need for mechanical assistance.

As long as engineers are concerned with making the electrical sciences serve human needs, they will be involved with electromechanical phenomena.

1.0.1 Scope of Application

Because they serve so many useful functions in everyday situations, *transducers* are the most familiar illustration of applied electromechanical dynamics. These devices are essential to the operation of such diverse equipment as automatic washing machines, electric typewriters, and power circuit breakers in which they translate electrical signals into such useful functions as opening a switch. The switch can be conventional or it can open a circuit carrying 30,000 A while withstanding 400,000 V 2 msec later. The telephone receiver and high-fidelity speaker are familiar transducers; less familiar relatives are the high-power sonar antenna for underwater communication or the high-fidelity shake tables capable of vibrating an entire space vehicle in accordance with a recording of rocket noise.

Electromechanical transducers play an essential role in the automatic control of industrial processes and transportation systems, where the ultimate goal is to control a mechanical variable such as the thickness of a steel sheet or the speed of a train. Of course, a transducer can also be made to translate mechanical motion into an electrical signal. The cartridge of a phonograph pickup is an example in this category, as are such devices as telephone transmitters, microphones, accelerometers, tachometers and dynamic pressure gages.

Not all transducers are constructed to provide mechanical input or output. The (electro)mechanical filter is an example of a signal-processing device that takes advantage of the extremely high Q of mechanical circuits at relatively low frequencies. Filters, delay lines, and logic devices capable of

performing even above 30 MHz are currently the object of research on electromechanical effects found in piezoelectric and piezomagnetic materials.

Primary sources of energy are often found in mechanical form in the kinetic energy of an expanding heated gas and in the potential energy of water at an elevation. Electromechanics has always played a vital role in obtaining large amounts of electric power from primary sources. This is accomplished by using large magnetic field-type devices called *rotating machines*. Today a single generator can produce 1000 MW (at a retail price of 2 cents/kWh this unit produces an income of \$20,000/h), and as electric utility systems grow larger generating units (with attendant problems of an unprecedented nature) will be needed. This need is illustrated by the fact that in 1960 the national peak load in the United States was 138,000 MW, whereas it is expected that in 1980 it will be 493,000 MW, an increase of more than 250 per cent in 20 years.

A large part of this electric power will be used to drive electric motors of immense variety to do a multitude of useful tasks, from moving the hands of an electric clock at a fraction of a watt to operating a steel rolling mill at 20 MW.

Because of our need for great amounts of energy, it is in the national interest to seek ways of producing it more efficiently (to conserve natural resources) and with less costly equipment (to conserve capital). The *magneto-hydrodynamic* generator, which employs an expanding heated gas as the moving conductor, shows some promise of meeting one or both of these objectives. Another possibility is the use of the interaction between charged particles and a flowing, nonconducting gas to achieve *electrohydrodynamic* power generation. Versions of this machine are similar in principle to the *Van de Graaff* generator which is currently producing extremely high voltages (20 million volts) for a variety of purposes, including medical treatment, physical research, and irradiation of various substances.

The efficient and economical conversion of mechanical energy to electrical form is not only of great interest to the rapidly expanding utility industry but is also of extreme importance to the space program, in which sources of electric power must satisfy new engineering requirements imposed by the environment, with obvious limitations on weight and size and with stringent requirements on reliability.

Electromechanical devices provide *power amplification* of signals for purposes similar to those involving electronic amplifiers; for example, in control systems in which large amounts of power (up to about 20 MW) must be produced with high fidelity over a bandwidth from zero to a few Hertz *dc rotating machines* are used. From this the impression is obtained that electromechanical amplifiers function only at low frequencies; but there are

electromechanical devices that provide amplification in the gigacycle-per-second range—*electron beam devices* which, like other physical electronic devices, depend on the small mass of the electron for high-frequency operation.

In current research concerned with controlled thermonuclear fusion the plasma can be regarded for some purposes as a highly conducting gas elevated to such a high temperature that it cannot be contained by solid boundaries. Thus proposed thermonuclear devices attempt to contain the plasma in a magnetic bottle. This illustrates another important application of electromechanical dynamics—the *orientation, levitation, or confinement* of mechanical media. More conventional examples in this category are those that use magnetic or electric fields to levitate the rotor of a gyroscope, to suspend the moving member of an accelerometer, or to position a model in a wind tunnel. Metallurgists employ ac magnetic fields to form a crucible for molten metals that must be free of contamination, and electric fields are proposed for orienting cyrogenic propellants in the zero-gravity environment of space. The use of electric and magnetic fields in shaping malleable metals and solidifying liquids has just begun.

The *propulsion* of vehicles represents still another application of electromechanics. Even when the primary source of energy is a rotating shaft from a reciprocating engine or a turbine, as in a locomotive or ship, the problem of transmitting and controlling the power to the wheels or propeller is simplified by converting the power to electrical form with a generator and installing electric motors to propel the vehicle. An important addition to this class of vehicles would be the electric car in which energy is stored in batteries and the wheels are driven by electric motors. Less familiar electromechanical propulsion schemes are being developed, largely for space applications, which make use of magnetohydrodynamic or electrohydrodynamic acceleration of matter to provide thrust. In this regard the particle accelerators required in high-energy physics research should be recognized as electromechanical devices.

1.0.2 Objectives

It should be apparent from the discussion of the preceding section that electromechanical dynamics covers a broad range of applications, many of which represent highly developed technologies, whereas others are the subject of research or development. In either case a single application could be the subject of an entire book and in many cases books already exist. Our objective here is to lay a cohesive and unified foundation by treating those concepts and techniques that are fundamental to an understanding of a wide range of electromechanical phenomena. As a consequence, we do not dwell at length on any area of application.

With our basic unified approach it is often difficult to distinguish between those aspects of electromechanics that may be considered research in the scientific sense and those that represent engineering applications. For example, there are many practical uses for a magnetohydrodynamic flow meter, yet the type of theoretical model needed in its study is also pertinent to an understanding of the origin of the earth's magnetic field as it is generated by motion of the molten interior of the earth. In fact, a study of magnetohydrodynamics involves models that are germane to an engineering problem such as the levitation of a molten metal, an applied physics problem such as plasma confinement, or a problem of astrophysical interest such as the dynamics of stellar structures.

The subject of electromechanical dynamics, as we approach it in the following chapters, provides a foundation for a range of interests that extends from the purely scientific to engineering applications and from interactions that occur in systems that can be represented by lumped parameters to those that need continuum representations.

The selection of appropriate mathematical models for electromechanical systems is a process that requires the maturity and insight that can result only from experience with electromechanical phenomena. Of course, the model chosen depends on the nature of the system being studied and the accuracy required. We shall not try to develop a formalism for the largely intuitive process of modeling but rather shall study representative systems with a variety of mathematical models to illustrate the principal phenomena that result from electromechanical interactions. In the course of this study the student should develop facility with the basic models and the mathematical tools used in their analysis and should acquire the insight into the interrelations among the physical phenomena that is necessary for him to be able to develop mathematical models on his own.

1.1 ELECTROMAGNETIC THEORY

The mathematical description of the electrical part of any electromechanical system is based on electromagnetic theory. We therefore assume that the reader is familiar with the basic theory and in particular with magnetostatics and electrostatics.

The subject of electromechanics necessarily includes the behavior of electromagnetic fields in the presence of moving media. In this introductory chapter it therefore seems appropriate to review the laws of electricity and magnetism and to include a discussion of those extensions of the theory required to account for the effects of moving media. This review, however, would represent a digression from our main purpose—the study of electromechanical dynamics. Consequently a discussion is presented in Appendix B

for completeness. We can get well into the study of electromechanical dynamics with a few simple extensions of magnetostatic and electrostatic theory. Therefore we cite the electromagnetic equations that form the basis for our study and start to use them immediately. The equations can be accepted as postulates, justified by their relation to ordinary magnetostatic and electrostatic theory and by the fact that they give adequate representation of the electromechanical systems we shall study. As our work progresses from the lumped-parameter models in Chapters 2 to 5 to situations requiring continuum models, the physical significance of the field equations in electromechanical interactions will be more apparent. It is at that point that a meaningful discussion can be made of the most significant effects of moving media on electromagnetic fields, and the reader may find that a study of Appendix B will be most helpful at that time.

1.1.1 Differential Equations

The symbols and units of electromagnetic quantities are defined in Table 1.1. At the outset, we consider two limiting cases of the electromagnetic field equations, which define the dynamics of quasi-static (almost static) magnetic and electric field systems. In spite of the restrictions implied by these limits, our models are adequate for virtually all electromechanical systems of technical importance. A discussion of the quasi-static approximations, which shows how both limiting cases come from the more general electromagnetic theory, is given in Appendix B.

1.1.1a Magnetic Field Systems

The electromagnetic field and source quantities in a magnetic field system are related by the following partial differential equations:

$$\nabla \times \mathbf{H} = \mathbf{J}_f, \quad (1.1.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (1.1.2)$$

$$\nabla \cdot \mathbf{J}_f = 0, \quad (1.1.3)$$

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}), \quad (1.1.4)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad (1.1.5)$$

Thus in our magnetic field system, even with time-varying sources and deforming media, the magnetic field intensity $\mathbf{H}$ and flux density $\mathbf{B}$ are determined as if the system were magnetostatic. Then the electric field intensity $\mathbf{E}$ is found from the resulting flux density by using (1.1.5). This is the origin of the term quasi-static magnetic field system. In addition to these equations, we need constituent relations that describe how the physical

Table 1.1 Symbols and Units of Electromagnetic Quantities

Symbol	Field Variable Name	MKS Rationalized Units
H	Magnetic field intensity	A/m
J_f	Free current density	A/m ²
K_f	Free surface current density	A/m
B	Magnetic flux density	Wb/m ²
M	Magnetization density	A/m
E	Electric field intensity	V/m
D	Electric displacement	C/m ²
ρ_f	Free charge density	C/m ³
σ_f	Free surface charge density	C/m ²
P	Polarization density	C/m ²
F	Force density	N/m ³
μ_0	Permeability of free space	$4\pi \times 10^{-7}$ H/m
ϵ_0	Permittivity of free space	8.854×10^{-12} F/m

properties of the materials affect the field and source quantities. The magnetization density **M** is introduced to account for the effects of magnetizable materials. The most common constitutive law for **M** takes the form

$$\mathbf{M} = \chi_m \mathbf{H}, \quad (1.1.6)$$

where χ_m is the magnetic susceptibility. An alternative way of expressing this relation is to define the permeability $\mu = \mu_0(1 + \chi_m)$, where μ_0 is the permeability of free space

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m}, \quad (1.1.7)$$

in which case it follows from (1.1.4) that the constitutive law of (1.1.6) can also be written as

$$\mathbf{B} = \mu \mathbf{H}. \quad (1.1.8)$$

We shall make considerable use of this simple linear model for magnetizable materials.

Free currents in a stationary material most often arise from conduction induced by the electric field according to Ohm's law:

$$\mathbf{J}_f = \sigma \mathbf{E}, \quad (1.1.9)$$

where σ is the conductivity (mhos/m). A similar constitutive law relates the surface current density **K_f** to the electric field intensity **E_t** tangential to the surface

$$\mathbf{K}_f = \sigma_s \mathbf{E}_t, \quad (1.1.10)$$

where σ_s is the surface conductivity (mhos). These constitutive laws for the

conduction process represent macroscopic models for the migration of charges in materials under the influence of an electric field.

Ideally, quasi-static magnetic field systems are characterized by perfectly conducting ($\sigma \rightarrow \infty$) current loops, in which case static conditions ($\partial/\partial t = 0$) result in zero electric field intensity. All practical conductors (except superconductors) have finite conductivity; consequently, a system is modeled as a magnetic field system when the electrical conductivity σ for a current loop is high enough to cause only small departures from the ideal. Thus in Chapter 2 iron structures with coils of wire wound around them are represented as ideal (electrically lossless) magnetic field systems in which the winding resistance is included as an external resistance in series with the winding terminals.

1.1.1b Electric Field Systems

The electromagnetic field and source quantities in an electric field system are related by the following partial differential equations:

$$\nabla \times \mathbf{E} = 0, \quad (1.1.11)$$

$$\nabla \cdot \mathbf{D} = \rho_f, \quad (1.1.12)$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \quad (1.1.13)$$

$$\nabla \cdot \mathbf{J}_f = - \frac{\partial \rho_f}{\partial t}, \quad (1.1.14)$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}. \quad (1.1.15)$$

Equations 1.1.11 to 1.1.13 describe the fields in an electrostatic system. Hence in our electric field system, even with time-varying sources and geometry, the electric field intensity $\mathbf{E}$ and electric displacement $\mathbf{D}$ are determined as though the system were static. Then the current density $\mathbf{J}_f$ is determined by (1.1.14), which expresses conservation of charge. In turn, the magnetic field intensity $\mathbf{H}$ (if it is of interest) is found from (1.1.15). It is because of the basically electrostatic relationship between the electric field intensity and the free charge density that these equations define the dynamics of a quasi-static electric field system.

Ideally, a quasi-static electric field system is characterized by a set of perfectly conducting ($\sigma \rightarrow \infty$) equipotentials separated by perfectly insulating ($\sigma \rightarrow 0$) dielectrics, in which case static conditions ($\partial/\partial t = 0$) result in no current density $\mathbf{J}_f$, hence no magnetic field intensity $\mathbf{H}$. Of course, real dielectrics have finite conductivity; thus a system is representable as an electric field system when the electrical conductivity is low enough to cause only a small departure from the ideal. In terms of the lumped-parameter representation to be introduced in Chapter 2, an electric field system is modeled as an

ideal circuit consisting of equipotentials separated by perfect insulators with resistances connected externally between terminals to account for the finite conductivity of the dielectric.

In this book the constituent relation for the conduction process usually takes the form of (1.1.9) or (1.1.10). In electric field systems, however, there can be appreciable net charge density, and we must be careful to distinguish between a *net flow of charge*, which occurs in electrically neutral conductors such as metals, and a *flow of net charge*, which occurs in situations such as the drift of negative charge in a vacuum tube. To allow for this differentiation when it is needed a more general form of the conduction constituent relation is used:

$$\mathbf{J}_f = (\rho_{f+}\mu_+ + \rho_{f-}\mu_-)\mathbf{E}, \quad (1.1.16)$$

where ρ_{f+} and ρ_{f-} are the densities of the two species of moving charges and μ_+ and μ_- are the respective mobilities in the field intensity $\mathbf{E}$. When the charge densities and mobilities are constants, (1.1.16) reduces to (1.1.9). In some electric field systems ρ_{f+} and ρ_{f-} are not constant, and (1.1.16) allows us to include the variable charge densities in our conduction model. As questions appear in this regard, it will be helpful to refer to Sections B.1.2 and B.3.3.

To account for the polarization density $\mathbf{P}$ of a dielectric material, we most often use the linear relation

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E}, \quad (1.1.17)$$

where ϵ_0 is the permittivity of free space

$$\epsilon_0 = 8.854 \times 10^{-12} \text{ F/m} \quad (1.1.18)$$

and χ_e is the electric susceptibility. In terms of the material permittivity, $\epsilon = \epsilon_0(1 + \chi_e)$ (1.1.17) can also be written

$$\mathbf{D} = \epsilon\mathbf{E}, \quad (1.1.19)$$

where (1.1.13) has been used.

1.1.2 Integral Equations

It is often necessary to have the electromagnetic equations in integral form; for example, boundary conditions are found from integral equations and terminal quantities—voltage and current—are found by integrating field quantities.

In stationary systems the contours, surfaces, and volumes are all fixed in space and the transition from differential to integral equations is simply a matter of using the appropriate integral theorems. In electromechanical dynamics we need integral equations for contours, surfaces, and volumes

that are deforming, and the resulting integral equations are different from those found in stationary systems. The formalism of integrating differential equations in the presence of motion is presented in Section B.4. The results are presented here essentially as postulates.

1.1.2a Magnetic Field Systems

The integral forms of (1.1.1) to (1.1.3) and (1.1.5) are

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot \mathbf{n} \, da, \quad (1.1.20)$$

$$\oint_S \mathbf{B} \cdot \mathbf{n} \, da = 0, \quad (1.1.21)$$

$$\oint_S \mathbf{J}_f \cdot \mathbf{n} \, da = 0, \quad (1.1.22)$$

$$\oint_C \mathbf{E}' \cdot d\mathbf{l} = - \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da. \quad (1.1.23)$$

The contours C , surfaces S , and unit normal vectors $\mathbf{n}$ are defined in the conventional manner, as shown in Fig. 1.1.1. The surfaces of integration S

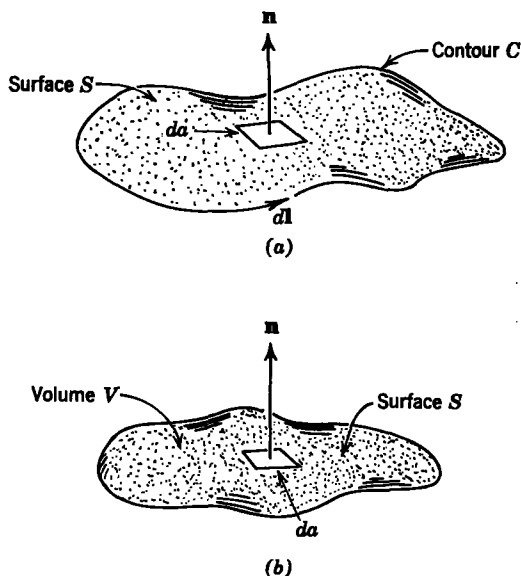


Fig. 1.1.1 (a) Surface S enclosed by the contour C , showing the right-handed relationship between the normal vector $\mathbf{n}$ and the line element $d\mathbf{l}$; (b) surface S enclosing a volume V . The normal vector $\mathbf{n}$ is directed outward, as shown.

for (1.1.21) and (1.1.22) enclose a volume V , whereas those of (1.1.20) and (1.1.23) are enclosed by a contour C .

Equations 1.1.20 to 1.1.23 are valid even when the contours and surfaces are deforming, as demonstrated in Appendix B. Note that in (1.1.23) the electric field intensity is written as $\mathbf{E}'$, and it is this value that would be measured by an observer attached to the deforming contour at the point in question. As demonstrated in Section B.4.1, when $\mathbf{E}' = \mathbf{E} \times (\mathbf{v} \times \mathbf{B})$, where $\mathbf{v}$ is the local velocity of the contour, (1.1.23) results from (1.1.5). More is said about the relation between quantities measured by observers in relative motion in Chapter 6.

In describing magnetic field systems, in addition to (1.1.20) to (1.1.23), we need constituent relations such as (1.1.8) and (1.1.9). We must keep in mind that these constituent relations are defined for stationary media. When there is motion, these equations still hold, but only for an observer moving with the medium. Thus we know that a perfect conductor can support no electric field intensity $\mathbf{E}'$. When the contour of (1.1.23) is fixed to a perfect conductor, the contribution to the contour integral from that portion in the conductor is zero, whether the conductor is moving or not. This is because $\mathbf{E}'$ is the quantity measured by an observer moving with the contour (conductor).

1.1.2b Electric Field Systems

The integral forms of (1.1.11) to (1.1.15) are

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0, \quad (1.1.24)$$

$$\oint_S \mathbf{D} \cdot \mathbf{n} \, da = \int_V \rho_f \, dV, \quad (1.1.25)$$

$$\oint_S \mathbf{J}'_f \cdot \mathbf{n} \, da = -\frac{d}{dt} \int_V \rho_f \, dV, \quad (1.1.26)$$

$$\oint_C \mathbf{H}' \cdot d\mathbf{l} = \int_S \mathbf{J}'_f \cdot \mathbf{n} \, da + \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{n} \, da. \quad (1.1.27)$$

These equations are valid for moving and deforming contours C , surfaces S , and volumes V (see Fig. 1.1.1).

Equations 1.1.24 and 1.1.25 are the same as those used to find $\mathbf{E}$ and $\mathbf{D}$ in an electrostatics problem. The current density and magnetic field intensity have been written in (1.1.26) and (1.1.27) as $\mathbf{J}'_f$ and $\mathbf{H}'$ to indicate that they are the values that would be measured by an observer moving with the contour or surface at the point in question. It is shown in Section B.4.2 that

(1.1.26) and (1.1.27) result from integrating (1.1.14) and (1.1.15) when $\mathbf{J}'_f = \mathbf{J}_f - \rho_f \mathbf{v}$ and $\mathbf{H}' = \mathbf{H} - \mathbf{v} \times \mathbf{D}$, where $\mathbf{v}$ is the local velocity of the contour or surface.

1.1.3 Electromagnetic Forces

The force experienced by a test charge q moving with velocity $\mathbf{v}$ is

$$\mathbf{f} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}. \quad (1.1.28)$$

This is referred to as the Lorentz force and provides a definition of the fields $\mathbf{E}$ and $\mathbf{B}$. For this case of a single moving charge the quantity $q\mathbf{v}$ constitutes a current. Hence the first term in (1.1.28) is the force on a static charge, whereas the second is the force on a current.

In a continuum theory in which we are concerned with a charge density ρ_f and a current density $\mathbf{J}_f$, forces are stated in terms of a force density

$$\mathbf{F} = \rho_f \mathbf{E} + \mathbf{J}_f \times \mathbf{B}. \quad (1.1.29)$$

Free charge and free current densities are used in (1.1.29) to make it clear that this expression does not account for forces due to polarization and magnetization. The terms in (1.1.29) provide a continuum representation of the terms in (1.1.28). The averaging process required to relate the force density of (1.1.29) to the Lorentz force is discussed in Sections B.1.1 and B.1.3. For our present purposes we accept these relations as equivalent and reserve discussion of the conditions under which this assumption is valid for Chapter 8.

In the class of problems undertaken in this book one or the other of the force densities in (1.1.29) is negligible. Hence in the magnetic field systems to be considered the force density is

$$\mathbf{F} = \mathbf{J}_f \times \mathbf{B}, \quad (1.1.30)$$

whereas in the electric field systems

$$\mathbf{F} = \rho_f \mathbf{E}. \quad (1.1.31)$$

In any particular example the validity of these approximations can be tested after the analysis has been completed by evaluating the force that has been ignored and comparing it with the force used in the model.

1.2 DISCUSSION

The equations summarized in Table 1.2 are those needed to describe the electrical side of electromechanical dynamics as presented here. We find that they are of far-reaching physical significance. Nonetheless, they are approximate and their regions of validity should be understood. Furthermore, their

Table 1.2 Summary of Quasi-Static Electromagnetic Equations

	Differential Equations		Integral Equations	
Magnetic field system	$\nabla \times \mathbf{H} = \mathbf{J}_f$	(1.1.1)	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot \mathbf{n} \, da$	(1.1.20)
	$\nabla \cdot \mathbf{B} = 0$	(1.1.2)	$\oint_S \mathbf{B} \cdot \mathbf{n} \, da = 0$	(1.1.21)
	$\nabla \cdot \mathbf{J}_f = 0$	(1.1.3)	$\oint_S \mathbf{J}_f \cdot \mathbf{n} \, da = 0$	(1.1.22)
	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	(1.1.5)	$\oint_C \mathbf{E}' \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da$	(1.1.23)
			where $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$	
Electric field system	$\nabla \times \mathbf{E} = 0$	(1.1.11)	$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	(1.1.24)
	$\nabla \cdot \mathbf{D} = \rho_f$	(1.1.12)	$\oint_S \mathbf{D} \cdot \mathbf{n} \, da = \int_V \rho_f \, dV$	(1.1.25)
	$\nabla \cdot \mathbf{J}_f = -\frac{\partial \rho_f}{\partial t}$	(1.1.14)	$\oint_S \mathbf{J}'_f \cdot \mathbf{n} \, da = -\frac{d}{dt} \int_V \rho_f \, dV$	(1.1.26)
	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$	(1.1.15)	$\oint_C \mathbf{H}' \cdot d\mathbf{l} = \int_S \mathbf{J}'_f \cdot \mathbf{n} \, da + \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{n} \, da$	(1.1.27)
			where $\mathbf{J}'_f = \mathbf{J}_f - \rho_f \mathbf{v}$ $\mathbf{H}' = \mathbf{H} - \mathbf{v} \times \mathbf{D}$	

relation to more general electromagnetic theory should also be known. Both topics are discussed in Appendix B. A study of that material may be more appropriate as questions are raised in the course of the developments to follow.

With the equations in Table 1.2 accepted on a postulational basis, we can—and should—proceed forthwith to study electromechanical dynamics.

Chapter 2

LUMPED ELECTROMECHANICAL ELEMENTS

2.0 INTRODUCTION

The purpose of this chapter is to present the techniques of making mathematical models (writing differential equations) for lumped-parameter electromechanical systems. In the context used here lumped-parameter systems are defined as follows: the electromagnetic fields are quasi-static and electrical terminal properties can be described as functions of a finite number of electrical variables. Also, the mechanical effects can be described by a finite number of mechanical variables. Thus the general feature of lumped-parameter electromechanical systems is that field equations can be integrated throughout space to obtain ordinary differential equations.

Electrical parts of the systems are treated by circuit theory generalized to include the effects of electromechanical coupling; the mechanical parts of the systems are treated by the techniques of rigid body mechanics with electromechanical forces included.

The approach followed here is best illustrated by considering the block diagram in Fig. 2.0.1 in which an electromechanical system is separated for analytical purposes into a purely electrical part, a purely mechanical part, and a coupling part. The equations that describe the electrical part of the system are based on Kirchhoff's laws; the equations for the mechanical part of the system are obtained from Newton's laws and the continuity of space. Both sets of equations contain electromechanical coupling terms that arise from the interconnection of the coupling system.

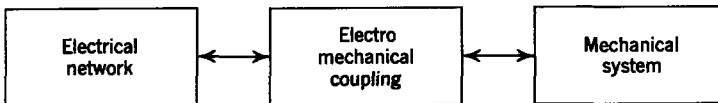


Fig. 2.0.1 An electromechanical system.

In what follows we review concepts of circuit theory and derive lumped parameters in a general way to include electromechanical coupling terms. We then review the concepts of rigid-body mechanics. Electromechanical coupling is discussed in Chapter 3.

2.1 CIRCUIT THEORY

The mathematical description of a circuit essentially involves two steps. First, we must be able to describe mathematically the physical properties of each element in the circuit in order to produce expressions for the terminal properties of the elements. Second, we must combine the equations for the elements in a manner prescribed by the interconnections of the elements. This step is performed by using Kirchhoff's laws and the topology of the circuit.* Thus we need only to generalize the description of circuit elements to include the effects of electromechanical interactions.

Conventional circuit theory is the special case of stationary systems in which quasi-static electromagnetic field theory applies. All the concepts of circuit theory can be derived from field theory†; for example, Kirchhoff's current law is derived from the conservation of charge. When we postulate a node that is an interconnection of wires at which no charge can accumulate, the conservation of charge [see (1.1.22) or (1.1.26) with $\rho_f = 0$] becomes

$$\oint_S \mathbf{J}_f \cdot \mathbf{n} \, da = 0, \quad (2.1.1)$$

where the surface S encloses the node. Because current is restricted to the wires, (2.1.1) yields Kirchhoff's current law

$$\sum_k i_k = 0, \quad (2.1.2)$$

where i_k is the current flowing away from the node on the k th wire.

Kirchhoff's voltage law is obtained by recognizing that a voltage is uniquely defined only in a region in which the time rate of change of magnetic flux density is negligible. Thus either (1.1.23) or (1.1.24) becomes

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0. \quad (2.1.3)$$

This leads to the Kirchhoff voltage equation which requires that the sum of the voltage drops around a closed loop (contour C) be zero,

$$\sum_k v_k = 0, \quad (2.1.4)$$

* E. A. Guillemin, *Introductory Circuit Theory*, Wiley, New York, 1953, Chapters 2 and 4.

† Electrical Engineering Staff, M.I.T., *Electric Circuits*, Technology Press and Wiley, New York, 1943, Chapters 1 and 2.

where v_k is the voltage drop across the k th element in the loop taken in the direction of summation.*

In conventional circuit theory there are three basic types of passive elements: (a) resistances that dissipate electric energy as heat; (b) inductances that store magnetic energy; and (c) capacitances that store electric energy. It is a fact of life that electromechanical coupling of practical significance occurs in elements with appreciable electric or magnetic energy storage. Consequently, we shall consider electromechanical effects in circuit elements that are generalizations of the inductances and capacitances of circuit theory. To be sure, our systems have resistances, but they are treated as purely electrical circuit elements and considered as external to the coupling network.

We proceed now to generalize the concepts of inductance and capacitance to include electromechanical effects. As stated before, we wish to obtain terminal equations suitable for inclusion in a Kirchhoff-law description of a circuit.

2.1.1 Generalized Inductance

From a field point of view an inductor is a quasi-static magnetic field system, as defined in Section 1.1.1*a*. Thus we start with the field description of a quasi-static magnetic field system and derive the terminal characteristics when parts of the system are in motion.

First it is essential to recognize that in an ideal, lossless magnetic field system there is a perfectly conducting path between the two terminals of each terminal pair, as illustrated schematically in Fig. 2.1.1. We assume that the

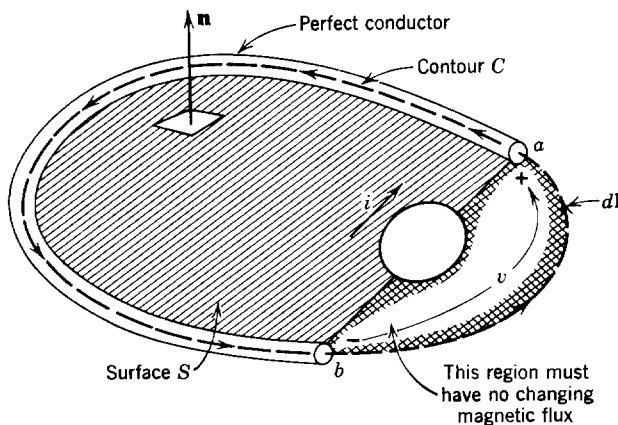


Fig. 2.1.1 Configuration for defining terminal voltage.

* For a discussion of the definition and use of the concept of voltage see Section B.1.4.

terminal pair is excited by the current source i and that the terminal pair is in a region of space in which the time rate of change of magnetic flux density is negligible. This restriction is necessary if we are to be able to describe a terminal voltage unambiguously. The perfect conductor that connects the two terminals is often wound into a coil and the coil may encircle an iron core. The drawing in Fig. 2.1.1 is simplified to illustrate the principles involved.

We must include the possibility that the perfect conductor in Fig. 2.1.1 is moving. We define a contour C that passes through and is fixed to the perfect conductor. That portion of the contour which goes from b to a outside the perfect conductor is fixed and in a region of negligible magnetic flux density. The *terminal voltage* v (see Fig. 2.1.1) is defined in the usual way* as

$$v = - \int_b^a \mathbf{E} \cdot d\mathbf{l}; \quad (2.1.5)$$

it is understood that this line integral is evaluated along the path from b to a that is external to the perfect conductor. We now consider the line integral

$$\oint_C \mathbf{E}' \cdot d\mathbf{l}$$

around the contour of Fig. 2.1.1. The electric field intensity $\mathbf{E}'$ is the field that an observer will measure when he is fixed with respect to the contour. The contour is fixed to the perfect conductor, and, by definition, a perfect conductor can support no electric field.† Consequently, we reach the conclusion that

$$\oint_C \mathbf{E}' \cdot d\mathbf{l} = \int_b^a \mathbf{E} \cdot d\mathbf{l} = -v. \quad (2.1.6)$$

Thus Faraday's law, (1.1.23) of Table 1.2, yields the terminal voltage

$$v = \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da, \quad (2.1.7)$$

where the surface S is enclosed by the contour C in Fig. 2.1.1 and the positive direction of the normal vector $\mathbf{n}$ is defined by the usual right-hand rule, as shown.

Equation 2.1.7 indicates why the external path from b to a in Fig. 2.1.1 must be in a region of negligible time rate of change of magnetic flux density. If it is not, the terminal voltage will depend on the location of the external

* See Section B.1.4.

† A more complete discussion of conductors in motion is given in Chapter 6.

contour and will not be defined unambiguously. For convenience we define the *flux linkage* λ of the circuit as

$$\lambda = \int_S \mathbf{B} \cdot \mathbf{n} \, da \quad (2.1.8)$$

and rewrite (2.1.7) as

$$v = \frac{d\lambda}{dt}. \quad (2.1.9)$$

In a quasi-static magnetic-field system the magnetic flux density is determined by (1.1.20) to (1.1.22) of Table 1.2 and a constitutive law,

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot \mathbf{n} \, da, \quad (1.1.20)$$

$$\oint_S \mathbf{B} \cdot \mathbf{n} \, da = 0, \quad (1.1.21)$$

$$\oint_S \mathbf{J}_f \cdot \mathbf{n} \, da = 0, \quad (1.1.22)$$

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}). \quad (1.1.4)$$

(The differential forms of these equations can also be employed.) In the solution of any problem the usual procedure is to use (1.1.22) first to relate the terminal current to current density in the system and then (1.1.20), (1.1.21), and (1.1.4) to solve for the flux density $\mathbf{B}$. The resulting flux density is a function of terminal current, material properties (1.1.4), and system geometry. The use of this result in (2.1.8) shows that the flux linkage λ is also a function only of terminal current, material properties, and system geometry.

We are interested in evaluating terminal voltage by using (2.1.9); thus we are interested in time variations of flux linkage λ . If we assume that the system geometry is fixed, except for one movable part whose position can be described instantaneously by a displacement x with respect to a fixed reference, and we further assume that $\mathbf{M}$ is a function of field quantities alone (and therefore a function of current), we can write

$$\lambda = \lambda(i, x). \quad (2.1.10)$$

In this expression we have indicated explicit functional dependence only on those variables (i and x) that may be functions of time.

We can now use (2.1.10) in (2.1.9) and expand the time derivative to obtain

$$v = \frac{d\lambda}{dt} = \frac{\partial \lambda}{\partial i} \frac{di}{dt} + \frac{\partial \lambda}{\partial x} \frac{dx}{dt}. \quad (2.1.11)$$

This expression illustrates some general terminal properties of magnetic

field systems. We note that the first term on the right of (2.1.11) is proportional to di/dt and is the result of changing current. This term can exist when the system is mechanically stationary and is often referred to as a *transformer voltage*. The second term on the right of (2.1.11) is proportional to dx/dt , which is a mechanical speed. This term exists only when there is relative motion in the system and is conventionally referred to as a *speed voltage*. No matter how many terminal pairs or mechanical displacements a system may have, the voltage at each terminal pair will have terms of the two types contained in (2.1.11).

If we now restrict our system (with one electrical terminal pair and one mechanical displacement) to materials whose magnetization densities are linear with field quantities, we have an *electrically linear* system whose flux linkage can be expressed in terms of an inductance L as

$$\lambda = L(x)i. \quad (2.1.12)$$

This system is electrically linear because the flux linkage is a linear function of current. The variation of flux linkage with geometry, as indicated in general in (2.1.10), is included in (2.1.12) in the function $L(x)$. When the flux linkage is written in the form of (2.1.12), the terminal voltage becomes

$$v = L(x) \frac{di}{dt} + i \frac{dL}{dx} \frac{dx}{dt}. \quad (2.1.13)$$

Once again the first term on the right is the transformer voltage and the second term is the speed voltage.

In the special case of fixed geometry (x constant) the second term on the right of (2.1.13) goes to zero and we obtain

$$v = L \frac{di}{dt}, \quad (2.1.14)$$

which is the terminal relation of an inductance that is conventional in linear circuit theory.

Electromechanical systems often have more than one electrical terminal pair and more than one mechanical displacement. For such a situation the process described is still valid. To illustrate this generalization assume a quasi-static magnetic field system with N electrical terminal pairs and M mechanical variables that are functions of time. There are N electrical currents,

$$i_1, i_2, \dots, i_N,$$

and M mechanical displacements,

$$x_1, x_2, \dots, x_M.$$

Because this is a quasi-static magnetic field system, there is a perfectly conducting path between the two terminals of each terminal pair, as illustrated in Fig. 2.1.1. Thus the voltage for any terminal pair is determined by using the contour for that terminal pair with (2.1.6). Then the flux linkage for any terminal pair (say the k th) is given by (2.1.8):

$$\lambda_k = \int_{S_k} \mathbf{B} \cdot \mathbf{n} \, da, \quad (2.1.15)$$

where S_k is the surface enclosed by the contour used with (2.1.6) to evaluate voltage v_k at the k th terminal pair. The voltage v_k is then given by (2.1.7) as

$$v_k = \frac{d\lambda_k}{dt}. \quad (2.1.16)$$

The fields in this more general situation are again described by (1.1.20) to (1.1.22) and (1.1.4). Consequently, the generalization of (2.1.10) is

$$\begin{aligned} \lambda_k &= \lambda_k(i_1, i_2, \dots, i_N; x_1, x_2, \dots, x_M), \\ k &= 1, 2, \dots, N. \end{aligned} \quad (2.1.17)$$

We can now write the generalization of (2.1.11) by using (2.1.17) in (2.1.16) to obtain

$$\begin{aligned} v_k &= \sum_{j=1}^N \frac{\partial \lambda_k}{\partial i_j} \frac{di_j}{dt} + \sum_{j=1}^M \frac{\partial \lambda_k}{\partial x_j} \frac{dx_j}{dt}, \\ k &= 1, 2, \dots, N. \end{aligned} \quad (2.1.18)$$

Once again the terms in the first summation are referred to as transformer voltages and the terms in the second summation are referred to as speed voltages.

The preceding development has indicated the formalism by which we obtain lumped-parameter descriptions of quasi-static magnetic field systems. We have treated ideal lossless systems. In real systems losses are primarily resistive losses in wires and losses in magnetic materials.* Even though they may be quite important in system design and operation (efficiency, thermal limitations, etc.), they usually have little effect on the electromechanical interactions. Consequently, the effects of losses are accounted for by electrical resistances external to the lossless electromechanical coupling system.

* Losses in magnetic materials result from hysteresis and eddy currents. For a discussion of these effects and their mathematical models see Electrical Engineering Staff, M.I.T., *Magnetic Circuits and Transformers*, Technology Press and Wiley, New York, 1943, Chapters 5, 6, and 13. Eddy currents are discussed in Chapter 7 of this book.

Example 2.1.1. As an example of the calculation of lumped parameters, consider the magnetic field system of Fig. 2.1.2. It consists of a fixed structure made of highly permeable magnetic material with an excitation winding of N turns. A movable plunger, also made of highly permeable magnetic material, is constrained by a nonmagnetic sleeve to move in the x -direction. This is the basic configuration used for tripping circuit breakers, operating valves, and other applications in which a relatively large force is applied to a member that moves a relatively small distance.*

We wish to calculate the flux linkage λ at the electrical terminal pair (as a function of current i and displacement x) and the terminal voltage v for specified time variation of i and x .

To make the analysis of the system of Fig. 2.1.2 more tractable but still quite accurate it is conventional to make the following assumptions:

1. The permeability of the magnetic material is high enough to be assumed infinite.
2. The air-gap lengths g and x are assumed small compared with transverse dimensions $g \ll w$, $x \ll 2w$, so that fringing at the gap edges can be ignored.
3. Leakage flux is assumed negligible; that is, the only appreciable flux passes through the magnetic material except for gaps g and x .

Needed to solve this problem are the quasi-static magnetic field equations (1.1.20) through (1.1.22) and (1.1.4).

We first assume that the terminal current is i . Then by using (1.1.22) we establish that the current at each point along the winding is i . Next, we recognize that the specification of infinitely permeable magnetic material implies that we can write (1.1.4) as

$$\mathbf{B} = \mu \mathbf{H}$$

with $\mu \rightarrow \infty$. Thus with finite flux density $\mathbf{B}$ the field intensity $\mathbf{H}$ is zero inside the magnetic

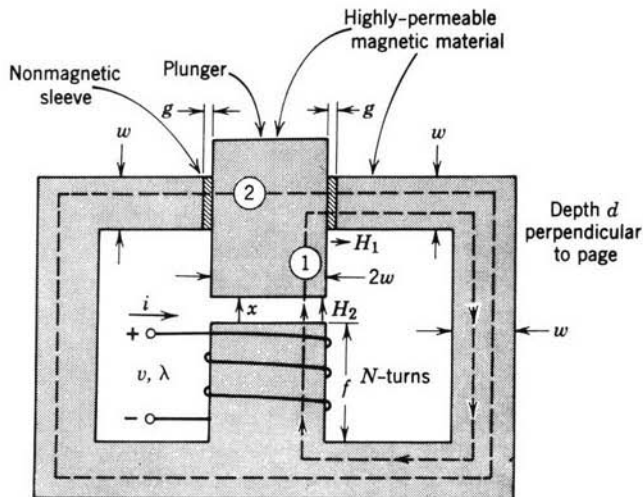


Fig. 2.1.2 A magnetic field system.

* A. E. Knowlton, ed., *Standard Handbook for Electrical Engineers*, 9th ed. McGraw-Hill, New York, 1957, Section 5-39 through 5-52.

material. Thus the only nonzero $\mathbf{H}$ occurs in the air gaps g and x , where $\mathbf{M} = 0$, and (1.1.4) becomes

$$\mathbf{B} = \mu_0 \mathbf{H}.$$

The use of (1.1.20) with contour (2) in Fig. 2.1.2 shows that the field intensities in the two gaps g are equal in magnitude and opposite in direction. This is expected from the symmetry of the system. Denoting the magnitude of the field intensity in the gaps g as H_1 and the field intensity in gap x by H_2 , we can integrate (1.1.20) around contour (1) in Fig. 2.1.2 to obtain

$$H_1 g + H_2 x = Ni, \quad (a)$$

where H_2 is taken positive upward and H_1 is taken positive to the right. We now use (1.1.21) with a surface that encloses the plunger and passes through the gaps to obtain

$$\mu_0 H_1 (2wd) - \mu_0 H_2 (2wd) = 0. \quad (b)$$

We combine (a) and (b) to obtain

$$H_1 = H_2 = \frac{Ni}{g + x}.$$

The flux through the center leg of the core is simply the flux crossing the air gap x and is

$$\phi = \mu_0 H_2 (2wd) = \frac{2wd\mu_0 Ni}{g + x}.$$

In the absence of leakage flux this same flux links the N -turn winding N times; that is, when we evaluate

$$\int_S \mathbf{B} \cdot \mathbf{n} \, da$$

over a surface enclosed by the wire of the N -turn winding, we obtain the flux linkage λ as

$$\lambda = N\phi = \frac{2wd\mu_0 N^2 i}{g + x}. \quad (c)$$

Note that because λ is a linear function of i the system is electrically linear and we can write (c) as

$$\lambda = L(x)i, \quad (d)$$

where

$$L(x) = \frac{2wd\mu_0 N^2}{g + x}. \quad (e)$$

When we assume that the current i and displacement x are specified functions of time, we can use (d) with (2.1.13) to evaluate the terminal voltage as

$$v = \frac{2wd\mu_0 N^2}{g + x} \frac{di}{dt} - \frac{2wd\mu_0 N^2 i}{(g + x)^2} \frac{dx}{dt}.$$

The first term is the transformer voltage that will exist if x is fixed and i is varying. The second term is the speed voltage that will exist if i is constant and x is varying.

Example 2.1.2. As a second example, consider the system in Fig. 2.1.3 which has two electrical terminal pairs and the mechanical displacement is rotational. This system consists

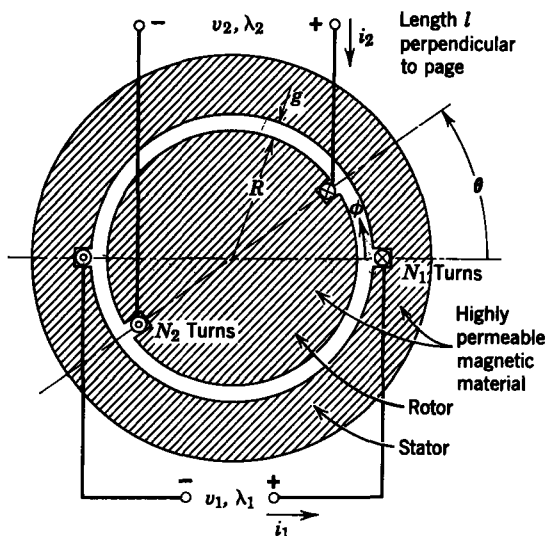


Fig. 2.1.3 Doubly excited magnetic field system.

of a fixed annular section of highly permeable magnetic material that is concentric with a cylindrical piece of the same material of the same axial length. Mounted in axial slots in the material are coils labeled with the numbers of turns and current directions. The angular position of the inner structure (rotor) relative to the outer structure (stator) is indicated by an angle θ which can vary with time. Current is fed to the coil on the rotor through sliding contacts (brushes that make contact with slip rings).

The system in Fig. 2.1.3 represents the basic method of construction of many rotating machines. In our solution we discuss how this configuration is used with some variations to achieve the lumped parameters desired for rotating-machine operation.

We wish to calculate the two flux linkages λ_1 and λ_2 as functions of the currents i_1 and i_2 and the angular displacement θ . The voltages at the two terminal pairs are also to be found, assuming that i_1 , i_2 , and θ are specified functions of time.

Electromechanical systems of the type illustrated in Fig. 2.1.3 are normally constructed with relative dimensions and materials that allow reasonably accurate calculation of lumped parameters when the following assumptions are made:

1. The permeability of the magnetic material is high enough to be assumed infinite.
2. The radial air-gap length g is small enough compared with the radius R and axial length l to allow the neglect of fringing fields at the ends and of radial variation of magnetic field intensity in the air gap.
3. The slots containing the windings are small enough both radially and circumferentially to perturb the fields a negligible amount; that is, the coils are considered to be infinitely thin.

Equations 1.1.20 and 1.1.21 are used to write the radial fields in the air gap in terms of the angular variable ϕ defined in Fig. 2.1.3. If we define the magnetic field as positive when

directed radially outward and consider $0 < \theta < \pi$,

$$H_r = \frac{N_1 i_1 - N_2 i_2}{2g}, \quad \text{for } 0 < \phi < \theta,$$

$$H_r = \frac{N_1 i_1 + N_2 i_2}{2g}, \quad \text{for } \theta < \phi < \pi,$$

$$H_r = -\frac{N_1 i_1 - N_2 i_2}{2g}, \quad \text{for } \pi < \phi < \pi + \theta,$$

$$H_r = -\frac{N_1 i_1 + N_2 i_2}{2g}, \quad \text{for } \pi + \theta < \phi < 2\pi.$$

The flux linkages with the two windings can be found from the integrals

$$\lambda_1 = \int_0^\pi N_1 \mu_0 H_r l R d\phi,$$

$$\lambda_2 = \int_\theta^{\pi+\theta} N_2 \mu_0 H_r l R d\phi.$$

Evaluation of these integrals yields

$$\lambda_1 = L_1 i_1 + L_m i_2,$$

$$\lambda_2 = L_m i_1 + L_2 i_2,$$

where

$$L_1 = N_1^2 L_0, \quad L_2 = N_2^2 L_0,$$

$$L_m = L_0 N_1 N_2 \left(1 - \frac{2\theta}{\pi}\right), \quad \text{for } 0 < \theta < \pi,$$

$$L_0 = \frac{\mu_0 l R \pi}{2g}.$$

Similar arguments show that for $-\pi < \theta < 0$ the terminal relations have the same form except that

$$L_m = L_0 N_1 N_2 \left(1 + \frac{2\theta}{\pi}\right)$$

Note that only the mutual inductance L_m is a function of angular displacement θ because the geometry seen by each coil individually does not change with θ ; thus the self-inductances are constants. The mutual inductance is sketched as a function of θ in Fig. 2.1.4.

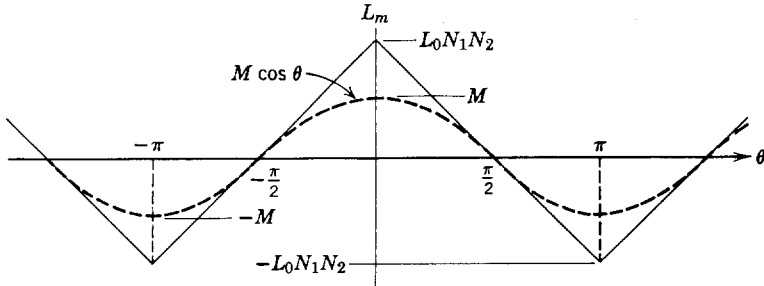


Fig. 2.1.4 Mutual inductance L_m as a function of θ .

In the design of rotating machines, especially for operation on alternating currents, it is desirable to have a system similar to that in Fig. 2.1.3 but to modify it in such a way that the mutual inductance varies cosinusoidally with θ ($L_m = M \cos \theta$). This is accomplished by putting additional slots and windings at different positions around the periphery of both members. By using a proper distribution of slots and numbers of turns the dependence of L_m can be made the cosinusoidal function shown by the dashed curve in Fig. 2.1.4. In many later examples we assume that this design process has been followed.

When the two currents i_1 and i_2 and the angular position θ are functions of time and the mutual inductance is expressed as $L_m = M \cos \theta$, we can write the terminal voltages as

$$v_1 = \frac{d\lambda_1}{dt} = L_1 \frac{di_1}{dt} + M \cos \theta \frac{di_2}{dt} - i_2 M \sin \theta \frac{d\theta}{dt},$$

$$v_2 = \frac{d\lambda_2}{dt} = L_2 \frac{di_2}{dt} + M \cos \theta \frac{di_1}{dt} - i_1 M \sin \theta \frac{d\theta}{dt}.$$

Note that the first term in each expression is a derivative with a constant coefficient, whereas the last two terms are derivatives with time-varying coefficients.

So far we have described lumped-parameter magnetic field systems by expressing the flux linkages as functions of the currents and displacements. Although this is a natural form for deriving lumped parameters, we shall find it convenient later to express lumped parameters in different forms. For example, we often use the lumped parameters in a form that expresses the current as a function of flux linkages and displacements. Assuming that the flux linkages are known as functions of currents and displacements, we are merely required to solve a set of simultaneous algebraic equations.

Consider the situation of an electrically linear system with one electrical terminal pair and one displacement. The flux linkage for this system has been described in (2.1.12) and it is a simple matter to solve this expression for i to obtain

$$i = \frac{\lambda}{L(x)}. \quad (2.1.19)$$

For an electrically linear system with two electrical terminal pairs we can write the expressions for the two flux linkages as

$$\lambda_1 = L_1 i_1 + L_m i_2, \quad (2.1.20)$$

$$\lambda_2 = L_m i_1 + L_2 i_2. \quad (2.1.21)$$

It is assumed that the three inductances are functions of the mechanical displacements. Solutions of these equations for i_1 and i_2 are

$$i_1 = \frac{L_2}{L_1 L_2 - L_m^2} \lambda_1 - \frac{L_m}{L_1 L_2 - L_m^2} \lambda_2, \quad (2.1.22)$$

$$i_2 = -\frac{L_m}{L_1 L_2 - L_m^2} \lambda_1 + \frac{L_1}{L_1 L_2 - L_m^2} \lambda_2. \quad (2.1.23)$$

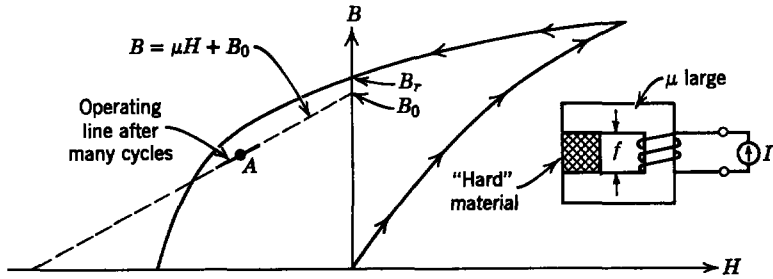


Fig. 2.1.5 Hard magnetic material magnetized and then demagnetized by the current I .

For the more general (and possibly not electrically linear) case for which flux linkages are expressed by (2.1.17) we can, at least in theory, solve these N simultaneous equations to obtain the general functional form

$$\begin{aligned} i_k &= i_k(\lambda_1, \lambda_2, \dots, \lambda_N; x_1, x_2, \dots, x_M), \\ k &= 1, 2, \dots, N. \end{aligned} \quad (2.1.24)$$

Although we have generalized the concept of inductance to the point at which we can describe an electrically nonlinear system, the full, nonlinear expressions are rarely used in the analysis of electromechanical devices. This is so because devices that involve mechanical motion normally have air gaps (where the material is electrically linear) and the fields in the air gaps predominate (see the last two examples). It is worthwhile, however, to understand the origins of lumped parameters and how to describe nonlinear systems because there are some devices in which nonlinearities predominate.

In our examples of magnetic field systems we have considered only a "soft" magnetic material with a flux density (or more accurately a magnetization density $\mathbf{M}$) that is ideally a linear function of a magnetic field intensity. A different type of magnetic material often used in electromechanical systems is the "hard" or permanent magnet material. In these materials the flux density is, in general, not given by $\mathbf{B} = \mu\mathbf{H}$. Figure 2.1.5 shows the type of curve that would result if a hard magnetic material were subjected to a field intensity H by means of an extremely large current applied as shown.* When the current (i.e., H) is removed, there is a residual flux density B_r . Then, if the sample is removed from the magnetic circuit and subjected to other magnetic fields (of limited strength), the B - H curve settles down to operate about some point such as A in Fig. 2.1.5. To a good approximation we can often model this relation by the straight line shown in Fig. 2.1.5.

$$B = \mu H + B_0. \quad (2.1.25)$$

* See, for example, Electrical Engineering Staff, M.I.T., *op. cit.*, Chapter 4.

The calculation of terminal relations is now the same as described, except that in the permanent magnet (2.1.25) is used rather than (1.1.8).

Example 2.1.3. The system shown in Fig. 2.1.2 is excited electrically by removing the N turns and placing a permanent magnet of length f in the magnetic circuit. Then the analysis of Example 2.1.1 is altered by the integration of (1.1.20) around the contour 1. If, in the section of length f , the material is characterized by (2.1.25), (a) of Example 2.1.1 is replaced by

$$H_1 g + H_2 x + \frac{fB}{\mu} = \frac{fB_0}{\mu}, \quad (a)$$

where B is the flux density in the magnet. Equation 1.1.21, however, again shows that B is the same in the magnet as it is in each of the air gaps. Hence

$$B = \mu_0 H_1 = \mu_0 H_2 \quad (b)$$

and (a) shows that

$$B = \frac{fB_0}{(\mu/\mu_0)(g+x) + f}. \quad (c)$$

Note from (a) that we can replace the permanent magnet with an equivalent current source I driving N turns, as shown in Fig. 2.1.2, but in which

$$i = I = \frac{fB_0}{N\mu},$$

and in the magnetic circuit there is a magnetic material of permeability μ and length f . This model allows us to compute forces of electrical origin for systems involving permanent magnets on the same basis as those excited by currents through electrical terminal pairs.

2.1.2 Generalized Capacitance

To derive the terminal characteristics of lumped-parameter electric field systems we start with the quasi-static equations given in Table 1.2. The equations we need are (1.1.24) to (1.1.26) and (1.1.13):

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0, \quad (1.1.24)$$

$$\oint_S \mathbf{D} \cdot \mathbf{n} \, da = \int_V \rho_f \, dV, \quad (1.1.25)$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \quad (1.1.13)$$

$$\oint_S \mathbf{J}'_f \cdot \mathbf{n} \, da = - \frac{d}{dt} \int_V \rho_f \, dV. \quad (1.1.26)$$

We can use the differential forms of these equations alternatively, although the integral forms are more appropriate for the formalism of this section.

It is essential to recognize that an ideal, lossless electric field system consists of a set of equipotential bodies with no conducting paths between them.

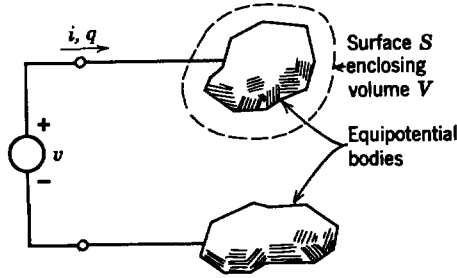


Fig. 2.1.6 A simple electric field system.

Terminals are brought out so that excitation may be applied to the equipotential bodies. It is conventional to select one equipotential body as a reference and designate its voltage as zero. The potentials of the other bodies are then specified with respect to the reference.

As a simple example of finding the terminal relations for an electric field system, consider the two equipotential bodies in Fig. 2.1.6. We assume that the voltage v is impressed between the two equipotential bodies and wish to find the current i . We choose a surface S (see Fig. 2.1.6) which encloses only the upper equipotential body and apply the conservation of charge (1.1.26). The only current density on the surface S occurs where the wire cuts through it. At this surface there is no free charge density ρ_f , hence $J_f = J'_f$, so that

$$\oint_S \mathbf{J}'_f \cdot \mathbf{n} \, da = -i. \quad (2.1.26)$$

The minus sign results because the normal vector $\mathbf{n}$ is directed outward from the surface.

The total charge q on the upper equipotential body in Fig. 2.1.6 is

$$q = \int_V \rho_f \, dV, \quad (2.1.27)$$

where V is a volume that includes the body and is enclosed by the surface S . Use of the conservation of charge (1.1.26) with (2.1.26) and (2.1.27) yields the terminal current

$$i = \frac{dq}{dt}. \quad (2.1.28)$$

Equation 2.1.28 simply expresses the fact that a current i leads to an accumulation of charge on the body. For a quasi-static system in which we impose voltage constraints the field quantities and the charge density ρ_f are determined by (1.1.24), (1.1.25) and (1.1.13), and all are functions of the

applied voltages, the material properties (polarization), and the geometry. Thus, because (2.1.27) is an integral over space, the charge q is a function of the applied voltages, material properties, and geometry.

If we again consider the system in Fig. 2.1.6 and specify that the time variation of the geometry is uniquely specified by a mechanical displacement x with respect to a fixed reference, we can write the charge in the general functional form

$$q = q(v, x). \quad (2.1.29)$$

In writing the charge in this way we have indicated explicit functional dependence on only those variables (v and x) that may be functions of time.

We can now use (2.1.29) in (2.1.28) to obtain the terminal current as

$$i = \frac{\partial q}{\partial v} \frac{dv}{dt} + \frac{\partial q}{\partial x} \frac{dx}{dt}. \quad (2.1.30)$$

The first term exists only when the voltage is changing with time and the second term exists only when there is relative mechanical motion.

If we consider a system whose polarization density $\mathbf{P}$ is a linear function of field quantities, the system is *electrically linear* and the functional dependence of (2.1.29) can be written in the form

$$q = C(x)v. \quad (2.1.31)$$

The capacitance C contains the dependence on geometry. For a system whose charge is expressible by (2.1.31) the terminal current is

$$i = C(x) \frac{dv}{dt} + v \frac{dC}{dx} \frac{dx}{dt}. \quad (2.1.32)$$

In the special case in which the geometry does not vary with time (x is constant), (2.1.32) reduces to

$$i = C \frac{dv}{dt}, \quad (2.1.33)$$

which is the terminal equation used for a capacitance in linear circuit theory.

Now that we have established the formalism for calculating the terminal properties of lumped-parameter electric field systems by treating the simplest case, we can, as we did for the magnetic field system in the preceding section, generalize the equations to describe systems with any number of electrical terminal pairs and mechanical displacements. We assume a system with $N + 1$ equipotential bodies. We select one of them as a reference (zero) potential and apply voltages to the other N terminals. There are then N terminal voltages,

$$v_1, v_2, \dots, v_N.$$

We assume that there are M mechanical displacements that uniquely specify the time variation of the geometry

$$x_1, x_2, \dots, x_M.$$

Although we write these displacements as if they were translational, they can equally well be rotational (angular).

For the integration of (2.1.26) we select a surface S_k that encloses the k th equipotential body. Then (2.1.27) is integrated over the enclosed volume V_k and the conservation of charge expression (1.1.26) is used to express the current into the terminal connected to the k th equipotential as

$$i_k = \frac{dq_k}{dt}, \quad (2.1.34)$$

where

$$q_k = \int_{V_k} \rho_f dV. \quad (2.1.35)$$

Because the system is quasi-static, the fields are functions only of the applied voltages, the material properties, and the displacements. Thus we can generalize the functional form of (2.1.29) for our multivariable problem to

$$\begin{aligned} q_k &= q_k(v_1, v_2, \dots, v_N; x_1, x_2, \dots, x_M), \\ k &= 1, 2, \dots, N. \end{aligned} \quad (2.1.36)$$

From (2.1.36) and (2.1.34), the k 'th terminal current follows as

$$\begin{aligned} i_k &= \sum_{j=1}^N \frac{\partial q_k}{\partial v_j} \frac{dv_j}{dt} + \sum_{j=1}^M \frac{\partial q_k}{\partial x_j} \frac{dx_j}{dt}, \\ k &= 1, 2, \dots, N. \end{aligned} \quad (2.1.37)$$

If we specify that our multivariable system is electrically linear (a situation that occurs when polarization $\mathbf{P}$ is a linear function of electric field intensity) we can write the function of (2.1.36) in the form

$$\begin{aligned} q_k &= \sum_{j=1}^N C_{kj}(x_1, x_2, \dots, x_M) v_j, \\ k &= 1, 2, \dots, N. \end{aligned} \quad (2.1.38)$$

Equations 2.1.36 and 2.1.38 can be inverted to express the voltages as functions of the charges and displacements. This process was illustrated for magnetic field systems by (2.1.19) through (2.1.24).

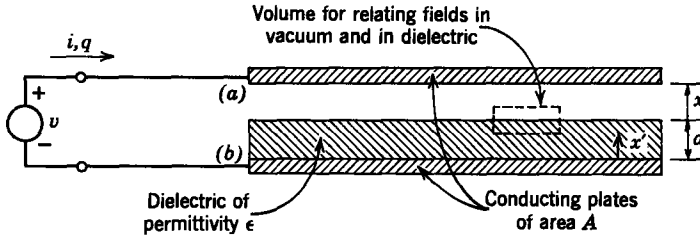


Fig. 2.1.7 A parallel-plate capacitor.

Example 2.1.4. Consider the simple parallel-plate capacitor of Fig. 2.1.7. It consists of two rectangular, parallel highly conducting plates of area A . Between the plates is a rectangular slab of dielectric material with constant permittivity ϵ ,

$$\mathbf{D} = \epsilon \mathbf{E}.$$

The lower plate and the dielectric are fixed and the upper plate can move and has the instantaneous position x with respect to the top of the dielectric. The transverse dimensions are large compared with the plate separation. Thus fringing fields can be neglected. The terminal voltage is constrained by the source v which is specified as a function of time.

We wish to calculate the instantaneous charge on the upper plate and the current to the upper plate.

To solve this problem we need the given relation between $\mathbf{D}$ and $\mathbf{E}$, (1.1.24) and (1.1.25), and the definition of the voltage of point a with respect to point b

$$v = - \int_b^a \mathbf{E} \cdot d\mathbf{l}$$

With the neglect of fringing fields, the field quantities $\mathbf{D}$ and $\mathbf{E}$ will have only vertical components. We take them both as being positive upward. In the vacuum space

$$D_v = \epsilon_0 E_v$$

and in the dielectric

$$D_d = \epsilon E_d.$$

We assume that the dielectric has no free charge; consequently, we use (1.1.25) with a rectangular box enclosing the dielectric-vacuum interface as illustrated in Fig. 2.1.7 to obtain

$$\epsilon_0 E_v = \epsilon E_d.$$

We now use the expression for the voltage to write

$$v = - \int_0^d \frac{\epsilon_0}{\epsilon} E_v dx' - \int_d^{x+d} E_v dx'.$$

Integration of these expressions yields the vacuum electric field intensity

$$E_v = - \frac{v}{x + (\epsilon_0/\epsilon)d}.$$

We now use (1.1.25) with a rectangular surface enclosing the upper plate to obtain

$$q = A\epsilon_0 E_v = \frac{\epsilon_0 A v}{x + (\epsilon_0/\epsilon)d}$$

As would be expected from the linear constitutive law used in the derivation, the system is electrically linear. The charge can be expressed as

$$q = C(x)v,$$

where

$$C(x) = \frac{\epsilon_0 A}{x + (\epsilon_0/\epsilon)d}.$$

When voltage v and displacement x are specified functions of time, we can write the terminal current as

$$i = \frac{dq}{dt} = \frac{\epsilon_0 A}{x + (\epsilon_0/\epsilon)d} \frac{dv}{dt} - \frac{\epsilon_0 A v}{[x + (\epsilon_0/\epsilon)d]^2} \frac{dx}{dt}.$$

Note that the first term will exist when the geometry (x) is fixed and the voltage is varying and that the second term will exist when the voltage is constant and the geometry is varying. This illustrates once again how mechanical motion can generate a time-varying current.

Example 2.1.5. As an example of a multiply excited electric field system, consider the system in Fig. 2.1.8 which is essentially a set of three parallel-plate capacitors immersed in vacuum. Of the three plates, one is fixed and two are movable, as indicated. The dimensions and variables are defined in Fig. 2.1.8.

We wish to calculate the charges q_1 and q_2 on the two movable plates and the currents i_1 and i_2 to those plates. We neglect fringing at the edges of the plates and assume that the voltages v_1 and v_2 are specified.

We designate the electric field intensities in the three regions as E_1 , E_2 , and E_m , with the positive directions as indicated in Fig. 2.1.8. The electric field intensities have constant

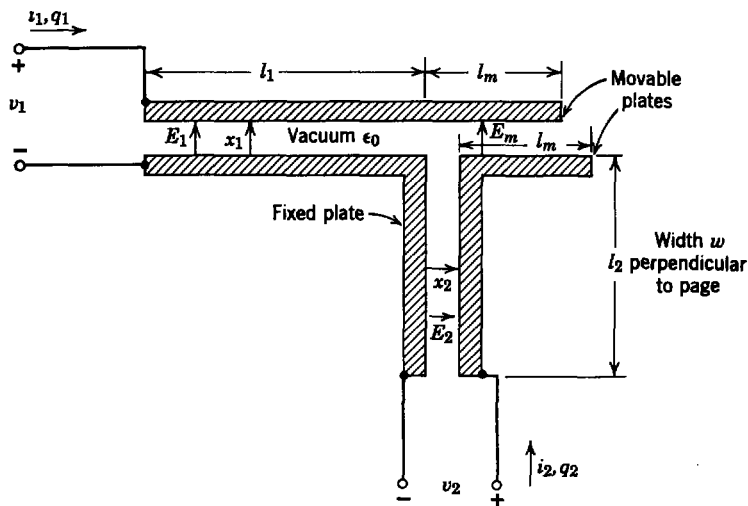


Fig. 2.1.8 A multiply excited electric field system.

magnitudes in the three regions; consequently, our definition of voltage applied in the three regions yields

$$E_1 = -\frac{v_1}{x_1},$$

$$E_2 = -\frac{v_2}{x_2},$$

$$E_m = \frac{v_2 - v_1}{x_1}.$$

We now use (1.1.25) with a rectangular surface enclosing the top plate to obtain

$$q_1 = -l_1 w \epsilon_0 E_1 - (l_m - x_2) w \epsilon_0 E_m$$

and with a rectangular box enclosing the right-hand movable plate to obtain

$$q_2 = -l_2 w \epsilon_0 E_2 + (l_m - x_2) w \epsilon_0 E_m.$$

These two expressions are electrically linear and can be written in the forms

$$q_1 = C_1 v_1 - C_m v_2, \quad (a)$$

$$q_2 = -C_m v_1 + C_2 v_2, \quad (b)$$

where

$$C_1 = \frac{\epsilon_0 w [l_1 + (l_m - x_2)]}{x_1}, \quad (c)$$

$$C_2 = \epsilon_0 w \left(\frac{l_2}{x_2} + \frac{l_m - x_2}{x_1} \right), \quad (d)$$

$$C_m = \frac{\epsilon_0 w (l_m - x_2)}{x_1}. \quad (e)$$

With v_1 , v_2 , x_1 , and x_2 given as specified functions of time, we can write the terminal currents as

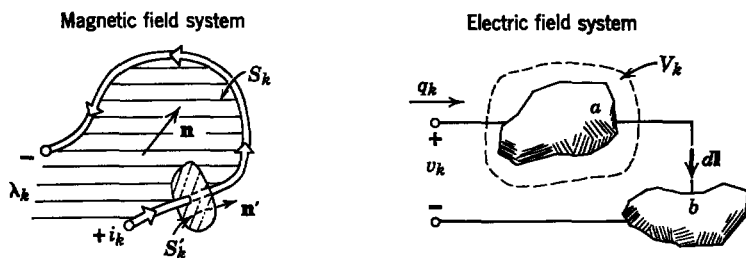
$$i_1 = C_1 \frac{dv_1}{dt} - C_m \frac{dv_2}{dt} + v_1 \left(\frac{\partial C_1}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial C_1}{\partial x_2} \frac{dx_2}{dt} \right) - v_2 \left(\frac{\partial C_m}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial C_m}{\partial x_2} \frac{dx_2}{dt} \right),$$

$$i_2 = -C_m \frac{dv_1}{dt} + C_2 \frac{dv_2}{dt} - v_1 \left(\frac{\partial C_m}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial C_m}{\partial x_2} \frac{dx_2}{dt} \right) + v_2 \left(\frac{\partial C_2}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial C_2}{\partial x_2} \frac{dx_2}{dt} \right).$$

A comparison of these results with those of the preceding example illustrates how quickly the expressions become longer and more complex as the numbers of electrical terminal pairs and mechanical displacements are increased.

2.1.3 Discussion

In the last two sections we specified the process by which we can obtain the electrical terminal properties of lumped-parameter, magnetic field and electric field systems. The general forms of the principal equations are summarized in Table 2.1. The primary purpose of obtaining terminal relations is to be able to include electromechanical coupling terms when writing circuit

Table 2.1 Summary of Terminal Variables and Terminal Relations**Definition of Terminal Variables****Flux**

$$\lambda_k = \int_{S_k} \mathbf{B} \cdot \mathbf{n} \, da$$

Current

$$i_k = \int_{S_k'} \mathbf{J}_f \cdot \mathbf{n}' \, da$$

Charge

$$q_k = \int_{V_k} \rho_f \, dV$$

Voltage

$$v_k = \int_a^b \mathbf{E} \cdot d\mathbf{l}$$

Terminal Conditions

$$v_k = \frac{d\lambda_k}{dt}$$

$$i_k = \frac{dq_k}{dt}$$

$$\lambda_k = \lambda_k(i_1 \cdots i_N; \text{geometry})$$

$$q_k = q_k(v_1 \cdots v_N; \text{geometry})$$

$$i_k = i_k(\lambda_1 \cdots \lambda_N; \text{geometry})$$

$$v_k = v_k(q_1 \cdots q_N; \text{geometry})$$

equations. After a review of rigid-body mechanics and a look at some energy considerations, we shall address ourselves to the problem of writing coupled equations of motion for electromechanical systems.

2.2 MECHANICS

We now discuss lumped-parameter modeling of the mechanical parts of systems. In essence, we shall consider the basic notions of rigid-body mechanics, including the forces of electric origin.

Just as in circuit theory, there are two steps in the formulation of equations of motion for rigid-body mechanical systems. First, we must specify the kinds of elements and their mathematical descriptions. This is analogous to defining terminal relations for circuit elements in circuit theory. Next, we must specify the laws that are used for combining the mathematical descriptions of elements into equations of motion. In mechanics these are Newton's second law and the continuity of space (often called geometrical compatibility) and they are analogous to Kirchhoff's laws in circuit theory.

In the two sections to follow we first define mechanical elements and then specify the laws and illustrate the formulation of equations of motion.

2.2.1 Mechanical Elements

In general, a mechanical system, which, for the moment, we define as an interconnected system of ponderable bodies in relative motion, will exhibit kinetic energy storage in moving masses, potential energy storage due to gravitational fields or elastic deformations, and mechanical losses due to friction of various types. Any component of a system will exhibit all three effects to varying degrees; as a practical matter of analysis and design, however, we can often represent mechanical systems as interconnections of ideal elements, each of which exhibits only one of these effects. This process is analogous to that of defining ideal circuit elements in circuit theory and has its justification in derivations from continuum mechanics. These derivations are not made in a formal way; however, in our treatment of continuum mechanics in later chapters the connections between the continuum theory and ideal, lumped-parameter, mechanical elements will become clear.

We consider three types of ideal mechanical elements: (a) elements that store kinetic energy—for translational systems these are masses and for rotational systems they are moments of inertia; (b) elements that store potential energy—for both translational and rotational systems these are springs; (c) elements that dissipate mechanical energy as heat—for both translational and rotational systems these are mechanical dampers. We define the nomenclature by which we describe each element before we combine them into systems. In the process we introduce the concept of mechanical circuits and mechanical circuit elements, which are simply pictorial representations of mathematical relations somewhat analogous to the circuits of electrical systems.* Mechanical circuits are of value for two reasons: they provide a formalism for writing the equations of motion and they emphasize the concept of mechanical terminal pairs.

The mechanical variables used in describing mechanical elements and systems are force and displacement. The displacement is always measured with respect to a reference position. In a way analogous to that used in circuit theory, we define ideal mechanical sources.

First, we define a position source as in Fig. 2.2.1. Here we represent a simple physical situation in which the motion of the two objects (Fig. 2.2.1a) is restricted to the vertical. Thus there are two *mechanical nodes* whose positions x_1 and x_2 are measured with respect to a fixed reference. The

* M. F. Gardner, and J. L. Barnes, *Transients in Linear Systems*, Vol. I, Wiley, New York, 1942, Chapter 2. (Various conventions are used for drawing mechanical circuits. One alternative is given in the above.)

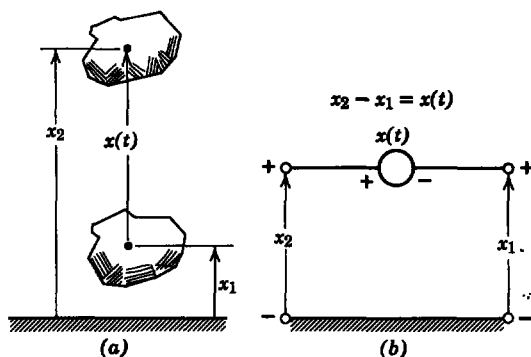


Fig. 2.2.1 A position source: (a) physical; (b) schematic.

position source $x(t)$ constrains the relative positions of the two nodes to

$$x_2 - x_1 = x(t); \quad (2.2.1)$$

the sign is determined by the $+$ and $-$ signs associated with the source. In Fig. 2.2.1b we give the circuit representation of the position source. The circuit is a pictorial representation of the scalar equation (2.2.1) and as such is completely analogous to the representations in circuit theory. Because (2.2.1) is valid regardless of other mechanical elements attached to the nodes, the ideal position source can supply an arbitrary amount of force.

In a similar way we can define a velocity source $v(t)$ that constrains the relative velocity of the two nodes to

$$\frac{dx_2}{dt} - \frac{dx_1}{dt} = v(t). \quad (2.2.2)$$

The circuit is as shown in Fig. 2.2.1b with x replaced by v .

A different kind of ideal mechanical source is a force source for which nomenclature is given in Fig. 2.2.2. In the physical representation of Fig. 2.2.2a motion is constrained to the vertical and the force $f(t)$ is vertical. The position of the arrow indicates the direction of positive force, and the convention we use here is that, with the arrow as shown and a positive $f(t)$, the force tends to push the two mechanical nodes apart exactly as if one were standing on node x_1 and pushing upward on node x_2 with the hands. The circuit representation of this force source is given in Fig. 2.2.2b. In the circuit our convention is that with the arrow as shown [$f(t)$ positive] the force tends to increase x_2 and decrease x_1 .

The sources of Figs. 2.2.1 and 2.2.2 have been specified for translational systems. We can also specify analogous sources and circuits for rotational

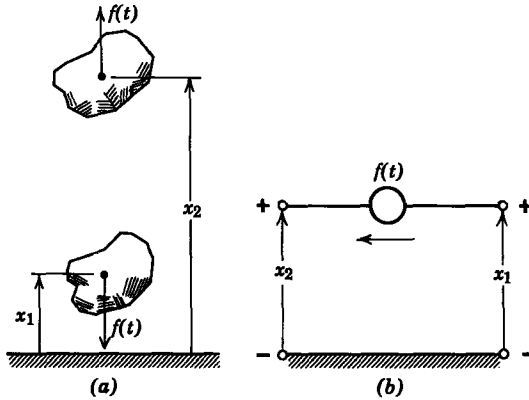


Fig. 2.2.2 A force source: (a) physical; (b) schematic.

systems. These extensions of our definitions should be clear from Figs. 2.2.3 and 2.2.4, in which θ_1 and θ_2 measure the angular displacements of the two disks, with respect to fixed references, and the torque T is applied between them.

In the examples we shall consider we shall encounter either pure translation or rotation about a fixed axis. Consequently, the geometry of motion as described so far is adequate for our purposes. We now describe ideal, passive, lumped-parameter, mechanical elements.

2.2.1a The Spring*

An *ideal* spring is a device with negligible mass and mechanical losses whose deformation is a single-valued function of the applied force. A *linear*

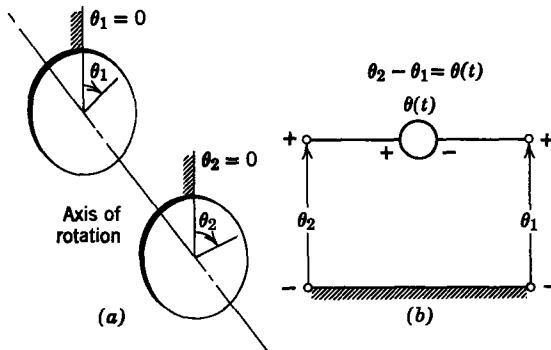


Fig. 2.2.3 An angular position source: (a) physical; (b) schematic.

* For a comprehensive treatise on the subject see A. M. Wahl, *Mechanical Springs*, 2nd ed., McGraw-Hill, New York, 1963.

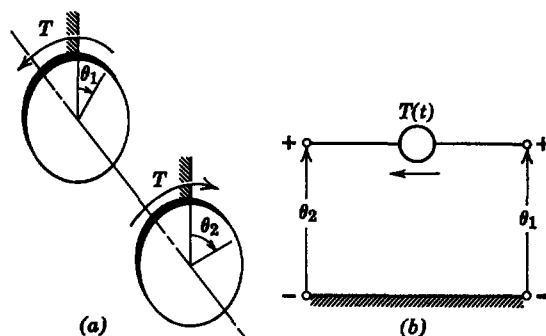


Fig. 2.2.4 A torque source: (a) physical; (b) schematic.

ideal spring has deformation *proportional* to force. In our treatment we are concerned almost exclusively with linear springs. We represent a spring physically as in Fig. 2.2.5a and in mechanical circuits as the symbol of Fig. 2.2.5b. The force f at one end of the spring must always be balanced by an equal and opposite force f at the other end. The force is thus transmitted through the spring much as current is transmitted through an inductance. In the circuit of Fig. 2.2.5b the applied force f is represented as a force source.

The spring of Fig. 2.2.5 has a spring constant K and the force is a linear function of the *relative* displacement of the two ends of the spring. Thus

$$f = K(x_2 - x_1 - l), \quad (2.2.3)$$

where l is the value of the relative displacement for which the force is zero.

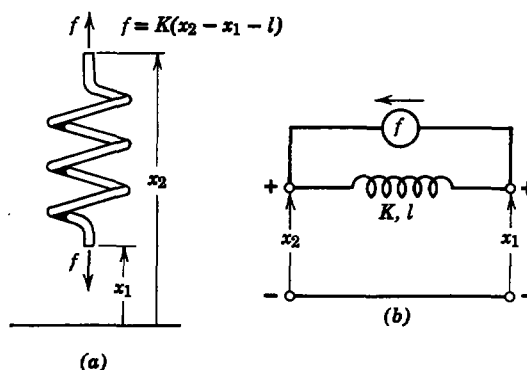


Fig. 2.2.5 A linear ideal spring for translational motion held in equilibrium by a force f : (a) physical system; (b) circuit.

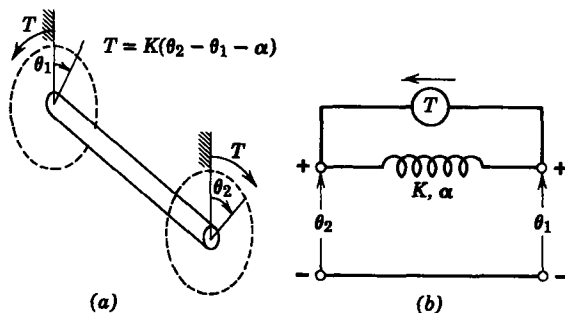


Fig. 2.2.6 A linear ideal torsional spring: (a) physical system; (b) circuit.

It is always possible, although in many cases not convenient, to define reference positions for measuring x_1 and x_2 such that $l = 0$.

We can also have linear ideal torsional springs in rotational systems. The mathematical and circuit representations are analogous to those of a translational spring and are evident in Fig. 2.2.6. The torque is a linear function of the relative angular displacement of the two ends

$$T = K(\theta_2 - \theta_1 - \alpha). \quad (2.2.4)$$

Note that the K in (2.2.3) has different dimensions than the K in (2.2.4).

2.2.1b The Mechanical Damper

The mechanical damper is analogous to electrical resistance in that it dissipates energy as heat. An *ideal* damper is a device that exhibits no mass or spring effect and exerts a force that is a function of the relative velocity between its two nodes. A *linear* ideal damper has a force proportional to the relative velocity of the two nodes. In all cases a damper produces a force that opposes the relative motion of the two nodes.

A linear damper (often called a viscous damper) is usually constructed in such a way that friction forces result from the viscous drag of a fluid under laminar flow conditions.* Two examples of viscous dampers, one for linear and one for rotary motion, are shown in Fig. 2.2.7 along with the mechanical circuits. Note that the force f (or torque T) is the force (or torque) that must be applied by an external agent to produce a positive relative velocity of the two nodes. For the linear-motion damper the terminal relation is

$$f = B \frac{d}{dt}(x_2 - x_1) \quad (2.2.5)$$

* For more detail on viscous laminar flow see Chapter 14.

and for the rotary-motion damper it is analogous (Fig. 2.2.7):

$$T = B \frac{d}{dt} (\theta_2 - \theta_1). \quad (2.2.6)$$

Note that the damping constant B has different dimensions for the two systems. In each case both displacements are measured with respect to references that are fixed.

Mechanical friction occurs in a variety of situations under many different physical conditions. Sometimes friction is unwanted but must be tolerated and accounted for analytically, as, for example, in bearings, sliding electrical contacts, and the aerodynamic drag on a moving body. In other cases friction is desired and is designed into equipment. Examples are vibration dampers and shock absorbers. Although in some cases a linear model is a useful approximation, in many others it is inadequate. The subject of friction

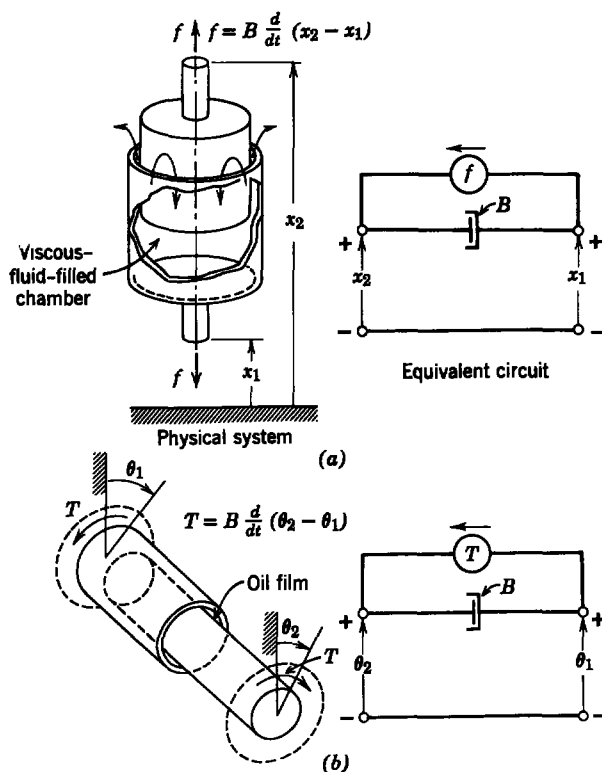


Fig. 2.2.7 Mechanical dampers: (a) translational system; (b) rotational system.

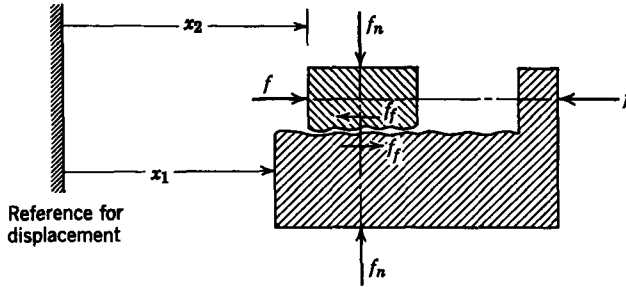


Fig. 2.2.8 Coulomb friction between members in contact.

is lengthy and complex*; most practical devices, however, can be modeled as described or by one of two nonlinear models that we now discuss.

The first of these additional models is coulomb friction which is characteristic of sliding contacts between dry materials. See Fig. 2.2.8 in which the blocks are assumed to have negligible mass. If we apply constant, equal and opposite, normal forces f_n , as shown, and then apply equal and opposite forces f , as shown, the blocks may or may not move relative, depending on the friction coefficient of the surface. If we vary the force f , which must be balanced by the friction force f_f for steady motion, and measure the resultant steady relative velocity, we can plot the friction coefficient (f_f/f_n) as a function of relative velocity (see Fig. 2.2.9). The quantity μ_s is the coefficient of static friction and μ_d is the coefficient of sliding friction. When we

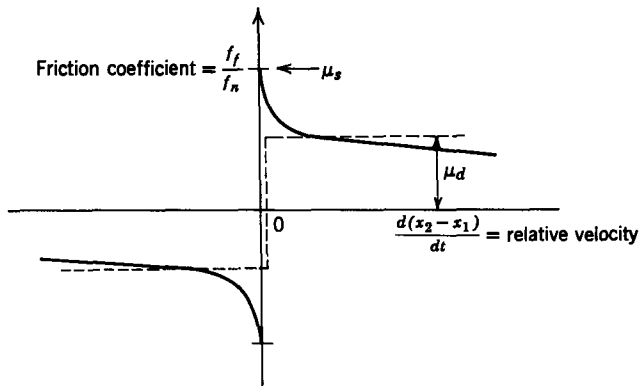


Fig. 2.2.9 Typical coulomb friction characteristic.

* See, for example, G. W. Van Stanten, *Introduction to a Study of Mechanical Vibration*, 3rd ed., Macmillan, New York, 1961, Chapter 14.

approximate this curve by the piecewise linear dashed line of Fig. 2.2.9, we can represent coulomb friction mathematically by the relation

$$f = \frac{f_n \mu_a (d/dt)(x_2 - x_1)}{|(d/dt)(x_2 - x_1)|}. \quad (2.2.7)$$

It is important to remember that coulomb friction, like all other forms of friction, produces a force that tends to oppose the relative motion of the nodes in the system.

Coulomb friction can also occur in rotational systems, in which case an expression analogous to (2.2.7) can be used for a mathematical description.

The final model of friction that we shall consider is that resulting primarily from the drag of a viscous fluid in turbulent flow.* This type of friction can be represented with fair accuracy by a model that makes the force (or torque) proportional to the square of relative velocity (or relative angular velocity). Such an expression is

$$f = \pm B_s \left[\frac{d}{dt} (x_2 - x_1) \right]^2. \quad (2.2.8)$$

Once again the force produced by the friction opposes the relative mechanical motion. Two examples in which square-law damping occurs are given in Fig. 2.2.10.

The linear motion damper of Fig. 2.2.10a has the configuration characteristic of dashpots for making time-delay relays and for automobile shock absorbers. The square-law rotational damping of Fig. 2.2.10b occurs frequently in high-speed rotating machines in which the fluid is air or some other gaseous coolant.

2.2.1c The Mass

The final ideal mechanical element we need to consider is the element that stores kinetic energy but has no spring or damping effects. For translational

* See, for example, H. Schlichting, *Boundary Layer Theory*, 4th ed., McGraw-Hill, New York, 1960, Chapter 21.

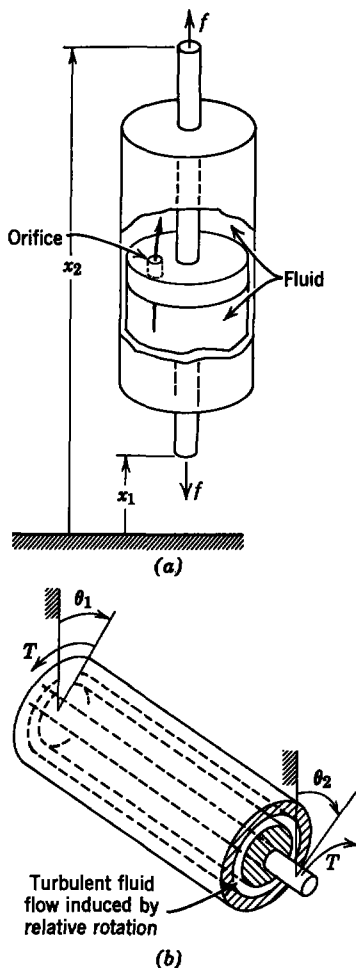


Fig. 2.2.10 Typical square-law dampers: (a) orifice and piston damper; (b) damping due to rotation.

systems this is a mass and for rotational systems a moment of inertia. After we have reviewed the plane motion of a point mass, we shall generalize to translation of a rigid body of finite size and to rotation of a rigid body about a fixed axis.

The motion of a point that has associated with it a constant amount of mass M is described by Newton's second law*:

$$\mathbf{f} = M \frac{d\mathbf{v}}{dt}, \quad (2.2.9)$$

where $\mathbf{f}$ is the force vector acting on the mass point and $\mathbf{v}$ is the absolute vector velocity of the point. It should be clearly recognized that $\mathbf{v}$ must be measured in relation to a fixed or nonaccelerating point or frame of reference. Such a reference system is called an *inertial reference*.

In motion occurring on the surface of the earth the earth is often considered to be approximately an inertial reference and $\mathbf{v}$ is measured in relation to the earth. When dealing with the motion of long-range missiles, orbital vehicles, and spacecraft, the earth cannot be considered to be a nonaccelerating reference and velocities are then measured with respect to the fixed stars.

When the velocity of a mass begins to approach the velocity of light, the Newtonian equation of motion in the form of (2.2.9) becomes invalid because the mass of a given particle of matter increases with velocity, according to the theory of relativity.† The present considerations are limited to velocity levels small compared with the velocity of light, so that Newton's law (2.2.9) will apply. This is consistent with the approximations we have made in defining the quasi-static electromagnetic field equations.

If we now consider the two-dimensional motion of a point mass M , we can define an orthogonal inertial reference system as in Fig. 2.2.11a and write Newton's second law in component form as

$$f_x = M \frac{d^2x}{dt^2}, \quad (2.2.10)$$

$$f_y = M \frac{d^2y}{dt^2}. \quad (2.2.11)$$

We can also represent these two equations by the mechanical circuit of Fig. 2.2.11b. Each degree of freedom of a point mass M is represented by a node with mass M . A force that tends to increase the displacement of a

* For a review of the fundamental concepts involved in Newton's laws, see for example R. R. Long, *Engineering Science Mechanics*, Prentice-Hall, Englewood Cliffs, N.J., 1963, Chapter 2.

† Long, *ibid.*, Chapter 10.

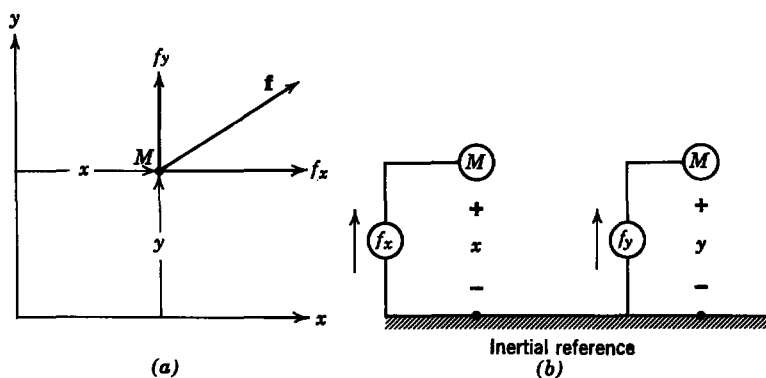


Fig. 2.2.11 Plane motion of a point mass: (a) physical; (b) circuit representation.

node is represented by an arrow pointing toward the node. The extension of these representations to three-dimensional motion is straightforward.

In order to obtain a representation for a mass element more general than a point mass, we need to review briefly the dynamics of rigid bodies. We consider first the translational motion of a rigid body.

A rigid body, by definition, is one in which any line drawn in or on the body remains constant in length and all angles drawn in or on the body remain constant. In Fig. 2.2.12 we represent a rigid body with a mass density ρ (kilograms per cubic meter) that may vary from point to point in the body but remains constant in time at any point in the body. We define an inertial coordinate system (x, y, z) and specify the position vector $\mathbf{r}$ of an arbitrary point p that is fixed in the body.

The instantaneous acceleration of the point p is

$$\mathbf{a}_p = \frac{d^2 \mathbf{r}}{dt^2}. \quad (2.2.12)$$

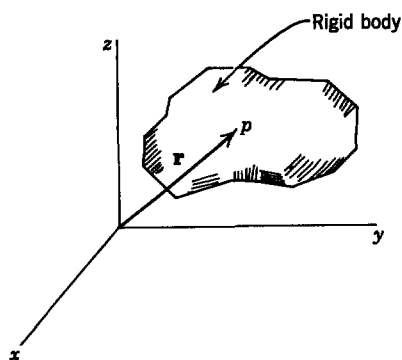


Fig. 2.2.12 Geometry for analyzing the translation of a rigid body.

Thus the infinitesimal element of mass ρdV at point p will have an acceleration force

$$d\mathbf{f}_n = \rho dV \frac{d^2\mathbf{r}}{dt^2}. \quad (2.2.13)$$

The force $d\mathbf{f}_n$ consists of two components: $d\mathbf{f}$ from sources external to the body and $d\mathbf{f}_i$ from sources within the body. Thus (2.2.13) can be written as

$$d\mathbf{f} + d\mathbf{f}_i = \rho dV \frac{d^2\mathbf{r}}{dt^2}. \quad (2.2.14)$$

The mass density associated with each point p in the rigid body is constant. Hence, when we integrate this expression throughout the volume of the body and recognize that the internal forces integrate to zero,* we obtain the result

$$\mathbf{f} = M \frac{d^2\mathbf{r}_m}{dt^2}, \quad (2.2.15)$$

where $\mathbf{f}$ is the total external force applied to the body,

$$M = \int_V \rho dV \text{ is a constant and is the total mass of the body,}$$

$$\mathbf{r}_m = \frac{\int_V \rho \mathbf{r} dV}{M} \text{ is the position vector of the center of mass of the body.}$$

From the result of (2.2.15) it is clear that the translational motion of a rigid body can be described completely by treating the body as if all the mass were concentrated at the center of mass. Consequently, (2.2.10) and (2.2.11)

* To illustrate that the internal forces integrate to zero consider an ensemble of N interacting particles. An internal force is applied between two particles: $\mathbf{f}_{ij} = -\mathbf{f}_{ji}$, where $\mathbf{f}_{ij}$ is the force on the i th particle due to the source connected between the i th and j th particles. The total internal force on the i th particle is

$$\mathbf{f}_i = \sum_{\substack{j=1 \\ j \neq i}}^N \mathbf{f}_{ij}.$$

The total internal force on the ensemble is

$$\mathbf{f} = \sum_{i=1}^N \mathbf{f}_i = \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \mathbf{f}_{ij}.$$

Because $\mathbf{f}_{ij} = -\mathbf{f}_{ji}$, we conclude that each term in this summation is canceled by an equal and opposite term and the net internal force is zero. If we let $N \rightarrow \infty$, the nature of the result is unchanged; thus we conclude that the integral of internal forces over the volume of a rigid body must be zero.

and Fig. 2.2.11, which were defined for a point mass, hold equally well for describing the motion of the center of mass of a rigid body.

Our rotational examples involve rotation about a fixed axis only. Thus we treat only the mechanics of rigid bodies rotating about fixed axes.* For this purpose we consider the system of Fig. 2.2.13. The body has mass density ρ that may vary with space in the body but at a point p fixed in the material is constant. We select a rectangular coordinate system whose z -axis coincides with the axis of rotation. The instantaneous angular velocity of the body is

$$\boldsymbol{\omega} = \mathbf{i}_z \frac{d\theta}{dt}. \quad (2.2.16)$$

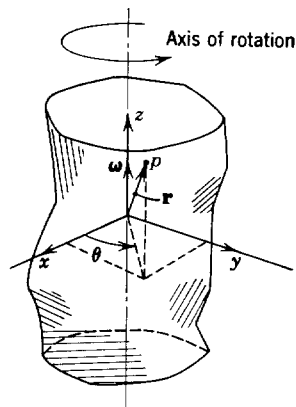


Fig. 2.2.13 Rigid-body rotation.

At the point p with coordinates (x, y, z) the element of mass ρdV will have the instantaneous velocity

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}, \quad (2.2.17)$$

where $\mathbf{r} = \mathbf{i}_x x + \mathbf{i}_y y + \mathbf{i}_z z$ is the radius vector from the origin to the mass element.

The acceleration force on this mass element is

$$d\mathbf{f}_n = \rho dV \frac{d\mathbf{v}}{dt} = \rho dV \left[\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \right], \quad (2.2.18)$$

where $d\mathbf{f}_n$ contains both internal and external forces [see (2.2.14)] and the last term has been written by using (2.2.17). We use the identity for the triple vector product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \quad (2.2.19)$$

to write (2.2.18) in the form

$$d\mathbf{f}_n = \rho dV \left[\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega}(\boldsymbol{\omega} \cdot \mathbf{r}) - \mathbf{r}(\boldsymbol{\omega} \cdot \boldsymbol{\omega}) \right]. \quad (2.2.20)$$

To find the acceleration torque on this mass element we write

$$d\mathbf{T}_n = \mathbf{r} \times d\mathbf{f}_n. \quad (2.2.21)$$

* For a treatment of the general case of simultaneous translation and rotation in three dimensions, see, for example, Long, *op. cit.*, Chapter 6.

Using (2.2.20) in (2.2.21) and simplifying, we find that

$$d\mathbf{T} + d\mathbf{T}_i = \rho dV \left\{ \mathbf{i}_x(x^2 + y^2) \frac{d^2\theta}{dt^2} - \mathbf{i}_x \left[xz \frac{d^2\theta}{dt^2} - yz \left(\frac{d\theta}{dt} \right)^2 \right] - \mathbf{i}_y \left[yz \frac{d^2\theta}{dt^2} + xz \left(\frac{d\theta}{dt} \right)^2 \right] \right\}, \quad (2.2.22)$$

where $d\mathbf{T}$ = the torque from external sources,

$d\mathbf{T}_i$ = the torque from internal sources.

To find the total acceleration torque on the body we must integrate (2.2.22) throughout the volume V of the body. For this purpose we find it convenient to define the moment of inertia about the z -axis as

$$J_z = \int_V (x^2 + y^2) \rho dV \quad (2.2.23)$$

and the products of inertia*

$$J_{xz} = \int_V xz \rho dV, \quad (2.2.24)$$

$$J_{yz} = \int_V yz \rho dV. \quad (2.2.25)$$

When we use these parameters and integrate (2.2.22) throughout the volume V of the body, the internal torque integrates to zero [see (2.2.14) and (2.2.15) and the associated footnote for arguments similar to those required here], and we obtain for the total acceleration torque $\mathbf{T}$ applied by external sources

$$\mathbf{T} = \mathbf{i}_z J_z \frac{d^2\theta}{dt^2} - \mathbf{i}_x \left[J_{xz} \frac{d^2\theta}{dt^2} - J_{yz} \left(\frac{d\theta}{dt} \right)^2 \right] - \mathbf{i}_y \left[J_{yz} \frac{d^2\theta}{dt^2} + J_{xz} \left(\frac{d\theta}{dt} \right)^2 \right]. \quad (2.2.26)$$

With the restriction to rotation about a fixed axis, only the first term in this expression affects the dynamics of the body. Thus we write the z component of (2.2.26) as

$$T_z = J_z \frac{d^2\theta}{dt^2} \quad (2.2.27)$$

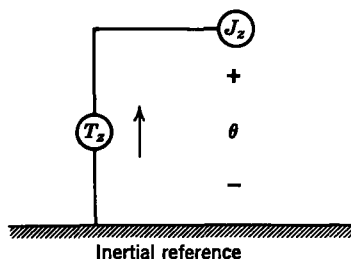


Fig. 2.2.14 Circuit representation of a moment of inertia.

and represent this element in a mechanical circuit as in Fig. 2.2.14. Note that this circuit has exactly the same form as that adopted earlier for representing mass in a translational system (see Fig. 2.2.11).

* See, for example, I. H. Shames, *Engineering Mechanics*, Prentice-Hall, Englewood Cliffs, N.J., 1960, pp. 187–188.

The last two terms on the right of (2.2.26) (the x - and y -components) represent a torque that must be applied by bearings and support structure to maintain the axis of rotation fixed. It should be clear from the definitions of the products of inertia in (2.2.24) and (2.2.25) that certain axes of symmetry make these products of inertia zero. Such axes are called principal axes.* When rotation occurs about a principal axis (the body is dynamically balanced), no bearing torque is necessary to maintain the axis of rotation fixed.

Now that we have completed the definitions of the elements that will make up the mechanical parts of our electromechanical systems, we have only to describe how we combine elements to obtain complete equations of motion.

2.2.2 Mechanical Equations of Motion

When electrical circuit elements are interconnected, Kirchhoff's loop and node relations must be satisfied. The sum of the voltage differences around any loop must be zero and the sum of all currents into any node must be zero.

Similar relations must hold in networks composed of interconnected ideal mechanical elements. Consider first a *mechanical node*. A mechanical node is a location in a mechanical system which has a certain position relative to a reference position and a particular mass associated with it. A node appears in the mechanical circuit as a circle in which two or more terminals of ideal elements are connected. Figure 2.2.15 shows a simple mechanical system with one node p having mass M connected to two springs, a damper, and a force source. The motion of this system is completely specified by one coordinate x which is the instantaneous position of node p . The system is said to have one *degree of freedom*. In general, the number of degrees of freedom in a mechanical system equals the number of nodes it has.

The forces acting in the passive elements of Fig. 2.2.15a are all vertical and may be described by the scalar quantities f_1 , f_2 , and f_3 and shown also in the free-body diagram of the mass shown in Fig. 2.2.15b and in the circuit of Fig. 2.2.15c. The convention used here is that forces are drawn as *applied to* the node. The arrow directions on f_1 , f_2 , and f_3 are such that when they are positive they tend to *decrease* the displacement of the node. Hence they act out of the node in the circuit. This convention is adopted because we can express f_1 and f_2 [see (2.2.3)] as

$$f_1 = K_1(x - l_1) \quad (2.2.28)$$

$$f_2 = K_2(x - l_2) \quad (2.2.29)$$

and can express f_3 [see (2.2.5)] as

$$f_3 = B \frac{dx}{dt}. \quad (2.2.30)$$

* *Ibid.*, pp. 195–197.

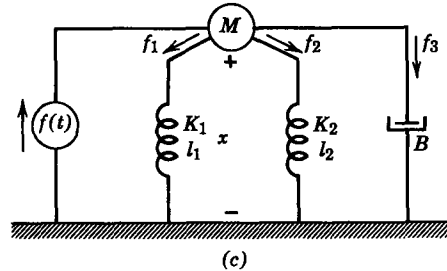
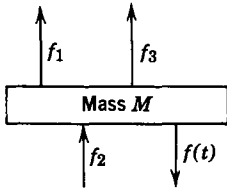
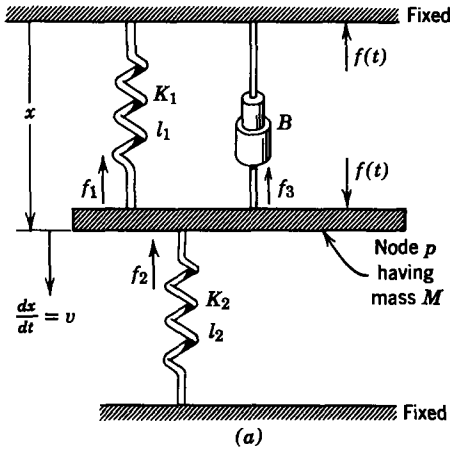


Fig. 2.2.15 System showing forces at a node: (a) system; (b) free body diagram of mass M ; (c) circuit.

Newton's second law (2.2.10) requires that the *algebraic sum of all forces applied to a node in the positive x -direction must equal the acceleration force for the mass of the node*. For the example in Fig. 2.2.15 this requires that

$$f(t) - f_1 - f_2 - f_3 = M \frac{d^2x}{dt^2}. \quad (2.2.31)$$

Note that forces acting in the $+x$ -direction “flow” into the node in Fig. 2.2.15c. Substitution from (2.2.28) through (2.2.30) into (2.2.31) yields the differential equation

$$f(t) - K_1(x - l_1) - K_2(x - l_2) - B \frac{dx}{dt} = M \frac{d^2x}{dt^2}. \quad (2.2.32)$$

Thus, if the system constants are known and $f(t)$ is specified, this differential equation can be solved to find $x(t)$; (2.2.32) is the equation of motion for the system in Fig. 2.2.15.

We can generalize (2.2.31) to describe any mechanical node with mass M and displacement x and with n forces applied by sources and passive elements.

$$\sum_{i=1}^n f_i = M \frac{d^2 x}{dt^2}. \quad (2.2.33)$$

We must exercise caution to include the correct sign on each force in the summation.

In a rotational system the nodes have moments of inertia; thus the summation of torques applied to a node must equal the acceleration torque of the moment of inertia associated with the node. For a node with n torques applied the expression is

$$\sum_{i=1}^n T_i = J \frac{d^2 \theta}{dt^2}. \quad (2.2.34)$$

Once again care must be exercised in attaching the correct sign to each torque in the summation.

As an example of the application of (2.2.34), consider the rotational system in Fig. 2.2.16a for which the mechanical circuit is shown in Fig. 2.2.16b.

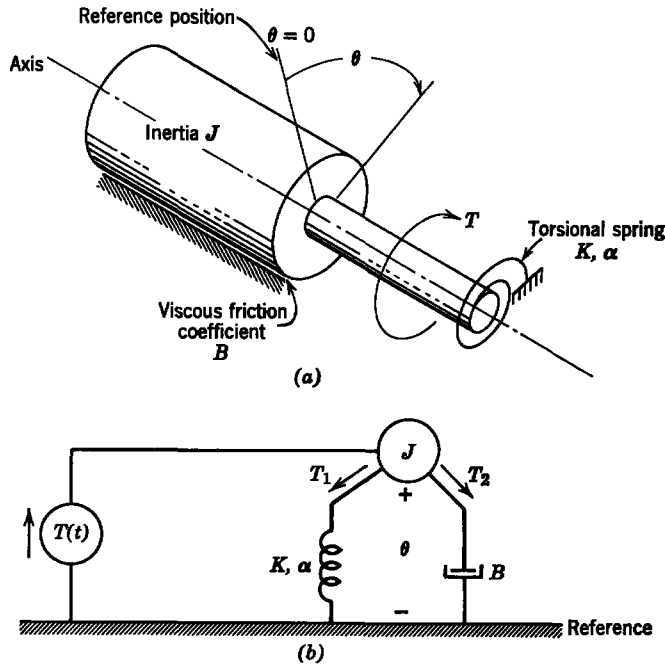


Fig. 2.2.16 Rotational mechanical system: (a) system; (b) mechanical circuit.

We represent the torques *applied to* the node in the $-\theta$ -direction by the passive elements as T_1 and T_2 . Reference to (2.2.4) and (2.2.6) shows that these torques can be expressed as

$$T_1 = K(\theta - \alpha), \quad (2.2.35)$$

$$T_2 = B \frac{d\theta}{dt}. \quad (2.2.36)$$

By use of these expressions with the source and arrow directions in Fig. 2.2.16*b* (2.2.34) yields the equation of motion

$$T(t) - K(\theta - \alpha) - B \frac{d\theta}{dt} = J \frac{d^2\theta}{dt^2}. \quad (2.2.37)$$

Once again, torques acting in the $+\theta$ -direction “flow” into the node. When the constants are known and $T(t)$ is specified, this differential equation can be used to find the response $\theta(t)$.

An important point should be made here. In a mechanical system the reference or “ground” is usually considered fixed; this is necessary if any masses are involved. If the reference is fixed and elements are attached to the reference, it implies that forces not usually considered are available to prevent the ground point from moving. The surface of the earth, for example, is often taken as a reference, and although the earth will move a certain amount when a net force is exerted on it this movement is extremely small and can be neglected.

Equations 2.2.33 and 2.2.34, which are analogous for our forms of mechanical equivalent circuits to Kirchhoff’s current law, are used almost exclusively for formulating equations of motion for mechanical systems. A second relation for mechanical systems analogous to Kirchhoff’s voltage law is seldom used in a formal way but must at all times be satisfied. This second relation is called geometrical compatibility or continuity of space. To illustrate this concept with an example we use the system in Fig. 2.2.17. This system has three mechanical nodes (p , q , and r) whose velocities (v_1 , v_2 , and v_3) are measured in relation to the fixed point g .

The mechanical circuit of Fig. 2.2.17*b* has two independent mechanical loops. From examination of Fig. 2.2.17*a* it is evident that the velocity of p must be equal to the velocity of q , plus the velocity of p relative to q .

$$v_1 + (v_2 - v_1) - v_2 = 0. \quad (2.2.38)$$

This condition, which is identically satisfied, states that in the left-hand loop of Fig. 2.2.17*b* the sum of the velocity differences around the loop must be zero. Similarly, for the right-hand loop of Fig. 2.2.17*b*

$$v_2 + (v_3 - v_2) - v_3 = 0. \quad (2.2.39)$$

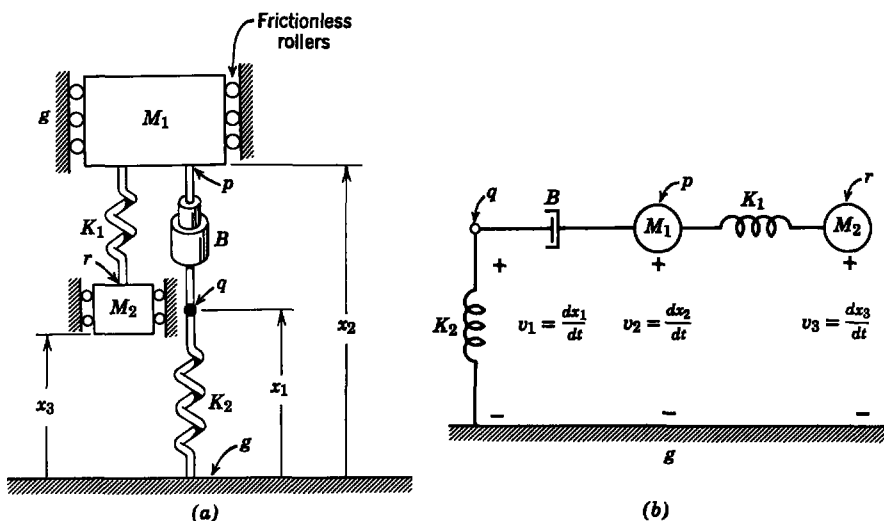


Fig. 2.2.17 Mechanical system and its network diagram: (a) system; (b) mechanical equivalent circuit.

We can generalize from this example to state that for a mechanical loop with n elements geometrical compatibility requires that

$$\sum_{i=1}^n (v_{i+1} - v_i) = 0, \quad (2.2.40)$$

where $(v_{i+1} - v_i)$ is the velocity difference across the i th element taken positive in the direction of summation. An equivalent relation for rotational systems can be obtained by summing angular velocity differences around a loop.

In establishing the loop equations it is preferable to work with *velocity differences* as above. If displacements are used, the geometric compatibility equations will contain constant terms, such as unstretched lengths of springs, and will be more complicated than (2.2.40).

As stated earlier, the geometric compatibility relation is not often explicitly used in formulating equations of motion. Nonetheless, it must be satisfied.

Example 2.2.1. Consider the simplified model of an automobile suspension system shown in Fig. 2.2.18a. The tire surface is excited by variations in the road surface. We wish to formulate equations suitable for determining the motion of the auto mass M_1 and the force on the tires.

The mechanical circuit is shown in Fig. 2.2.18b. The road acts as a position source applied to node 1, as shown. We assume that the references for measuring displacements of the two springs are chosen so that the equilibrium lengths are zero. In the circuit of

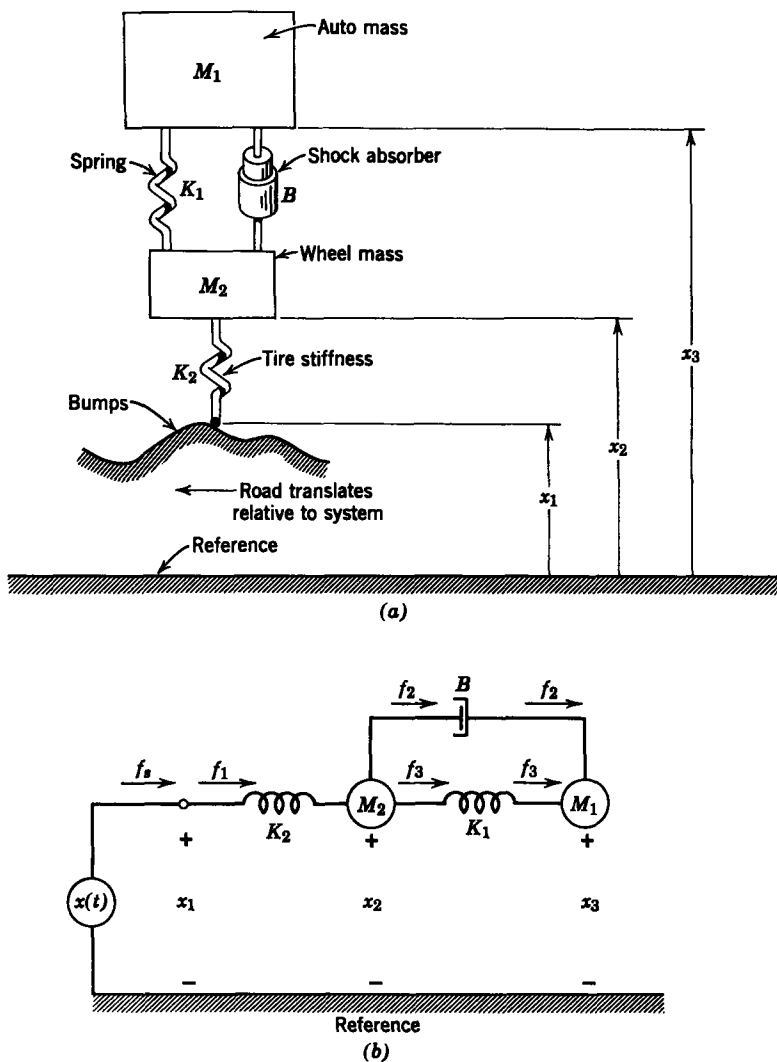


Fig. 2.2.18 One-dimensional model of auto suspension: (a) system; (b) equivalent circuit.

Fig. 2.2.18b three passive ideal mechanical elements (excluding masses) appear. The equations for these ideal elements are

$$f_1 = K_2(x_1 - x_2), \quad (a)$$

$$f_2 = B \left(\frac{dx_2}{dt} - \frac{dx_3}{dt} \right), \quad (b)$$

$$f_3 = K_1(x_2 - x_3). \quad (c)$$

These equations may be combined at nodes 2 and 3 according to (2.2.33) to eliminate the forces at those nodes (which are not of interest).

$$K_2(x_1 - x_2) = B \left(\frac{dx_2}{dt} - \frac{dx_3}{dt} \right) + K_1(x_2 - x_3) + M_2 \frac{d^2 x_2}{dt^2}, \quad (d)$$

$$B \left(\frac{dx_2}{dt} - \frac{dx_3}{dt} \right) + K_1(x_2 - x_3) = M_1 \frac{d^2 x_3}{dt^2}. \quad (e)$$

With the specified position source

$$x_1 = x(t), \quad (f)$$

(d) and (e) can be solved for x_2 and x_3 . Then the force f_s applied to the tires by the road can be found from (a) as

$$f_s = f_1 = K_2(x_1 - x_2). \quad (g)$$

Note that the forces acting on the reference in the network diagram *do not balance* but equal f_s . It is presumed that the force transmitted to the earth by the automobile tires will not move the earth.

2.3 DISCUSSION

In this chapter we have laid the foundation for studying lumped-parameter electromechanics by reviewing the derivations of lumped electric circuit elements, by generalizing the derivations to include the effects of mechanical motion, and by reviewing the basic definitions and techniques of rigid-body mechanics. The stage is now set to include the electromechanical coupling network of Fig. 2.0.1 and to study some general properties of electromechanical systems, including the techniques for obtaining complete equations of motion.

PROBLEMS

2.1. A piece of infinitely permeable magnetic material completes the magnetic circuit in Fig. 2P.1 in such a way that it is free to move in the x - or y -direction. Under the assumption that the air gaps are short compared with their cross-sectional dimensions (i.e., that the fields are as shown), find $\lambda(x, y, i)$. For what range of x and y is this expression valid?

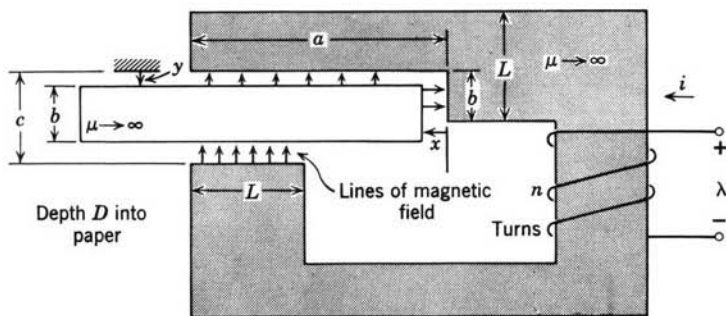


Fig. 2P.1

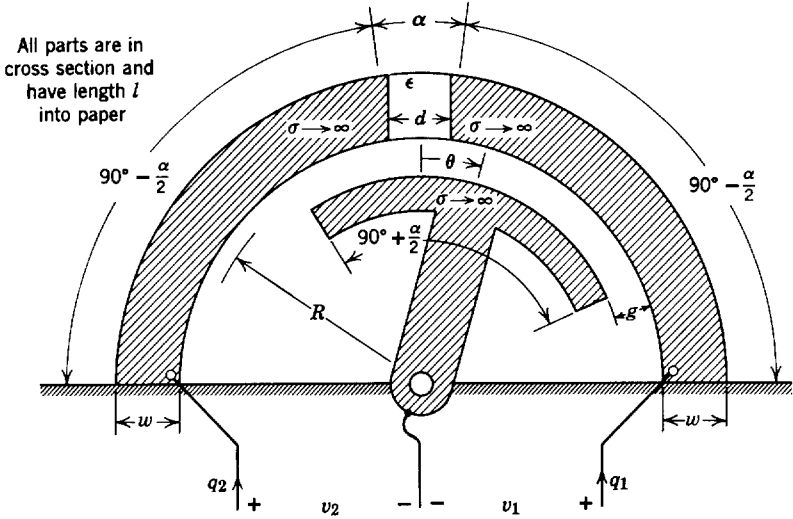


Fig. 2P.2

2.2. Three pieces of infinitely conducting material are arranged as shown in Fig. 2P.2. The two outer pieces are stationary and are separated by a block of insulating material of permittivity ϵ . The inner piece is free to rotate an angle θ . The gap g is much less than the average radius R , which implies that the fields are approximately those of a plane-parallel geometry. Neglect the fringing fields. Find $q_1(v_1, v_2, \theta)$, $q_2(v_1, v_2, \theta)$.

2.3. The cross section of a cylindrical solenoid used to position the valve mechanism of a hydraulic control system is shown in Fig. 2P.3. When the currents i_1 and i_2 are equal, the plunger is centered horizontally ($x = 0$). When the coil currents are unbalanced, the plunger moves a distance x . The nonmagnetic sleeves keep the plunger centered radially. The

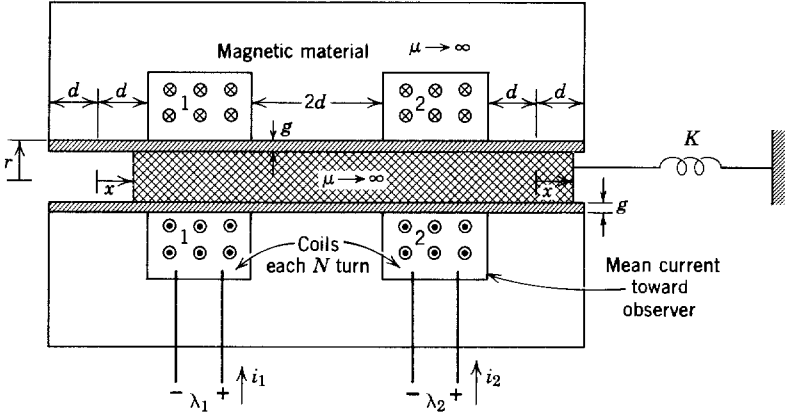


Fig. 2P.3

displacement x is limited to the range $-d < x < d$. Show that the electrical terminal relations are

$$\lambda_1 = L_{11}i_1 + L_{12}i_2,$$

$$\lambda_2 = L_{12}i_1 + L_{22}i_2,$$

where

$$L_{11} = L_0 \left[3 - 2 \left(\frac{x}{d} \right) - \left(\frac{x}{d} \right)^2 \right],$$

$$L_{22} = L_0 \left[3 + 2 \left(\frac{x}{d} \right) - \left(\frac{x}{d} \right)^2 \right],$$

$$L_{12} = L_0 \left[1 - \left(\frac{x}{d} \right)^2 \right].$$

What is L_0 in terms of the system geometry?

- 2.4. (a) Write the differential equation governing the motion of mass M acted on by the force source f and the linear damper with coefficient B (Fig. 2P.4).
 (b) Calculate and make a dimensioned sketch of dx/dt and x as functions of time for $t > 0$ when the force source is the impulse ($u_0 =$ unit impulse) $f = I_0 u_0(t)$. (This is like hitting the mass with a hammer.)

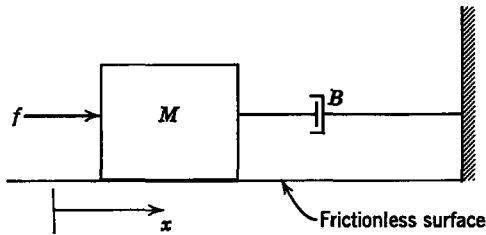


Fig. 2P.4

- 2.5. (a) Find the response $x(t)$ of the system shown in Fig. 2P.5a to a driving force $f(t)$ which is
 (1) an impulse $f(t) = I_0 u_0(t)$,
 (2) a step $f(t) = F_0 u_{-1}(t)$.

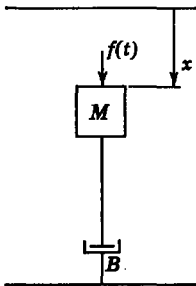


Fig. 2P.5a

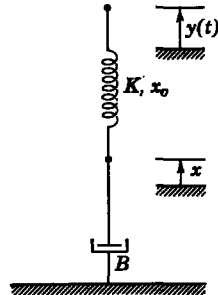


Fig. 2P.5b

(b) Find the response $x(t)$ of the system shown in Fig. 2P.5b to a driving displacement $y(t)$ which is

- (1) $y(t) = Au_0(t),$
- (2) $y(t) = Y_0\mu_{-1}(t).$

2.6. The mechanical system shown in Fig. 2P.6 is set into motion by a forcing function $f(t)$. This motion is translational only. The masses M_2 and M_3 slip inside the cans as shown. Note that the upper can is attached to the mass M_1 .

- (a) Draw the mechanical circuit with nodes and parameters designated.
- (b) Write three differential equations in $x_1, x_2,$ and x_3 to describe the motion.

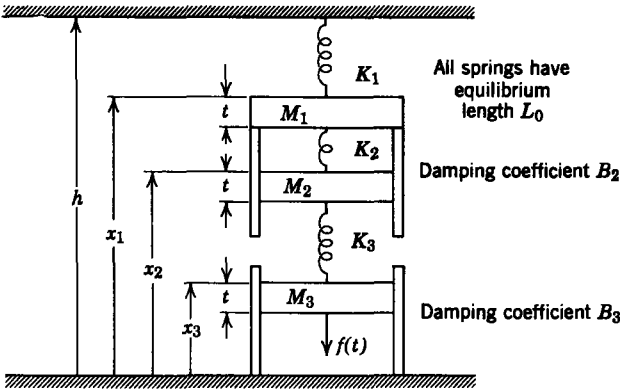


Fig. 2P.6

2.7. In the system in Fig. 2P.7 the two springs have zero force when both x_1 and x_2 are zero. A mechanical force f is applied to node 2 in the direction shown. Write the equations governing the motion of the nodes 1 and 2. What are the natural frequencies involved?

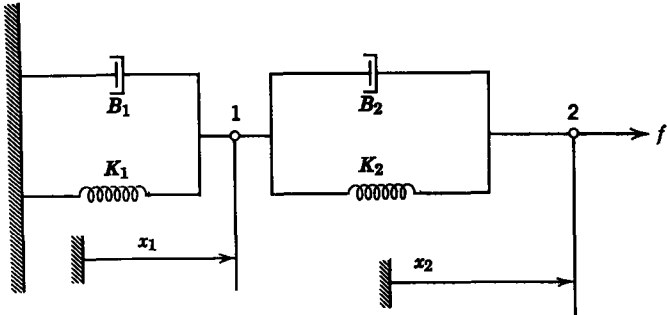


Fig. 2P.7

2.8. The velocity of the point P shown in Fig. 2P.8 is

$$\mathbf{v} = \mathbf{i}_r \frac{dr}{dt} + \mathbf{i}_\theta r \frac{d\theta}{dt}.$$

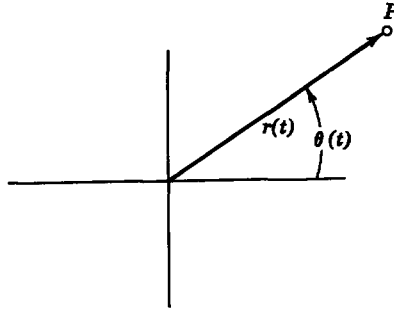


Fig. 2P.8

Show that the acceleration is

$$\frac{d\mathbf{v}}{dt} = \mathbf{i}_r \left[\frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \right] + \mathbf{i}_\theta \left(r \frac{d^2 \theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} \right),$$

where

$$-r \left(\frac{d\theta}{dt} \right)^2 = \text{centripetal acceleration},$$

$$2 \frac{dr}{dt} \frac{d\theta}{dt} = \text{Coriolis acceleration}.$$

Hint. Remember in carrying out the time derivatives that $\mathbf{i}_r$ and $\mathbf{i}_\theta$ are functions of time. In fact, you will wish to show that

$$\frac{d\mathbf{i}_r}{dt} = \mathbf{i}_\theta \frac{d\theta}{dt}, \quad \frac{d\mathbf{i}_\theta}{dt} = -\mathbf{i}_r \frac{d\theta}{dt}.$$

Chapter 3

LUMPED-PARAMETER ELECTROMECHANICS

3.0 INTRODUCTION

Having reviewed the derivations of lumped electric circuit elements and rigid-body mechanical elements and generalized these concepts to allow inclusion of electromechanical coupling, we are now prepared to study some of the consequences of this coupling.

In the analysis of lumped-parameter electromechanical systems experience has shown that sufficient accuracy is obtained in most cases by making a lossless model of the coupling system. Thus energy methods are used to provide simple and expeditious techniques for studying the coupling process.

After introducing the method of calculating the energy stored in an electromechanical coupling field, we present energy methods for obtaining forces of electric origin. We shall then study the energy conversion process in coupling systems and finally discuss the formalism of writing equations of motion for complete electromechanical systems. The techniques for analyzing the dynamic behavior of lumped-parameter electromechanical systems are introduced and illustrated in Chapter 5.

3.1 ELECTROMECHANICAL COUPLING

There are four technically important forces of electric origin.

1. The force resulting from an electric field acting on free charge.
2. The force resulting from an electric field acting on polarizable material.
3. The force resulting from a magnetic field acting on a moving free charge (a current).
4. The force resulting from a magnetic field acting on magnetizable material.

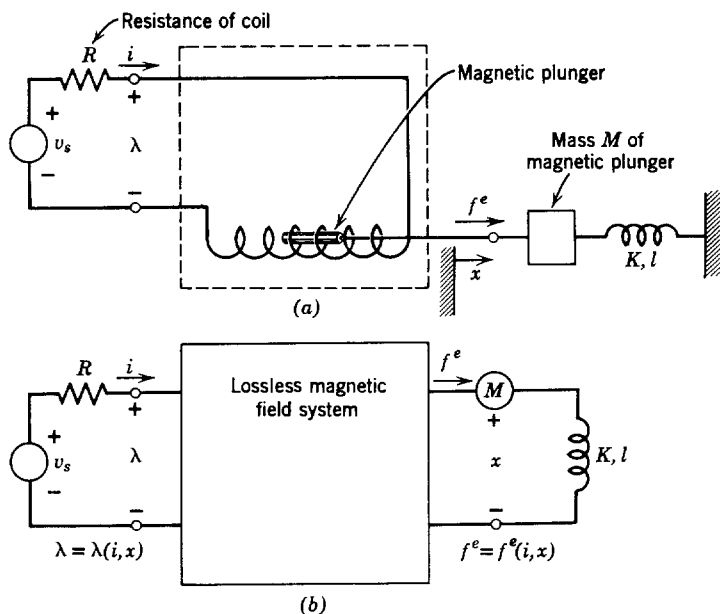


Fig. 3.1.1 (a) A magnetic field electromechanical system; (b) its representation in terms of terminal pairs. Note that the coupling network does not include mechanical energy storages (M) or electrically dissipative elements (R).

Because of the restriction of our treatment to quasi-static systems, the fields that give rise to forces in a particular element are electric or magnetic, but not both. Thus we can consider separately the forces due to electric fields and the forces due to magnetic fields.

To illustrate how the coupling can be taken into account suppose the problem to be considered is the magnetic field system shown in Fig. 3.1.1. The electromechanical coupling occurs between one electrical terminal pair with the variables i and λ and one mechanical terminal pair composed of the node x acted on by the electrical force f^e . It has been demonstrated in Sections 2.1.1 and 2.1.2 that the electrical terminal variables are related by an electrical terminal relation expressible in the form

$$\lambda = \lambda(i, x). \quad (3.1.1)$$

This relation tells us the value of λ , given the values of i and x . We can say, given the *state* (i, x) of the magnetic field system enclosed in the box, that the value of λ is known.

We now make a crucial assumption, motivated by the form of the electrical equation: given the current i and position x , the force of electric origin has a certain single value

$$f^e = f^e(i, x); \quad (3.1.2)$$

that is, the force f^e exerted by the system in the box on the mechanical node is a function of the state (i, x) . This is reasonable if the box includes only those elements that store energy in the magnetic field. Hence all purely electrical elements (inductors that do not involve x , capacitors, and resistors) and purely mechanical elements (all masses, springs, and dampers) are connected to the terminals externally.

Note that f^e is defined as the force of electrical origin applied to the mechanical node in a direction that tends to increase the relative displacement x . Because (3.1.1) can be solved for i to yield

$$i = i(\lambda, x), \quad (3.1.3)$$

the force f^e can also be written as

$$f^e = f^e(\lambda, x). \quad (3.1.4)$$

It is well to remember that the functions of (3.1.2) and (3.1.4) are different because the variables are different; however, for a particular set of i, λ, x the force f^e will have the same numerical value regardless of the equation used.

In a similar way the mechanical force of electric origin for an electric field system (see Fig. 3.1.2) can be written as

$$f^e = f^e(q, x) \quad (3.1.5)$$

or

$$f^e = f^e(v, x). \quad (3.1.6)$$

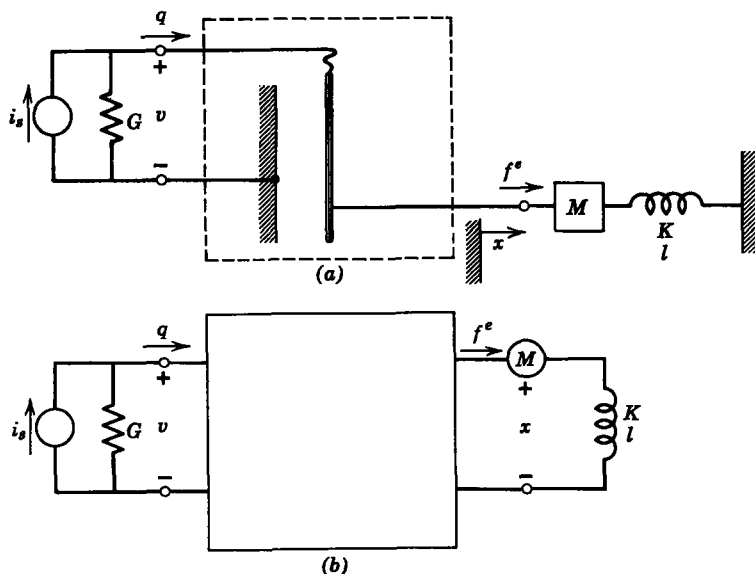


Fig. 3.1.2 (a) An electric field electromechanical system; (b) its representation in terms of terminal pairs. Note that the coupling network does not include mechanical energy storage elements (M) or electrically dissipative elements (G).

When the mechanical motion is rotational, the same ideas apply. We replace force f^e by torque T^e and displacement x by angular displacement θ .

Although the systems of Figs. 3.1.1 and 3.1.2 have only one electrical and one mechanical terminal pair, the discussion can be generalized to any arbitrary number of terminal pairs. For instance, if an electric field system has N electrical terminal pairs and M mechanical terminal pairs for which the terminal relations are specified by (2.1.36), then (3.1.6) is generalized to

$$\begin{aligned} f_i^e &= f_i^e(v_1, v_2, \dots, v_N; x_1, x_2, \dots, x_M), \\ i &= 1, 2, \dots, M, \end{aligned} \quad (3.1.7)$$

where the subscript i denotes the mechanical terminal pair at which f_i^e is applied to the external system by the coupling field. The other forms of f^e can be generalized in the same way.

The next question to be considered is how to determine the force f^e for a particular system. One method is to solve the field problem, find force densities, and then perform a volume integration to find the total force. This process, described in Chapter 8, supports our assumption that f^e has the form of (3.1.2) and (3.1.5). It is often impractical, however, to solve the field problem. A second method of determining f^e is experimental; that is, if the device exists, we can measure f^e as a function of the variables (i and x , λ and x , v and x , or q and x) on which it depends, plot the results, and fit an analytical curve to obtain a function in closed form. This method also has obvious disadvantages.

It is shown in the next section that when the electrical terminal relations are known and the coupling system can be represented as lossless the force f^e can be found analytically. Because electrical lumped parameters are usually easier to calculate and/or measure than mechanical forces, this often provides the most convenient way of determining the mechanical forces of electric origin f_i^e .

3.1.1 Energy Considerations

It will be useful to study some of the general properties of lossless electric and magnetic field energy storages that are functions of geometry. In these considerations we use the conservation of energy (first law of thermodynamics) repeatedly.

As an example, consider again the magnetic field system of Fig. 3.1.1. The system *symbolically* enclosed in the box contains only a magnetic field whose value and therefore energy storage is affected by both electrical and mechanical variables. This coupling network is assumed to be lossless, which means that energy put into the system by the electrical and mechanical terminal pairs is stored in the magnetic field and can be recovered completely

through the terminals. Such a system is often called conservative. We use lossless and conservative as synonyms.

When the total energy stored in the magnetic field is denoted by W_m , the conservation of power for the system can be written as

$$\frac{dW_m}{dt} = i \frac{d\lambda}{dt} - f^e \frac{dx}{dt}. \quad (3.1.8)$$

The term dW_m/dt is the time rate of increase in magnetic energy stored, the term $i(d\lambda/dt)$ is the power input at the electrical terminals, and $[-f^e(dx/dt)]$ is the power input at the mechanical terminals. The minus sign on the mechanical power results because f^e is defined as acting on (into) the mechanical node.

Multiplication of (3.1.8) by dt yields an equation for conservation of energy

$$dW_m = i d\lambda - f^e dx. \quad (3.1.9)$$

From (3.1.3) and (3.1.4), it is evident that only two of the four variables (i, λ, f^e, x) can be set independently without violating the internal physics of the system. There are further restrictions that the external mechanical and electrical systems impose on the terminal pairs of the box (mechanical and electrical circuit equations). If, however, we think of the coupling network as being temporarily disconnected from the electrical and mechanical circuits, we can choose two *independent variables*, say (λ, x) , which through the terminal relations stipulate i and f^e . Our choice of λ and x is motivated by (3.1.9), which shows how incremental changes in these variables are related to incremental changes in the magnetic stored energy W_m . The evaluation of the change in W_m when λ and/or x are varied by finite amounts requires an integration of (3.1.9). This is a line integration through variable space. For the example being considered (Fig. 3.1.1) there are two independent variables (λ, x) ; thus variable space is two-dimensional, as illustrated in Fig. 3.1.3. Independence of variables is indicated by orthogonality of axes. Suppose it is

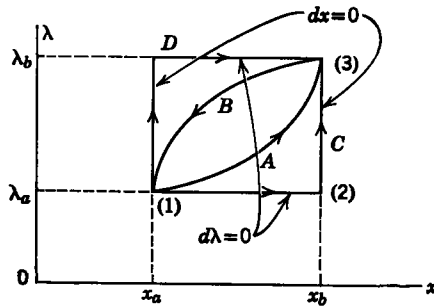


Fig. 3.1.3 Two-dimensional variable space.

desired to find the change in stored energy when the independent variables are changed from the point (λ_a, x_a) to the point (λ_b, x_b) . To evaluate a line integral we must specify the path of integration; an infinite number of possible paths exist between the two points. A property of a conservative system, however, is that its stored energy is a function of its *state* (i.e., of the particular values of λ and x that exist) and does not depend on what succession of variable values or what path through variable space was used to reach that state. A consequence of this property is that if the system variables are made to change along path *A* from (λ_a, x_a) to (λ_b, x_b) in Fig. 3.1.3 and then along path *B* back to (λ_a, x_a) , the net change in stored energy W_m during the process is zero.

In a conservative system the change in stored energy between any two points in variable space is independent of the path of integration. Thus we can select the path that makes the integration easiest. As an example, consider the evaluation of the change in energy between points (λ_a, x_a) and (λ_b, x_b) in Fig. 3.1.3. Along segment 1–2, $d\lambda = 0$; and along segment 2–3, $dx = 0$. Thus, using path *C*, integration of (3.1.9) takes on the particular form

$$W_m(\lambda_b, x_b) - W_m(\lambda_a, x_a) = - \int_{x_a}^{x_b} f^e(\lambda_a, x) dx + \int_{\lambda_a}^{\lambda_b} i(\lambda, x_b) d\lambda. \quad (3.1.10)$$

If, alternatively, we wish to evaluate the integral along path *D* in Fig. 3.1.3, the result is

$$W_m(\lambda_b, x_b) - W_m(\lambda_a, x_a) = \int_{\lambda_a}^{\lambda_b} i(\lambda, x_a) d\lambda - \int_{x_a}^{x_b} f^e(\lambda_b, x) dx. \quad (3.1.11)$$

The energy difference as evaluated by (3.1.10) and (3.1.11) must, of course, be the same.

The integrations given in (3.1.10) and (3.1.11) have a simple physical significance. The integrations of (3.1.10) represent putting energy into the network in two successive steps. First we put the system together mechanically (integrate on x) while keeping λ constant. In general, this operation requires doing work against the force f^e , and this is the contribution of the first integral in (3.1.10) to the energy stored in the coupling network. Then we put energy in through the electrical terminals, keeping the geometry (x) fixed. The second integral is the energy supplied by an electrical source which provides the excitation λ . In (3.1.11) these successive steps are reversed in order.

We always define electrical terminal pairs that account for the excitation of all electric or magnetic fields in the system. Then, when the electrical terminal variables are zero ($\lambda_a = 0$ in the present example), we can say that there is no force of electrical origin. The difference between (3.1.10) and (3.1.11) with $\lambda_a = 0$ is crucial, for in the first the contribution of f^e to the integration is zero [$f^e(0, x) = 0$], whereas in the second we must know f^e to

carry out the integration; that is, by first integrating on the mechanical variables and then on the electrical variables we can determine W_m from the electrical terminal relations. Physically, this simply means that if we put the system together mechanically when no force is required we can account for all the energy stored by putting it in through the electrical terminal pairs.

An example of a system in which there will be energy stored in the network and a force of electrical origin even with no external electrical excitation is the permanent magnet device of Example 2.1.3. In that example, however, it was shown that we could replace the permanent magnet with an externally excited terminal pair; hence this case imposes no restriction on our development.

We can also study electric field systems using the conservation of energy. For the example in Fig. 3.1.2, with the electrical energy stored in the system denoted by W_e , the conservation of power can be written as

$$\frac{dW_e}{dt} = v \frac{dq}{dt} - f^e \frac{dx}{dt}, \quad (3.1.12)$$

and multiplication by dt yields the conservation of energy

$$dW_e = v dq - f^e dx. \quad (3.1.13)$$

A comparison of (3.1.9) and (3.1.13) shows that the description of lossless magnetic field systems can be used directly for electric field systems by replacing W_m by W_e , i by v , and λ by q . All the mathematical processes are exactly the same.

These examples are systems with one electrical and one mechanical terminal pair. The results can be extended to systems with any arbitrary number of terminal pairs; for example, consider an electric field system with N electrical terminal pairs and M rotational mechanical terminal pairs. Then the conservation of energy can be written as

$$\frac{dW_e}{dt} = \sum_{i=1}^N v_i \frac{dq_i}{dt} - \sum_{i=1}^M T_i^e \frac{d\theta_i}{dt}, \quad (3.1.14)$$

where v_i and q_i are the voltage and charge associated with the i th electrical terminal pair, T_i^e and θ_i are the torque and angular displacement at the i th mechanical terminal pair, and W_e represents the total electric energy stored in the system.

Multiplication of (3.1.14) by dt yields

$$dW_e = \sum_{i=1}^N v_i dq_i - \sum_{i=1}^M T_i^e d\theta_i. \quad (3.1.15)$$

For this system there will be N electrical terminal relations of the general form

$$v_i = v_i(q_1, q_2, \dots, q_N; \theta_1, \theta_2, \dots, \theta_M); i = 1, 2, \dots, N \quad (3.1.16)$$

and M mechanical terminal relations

$$T_i^e = T_i^e(q_1, q_2, \dots, q_N; \theta_1, \theta_2, \dots, \theta_M); i = 1, 2, \dots, M. \quad (3.1.17)$$

As a result of the use of (3.1.16) and (3.1.17), (3.1.15) is expressed as a function of $(N + M)$ independent variables, the N charges and M angles. Thus the stored energy can be written in general as

$$W_e = W_e(q_1, q_2, \dots, q_N; \theta_1, \theta_2, \dots, \theta_M) \quad (3.1.18)$$

and W_e can be obtained by integrating (3.1.15) along any convenient path through the $(N + M)$ -dimensional variable space.

Further generalization of these ideas to magnetic field systems and translational mechanical terminal pairs is straightforward and is not carried out here (see Table 3.1). Example 3.1.1 illustrates the line integration that has been described.

3.1.2 Mechanical Forces of Electric Origin

Now that we have specified the formalism by which we calculate stored energy, we shall derive mechanical forces of electric origin by using the conservation of energy.

3.1.2a Force-Energy Relations

To start with a simple example, we consider again the magnetic field system of Fig. 3.1.1 which was described mathematically by (3.1.3), (3.1.4), and (3.1.9). From these expressions it is clear that the magnetic stored energy W_m is expressible as a function of the two independent variables λ and x .

$$W_m = W_m(\lambda, x). \quad (3.1.19)$$

We shall find that if the system is to be conservative the energy must be a single-valued function of the independent variables (λ, x) with finite second partial derivatives. Making this restriction on W_m we can formally take the total differential of (3.1.19) to obtain

$$dW_m = \frac{\partial W_m}{\partial \lambda} d\lambda + \frac{\partial W_m}{\partial x} dx, \quad (3.1.20)$$

where the partial derivatives are taken by using λ and x as independent variables. When (3.1.20) is subtracted from (3.1.9), the result is

$$0 = \left(i - \frac{\partial W_m}{\partial \lambda} \right) d\lambda - \left(f^e + \frac{\partial W_m}{\partial x} \right) dx. \quad (3.1.21)$$

Table 3.1 Energy Relations for an Electromechanical Coupling Network with N Electrical and M Mechanical Terminal Pairs*

Magnetic Field Systems		Electric Field Systems	
Conservation of Energy			
$dW_m = \sum_{j=1}^N i_j d\lambda_j - \sum_{j=1}^M f_j^e dx_j$	(a)	$dW_e = \sum_{j=1}^N v_j dq_j - \sum_{j=1}^M f_j^e dx_j$	(b)
$dW'_m = \sum_{j=1}^N \lambda_j di_j + \sum_{j=1}^M f_j^e dx_j$	(c)	$dW'_e = \sum_{j=1}^N q_j dv_j + \sum_{j=1}^M f_j^e dx_j$	(d)
Forces of Electric Origin, $j = 1, \dots, M$			
$f_j^e = - \frac{\partial W_m(\lambda_1, \dots, \lambda_N; x_1, \dots, x_M)}{\partial x_j}$	(e)	$f_j^e = - \frac{\partial W_e(q_1, \dots, q_N; x_1, \dots, x_M)}{\partial x_j}$	(f)
$f_j^e = \frac{\partial W'_m(i_1, \dots, i_N; x_1, \dots, x_M)}{\partial x_j}$	(g)	$f_j^e = \frac{\partial W'_e(v_1, \dots, v_N; x_1, \dots, x_M)}{\partial x_j}$	(h)
Relation of Energy to Coenergy			
$W_m + W'_m = \sum_{j=1}^N \lambda_j i_j$	(i)	$W_e + W'_e = \sum_{j=1}^N v_j q_j$	(j)
Energy and Coenergy from Electrical Terminal Relations			
$W_m = \sum_{j=1}^N \int_0^{\lambda_j} i_j(\lambda_1, \dots, \lambda_{j-1}, \lambda'_j, 0, \dots, 0; x_1, \dots, x_M) d\lambda'_j$	(k)	$W_e = \sum_{j=1}^N \int_0^{q_j} v_j(q_1, \dots, q_{j-1}, q'_j, 0, \dots, 0; x_1, \dots, x_M) dq'_j$	(l)
$W'_m = \sum_{j=1}^N \int_0^{i_j} \lambda_j(i_1, \dots, i_{j-1}, i'_j, 0, \dots, 0; x_1, \dots, x_M) di'_j$	(m)	$W'_e = \sum_{j=1}^N \int_0^{v_j} q_j(v_1, \dots, v_{j-1}, v'_j, 0, \dots, 0; x_1, \dots, x_M) dv'_j$	(n)

* The mechanical variables f_j and x_j can be regarded as the j th force and displacement or the j th torque T_j and angular displacement θ_j .

The variables λ and x are independent. Thus $d\lambda$ and dx can have arbitrary values, and the equation must be satisfied by requiring the coefficients of $d\lambda$ and dx to be zero:

$$i = \frac{\partial W_m(\lambda, x)}{\partial \lambda}, \quad (3.1.22)$$

$$f^e = - \frac{\partial W_m(\lambda, x)}{\partial x}. \quad (3.1.23)$$

If the stored energy is known, the electrical and mechanical terminal relations can now be calculated.

Equations 3.1.22 and 3.1.23 can be generalized to describe a system with arbitrary numbers of electrical and mechanical terminal pairs (see Table 3.1). To illustrate this generalization we consider again the electric field system of N electrical terminal pairs and M rotational terminal pairs which was described mathematically by (3.1.14) to (3.1.18). We now take the total differential of (3.1.18),

$$dW_e = \sum_{i=1}^N \frac{\partial W_e}{\partial q_i} dq_i + \sum_{i=1}^M \frac{\partial W_e}{\partial \theta_i} d\theta_i. \quad (3.1.24)$$

Subtraction of (3.1.24) from (3.1.15) yields

$$0 = \sum_{i=1}^N \left(v_i - \frac{\partial W_e}{\partial q_i} \right) dq_i - \sum_{i=1}^M \left(T_i^e + \frac{\partial W_e}{\partial \theta_i} \right) d\theta_i. \quad (3.1.25)$$

All N of the q_i 's and M of the θ_i 's are independent. Thus each coefficient of dq_i and $d\theta_i$ must be equal to zero:

$$v_i = \frac{\partial W_e}{\partial q_i}; \quad i = 1, 2, \dots, N, \quad (3.1.26)$$

$$T_i^e = - \frac{\partial W_e}{\partial \theta_i}; \quad i = 1, 2, \dots, M. \quad (3.1.27)$$

These expressions are generalizations of (3.1.22) and (3.1.23) to describe systems with arbitrary numbers of terminal pairs. They indicate that when the stored energy W_e is known as a function of the independent variables all terminal relations can be calculated (see Table 3.1).

It is usually easier in practice to determine the electrical terminal relations by calculation or measurement than it is to determine the mechanical terminal relations or the stored energy. We have seen that the electrical terminal relations are sufficient to evaluate the stored energy if we choose a path of integration in variable space that keeps electrical excitations zero while mechanical variables are brought to their final values. Once the stored energy is known, the force f^e can be calculated as a derivative of the stored

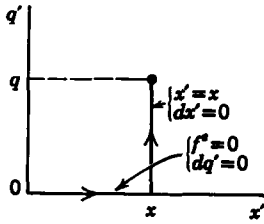


Fig. 3.1.4 Variable space for system of Fig. 3.1.2.

energy [see (3.1.23) or (3.1.27)]. Thus the properties of a coupling system can be determined completely if the electrical terminal relations are known and the system is represented by a conservative model.

To illustrate these ideas consider the electric field system of Fig. 3.1.2 for which the electrical terminal relation is

$$v = v(q, x). \quad (3.1.28)$$

The path of integration in the q - x plane to be used in evaluating stored energy W_e is shown in Fig. 3.1.4. If we use (3.1.13), the energy at point (q, x) is

$$W_e(q, x) = \int_0^x -f^e(0, x') dx' + \int_0^q v(q', x) dq'. \quad (3.1.29)$$

In this expression and in Fig. 3.1.4 the primes denote running variables and (q, x) represents the fixed end point of the line integration. The first term on the right of (3.1.29) is zero because f^e is the force of interaction between charges and electric fields, and with no charge ($q = 0$) f^e must be zero. Thus (3.1.29) can be written for this particular path of integration in the simpler form

$$W_e(q, x) = \int_0^q v(q', x) dq'. \quad (3.1.30)$$

This result can be generalized in a straightforward way to magnetic field coupling systems, rotational mechanical systems, and multiterminal-pair systems. The generalized force and energy relations are summarized in Table 3.1. This table is intended to illustrate the generality and interrelations of the equations. These general equations are *not* intended for use in the solution of most problems. The concepts and techniques are simple enough that it is good practice to start from the conservation of energy and derive the forces in each problem. In this way we can be certain that fundamental physical laws are satisfied.

Example 3.1.1. To illustrate the use of this technique consider again the electric field system that was treated earlier in Example 2.1.5 and represented by Fig. 2.1.8. That figure is reproduced here as Fig. 3.1.5 for convenience. The electrical terminal relations were derived in Example 2.1.5 and are expressible in the forms

$$v_1 = S_1(x_1, x_2)q_1 + S_m(x_1, x_2)q_2, \quad (a)$$

$$v_2 = S_m(x_1, x_2)q_1 + S_2(x_1, x_2)q_2, \quad (b)$$

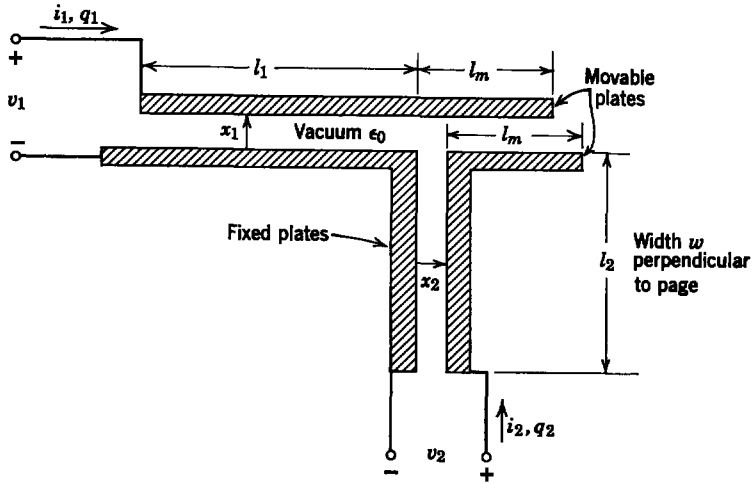


Fig. 3.1.5. Multiply excited electric field system.

where we have solved (a) and (b) of Example 2.1.5 for v_1 and v_2 , and therefore have

$$S_1 = \frac{C_2}{C_1 C_2 - C_m^2},$$

$$S_2 = \frac{C_1}{C_1 C_2 - C_m^2},$$

$$S_m = \frac{C_m}{C_1 C_2 - C_m^2};$$

C_1 , C_2 , and C_m are the functions of x_1 and x_2 given by (c), (d), and (e) of Example 2.1.5.

The system is first assembled mechanically with q_1 and q_2 zero, during which process no energy is put into the system. Next, charges q_1 and q_2 are brought to their final values with x_1 and x_2 fixed. This step requires an integration along a path in the q_1 - q_2 plane. The path chosen for this example is shown in Fig. 3.1.6. Along this path the running variables are related by

$$q'_2 = \frac{q_2}{q_1} q'_1;$$

thus the necessary integral takes the form

$$W_e(q_1, q_2, x_1, x_2) = \int_0^{q_1} \left[v_1 \left(q'_1, \frac{q_2}{q_1} q'_1, x_1, x_2 \right) dq'_1 + v_2 \left(q'_1, \frac{q_2}{q_1} q'_1, x_1, x_2 \right) \frac{q_2}{q_1} dq'_1 \right]. \quad (c)$$

Substitution of (a) and (b) into (c) and evaluation of the integral yields

$$W_e(q_1, q_2, x_1, x_2) = \frac{1}{2} S_1(x_1, x_2) q_1^2 + S_m(x_1, x_2) q_1 q_2 + \frac{1}{2} S_2(x_1, x_2) q_2^2. \quad (d)$$

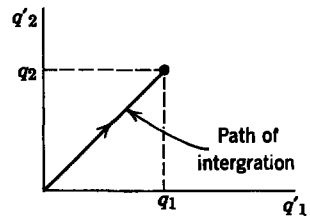


Fig. 3.1.6 Illustrating a path for line integration in variable space for Example 3.1.1.

From this expression we can now evaluate the mechanical forces of electric origin f_1^e and f_2^e (mechanical terminal relations); thus

$$f_1^e(q_1, q_2, x_1, x_2) = -\frac{\partial W^e}{\partial x_1} = -\frac{1}{2}q_1^2 \frac{\partial S_1}{\partial x_1} - q_1 q_2 \frac{\partial S_m}{\partial x_1} - \frac{1}{2}q_2^2 \frac{\partial S_2}{\partial x_1}, \quad (e)$$

$$f_2^e(q_1, q_2, x_1, x_2) = -\frac{\partial W^e}{\partial x_2} = -\frac{1}{2}q_1^2 \frac{\partial S_1}{\partial x_2} - q_1 q_2 \frac{\partial S_m}{\partial x_2} - \frac{1}{2}q_2^2 \frac{\partial S_2}{\partial x_2}. \quad (f)$$

Because S_1 , S_2 , and S_m are known as functions of x_1 and x_2 for this example, the derivatives in (e) and (f) can be calculated; this is straightforward differentiation, however, and is not carried out here.

3.1.2b Force-Coenergy Relations

So far in the magnetic field examples the flux linkage λ has been used as the independent variable, with current i described by the terminal relation. Similarly, in electric field examples charge q has been used as the independent variable, with voltage v described by the terminal relation. These choices were natural because of the form of the conservation of energy equations (3.1.9) and (3.1.13). Note that in Example 3.1.1 we were required to find $v_1(q_1, q_2)$ and $v_2(q_1, q_2)$. It would have been more convenient if we had been able to use $q_1(v_1, v_2)$ and $q_2(v_1, v_2)$, for this is the form these equations took in Example 2.1.5. We consider next how this can be done.

It should be possible to analyze systems using current as the independent electrical variable for magnetic field systems and voltage as the independent variable for electric field systems. In fact, it is often more convenient to make this choice. Alternatively, it is sometimes convenient to use a hybrid set of variables consisting of both currents and flux linkages in magnetic field systems and voltages and charges in electric field systems. Such hybrid sets of variables are used in Chapter 5.

To illustrate this change of independent variables consider once again the magnetic field system described in Fig. 3.1.1, with the restriction that the current i is to be used as the independent variable. The conservation of energy as expressed by (3.1.9) is still a fundamental relation:

$$dW_m = i d\lambda - f^e dx. \quad (3.1.9)$$

The electrical terminal relation is (3.1.1),

$$\lambda = \lambda(i, x), \quad (3.1.1)$$

and the mechanical terminal relation is (3.1.2),

$$f^e = f^e(i, x). \quad (3.1.2)$$

Equation 3.1.9 can be written in a form that involves di and dx by first using the rule of differentiation,

$$i d\lambda = d(\lambda i) - \lambda di. \quad (3.1.31)$$

Then the energy equation (3.1.9) is

$$dW'_m = \lambda di + f^e dx, \quad (3.1.32)$$

where

$$W'_m = \lambda i - W_m. \quad (3.1.33)$$

The energy equation (3.1.32) now has the required form in which changes in the function W'_m are accounted for by changes in the independent variables (i, x). The function $W'_m(i, x)$ is called the *coenergy* and is defined in terms of the energy $W_m(i, x)$ and terminal relations $\lambda(i, x)$ by (3.1.33).*

Remember that (3.1.32) physically represents conservation of energy for the coupling network. The form of this equation is similar to that of (3.1.9) and our arguments now parallel those of Section 3.1.2a. Because $W'_m = W'_m(i, x)$,

$$dW'_m = \frac{\partial W'_m}{\partial i} di + \frac{\partial W'_m}{\partial x} dx. \quad (3.1.34)$$

We subtract (3.1.34) from (3.1.32) to obtain

$$0 = \left(\lambda - \frac{\partial W'_m}{\partial i} \right) di + \left(f^e - \frac{\partial W'_m}{\partial x} \right) dx. \quad (3.1.35)$$

Because di and dx are independent (arbitrary),

$$\lambda = \frac{\partial W'_m(i, x)}{\partial i}, \quad (3.1.36)$$

$$f^e = \frac{\partial W'_m(i, x)}{\partial x}. \quad (3.1.37)$$

If the stored energy (hence coenergy) is known, the electrical and mechanical terminal relations can be calculated. Comparison of (3.1.37) and (3.1.23) shows the change in the form of the force expression when the electrical variable chosen as independent is changed from λ to i .

The result of (3.1.37) can be generalized to a system with any number of terminal pairs in a straightforward manner (see Table 3.1). For a magnetic field system with N electrical terminal pairs and M translational mechanical terminal pairs the conservation of energy equation becomes

$$dW_m = \sum_{j=1}^N i_j d\lambda_j - \sum_{j=1}^M f_j^e dx_j. \quad (3.1.38)$$

We now use the generalization of (3.1.31),

$$\sum_{j=1}^N i_j d\lambda_j = \sum_{j=1}^N d(i_j \lambda_j) - \sum_{j=1}^N \lambda_j di_j, \quad (3.1.39)$$

* This manipulation, which represents conservation of energy in terms of new independent variables, is called a Legendre transformation in classical mechanics and thermodynamics.

to replace the first term on the right-hand side of (3.1.38). Rearranging terms, we obtain

$$dW'_m = \sum_{j=1}^N \lambda_j di_j + \sum_{j=1}^M f_j^e dx_j, \quad (3.1.40)$$

where

$$W'_m = \sum_{j=1}^N i_j \lambda_j - W_m \quad (3.1.41)$$

and W'_m is the coenergy. The independent variables are $(i_1, i_2, \dots, i_N; x_1, x_2, \dots, x_M)$. We assume that the λ 's and W_m in (3.1.41) are written in terms of these variables, hence that W'_m is a function of these variables. Then

$$dW'_m = \sum_{j=1}^N \frac{\partial W'_m}{\partial i_j} di_j + \sum_{j=1}^M \frac{\partial W'_m}{\partial x_j} dx_j, \quad (3.1.42)$$

and when we subtract (3.1.42) from (3.1.40) and require that the coefficient of each di_j and each dx_j be zero

$$\lambda_j = \frac{\partial W'_m}{\partial i_j}; \quad j = 1, 2, \dots, N, \quad (3.1.43)$$

$$f_j^e = \frac{\partial W'_m}{\partial x_j}; \quad j = 1, 2, \dots, M. \quad (3.1.44)$$

This same process of generalization can be carried out for an electric field system (see Table 3.1); for instance, for the system of N electrical terminal pairs and M rotational mechanical terminal pairs for which the torque was found in (3.1.27) the use of the voltage as the independent variable instead of charge leads to the result

$$T_i^e = \frac{\partial W'_e(v_1, v_2, \dots, v_N; \theta_1, \theta_2, \dots, \theta_M)}{\partial \theta_i}, \quad (3.1.45)$$

where

$$W'_e = \sum_{j=1}^N v_j q_j - W_e. \quad (3.1.46)$$

This expression is obtained by a straightforward process of exactly the same form as that used for the general magnetic field system (3.1.38) to (3.1.44).

It is not necessary to find the coenergy by first determining the energy; for example, we can integrate (3.1.32) to find W'_m just as we integrated (3.1.9) to find W_m . In general, we evaluate W'_m by selecting a path of integration through variable space for (3.1.40) that changes the x_j 's with all electrical excitations zero and then changes electrical excitations with mechanical displacements held fixed.

For a better understanding of the meaning of coenergy consider the

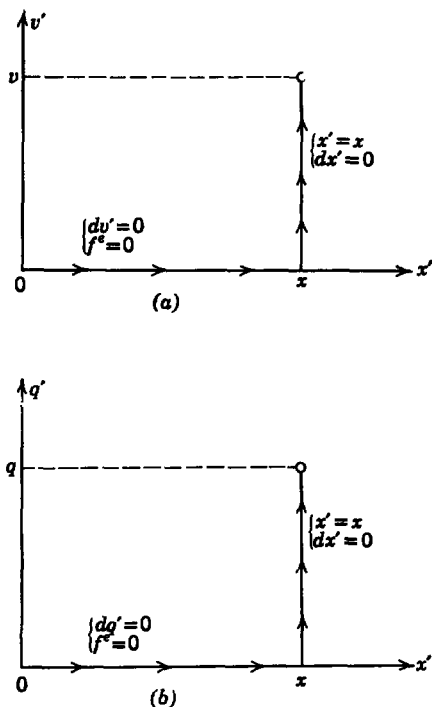


Fig. 3.1.7 Paths of integration in variable space: (a) for evaluating coenergy; (b) for evaluating energy.

simple electric field system presented earlier in Fig. 3.1.2. The coenergy is evaluated by the integration of

$$dW'_e = q dv + f^e dx. \quad (3.1.47)$$

[This is the energy equation (3.1.13) with $v dq = d(vq) - q dv$ and $W'_e = qv - W_e$.] We use the path of integration defined in Fig. 3.1.7a to reduce this integration to

$$W'_e = \int_0^v q(v', x) dv'. \quad (3.1.48)$$

In the case of electrical linearity

$$q(v, x) = C(x)v, \quad (3.1.49)$$

and (3.1.48) becomes

$$W'_e = \frac{1}{2} C v^2. \quad (3.1.50)$$

It follows that

$$f^e = \frac{\partial W'_e(v, x)}{\partial x} = \frac{1}{2} v^2 \frac{dC}{dx}. \quad (3.1.51)$$

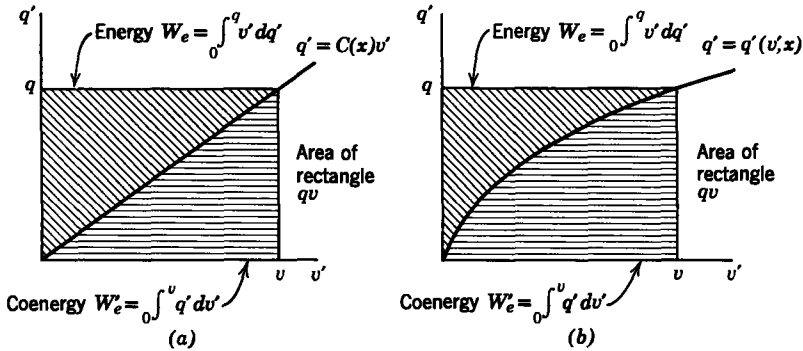


Fig. 3.1.8 Illustration of energy and coenergy: (a) electrically linear system; (b) electrically nonlinear system.

We can compare this result with what we find if we integrate (3.1.13) along the path of Fig. 3.1.7b to find the energy

$$W_e = \int_0^q v(q', x) dq', \quad (3.1.52)$$

which from (3.1.49) is

$$W_e(q, x) = \frac{q^2}{2C(x)}. \quad (3.1.53)$$

Now, when we use (3.1.49) to eliminate q from this expression, we see that the coenergy and energy are *numerically* equal. This is a consequence of the electrical linearity, as may be seen by observing Fig. 3.1.8a, in which (3.1.48) and (3.1.52) are the areas in the q' - v' plane indicated. (Remember that, by definition, in our system with one electrical terminal pair $W_e' + W_e = qv$.) When the areas are separated by a straight line (3.1.49), the integrals are obviously equal. On the other hand, when the areas are not separated by a straight line, the system is electrically nonlinear and energy and coenergy are not equal. An example of electrical nonlinearity is shown in Fig. 3.1.8b.

Energy and coenergy have the same numerical values in an electrically linear system. We have, however, consistently made use of the energy expressed as a function of (q, x) or (λ, x) and the coenergy expressed as a function of (v, x) or (i, x) . These functions are quite different in mathematical form, even when the system is electrically linear [compare (3.1.50) and (3.1.53)].

A word of caution is called for at this point. A partial derivative is taken with respect to one independent variable holding the other independent variables fixed. In order for this process to be correct, it is easiest to perform the differentiation when the function to be differentiated is written without explicit dependence on dependent variables. To be more specific, consider

the capacitance $C(x)$ of plane parallel plates with area A and spacing x (Fig. 3.1.2). Then

$$C(x) = \frac{A\epsilon}{x} \quad (3.1.54)$$

and (3.1.51) gives

$$f^e = - \frac{v^2 A \epsilon}{2x^2}. \quad (3.1.55)$$

The minus sign tells us that f^e acts on the upper plate (node) in the $(-x)$ direction. This we expect, for positive charges on the top plate are attracted by negative charges on the bottom plate. We can obtain the same result by using the energy and the translational form of (3.1.27).

$$f^e = - \frac{\partial W_e(q, x)}{\partial x}. \quad (3.1.56)$$

From (3.1.53) and (3.1.54)

$$f^e = - \frac{q^2}{2A\epsilon}. \quad (3.1.57)$$

In view of (3.1.49) and (3.1.54) this result and (3.1.55) are identical. Suppose, however, that we blindly apply (3.1.56) to the energy of (3.1.53) with q replaced by Cv . The magnitude of the resulting force will be correct, but the sign will be wrong. For electrically nonlinear systems the magnitude of the force will also be wrong if the partial differentiation is not carried out correctly.

The generalized force and coenergy equations are summarized in Table 3.1. This table is intended to illustrate the generality of the equations and their interrelations. The general equations are *not* recommended for use in solving problems. It is better to rederive the equations in each case to make certain that fundamental physical laws are satisfied. Equations (k) to (n) in Table 3.1 for evaluating energy and coenergy are written by using a path of integration that brings each electrical variable from zero to its final value in sequence $j = 1$ to $j = N$.

3.1.2c Reciprocity

The mathematical description of a conservative electromechanical coupling system must satisfy a *reciprocity* condition that is a generalization of the reciprocity conventionally discussed in electric circuit theory.* To illustrate reciprocity for a simple example, consider the magnetic field system of Fig. 3.1.1 for which the terminal relations are expressed as derivatives of stored

* E. A. Guillemin, *Introductory Circuit Theory*, Wiley, New York, 1953, pp. 148–150 and 429.

energy in (3.1.22) and (3.1.23):

$$i = \frac{\partial W_m(\lambda, x)}{\partial \lambda}, \quad (3.1.22)$$

$$f^e = - \frac{\partial W_m(\lambda, x)}{\partial x}. \quad (3.1.23)$$

We now differentiate (3.1.22) with respect to x and (3.1.23) with respect to λ . Then, because

$$\frac{\partial^2 W_m}{\partial \lambda \partial x} = \frac{\partial^2 W_m}{\partial x \partial \lambda},$$

the reciprocity relation results:

$$\frac{\partial i(\lambda, x)}{\partial x} = - \frac{\partial f^e(\lambda, x)}{\partial \lambda}. \quad (3.1.58)$$

The process used in obtaining the reciprocity condition (3.1.58) shows that the condition is necessary for the system to be conservative. This same condition can also be shown to be sufficient to ensure that the system is conservative. The proof requires a straightforward but involved integration and is not carried out here primarily because it is a standard inclusion in some thermodynamics texts.*

The reciprocity condition of (3.1.58) can be generalized to describe a conservative system with any number of terminal pairs. Consider again the electric-field system with N electrical terminal pairs and M rotational mechanical terminal pairs whose terminal relations are described by (3.1.26) and (3.1.27):

$$v_i = \frac{\partial W_e}{\partial q_i}; \quad i = 1, 2, \dots, N, \quad (3.1.26)$$

$$T_i^e = - \frac{\partial W_e}{\partial \theta_i}; \quad i = 1, 2, \dots, M. \quad (3.1.27)$$

When we take appropriate partial derivatives of these equations and recognize that the order of differentiation is immaterial, we obtain the general reciprocity conditions:

$$\frac{\partial v_i}{\partial q_j} = \frac{\partial v_j}{\partial q_i}, \quad i, j = 1, 2, \dots, N, \quad (3.1.59)$$

$$\frac{\partial T_i^e}{\partial \theta_j} = \frac{\partial T_j^e}{\partial \theta_i}, \quad i, j = 1, 2, \dots, M, \quad (3.1.60)$$

$$\frac{\partial v_i}{\partial \theta_j} = - \frac{\partial T_j^e}{\partial q_i}; \quad i = 1, 2, \dots, N, \quad (3.1.61)$$

$$j = 1, 2, \dots, M.$$

* See, for instance, W. P. Allis and M. A. Herlin, *Thermodynamics and Statistical Mechanics*, McGraw-Hill, 1952, pp. 6-9.

Note that for an electrically linear system (3.1.59) reduces to $C_{ij} = C_{ji}$ which is the usual form of the reciprocity relation for linear capacitive circuits.*

Although the reciprocity conditions must always be satisfied for a conservative system, they are not often used in the analysis and design of electromechanical systems. Their primary usefulness is twofold. First, they provide a rapid check on results to identify certain kinds of mathematical error; and, second, they provide a mathematical framework for identifying the classes of nonlinear functions with which we can approximate the terminal relations of multiterminal-pair, electrically nonlinear, systems. If the reciprocity conditions are not satisfied, the mathematical description will imply sources and/or sinks of energy in the coupling field that can lead to nonphysical results.

3.1.3 Energy Conversion

The fact that in lumped-parameter electromechanics we are dealing with lossless coupling systems in which stored energy is a state function (single-valued function of the independent variables) can be quite useful in assessing energy conversion properties of electromechanical systems. This is especially true of systems that operate cyclically. For any conservative coupling system we can write the conservation of energy as

$$\left[\begin{array}{c} \text{electrical energy} \\ \text{input} \end{array} \right] + \left[\begin{array}{c} \text{mechanical energy} \\ \text{input} \end{array} \right] = \left[\begin{array}{c} \text{change in} \\ \text{stored energy} \end{array} \right]. \quad (3.1.62)$$

For a complete cycle of operation, that is, for a situation in which the independent variables return to the values from which they started, the net change in stored energy is zero. Thus for a cyclic process (3.1.62) becomes

$$\left[\begin{array}{c} \text{net electrical} \\ \text{energy input} \\ \text{for one cycle} \end{array} \right] + \left[\begin{array}{c} \text{net mechanical} \\ \text{energy input} \\ \text{for one cycle} \end{array} \right] = 0. \quad (3.1.63)$$

We need to calculate only the electrical or mechanical energy input to find the net conversion of energy between electrical and mechanical forms.

Example 3.1.2. The device shown schematically in Fig. 3.1.9 is used to illustrate the energy conversion properties of a cyclically operating system.† It contains a cylindrical stator of highly permeable magnetic material with polar projections on which coils are wound. The two coils are connected in series in the polarity shown to form one electrical terminal pair. This machine also contains a rotor, made of highly permeable magnetic

* Guillemin, *loc. cit.*

† For more detail on this type of machine (called a two-pole, single-phase, salient-pole synchronous machine) see Section 4.2.

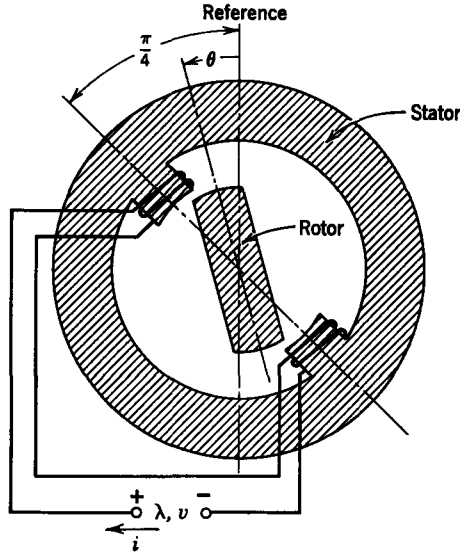


Fig. 3.1.9 A rotational magnetic field transducer.

material, which has the shape shown in end view in Fig. 3.1.9 and which can rotate about the axis with the instantaneous angle θ .

It is determined experimentally that the machine is electrically linear and that the electrical terminal relation can be approximated by the inductance

$$L = L_0 + L_2 \sin 2\theta, \quad (a)$$

where L_0 and L_2 are positive constants and $L_0 > L_2$. Note that this inductance is a maximum at $\theta = \pi/4$ and $\theta = 5\pi/4$, as we expected, because the air gaps between rotor and stator iron are smallest for these angles. Also, the inductance is a minimum for $\theta = -\pi/4$ and $\theta = 3\pi/4$, in which case the air gaps are largest. In practice, the rotor and stator are shaped so that the periodic variation of inductance with angle closely approaches the ideal of (a).

With the inductance thus specified, we can write the electrical terminal relation as

$$\lambda = Li = (L_0 + L_2 \sin 2\theta)i. \quad (b)$$

We can now use (b) to evaluate the magnetic coenergy by using (m) of Table 3.1,

$$W'_m = \frac{1}{2}(L_0 + L_2 \sin 2\theta)i^2, \quad (c)$$

and (g) in Table 3.1 to find the torque of electric origin,

$$T^e = \frac{\partial W'_m}{\partial \theta} = L_2 i^2 \cos 2\theta. \quad (d)$$

We can now represent the electromechanical coupling symbolically, as in Fig. 3.1.10. The box includes only the magnetic field energy storage of the machine. All purely electrical properties (winding resistance and losses in the magnetic material) and all purely mechanical properties (moment of inertia and friction) can be represented as lumped elements connected externally to the terminals of the coupling system.

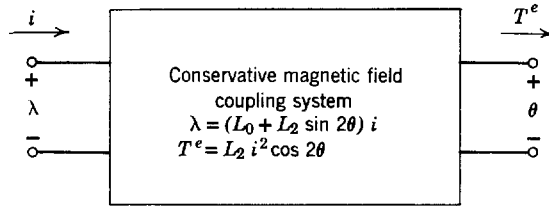


Fig. 3.1.10 Representation of the coupling field of the system in Fig. 3.1.9.

In an actual application there would be lumped electrical and mechanical elements, in addition to those inherent in the machine, connected to the coupling network. Our purpose here is to study the energy conversion properties of the coupling system; consequently, we will excite the terminals with ideal sources and there will be no need to consider passive elements connected to the terminals.

We now excite the electrical terminal pair of the coupling system with a sinusoidal current source

$$i = I \cos \omega t \quad (e)$$

and the mechanical terminal pair with the position source

$$\theta = \omega t, \quad (f)$$

where ω is a positive constant. With these terminal constraints and with steady-state operation, we wish to calculate the electromechanical energy conversion per cycle of operation.

Because they are constrained independently, current i and angle θ are the logical choices as independent variables. We can sketch the path of operation for one cycle in the i - θ plane, as shown in Fig. 3.1.11. Note that $\theta = 0$ and $\theta = 2\pi$ represent the same geometry; thus, although the trajectory in Fig. 3.1.11 does not close on itself, it nonetheless represents one cycle of operation in which the final physical state is the same as the initial physical state. The arrows indicate the direction that the operating point travels in the i - θ plane.

When we apply (3.1.63) to this system for a complete cycle of operation, we obtain,

$$\oint i d\lambda - \oint T^e d\theta = 0, \quad (g)$$

wherein $\oint$ indicates an integral around a closed cycle. The first term represents the net

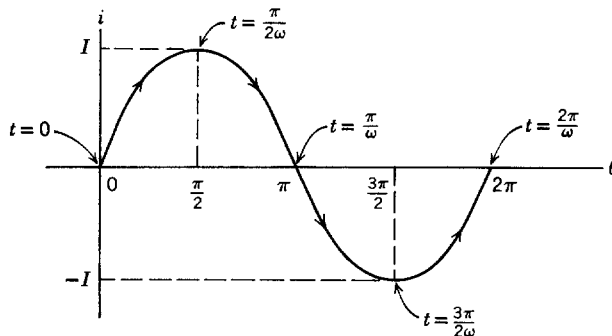


Fig. 3.1.11 Trajectory of operating point in i - θ plane.

electrical energy input over a cycle and the second (with the minus sign) represents the net mechanical energy input. Because there is no net change in stored energy, we need to calculate only the first or second term to find energy converted. To be thorough in our study we shall consider both terms.

We first look at the trajectory of the operating point in a λ - i plane. We can express it as two parametric equations (time is the parameter) by using (b) and (e):

$$\lambda = I(L_0 + L_2 \sin 2\omega t) \cos \omega t, \quad (h)$$

$$i = I \cos \omega t. \quad (i)$$

Alternatively, we can use trigonometric identities* to eliminate t from the two equations and obtain

$$\lambda = i \left[L_0 \pm \frac{2L_2 i}{I} \left(1 - \frac{i^2}{I^2} \right)^{1/2} \right]. \quad (j)$$

The double-valued character of this equation makes it easier to plot the trajectory by using the parametric equations (h) and (i). This trajectory is shown in Fig. 3.1.12, plotted for the relative parameter values

$$L_2 = \frac{1}{2}L_0.$$

Next, we can look at the trajectory of the operating point in the T^e - θ plane. We use (d), (e), and (f) to write

$$T^e = L_2 I^2 \cos^2 \theta \cos 2\theta. \quad (k)$$

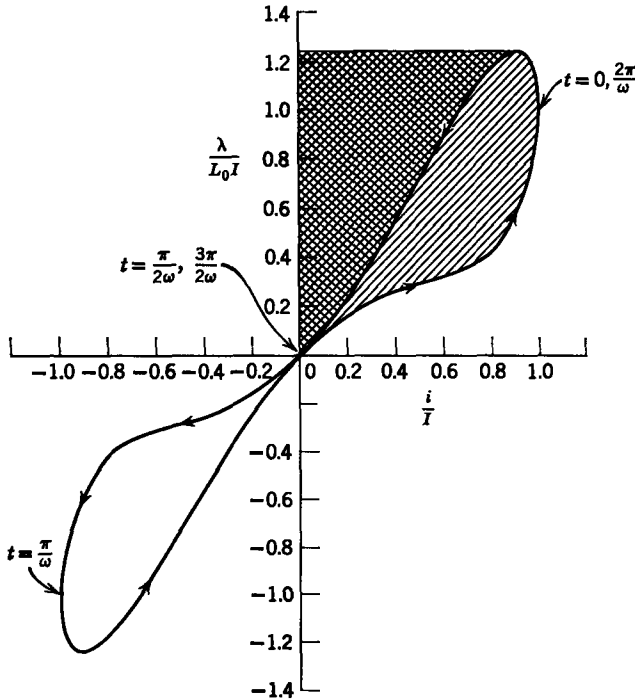


Fig. 3.1.12 Trajectory of operation in the λ - i plane for $L_2 = \frac{1}{2}L_0$.

* $\sin 2\omega t = 2 \sin \omega t \cos \omega t$; $\sin \omega t = \pm \sqrt{1 - \cos^2 \omega t}$.

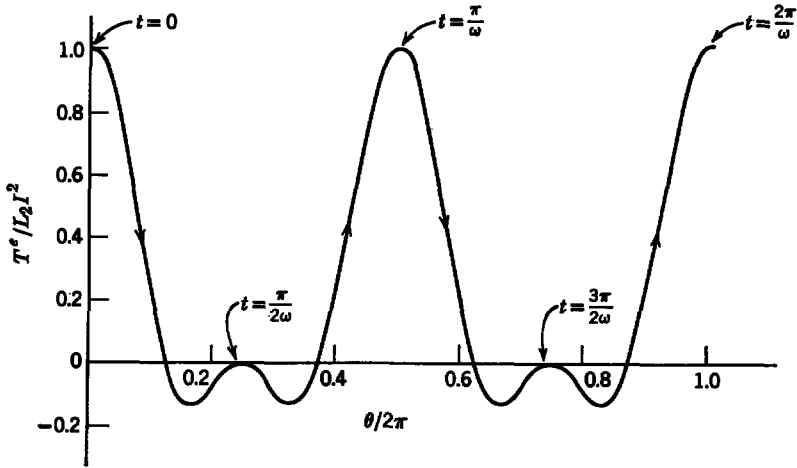


Fig. 3.1.13 Trajectory of operating point in T^e - θ plane.

This trajectory is shown in Fig. 3.1.13. Note once again that although the curve does not close on itself it represents a full cycle of operation because $\theta = 0$ and $\theta = 2\pi$ represent the same state. The direction of travel of the operating point is indicated on the curve.

We can now calculate the energy converted per cycle. First, evaluating

$$\oint i \, d\lambda = \text{net electrical input power,}$$

we can see graphically in Fig. 3.1.12 that the integral of $i \, d\lambda$ around the trajectory yields the area enclosed by the loop; furthermore, this area is positive. There is net conversion of energy from electrical to mechanical form. Under these conditions the machine is operating as a motor.

We can evaluate the energy converted per cycle by calculating the area enclosed by the loop in the first quadrant of Fig. 3.1.12 and multiplying the answer by two. This integral can best be performed parametrically by writing

$$i = I \cos \theta,$$

$$\lambda = I(L_0 + L_2 \sin 2\theta) \cos \theta,$$

$$d\lambda = (-IL_0 \sin \theta + 2L_2 I \cos 2\theta \cos \theta - L_2 I \sin 2\theta \sin \theta) d\theta.$$

Some trigonometric manipulation allows us to put $d\lambda$ in the form

$$d\lambda = I(-L_0 \sin \theta - 2L_2 \cos \theta + 4L_2 \cos^3 \theta - 2L_2 \cos \theta \sin^2 \theta) d\theta.$$

We can now write for the area of the loop in the first quadrant of Fig. 3.1.12

$$\begin{aligned} \frac{W_c}{2} = \int_{-\pi/2}^{\pi/2} i(\theta) \, d\lambda(\theta) &= \int_{-\pi/2}^{\pi/2} I^2 (-L_0 \sin \theta \cos \theta - 2L_2 \cos^2 \theta + 4L_2 \cos^4 \theta \\ &\quad - 2L_2 \cos^2 \theta \sin^2 \theta) d\theta, \end{aligned}$$

where W_c is the energy converted per cycle. Evaluation of this integral yields

$$W_c = \frac{\pi}{2} L_2 I^2.$$

We can also calculate the mechanical output energy per cycle from

$$\int_0^{2\pi} T^e d\theta = \int_0^{2\pi} L_2 I^2 \cos^2 \theta \cos 2\theta d\theta = \frac{\pi}{2} L_2 I^2,$$

which is equal to the electric input energy per cycle as it should be.

The ideas of energy bookkeeping illustrated by Example 3.1.2 can be extended to systems with arbitrary numbers of terminal pairs. For more than two variables the graphical representation of operation in variable space (Fig. 3.1.11) is difficult; it is possible, however, to represent the path of operation at each terminal pair (Figs. 3.1.12 and 3.1.13). Such techniques are especially suitable for systems that operate cyclically.

3.2 EQUATIONS OF MOTION

In the preceding sections of this chapter we have described in detail the various elements that make up lumped-parameter electromechanical systems. Our approach is to isolate the coupling system (either electric or magnetic field) and analyze its properties. We can then write Kirchhoff's laws for the electrical parts of the system by introducing electromechanical coupling effects through the terminal relations of the coupling system. Similarly, we write Newton's second law and continuity of space for the mechanical parts of the system, including electromechanical coupling effects in the terminal relations of the coupling system. We now present examples in which our objective is to write the complete equations of motion for electromechanical systems.

Example 3.2.1. We consider again the magnetic field system shown in Fig. 3.2.1. The electrical terminal relation of the coupling system was calculated in Example 2.1.1. Now we include the type of electrical and mechanical elements that will normally be present in applications of this transducer. The resistance R represents the winding resistance plus any additional series resistance in the external circuit. This system is of the form conventionally used to actuate relays, valves, etc.; consequently, the source $v_s(t)$ is usually a positive or negative step. The spring K is used to open the gap x to its maximum width when the current is zero. The linear damper B represents friction between the nonmagnetic sleeve and the plunger, although in some cases additional damping is added externally either to slow down the mechanical motion (as in a time-delay relay) or to reduce the bouncing that may occur when the plunger reaches $x = 0$.

In Example 2.1.1, with suitable assumptions, the flux linkages of this device were calculated to be

$$\lambda = \frac{L_0 i}{1 + x/g}, \quad (a)$$

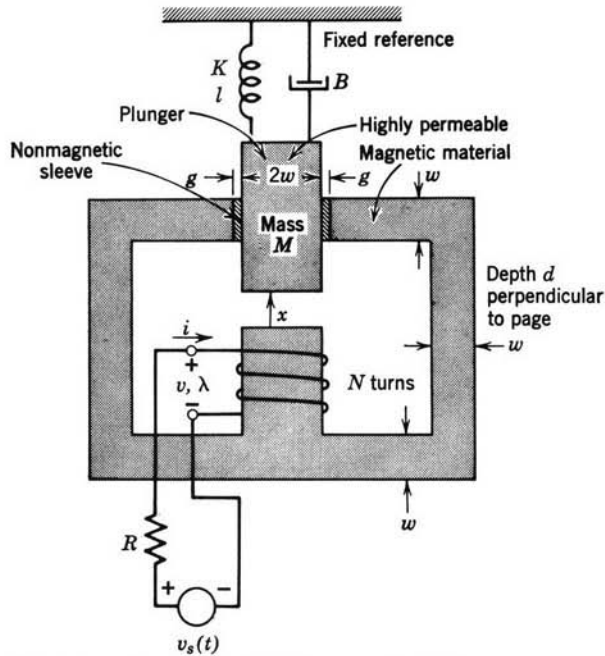


Fig. 3.2.1 A magnetic field electromechanical system.

where $L_0 = 2wd\mu_0 N^2/g$ is the coil inductance with the variable gap closed ($x = 0$). We wish to write the complete equations of motion.

We have a single electrical loop and a single mechanical node; consequently, we can write two equations in which the current i and displacement x are the dependent variables.

Applying Kirchhoff's voltage law to the electrical loop and using the terminal voltage of the coupling system as derived in Example 2.1.1, we obtain

$$v_s(t) = iR + \frac{L_0}{1 + x/g} \frac{di}{dt} - \frac{L_0}{g(1 + x/g)^2} \frac{dx}{dt}. \quad (b)$$

To write Newton's second law for the mechanical node we need the force of electric origin. We first write the magnetic coenergy [see (m) in Table 3.1] as

$$W'_m = \int_0^i \lambda(i', x) di'$$

and use (a) to write

$$W'_m = \frac{1}{2} \frac{L_0 i^2}{1 + x/g} \quad (c)$$

We now find the force by using (g) in Table 3.1.

$$f^e = -\frac{1}{2} \frac{L_0 i^2}{g(1 + x/g)^2}. \quad (d)$$

This is a force source applied to the mechanical node x .

We can now write Newton's law for the mechanical node as

$$-\frac{1}{2} \frac{L_0 i^2}{g(1+x/g)^2} = M \frac{d^2 x}{dt^2} + B \frac{dx}{dt} + K(x-l). \quad (e)$$

Equations b and e are the equations of motion for this system. Note that there are two equations with two dependent variables (unknowns) i and x . The driving function is the source voltage $v_s(t)$. If we specify the explicit variation of v_s with time and also specify initial conditions, we, at least in theory, can solve (b) and (e) for i and x . The dynamic behavior of this system is studied in Section 5.1.2.

In the above analysis no account has been taken of the two mechanical stops that limit the mechanical motion. It is easiest to include them as position sources; in practical cases, however, the stops may also have some elastic effects that result in bouncing of the plunger at the ends of its travel. If such effects are important, they can be included in a straightforward manner.

Example 3.2.2. In this example we wish to consider a system with more than one electrical terminal pair and more than one mechanical node. For this purpose we use the basic electric field coupling system of Example 2.1.5, shown in Fig. 3.2.2, along with suitable external electrical and mechanical elements.

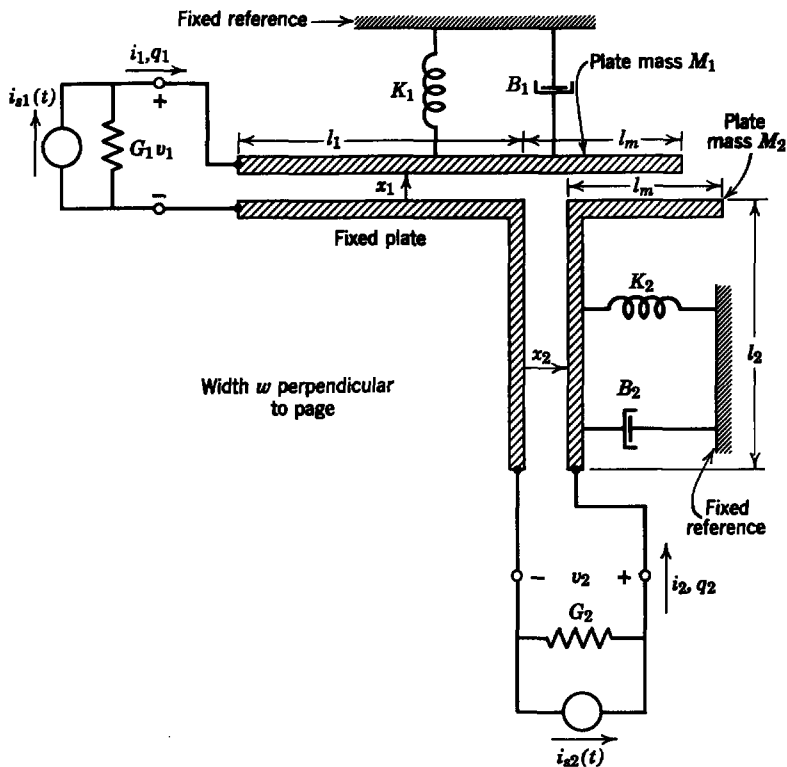


Fig. 3.2.2 Multiply excited electric field-coupled electromechanical system.

The electrical terminal relations were derived for this system in Example 2.1.5 and are

$$q_1 = C_1 v_1 - C_m v_2, \quad (a)$$

$$q_2 = -C_m v_1 + C_2 v_2, \quad (b)$$

where

$$C_1 = \frac{\epsilon_0 w [l_1 + (l_m - x_2)]}{x_1}, \quad (c)$$

$$C_2 = \epsilon_0 w \left(\frac{l_2}{x_2} + \frac{l_m - x_2}{x_1} \right), \quad (d)$$

$$C_m = \frac{\epsilon_0 w (l_m - x_2)}{x_1}. \quad (e)$$

We write Kirchhoff's current law for the two electrical nodes as

$$i_{s1}(t) = G_1 v_1 + \frac{dq_1}{dt}, \quad (f)$$

$$i_{s2}(t) = G_2 v_2 + \frac{dq_2}{dt}. \quad (g)$$

Using (a) to (e), we express these equations explicitly in terms of the unknowns as

$$i_{s1}(t) = G_1 v_1 + \frac{\epsilon_0 w [l_1 + (l_m - x_2)]}{x_1} \frac{dv_1}{dt} - \frac{\epsilon_0 w (l_m - x_2)}{x_1} \frac{dv_2}{dt} \\ - \frac{\epsilon_0 w [l_1 + (l_m - x_2)]}{x_1^2} v_1 \frac{dx_1}{dt} - \frac{\epsilon_0 w}{x_1} v_1 \frac{dx_2}{dt} + \frac{\epsilon_0 w (l_m - x_2)}{x_1^2} v_2 \frac{dx_1}{dt} + \frac{\epsilon_0 w}{x_1} v_2 \frac{dx_2}{dt} \quad (h)$$

$$i_{s2}(t) = G_2 v_2 - \frac{\epsilon_0 w (l_m - x_2)}{x_1} \frac{dv_1}{dt} + \epsilon_0 w \left(\frac{l_2}{x_2} + \frac{l_m - x_2}{x_1} \right) \frac{dv_2}{dt} + \frac{\epsilon_0 w (l_m - x_2)}{x_1^2} v_1 \frac{dx_1}{dt} \\ + \frac{\epsilon_0 w}{x_1} v_1 \frac{dx_2}{dt} - \frac{\epsilon_0 w (l_m - x_2)}{x_1^2} v_2 \frac{dx_1}{dt} - \epsilon_0 w \left(\frac{l_2}{x_2^2} + \frac{1}{x_1} \right) v_2 \frac{dx_2}{dt}. \quad (i)$$

Before we can write equations for the mechanical nodes we must calculate the forces of electric origin. Because we want the explicit electrical variables to be the voltages, we use (n) in Table 3.1 to evaluate the coenergy as

$$W'_e = \frac{1}{2} C_1 v_1^2 + C_m v_1 v_2 + \frac{1}{2} C_2 v_2^2. \quad (j)$$

We now use (h) in Table 3.1 to evaluate the forces

$$f_1^e = \frac{\partial W'_e}{\partial x_1} = \frac{1}{2} v_1^2 \frac{\partial C_1}{\partial x_1} + v_1 v_2 \frac{\partial C_m}{\partial x_1} + \frac{1}{2} v_2^2 \frac{\partial C_2}{\partial x_1}, \quad (k)$$

$$f_2^e = \frac{\partial W'_e}{\partial x_2} = \frac{1}{2} v_1^2 \frac{\partial C_1}{\partial x_2} + v_1 v_2 \frac{\partial C_m}{\partial x_2} + \frac{1}{2} v_2^2 \frac{\partial C_2}{\partial x_2}. \quad (l)$$

We carry out the indicated differentiations and include these two forces as sources in writing Newton's second law for the two mechanical nodes.

$$-\frac{1}{2}v_1^2 \frac{\epsilon_0 w [l_1 + (l_m - x_2)]}{x_1^3} - v_1 v_2 \frac{\epsilon_0 w (l_m - x_2)}{x_1^3} - \frac{1}{2}v_2^2 \frac{\epsilon_0 w (l_m - x_2)}{x_1^3} \\ = M_1 \frac{d^2 x_1}{dt^2} + B_1 \frac{dx_1}{dt} + K_1 x_1, \quad (m)$$

$$-\frac{1}{2}v_1^2 \frac{\epsilon_0 w}{x_1} - v_1 v_2 \frac{\epsilon_0 w}{x_1} - \frac{1}{2}v_2^2 \epsilon_0 w \left(\frac{l_2}{x_2^2} + \frac{1}{x_1} \right) = M_2 \frac{d^2 x_2}{dt^2} + B_2 \frac{dx_2}{dt} + K_2 x_2. \quad (n)$$

Equations (h), (i), (m), and (n) are the four equations of motion for the system in Fig. 3.2.2. Several important aspects of these equations should be examined. First, we note that all four equations are coupled, that is, each equation contains all four dependent variables. We also note that there is no external coupling between electrical terminal pairs and between mechanical terminal pairs; thus all the coupling occurs through the electric fields. We note further that the coupling between the two mechanical terminal pairs [see (m) and (n)] results in terms that are functions of mechanical positions and voltages. Thus these coupling terms appear essentially as nonlinear elements whose properties depend on the electrical variables (voltages).

3.3 DISCUSSION

In this chapter we have learned some of the general properties of conservative electromechanical coupling networks. In the process we have indicated techniques for finding mechanical forces of electric origin once electrical terminal relations are known. We have also introduced techniques for studying the energy conversion properties of coupling fields and illustrated the method of writing complete equations of motion for electromechanical systems. In Chapter 5 we complete our study of lumped-parameter electromechanical systems by introducing techniques for solving the equations of motion and by emphasizing some of the more important phenomena that occur in these systems.

PROBLEMS

3.1. A simple plunger-type solenoid for the operation of relays, valves, etc., is represented in Fig. 3P.1. Assume that it is a conservative system and that its electrical equation of state is

$$\lambda = \frac{L_0 i}{1 + x/a}.$$

- (a) Find the force that must be *applied to* the plunger to hold it in equilibrium at a displacement x and with a current i .

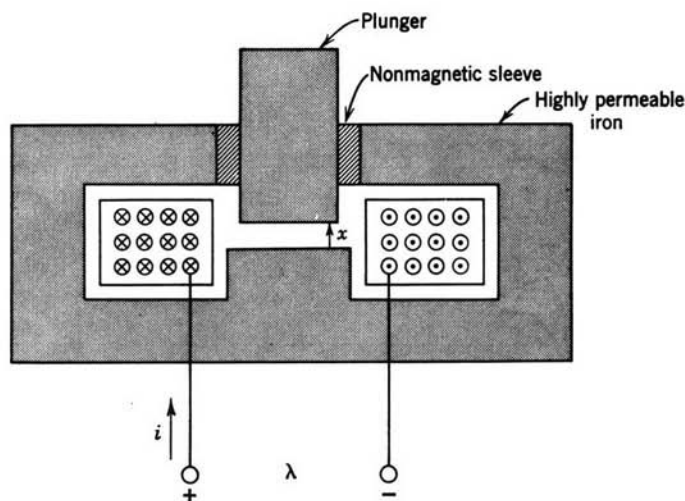


Fig. 3P.1

- (b) Make a labeled sketch of the force of part (a) as a function of x with constant i .
 (c) Make a labeled sketch of the force of part (a) as a function of x with constant λ .

3.2. An electrically linear electric field system with two electrical terminal pairs is illustrated in Fig. 3P.2. The system has the electrical equations of state $v_1 = S_{11}q_1 + S_{12}q_2$ and $v_2 = S_{21}q_1 + S_{22}q_2$. (See Example 3.1.1 for a physical case of this type.)

- (a) Calculate the energy input to the system over each of the three paths A , B , and C in the q_1 - q_2 plane illustrated in Fig. 3P.2b.
 (b) What is the relation between coefficients S_{12} and S_{21} to make these three values of energy the same?
 (c) Derive the result of (b) by assuming that the system is conservative and applying reciprocity.

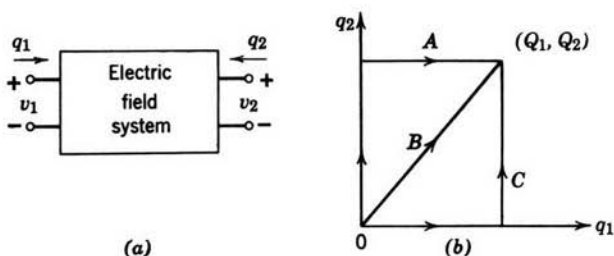


Fig. 3P.2

3.3. A slab of dielectric slides between plane parallel electrodes as shown. The dielectric obeys the constitutive law $\mathbf{D} = \alpha(\mathbf{E} \cdot \mathbf{E})\mathbf{E} + \epsilon_0\mathbf{E}$, where ϵ_0 is the permittivity of free space and α is a constant. Find the force of electrical origin on the slab. Your answer should take the form $f^e = f^e(v, x)$.

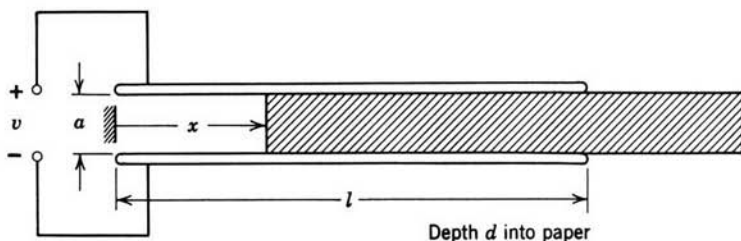


Fig. 3P.3

3.4. A magnetic circuit, including a movable plunger, is shown in Fig. 3P.4. The circuit is excited by an N -turn coil and consists of a perfectly permeable yoke and plunger with a variable air gap $x(t)$ and a fixed nonmagnetic gap d . The system, with the cross section shown, has a width w into the paper. The following parts lead to a mathematical formulation of the equations of motion for the mass M , given the excitation $I(t)$.

- Find the terminal relation for the flux $\lambda(i, x)$ linked by the electrical terminal pair. Ignore fringing in the nonmagnetic gaps. Note that the coil links the flux through the magnetic material N times.
- Find the energy $W_m(\lambda, x)$ stored in the electromechanical coupling. This should be done by making use of part (a).
- Use the energy function $W_m(\lambda, x)$ to compute the force of electrical origin f^e acting on the plunger.
- Write an electrical (circuit) equation of motion involving λ and x as the only dependent variables and $I(t)$ as a driving function.
- Write the mechanical equation of motion for the mass. This differential equation should have λ and x as the only dependent variables, hence taken with the result of (d) should constitute a mathematical formulation appropriate for analyzing the system dynamics.

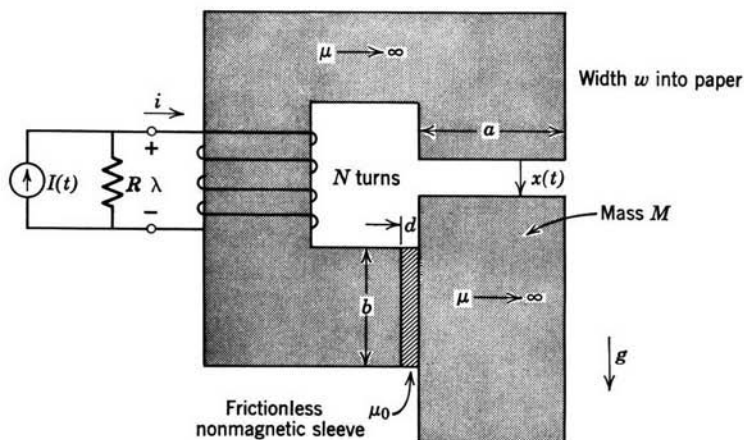


Fig. 3P.4

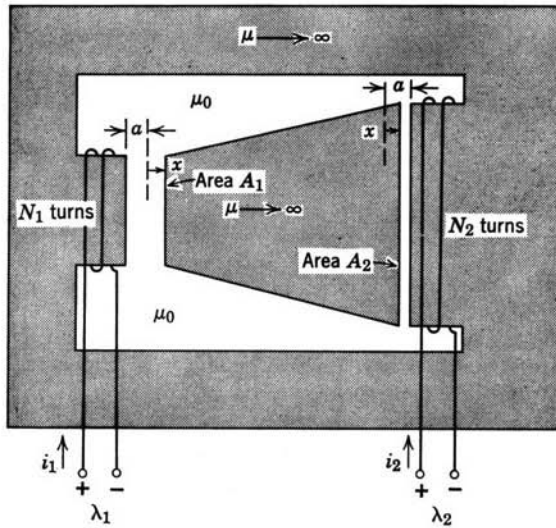


Fig. 3P.5

3.5. A magnetic circuit with a movable element is shown in Fig. 3P.5. With this element centered, the air gaps have the same length (a). Displacements from this centered position are denoted by x .

- Find the electrical terminal relations $\lambda_1(i_1, i_2, x)$ and $\lambda_2(i_1, i_2, x)$ in terms of the parameters defined in the figure.
- Compute the coenergy $W'_m(i_1, i_2, x)$ stored in the electromechanical coupling.

3.6. An electrically nonlinear magnetic field coupling network illustrated in Fig. 3P.6 has the equations of state

$$i = I_0 \left[\frac{\lambda/\lambda_0 + (\lambda/\lambda_0)^3}{1 + x/a} \right], \quad f^e = \frac{I_0}{a} \left[\frac{\frac{1}{2}\lambda^2/\lambda_0 + \frac{1}{4}\lambda^4/\lambda_0^3}{(1 + x/a)^2} \right],$$

where I_0 , λ_0 , and a are positive constants.

- Prove that this system is conservative.
- Evaluate the stored energy at the point λ_1, x_1 in variable space.

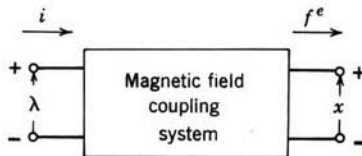


Fig. 3P.6

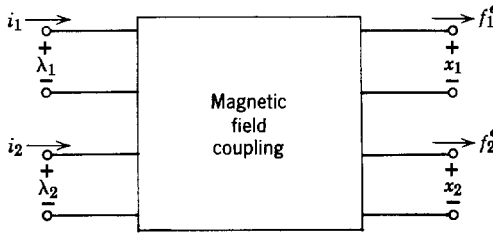


Fig. 3P.7

3.7. The electrical terminal variables of the electromechanical coupling network shown in Fig. 3P.7 are known to be $\lambda_1 = ax_1i_1^3 + bx_1x_2i_2$ and $\lambda_2 = bx_1x_2i_1 + cx_2i_2^3$, where a , b , and c are constants. What is the coenergy $W'_m(i_1, i_2, x_1, x_2)$ stored in the coupling network?

3.8. A schematic diagram of a rotating machine with a superconducting rotor (moment of inertia J) is shown in Fig. 3P.8. Tests have shown that $\lambda_1 = i_1L_1 + i_2L_m \cos \theta$ and $\lambda_2 = i_1L_m \cos \theta + i_2L_2$, where $\theta(t)$ is the angular deflection of the shaft to which coil (2) is attached. The machine is placed in operation as follows:

- With the (2) terminals open circuit and the shaft at $\theta = 0$, $I(t)$ is raised to I_0 .
- Terminals (2) are shorted to conserve the flux λ_2 regardless of $\theta(t)$ or $i_1(t)$.
- $I(t)$ is now made a given driving function.

Write the equation of motion for the shaft. Your answer should be one equation involving only $\theta(t)$ as an unknown. Damping can be ignored.

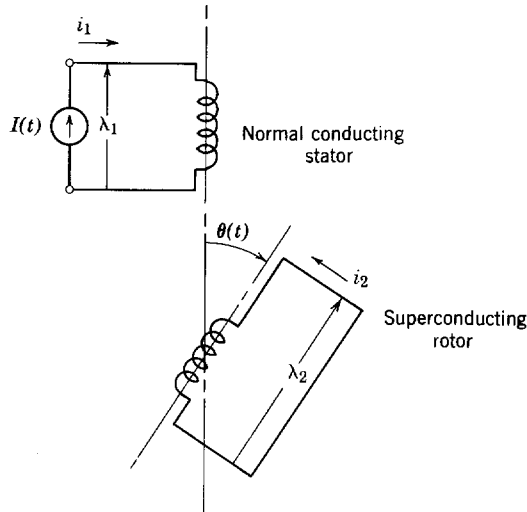


Fig. 3P.8

3.9. The electric terminal variables of the electromechanical coupling system shown in Fig. 3P.9 are known to be $\lambda_1 = ax_1^2i_1^3 + bx_2^2x_1i_2$ and $\lambda_2 = bx_2^2x_1i_1 + cx_2^2i_2^3$, where a , b , and c are constants.

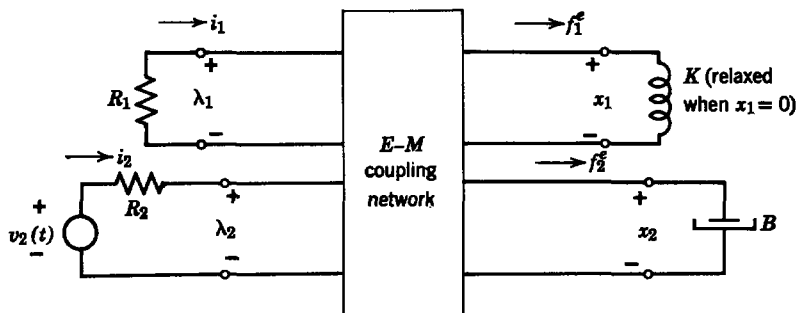


Fig. 3P.9

- What is the coenergy $W'_m(i_1, i_2, x_1, x_2)$ stored in the coupling network?
- Find the forces f_1^e and f_2^e .
- Write the *complete* set of equations for the system with the terminal constraints shown.

3.10. The following equations of state describe the conservative, magnetic field coupling system of Fig. 3P.10 for the ranges of variables of interest ($i_1 > 0, i_2 > 0$). $\lambda_1 = L_0 i_1 + A i_1 i_2 x$ and $\lambda_2 = A i_1^2 i_2 x + L_0 i_2$, where L_0 and A are positive constants.

- Find the force applied by the coupling system on the external mechanical circuit as a function of i_1, i_2 , and x .
- Write the *complete* set of differential equations for the system by using i_1, i_2 , and x as dependent variables.

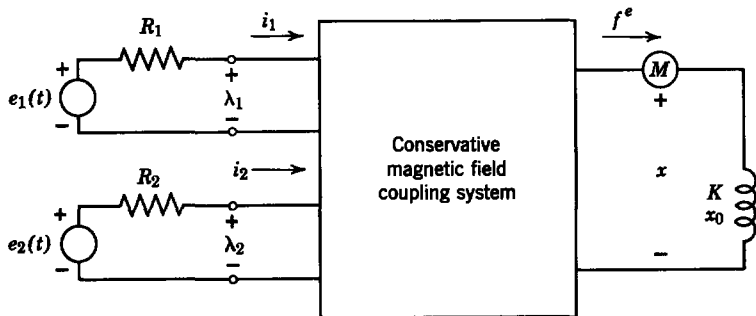


Fig. 3P.10

3.11. Two coils are free to rotate as shown in Fig. 3P.11. Each coil has a moment of inertia J . Measurements have shown that $\lambda_1 = L_1 i_1 + M i_2 \cos \theta \cos \psi$ and $\lambda_2 = M i_1 \cos \theta \cos \psi + L_2 i_2$, where L_1, L_2 , and M are constants. Because the system is electrically linear, we know that the coenergy $W'_m(i_1, i_2, \psi, \theta)$ is given by $W'_m = \frac{1}{2} L_1 i_1^2 + M \cos \theta \cos \psi i_1 i_2 + \frac{1}{2} L_2 i_2^2$. The coils are driven by the external circuits, where I_1 and I_2 are known functions of time.

- What are the torques of electrical origin T_1^e and T_2^e that the electrical system exerts on the coils?
- Write the *complete* equations of motion that define $\theta(t)$ and $\psi(t)$.

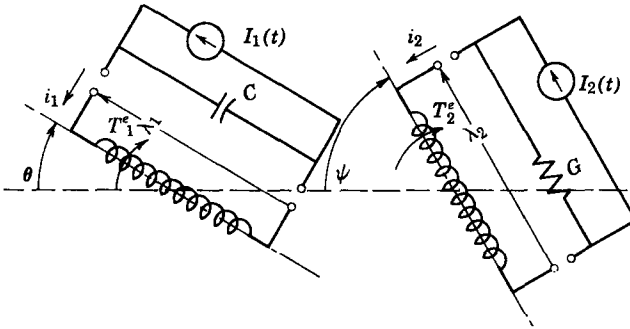


Fig. 3P.11

3.12. A magnetic field system has three electrical terminal pairs and two mechanical terminal pairs as shown in Fig. 3P.12. The system is electrically linear and may be described by the relations $\lambda_1 = L_{11}i_1 + L_{12}i_2 + L_{13}i_3$, $\lambda_2 = L_{21}i_1 + L_{22}i_2 + L_{23}i_3$, and $\lambda_3 = L_{31}i_1 + L_{32}i_2 + L_{33}i_3$. Each of the inductances L_{ij} ($i = 1, 2, 3; j = 1, 2, 3$) may be functions of the mechanical variables x_1 and x_2 . Prove that if the system is conservative, $L_{12} = L_{21}$, $L_{13} = L_{31}$, and $L_{23} = L_{32}$. To do this recall that for a conservative system the energy (or coenergy) does not depend on the path of integration but only on the end point.

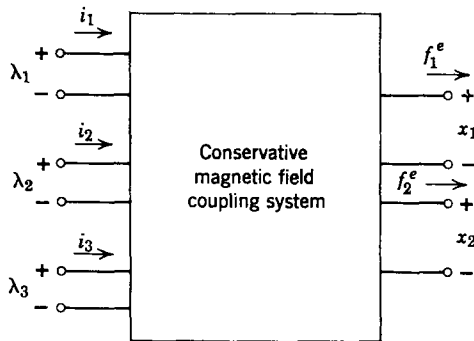


Fig. 3P.12

3.13. Electrostatic voltmeters are often constructed as shown in Fig. 3P.13a. N pairs of pie-shaped plates form the stator and rotor of a variable capacitor (Fig. 3P.13a shows six pairs of rotor plates and six pairs of stator plates). The rotor plates are attached to a conducting shaft that is free to rotate through an angle θ . In the electrostatic voltmeter a pointer is attached to this shaft so that the deflection θ is indicated on a calibrated scale (not shown).

- Determine $q(v, \theta)$, where q is the charge on the stator and v is the voltage applied between the rotor and stator. The device is constructed so that fringing fields can be ignored and the area of the plates is large compared with the cross section of the shaft. In addition, it is operated in a region of θ in which the plates overlap but not completely.
- Find the torque of electrical origin on the rotor.
- The shaft is attached to a torsional spring which has the deflection θ when

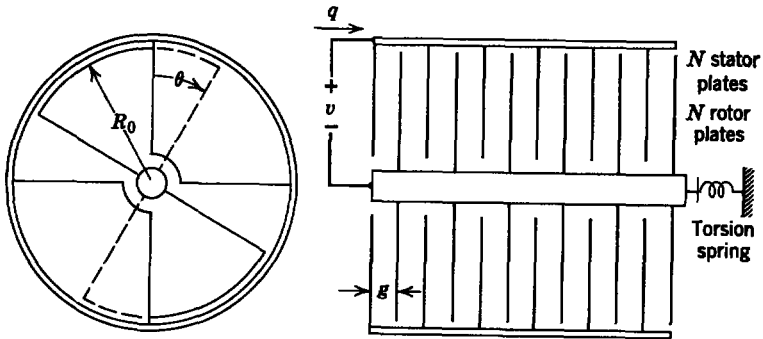


Fig. 3P.13a

subjected to a torque T_m , where θ and T_m are related by $T_m = K(\theta - \alpha)$. The shaft has a moment of inertia J and is subject to a damping torque $B d\theta/dt$. Write the torque equation for the shaft.

- (d) The circuit of Fig. 3P.13b is attached to the terminals. Write the electrical equation for the system. [The results of parts (c) and (d) should constitute two equations in two unknowns.]

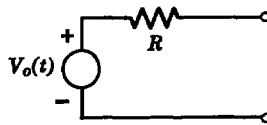


Fig. 3P.13b

- (e) A “zero adjust” knob on the electrostatic voltmeter is used to set α in such a way that a pointer attached to the shaft indicates 0 when $\theta = \alpha$. A constant voltage $v = V_0$ is attached to the terminals. What is the static deflection of the pointer $(\theta - \alpha)$ as a function of V_0 ?

3.14. A fixed cylindrical capacitor of length L is made of a solid perfectly conducting inner rod of radius a which is concentric with a perfectly conducting outer shell of radius b . An annular half cylinder (inner radius a , outer radius b) of dielectric with permittivity ϵ and length L is free to move along the axis of the capacitor as shown in Fig. 3P.14 (ignore fringing).

- Find the charge on the outer cylinder $q = q(v, x)$, where v is the voltage between the inner and outer conductors and x is the displacement of the half cylinder of dielectric (assume $L > x > 0$).
- Write the conservation of power for this system in terms of the terminal voltage and current, the electric energy stored, the force of electric origin, and the velocity of the dielectric.
- Find the electric energy stored in terms of q and x .
- Find the electric coenergy in terms of v and x .
- Find the force of electric origin exerted by the fields on the dielectric.

Suppose one end of the dielectric is attached to a spring of constant K , which is relaxed when $x = l$.

- Write the differential equation of motion for the dielectric, assuming that it has mass M and slides without friction.
- If a constant voltage V_0 is established between the conductors, find x .

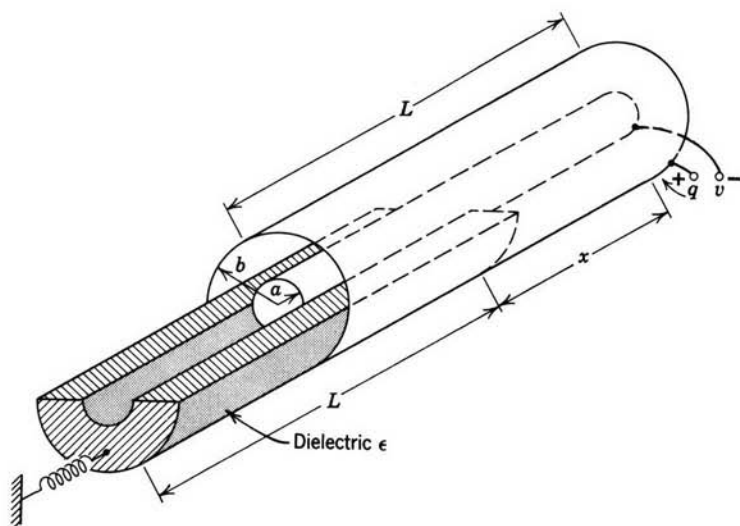


Fig. 3P.14

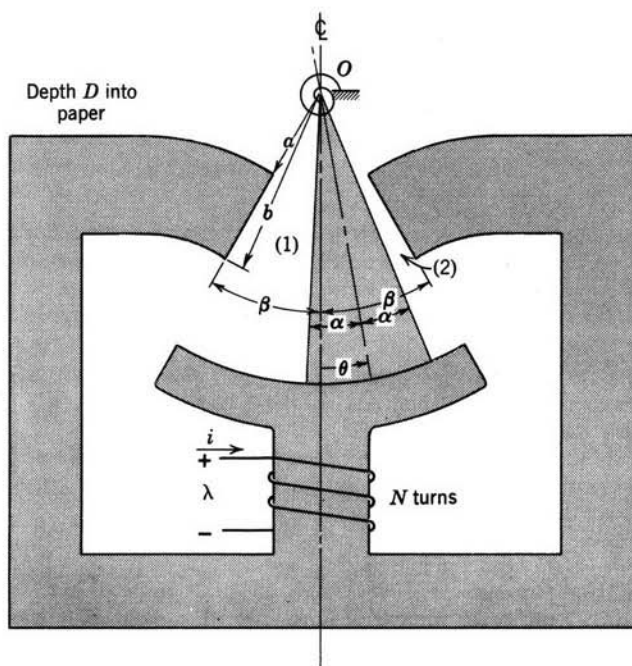


Fig. 3P.15

3.15. A magnetic transducer is shown in Fig. 3P.15. A wedge-shaped infinitely permeable piece of metal is free to rotate through the angle θ $(-(\beta - \alpha) < \theta < \beta - \alpha)$. The angle θ is the deflection of the wedge center line from the center line of the device. A magnetic field is produced in regions (1) and (2) by means of the infinitely permeable yoke and the N -turn winding.

- Find $\lambda(i, \theta)$. (You may assume that the fringing fields at the radii $r = a$ and $r = b$ from the origin O are of negligible importance.)
- Compute the magnetic coenergy stored in the electromechanical coupling $W'_m(i, \theta)$.
- Use the conservation of energy to find the torque T^e exerted by the magnetic field on the wedge.
- The wedge has a moment of inertia J about O and is constrained by a torsion spring that exerts the torque $T_m = K\theta$. Write the equation of motion for the wedge, assuming that i is a given function of time.
- If $i = I_0 = \text{constant}$, show that the wedge can be in static equilibrium at $\theta = 0$.

3.16. A plane electrode is free to move into the region between plane-parallel electrodes, as shown in Fig. 3P.16. The outer electrodes are at the same potential, whereas the inner electrode is at a potential determined by the constant voltage source V_0 in series with the output of an amplifier driven by a signal proportional to the displacement of the movable electrode itself. Hence the voltage of the inner electrode relative to that of the outer electrodes is $v = -V_0 + Ax$, where A is a given feedback gain. Find the force of electrical origin $f^e(x)$. (Note that this force is only a function of position, since the voltage is a known function of x .)

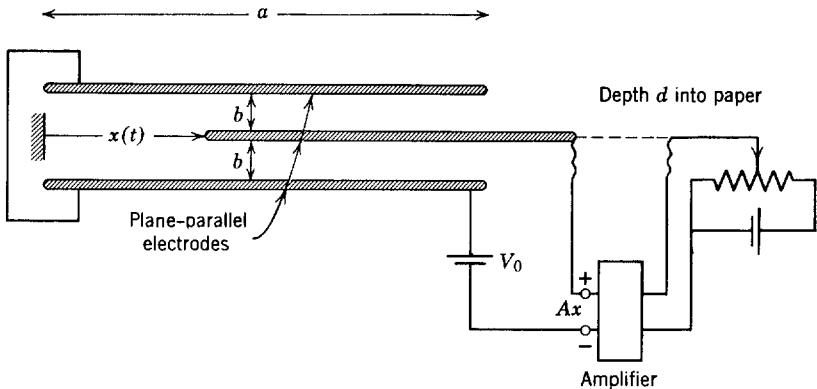


Fig. 3P.16

3.17. In Fig. 3P.17 we have a slab of magnetic material positioned between three pole faces. The nonlinear magnetic material is such that the constitutive relation is given by $\mathbf{B} = \alpha(\mathbf{H} \cdot \mathbf{H})\mathbf{H} + \mu_0\mathbf{H}$, where α is a known constant.

- Show that

$$\lambda_1 = L_0 \left(1 + \frac{d}{g} \right) i_1 + L_0 \beta \left(1 - \frac{x}{l} \right) i_1^3 + L_0 \frac{d}{g} i_2.$$

and

$$\lambda_2 = L_0 \frac{d}{g} i_1 + L_0 \beta \left(\frac{x - g}{l} \right) i_2^3 + L_0 \left(1 + \frac{d}{g} \right) i_2.$$

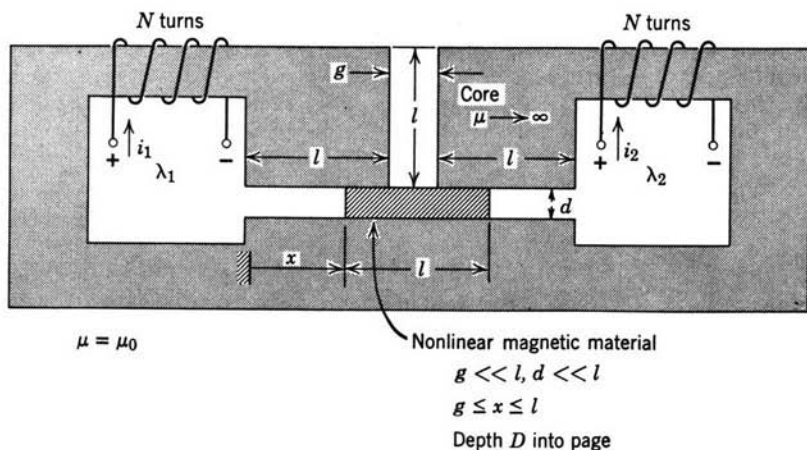


Fig. 3P.17

where

$$L_0 = \frac{\mu_0 N^2 l D}{d},$$

$$\beta = \frac{\alpha}{\mu_0} \left(\frac{N}{d} \right)^2,$$

and

$$g \leq x \leq l$$

(b) Determine an expression for the magnetic coenergy $W'_m = W'_m(i_1, i_2, x)$.

(c) What is the force of magnetic origin f^e acting on the nonlinear magnetic material?

3.18. A slab of dielectric material is positioned between three perfectly conducting plates shown in Fig. 3P.18. The dielectric is such that the displacement vector $\mathbf{D}$ is related to the electric field $\mathbf{E}$ through the relation $\mathbf{D} = \alpha(\mathbf{E} \cdot \mathbf{E})\mathbf{E} + \epsilon_0\mathbf{E}$, where α is a known positive constant. The slab and adjacent plates have a width (into paper) w .

(a) With the slab at the position x , find the electrical terminal relations. Ignore fringing fields and assume that the slab is always well within the plates

$$q_r = q_r(v_r, v_l, x) \text{ and } q_l = q_l(v_r, v_l, x).$$

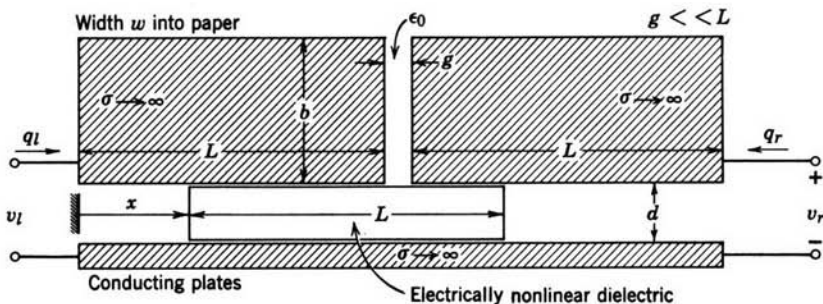


Fig. 3P.18

- (b) Find the electrical coenergy $W'_e(v_r, v_i, x)$ stored between the plates.
 (c) What is the force of electrical origin f^e on the slab of dielectric?

3.19. A perfectly conducting plate of length $2\alpha R$ and depth D is attached to the end of a conducting bar that rotates about the axis O , shown in Fig. 3P.19. A pair of conducting electrodes forms half cylinders, coaxial with the axis O . The gap $\Delta \ll R$. We ignore fringing fields in the present analysis.

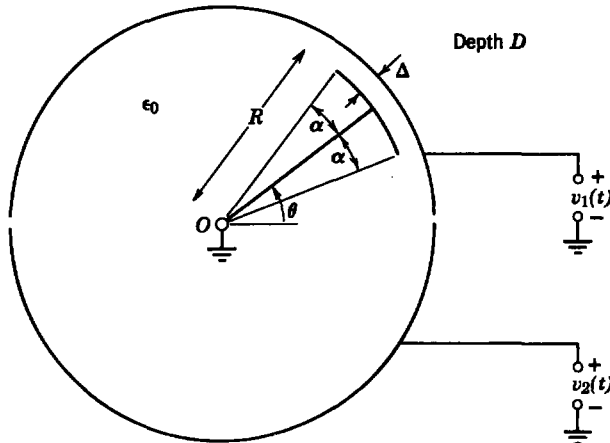


Fig. 3P.19

- (a) Make a dimensioned plot of the coenergy $W'_e(\theta, v_1, v_2)$ as a function of θ .
 (b) Make a dimensioned plot of the torque exerted by the electric fields on the rotor.
 (c) In terms of this torque, write a differential equation for $\theta(t)$. You may assume that the rotor has an inertia J but is not impeded by a viscous damping force.
- 3.20.** A parallel-plate capacitor has its bottom plate fixed and its top plate free to move vertically under the influence of the externally applied mechanical force f . A slab of the dielectric of thickness d is between the plates shown in Fig. 3P.20a. With plate area A and displacement x , the electrical terminal relation (neglecting fringing fields; see Example 2.1.4) is

$$q(v, x) = \frac{\epsilon A}{d(1 + \epsilon x/\epsilon_0 d)} v.$$

- (a) The capacitor is charged to a value of charge $q = Q$ and the terminals are open-circuited. Calculate, sketch, and label the externally applied force $f(Q, x)$ necessary to hold the plate in equilibrium and the terminal voltage $v(Q, x)$ as functions of x for the range $0 < x < 2(\epsilon_0/\epsilon)d$.
 (b) A battery of constant voltage V is connected to the terminals of the capacitor. Calculate, sketch, and label the externally applied force $f(V, x)$ necessary to hold the plate in equilibrium and the charge $q(V, x)$ as functions of x for the range $0 < x < 2(\epsilon_0/\epsilon)d$.
 (c) By the use of suitable electrical and mechanical sources the system of Fig. 3P.20a is made to traverse the closed cycle in q - x plane shown in Fig. 3P.20b in the direction indicated by the arrows. Calculate the energy converted per cycle and specify whether the conversion is from electrical to mechanical or vice versa.

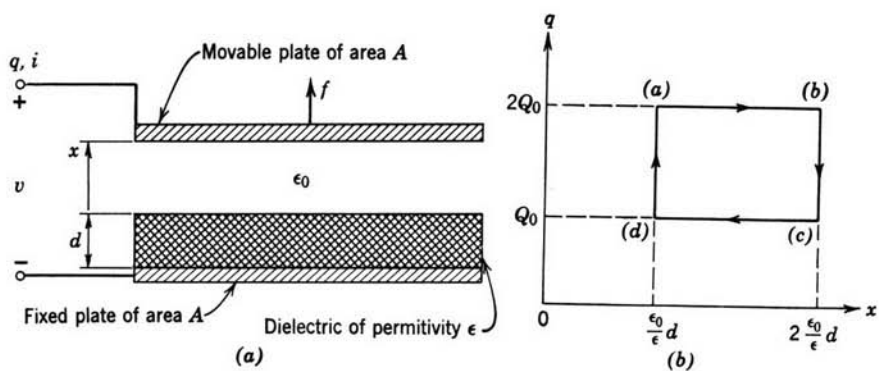


Fig. 3P.20

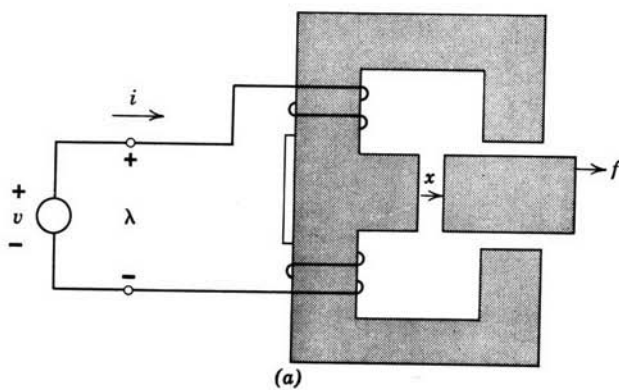


Fig. 3P.21

3.21. The magnetic field transducer illustrated schematically in Fig. 3P.21a has a movable plunger that is constrained to move only in the x -direction. The coupling field is conservative and electrically linear and has the electrical equation of state

$$\lambda = \frac{L_0 i}{1 + x/a}$$

where L_0 and a are positive constants (see Example 2.1.1).

- (a) For any flux linkage λ and position x find the external force f (see Fig. 3P.21a), which must be applied to hold the plunger in static equilibrium.

We now constrain the electrical terminal pair with a voltage source v and the movable plunger by a position source x in such a way that the system slowly traverses the closed cycle in the λ - x plane illustrated in Fig. 3P.21b.

- (b) Sketch and label current i as a function of flux linkage λ for the closed cycle of Fig. 3P.21b.
 (c) Sketch and label the force f applied by the position source as a function of x for closed cycle of Fig. 3P.21b.
 (d) Find the energy converted between electrical and mechanical forms for one traversal of the cycle of Fig. 3P.21b. Specify the direction of flow.

3.22. The system shown in Fig. 3P.22 consists of two thin perfectly conducting plates, one of which is free to move. The movable plate slides on a perfectly conducting plane. It has been proposed that energy could be converted from mechanical to electrical form by the following scheme:

The process is started by holding the plate at $x = X_b$ with the switch in position (1). An external mechanical system moves the plate to $x = X_a$ and holds it there. S is then put in

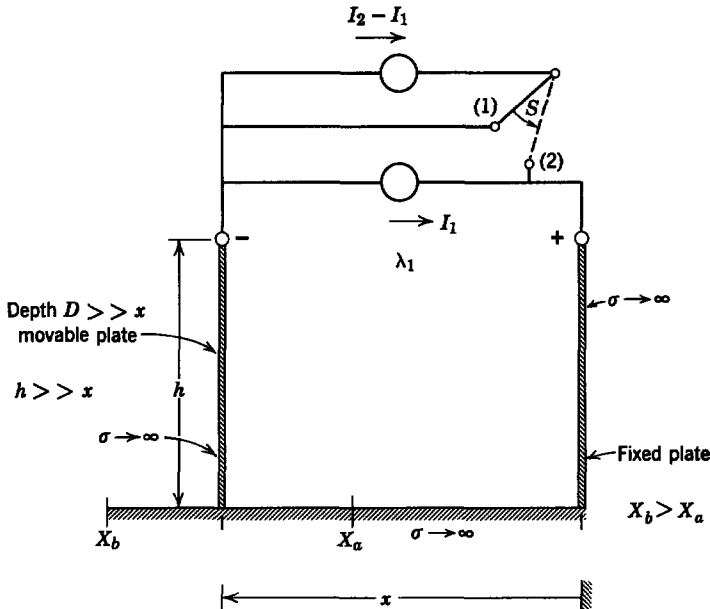


Fig. 3P.22

position (2) and the mechanical system moves the plate back to $x = X_0$ and holds it in place. S is then reset in position (1) and the cycle is repeated several times.

- (a) To convert energy *from mechanical to electrical* form during each cycle, how must I_1 and I_2 be related?
- (b) Sketch the path of operation in the i - x plane under the conditions of part (a) and compute the amount of energy converted from mechanical to electrical form during one cycle.
- (c) Sketch carefully the path of operation for one cycle in the λ - x plane under the conditions of part (a). Compute the amount of energy converted from mechanical to electrical form along each part of the path in the λ - x plane.

Chapter 4

ROTATING MACHINES

4.0 INTRODUCTION

The most numerous and the most widely used electromechanical device in existence is the magnetic field type rotating machine. Rotating machines occur in many different types, depending on the nature of the electrical and mechanical systems to be coupled and on the coupling characteristics desired. The primary purpose of most rotating machines is to convert energy between electrical and mechanical systems, either for electric power generation or for the production of mechanical power to do useful tasks. These machines range in size from motors that consume a fraction of a watt to large generators that produce 10^9 W. In spite of the wide variety of types and sizes and of methods of construction, which vary greatly, most rotating machines fall into two classes defined by their geometrical structures—namely smooth-air-gap and salient-pole. The analysis of the electromechanical coupling systems in rotating machines can thus be reduced to the analysis of two configurations, regardless of the size or type of machine. As is to be expected, some machines do not fit our classification; they are not numerous, however, and their analyses can be performed by making simple changes in the models and techniques presented in this chapter.

After defining the two classes of machine geometry (smooth-air-gap and salient-pole), we establish the conditions necessary for average power conversion and use them as a basis for defining different types of machine. We also derive the equations of motion for the different machine types and solve them in the steady state to describe the machines' principal characteristics. The behavior of machines under transient conditions is covered in Chapter 5.

Before starting the treatment of machines it is important to recognize several significant points. First, as is evident from the treatment, a rotating machine is but one specific embodiment of a more general class of electromechanical devices defined in Chapter 3, and, as such, is conceptually quite simple. In a practical configuration, such as a polyphase machine, the

number of terminal pairs is great enough to make the mathematical description seem lengthy. In no case should mathematical complexity be mistaken for conceptual difficulty. The analysis of rotating machines is conceptually simple and mathematically complex. As our treatment unfolds, it will become clear that there are geometrical and mathematical symmetries that imply simplification techniques. These techniques have been developed to a high degree of sophistication and are essential in the analysis of machine systems. Because our interest here is in the basic physical processes, we forego the special techniques and refer the reader to other texts.*

4.1 SMOOTH-AIR-GAP MACHINES

All rotating machines that fit in the smooth-air-gap classification can be represented schematically by a physical structure like that shown in end

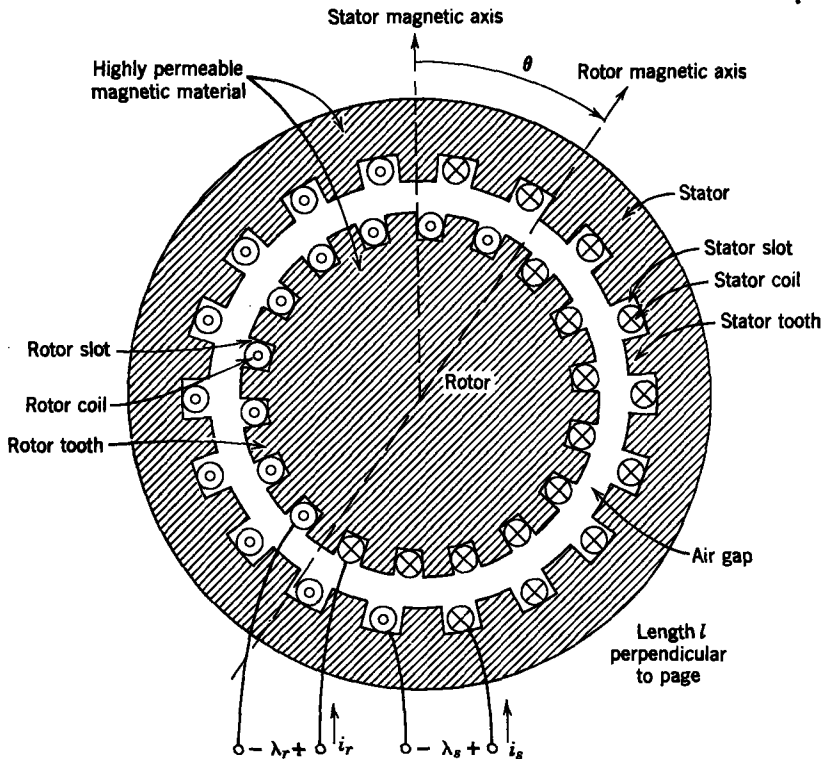


Fig. 4.1.1a Geometry of smooth-air-gap rotating machine showing distributed windings on stator and rotor of a single-phase machine.

* See, for example, D. C. White and H. H. Woodson, *Electromechanical Energy Conversion*, Wiley, New York, 1959, Chapters 4 and 7 to 10.

view in Fig. 4.1.1*a*. Pictures of a stator and a rotor that fall into this classification are shown in Figs. 4.1.2 and 4.1.3.

In the structure of Fig. 4.1.1*a* conductors are laid in axial slots that face the air gap. The number of conductors in each slot depends on the type and size of the machine and varies from 1 in large turbo-generators to 10 to 12 in small induction machines. The conductors of one circuit on one member are in series or series-parallel connection at the ends of the machine. (Note the end turns in Figs. 4.1.2 and 4.1.3.) The circuits are arranged so that a current in one winding will produce the antisymmetrical pattern about an axial plane indicated by the dots and crosses in Fig. 4.1.1*a*. This axial plane is the plane of symmetry of the magnetic field produced by the currents and is therefore called the magnetic axis. The stator and rotor magnetic axes are shown in Fig. 4.1.1.

The example in Fig. 4.1.1 has only one circuit (winding) on the stator and one circuit on the rotor. Most machines have more than one circuit on each member. In this case a slot will usually contain conductors from different circuits. Nonetheless, the description given fits each circuit on the rotor or stator.

The rotor is free to rotate and its instantaneous angular position θ is, by convention, the displacement of the rotor magnetic axis with respect to the stator magnetic axis.

The structure of Fig. 4.1.1*a* is called smooth air gap because it can be modeled mathematically with sufficient accuracy by assuming that the magnetic path seen by each circuit is independent of rotor position. Such a model neglects the effects of slots and teeth on magnetic path as the angle is changed. In a real machine (see Figs. 4.1.2 and 4.1.3) the slots and teeth are relatively smaller than those shown in Fig. 4.1.1*a*. Moreover, special construction techniques, such as skewing the slots of one member slightly with respect to a line parallel to the axis,* minimize these effects. In any case, the essential properties of a machine can be obtained with good accuracy by using a smooth-air-gap model, but slot effects are always present as second-order effects in machine terminal characteristics and as first-order problems to machine designers.

* For constructional details of rotating machines see, for example, A. E. Knowlton, ed., *Standard Handbook for Electrical Engineers*, 9th ed., McGraw-Hill, New York, 1957, Sections 7 and 8. This also includes numerous references to more detailed design treatments.

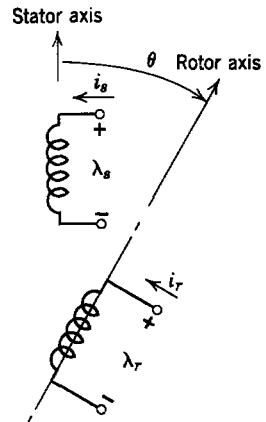


Fig. 4.1.1*b* Schematic representation of the inductors constituting the rotor and stator windings shown in (a).

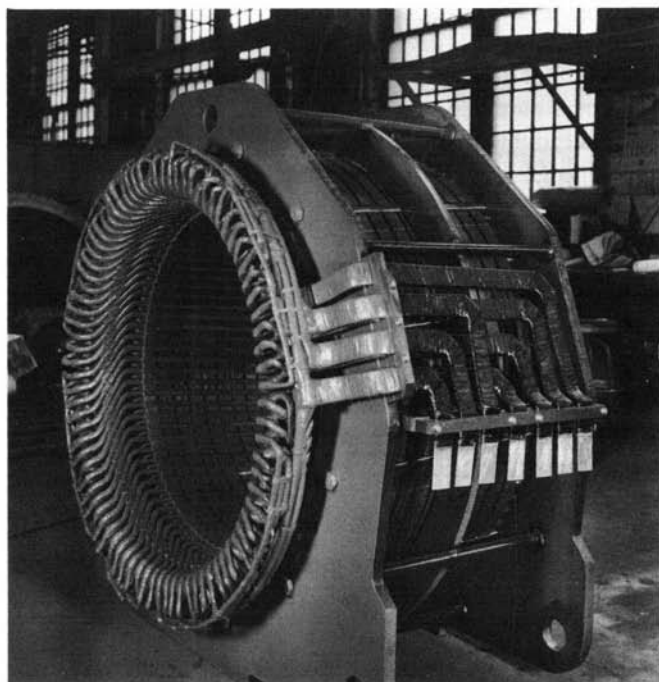


Fig. 4.1.2 Stator (armature) of an induction motor. This is an example of a smooth-air-gap stator. (Courtesy of Westinghouse Electric Corporation.)

4.1.1 Differential Equations

In terms of the conventions and nomenclature for lumped-parameter systems introduced in Chapter 3, the device in Fig. 4.1.1 has magnetic field electromechanical coupling with two electrical terminal pairs and one rotational mechanical terminal pair. Thus the coupling system can be represented symbolically, as in Fig. 4.1.4.

It is conventional practice in machine analysis to assume electrical linearity (no saturation in stator or rotor magnetic material)*; consequently, the electrical terminal relations can be written in terms of inductances that can be functions of the angle θ (see Section 2.1.1 of Chapter 2). The further assumption of a smooth air gap indicates that because the field produced by

* Magnetic saturation in machines is quite important, but it is conventionally treated as a perturbation of the results of an analysis such as we will do. See, for example, White and Woodson, *op. cit.*, pp. 532–535.

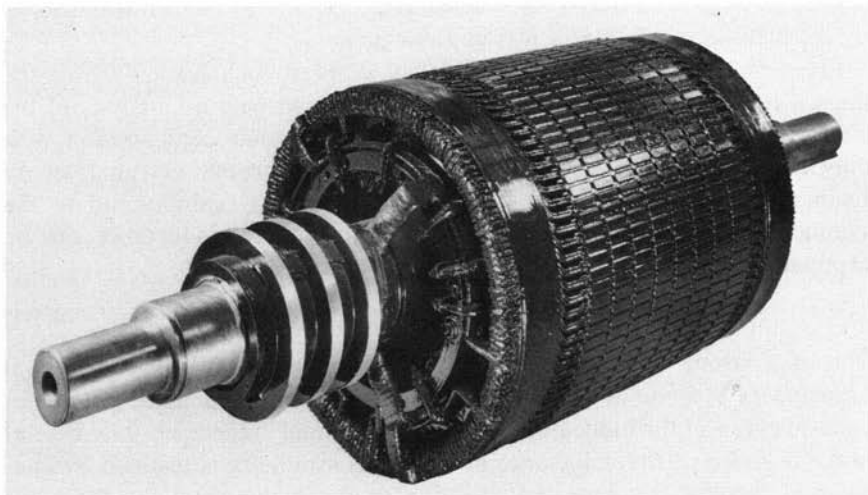


Fig. 4.1.3 Rotor of a wound-rotor induction machine. This is an example of a smooth-air-gap rotor. (Courtesy of Westinghouse Electric Corporation.)

each coil is unaffected by rotor position the self-inductances will be independent of rotor position and the mutual inductance will depend on rotor position. Hence the terminal relations for the coupling system of Fig. 4.1.1 as represented symbolically in Fig. 4.1.4 can be written as

$$\lambda_s = L_s i_s + L_{sr}(\theta) i_r, \quad (4.1.1)$$

$$\lambda_r = L_{sr}(\theta) i_s + L_r i_r, \quad (4.1.2)$$

$$T^e = i_s i_r \frac{dL_{sr}(\theta)}{d\theta}, \quad (4.1.3)$$

where L_s and L_r are the constant self-inductances, $L_{sr}(\theta)$ is the angular-dependent mutual inductance, and the variables $(\lambda_s, \lambda_r, i_s, i_r, T^e, \theta)$ are

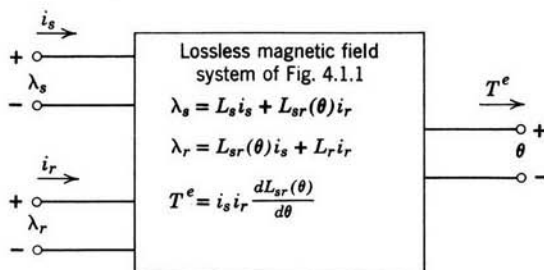


Fig. 4.1.4 Symbolic representation of coupling system in Fig. 4.1.1.

defined in Figs. 4.1.1 and 4.1.4. The torque T^e , given by (4.1.3), was derived by the method of Chapter 3 [(g) in Table 3.1].

From (4.1.1) to (4.1.3) it is clear that we need to know only how the mutual inductance L_{sr} varies with angle to proceed with an analysis of the electromechanical coupling in this machine. A similar configuration with only two slots on the stator and two slots on the rotor was analyzed in Example 2.1.2 of Chapter 2. With reference to that example and to the symmetries of the windings in Fig. 4.1.1a, the mutual inductance can be expressed in the general form

$$L_{sr}(\theta) = M_1 \cos \theta + M_3 \cos 3\theta + M_5 \cos 5\theta + \cdots \quad (4.1.4)$$

This is a cosine series containing only odd harmonics. Thus the mutual inductances at θ and at $-\theta$ are the same, the mutual inductance at $(\theta + \pi)$ is the negative of the inductance at θ , and the mutual inductance at $(-\theta - \pi)$ is the negative of the inductance at $-\theta$. This symmetry is justified by considering qualitatively how the flux due to stator current links the rotor winding as the rotor position is varied.

The winding distribution around the periphery of alternating current machines is normally designed to enhance the fundamental component of mutual inductance M_1 and to suppress all higher harmonics. The purpose of this design criterion is to minimize unwanted harmonic current generation in the machine. On the other hand, in the design of dc machines, other criteria are used and several of the harmonics of (4.1.4) are present in appreciable amounts. Nonetheless, it suffices for the purposes here to assume that the mutual inductance is represented by the space fundamental term only. Such an assumption simplifies the analyses, does not eliminate any fundamental properties of machines, and can be used as the basis for a complete analysis, if we assume that all harmonics are present.* Thus for the remainder of this analysis the mutual inductance $L_{sr}(\theta)$ is specified as

$$L_{sr}(\theta) = M \cos \theta. \quad (4.1.5)$$

and the three terminal relations (4.1.1) to (4.1.3) become

$$\lambda_s = L_s i_s + M i_r \cos \theta, \quad (4.1.6)$$

$$\lambda_r = M i_s \cos \theta + L_r i_r, \quad (4.1.7)$$

$$T^e = -i_s i_r M \sin \theta. \quad (4.1.8)$$

Before beginning a study of the energy conversion properties of the lossless coupling part of the machine in Fig. 4.1.1, it would be worthwhile to inquire into the circuit representation of a machine, including the essential parameters of the machine by itself, which are illustrated in the equivalent circuit of Fig. 4.1.5. On the electrical side the windings have resistances R_s and R_r ,

* For the general analysis see White and Woodson, *op. cit.*, Chapter 11.

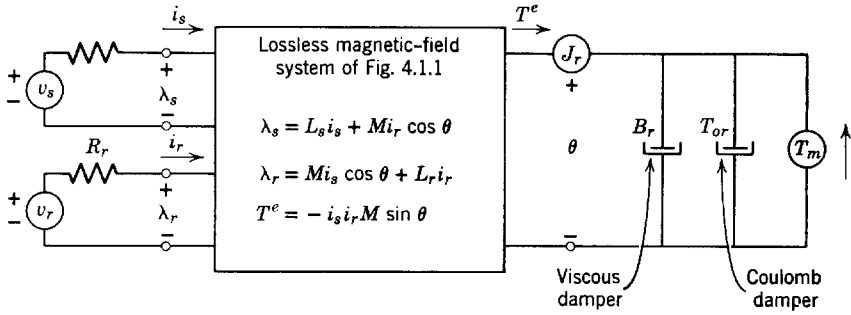


Fig. 4.1.5 Equivalent circuit for machine in Fig. 4.1.1.

which are treated as series resistances external to the lossless coupling system. There are additional losses in the iron (hysteresis and eddy-current losses) which are not included here because they are usually small enough to be treated as a simple perturbation.* On the mechanical side the essential parameters that must be included are the rotor inertia J_r and losses that occur because of friction in bearings and in sliding electrical contacts for exciting the rotor and because of windage losses due to rotation of the rotor in a gas, usually air or hydrogen. It is normally sufficient to represent these mechanical losses as a combination of viscous (B_r) and coulomb (T_{or}) friction, as shown in Fig. 4.1.5. The sources included in Fig. 4.1.5, v_s , v_r , and T_m , are general. Any or all of them may be independently set or they may be dependent on some variable. They may represent passive loads. In addition, they can be replaced by other sources, that is, the electrical terminal pairs can be excited by current sources and the mechanical terminal pair can be excited by a position or velocity source. The point of including the sources is to indicate that, in addition to the essential machine parameters, external circuits must be included before the machine can be made to operate usefully.

Using the equivalent circuit of Fig. 4.1.5, we can write the differential equations that describe the system:

$$v_s = R_s i_s + \frac{d\lambda_s}{dt} \quad (4.1.9)$$

$$v_r = R_r i_r + \frac{d\lambda_r}{dt} \quad (4.1.10)$$

$$T_m + T^e = J_r \frac{d^2\theta}{dt^2} + B_r \frac{d\theta}{dt} + T_{or} \frac{d\theta/dt}{|d\theta/dt|} \quad (4.1.11)$$

* See, for example, A. E. Fitzgerald and C. Kingsley, Jr., *Electric Machinery*, 2nd ed., McGraw-Hill, New York, 1961, Chapter 7. Although electrical losses can be treated as perturbations when analyzing the behavior of a machine, these losses are vitally important in determining the machine's rating because it is set by thermal limitations in transferring heat generated by losses out of the machine.

Once the sources (or loads) v_s , v_r , and T_m are defined, these three equations with the terminal relations (4.1.6) to (4.1.8) form a complete description of the dynamics of the system, including the machine and the electric and mechanical circuits connected to it.

4.1.2 Conditions for Conversion of Average Power

We next consider the problem of finding the conditions under which the electromechanical coupling system of the machine in Fig. 4.1.1 can convert average power between the electrical and mechanical systems. For this problem a steady-state analysis of the coupling system in Fig. 4.1.4 with ideal sources will suffice. Once the conditions are established, the analysis can be generalized to include nonideal sources and transient conditions.

For this problem the coupling system of Fig. 4.1.4 will be excited by the ideal sources indicated in Fig. 4.1.6. The specific time dependences of these sources are

$$i_s(t) = I_s \sin \omega_s t, \quad (4.1.12)$$

$$i_r(t) = I_r \sin \omega_r t, \quad (4.1.13)$$

$$\theta(t) = \omega_m t + \gamma, \quad (4.1.14)$$

where I_s , I_r , ω_s , ω_r , ω_m , and γ are positive constants and t is the time.

We now ask for the conditions under which the machine with the steady-state excitations of (4.1.12) to (4.1.14) can convert average power between the electrical and mechanical systems. To find these conditions we evaluate the instantaneous power p_m flowing from the coupling system into the position source

$$p_m = T^e \frac{d\theta}{dt} = T^e \omega_m. \quad (4.1.15)$$

Then substitution from (4.1.12) and (4.1.13) into (4.1.8) and of that result into (4.1.15) yield

$$p_m = -\omega_m I_s I_r M \sin \omega_s t \sin \omega_r t \sin (\omega_m t + \gamma). \quad (4.1.16)$$

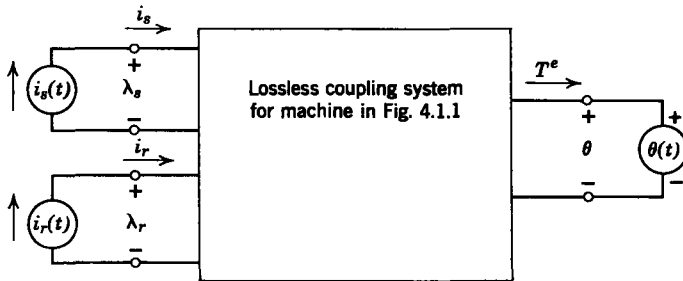


Fig. 4.1.6 Excitations used in derivation of conditions for average power conversion.

To ascertain under what conditions this power can have an average value, trigonometric identities* are used to put (4.1.16) in the form

$$p_m = -\frac{\omega_m I_s I_r M}{4} \{ \sin [(\omega_m + \omega_s - \omega_r)t + \gamma] + \sin [(\omega_m - \omega_s + \omega_r)t + \gamma] \\ - \sin [(\omega_m + \omega_s + \omega_r)t + \gamma] - \sin [(\omega_m - \omega_s - \omega_r)t + \gamma] \}. \quad (4.1.17)$$

Because a sinusoidal function of time has no average value, (4.1.17) can have a time-average value only when one of the coefficients of t is zero. These four conditions, which cannot in general be satisfied simultaneously, can be written in the compact form

$$\omega_m = \pm \omega_s \pm \omega_r; \quad (4.1.18)$$

for example, when

$$\omega_m = -\omega_s + \omega_r, \\ p_{m(av)} = -\frac{\omega_m I_s I_r M}{4} \sin \gamma,$$

and, when

$$\omega_m = \omega_s + \omega_r, \\ p_{m(av)} = \frac{\omega_m I_s I_r M}{4} \sin \gamma.$$

It is evident from these expressions that a necessary condition for average power conversion is the frequency condition of (4.1.18). Sufficient conditions for average power conversion are (4.1.18) and $\sin \gamma \neq 0$.

As a result of this analysis, we can state that the whole field of machine theory for smooth-air-gap machines is concerned with how to satisfy the frequency condition of (4.1.18) with the available electrical and/or mechanical sources to obtain the machine characteristics needed for a particular application. It is just this process that has led to the several different machine types presently used. The frequency relations provide the starting point in the invention of new machine types for unusual applications.

4.1.3 Two-Phase Machine

Before describing the different standard machine types, how they are excited to satisfy (4.1.18), and what their essential characteristics are, the smooth-air-gap model of Fig. 4.1.1 will be modified to allow a more realistic portrayal of the energy conversion properties of rotating machines.

It is evident by examination of (4.1.17) that when one of the four possible conditions of (4.1.18) is satisfied, the corresponding term in (4.1.17) becomes a constant, but the other three terms are still sinusoidal time functions

* $\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$; $\sin x \cos y = \frac{1}{2}[\sin(x - y) + \sin(x + y)]$.

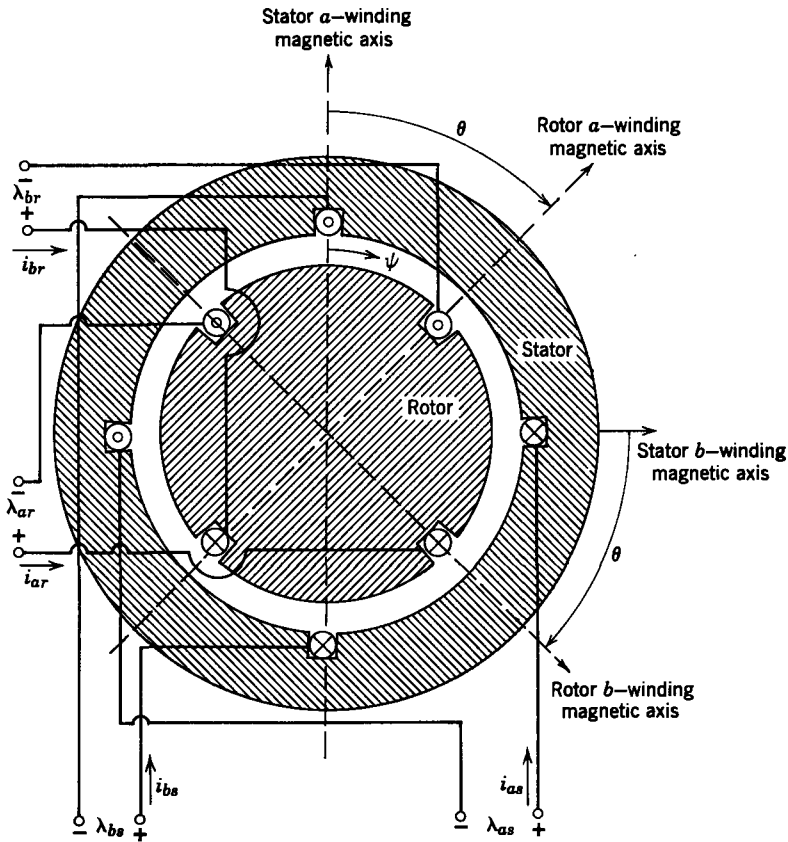


Fig. 4.1.7a Cross-sectional view of a two-phase machine with distributed windings represented by single-turn coils.

and each term represents an alternating power flow. These alternating power flows have no average values and can cause pulsations in speed and vibrations that are detrimental to machine operation and life. The alternating flow can be eliminated by adding one additional winding to both rotor and stator, as illustrated in Fig. 4.1.7a. The windings of Fig. 4.1.7a are represented as being concentrated in single slots for simplicity of illustration. In actual machines the windings are distributed like those of Fig. 4.1.1a. and a single slot can carry conductors from both windings. In Fig. 4.1.7 windings a on rotor and stator represent the original windings of Fig. 4.1.1. Windings b on rotor and stator are identical to the windings a in every respect, except that they are displaced mechanically 90° in the positive θ -direction.

The two additional windings in Fig. 4.1.7a require two additional electrical terminal pairs and, using the assumptions of constant self-inductances and

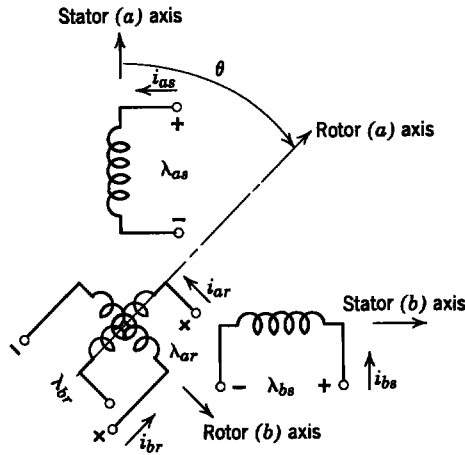


Fig. 4.1.7b Schematic representation of balanced two-phase machine in (a) showing relative orientations of magnetic axes.

sinusoidally varying mutual inductances discussed before, the terminal relations are now written as

$$\lambda_{as} = L_s i_{as} + M i_{ar} \cos \theta - M i_{br} \sin \theta, \quad (4.1.19)$$

$$\lambda_{bs} = L_s i_{bs} + M i_{ar} \sin \theta + M i_{br} \cos \theta, \quad (4.1.20)$$

$$\lambda_{ar} = L_r i_{ar} + M i_{as} \cos \theta + M i_{bs} \sin \theta, \quad (4.1.21)$$

$$\lambda_{br} = L_r i_{br} - M i_{as} \sin \theta + M i_{bs} \cos \theta, \quad (4.1.22)$$

$$T^e = M[(i_{ar} i_{bs} - i_{br} i_{as}) \cos \theta - (i_{ar} i_{as} + i_{br} i_{bs}) \sin \theta]. \quad (4.1.23)$$

Study of the relative winding geometry in Fig. 4.1.7a verifies the correctness of the mutual inductance terms in the electrical terminal relations. Once again, the torque T^e has been found by using the techniques of Chapter 3 [see (g) in Table 3.1].

The windings of Fig. 4.1.7 are called *balanced two-phase windings** because excitation with balanced two-phase currents will result in constant power conversion with no alternating components. To show this the terminal variables of the machine are constrained by the *balanced, two-phase*, current sources

$$i_{as} = I_s \cos \omega_s t, \quad (4.1.24)$$

$$i_{bs} = I_s \sin \omega_s t, \quad (4.1.25)$$

$$i_{ar} = I_r \cos \omega_r t, \quad (4.1.26)$$

$$i_{br} = I_r \sin \omega_r t \quad (4.1.27)$$

* More is said about phases in Section 4.1.7.

and by the angular position source

$$\theta = \omega_m t + \gamma. \quad (4.1.28)$$

The use of these terminal constraints with the instantaneous power given by (4.1.15) and the torque T^e given by (4.1.23) yields, after some trigonometric manipulation,*

$$p_m = -\omega_m M I_s I_r \sin [(\omega_m - \omega_s + \omega_r)t + \gamma]. \quad (4.1.29)$$

This power can have an average value only when the coefficient of t is zero, that is, when

$$\omega_m = \omega_s - \omega_r \quad (4.1.30)$$

for which condition (4.1.29) reduces to

$$p_m = -\omega_m M I_s I_r \sin \gamma. \quad (4.1.31)$$

This is still the instantaneous power out of the machine, but it is now constant in spite of the ac electrical excitation. Note further that (4.1.30) is one of the four conditions of (4.1.18). Thus the additional windings with proper excitation have produced only a single frequency condition (4.1.30) for average power conversion, and when this condition is satisfied the instantaneous power is constant and equal to the average value.

The other three conditions of (4.1.18) can be achieved individually in the machine of Fig. 4.1.7 with the excitations of (4.1.24) to (4.1.28) by changing the time phase of one stator current and/or one rotor current by 180° .

When the two stator currents or the two rotor currents are unbalanced in amplitude or the phase difference is changed from 90° , the pulsating power flow will again occur even when one of the conditions of (4.1.18) is satisfied. The analysis of these situations is straightforward trigonometry and is not carried out here.

4.1.4 Air-Gap Magnetic Fields

It is helpful for qualitative physical reasoning and for a more thorough understanding of the coupling mechanism occurring in rotating machines to think in terms of the magnetic fields that exist in the air gap. To develop these ideas consider again the machine in Fig. 4.1.7a but with only the stator excited by current sources. The assumption that rotor-to-stator mutual inductance varies sinusoidally with rotor position implies that the flux density produced in the air gap by a current in a winding varies sinusoidally with angular position; that is, a current in stator winding a will produce an air-gap flux density whose radial component is maximum along the magnetic axis (positive in one direction, negative in the other) and varies sinusoidally

* $\sin(x - y) = \sin x \cos y - \cos x \sin y$, $\cos(x + y) = \cos x \cos y - \sin x \sin y$.

between these extremes. We now assume that the stator windings of Fig. 4.1.7a are excited by the two-phase current sources of (4.1.24) and (4.1.25) and look at the space distribution of radial air-gap flux density produced by these currents at different instants of time. For this we use the angle ψ defined in Fig. 4.1.7a to indicate position in the air gap and we recognize that the air-gap flux density produced by current in a winding is proportional to the current producing it. Consequently, with reference to Fig. 4.1.7a, we can write for the instantaneous radial flux density due to current in stator winding a

$$B_{ra} = B_{rm} \cos \omega_s t \cos \psi \quad (4.1.32)$$

and for stator winding b

$$B_{rb} = B_{rm} \sin \omega_s t \sin \psi, \quad (4.1.33)$$

where B_{rm} is a constant related to the current amplitude I_s . These component flux densities and the resultant air-gap flux density are sketched for five values of time in Fig. 4.1.8. For the interval shown the resultant flux density (B_r) is sinusoidally distributed in space, has a constant amplitude (B_{rm}), and moves in the positive ψ -direction as time progresses. An extension of this process would show that these facts remain true for all time and that the resultant flux density makes one revolution in $(2\pi/\omega_s)$ sec or it rotates with an angular speed ω_s .

A similar argument for the rotor of Fig. 4.1.7a with the excitation of (4.1.26) and (4.1.27) shows that these rotor currents produce a flux density in the air gap that is sinusoidally distributed around the periphery, has constant amplitude, and has an angular velocity ω_r with respect to the rotor.

It is to be expected from simple considerations of the tendency of two magnets to align themselves, that a steady torque and therefore constant power conversion will occur when the rotor and stator fields are fixed in space relative to each other and the rotor is turning at constant angular velocity. To accomplish this a mechanical speed given by (4.1.30) is required. Thus the condition for average power conversion can be interpreted as establishing the condition under which the stator and rotor fields, both of which rotate with respect to the members carrying the excitation currents, are fixed in space relative to each other. Furthermore, we expect the torque (and average power) to be a function of the constant angle of separation of the axes of symmetry of the two fields. This variation with angle is indicated by the $\sin \gamma$ term in (4.1.31). Examination of (4.1.24) to (4.1.28) with the ideas introduced in Fig. 4.1.8 shows that γ is the angle by which the rotor magnetic field axis precedes the stator magnetic field axis around the air gap in the positive θ -direction. Thus the torque [or power in (4.1.31)] is proportional to $(-\sin \gamma)$.

It must be recognized that this analysis of air-gap magnetic fields is idealized. With excitation provided by finite-size coils in finite-size slots and

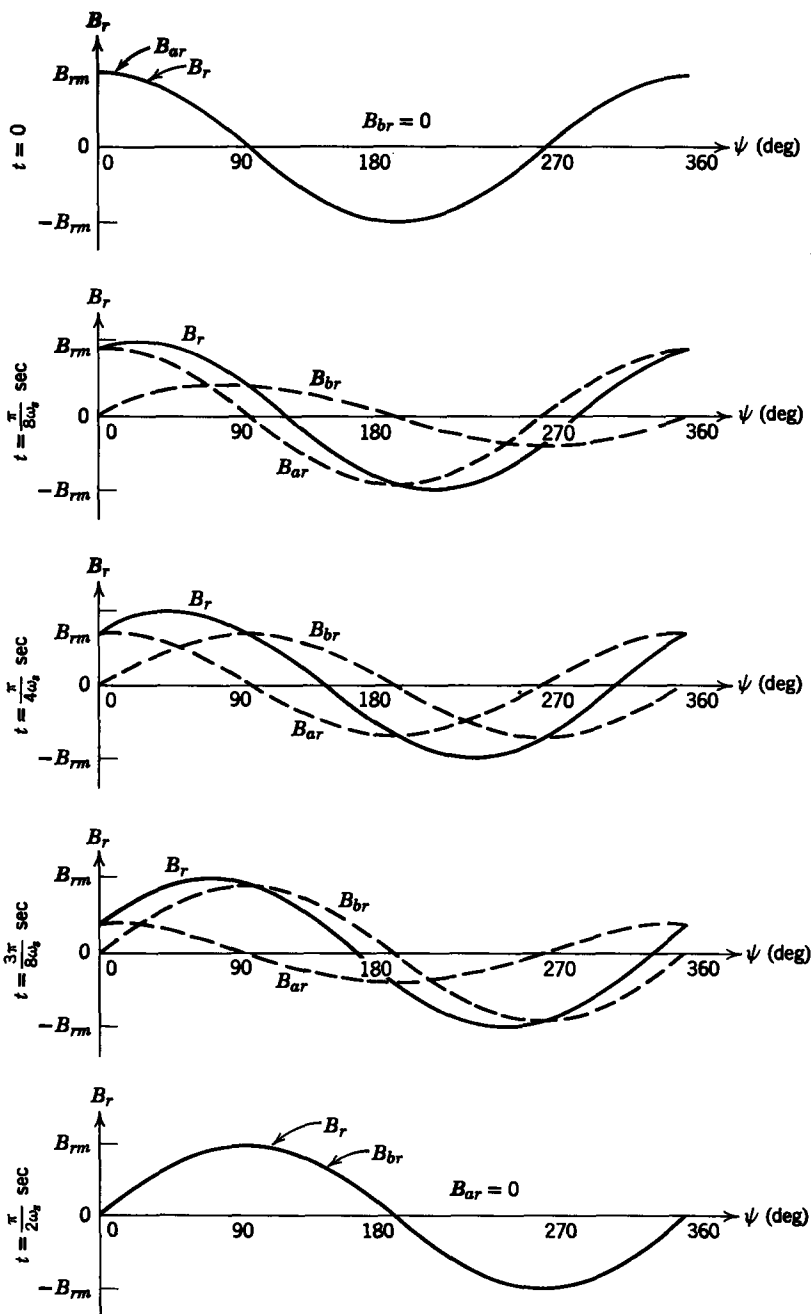


Fig. 4.1.8 Component (B_{ar} and B_{br}) and resultant (B_r) flux density distributions in a balanced two-phase machine due to balanced two-phase stator currents.

usually with equal numbers of turns in all slots, the air-gap flux density will not be exactly sinusoidal. Nonetheless, the simple picture presented is a design objective with ac machines and it yields remarkably accurate results for practical machines. (See part (c) of Prob. 4.4.)

We now reconsider the machine of Fig. 4.1.1 with the excitations of (4.1.12) to (4.1.14) and apply the rotating field ideas. Because the rotor-stator mutual inductance is a sinusoidal function of space, the air-gap flux density will still have a sinusoidal space distribution around the air gap. But now each field, rotor and stator, being excited by a sinusoidal current in a single coil keeps a fixed sinusoidal space distribution with respect to its exciting coil and varies periodically in amplitude. Such an alternating field can be represented by two constant-amplitude fields (B_{r+} and B_{r-}) rotating in opposite directions, as illustrated for one quarter cycle of the stator excitation in Fig. 4.1.9. Thus the stator excitation produces two such fields rotating with angular velocity $\pm\omega_s$ with respect to the stator. Similarly, the rotor excitation produces two fields rotating with angular velocity $\pm\omega_r$ with respect to the rotor. Thus each of the four conditions of (4.1.18) represents the situation in which the mechanical speed is adjusted to the proper value to make one component of rotor field fixed in space relative to one component of stator field. When one of these conditions is satisfied, the other three conditions are not, and the interactions of these other field components give rise to alternating torque and alternating power flow. As a consequence, the addition of the second set of windings in Fig. 4.1.7 can be interpreted as being for the purpose of eliminating those field components that do not produce average power conversion.

4.1.5 Discussion

Although these analyses have been made for special cases, the method is quite general; for instance, the analytical techniques are the same when the restriction of a space fundamental mutual inductance variation is removed and the more general form of (4.1.4) is used. Furthermore, the windings added to the machine of Fig. 4.1.1 to obtain the machine of Fig. 4.1.7 need not be identical to the original windings, nor must they be exactly in space quadrature with them. The only requirement for a correct analysis of the coupling mechanisms is that the electrical terminal relations be accurate representations of the physical system under study. The particular restrictions chosen here are representative of design objectives for practical rotating machines and of techniques used in their analysis.

The assumption has been made that stator circuits are excited by currents of the single frequency ω_s and rotor circuits are excited by currents of the single frequency ω_r . This analysis is easily generalized to any number of

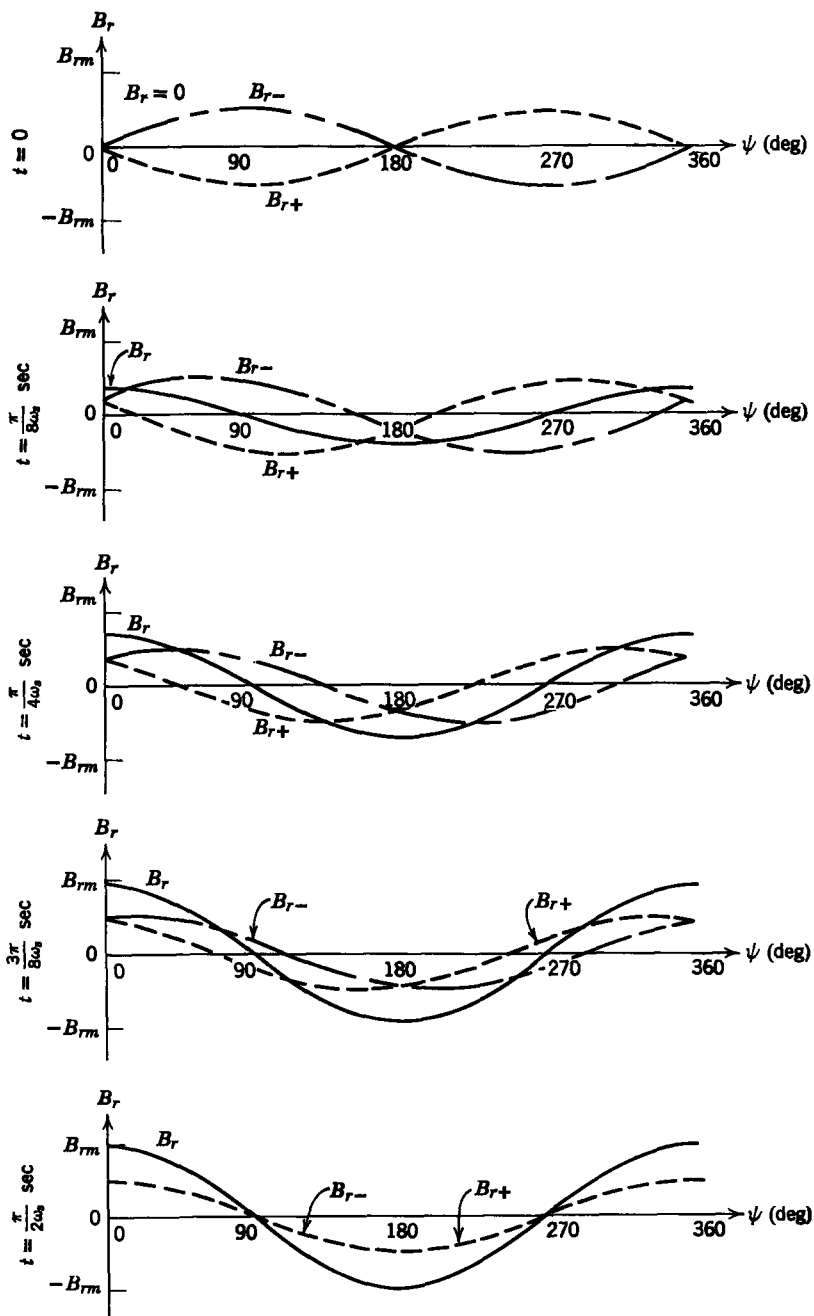


Fig. 4.1.9 Component (B_{r+} and B_{r-}) and resultant (B_r) flux density distributions in a single-phase machine with sinusoidal excitation current on the stator.

frequencies on each member (rotor and stator). The result is that for a component of current at a particular frequency to participate in the average power conversion process there must be a current on the other member whose frequency combines with the mechanical velocity and the other electrical frequency to satisfy one of the four conditions of (4.1.18).

The analyses have been done by assuming a steady-state problem with electrical current sources and a mechanical position source. The results are still valid when the sources are changed to voltage and torque sources or to dependent sources of any type. Moreover, transient problems can be analyzed by writing the complete differential equations for the machine and the electrical and mechanical circuits to which it is connected and using the techniques of Chapter 5. The steady-state analysis is representative of the final conditions usually reached in machine operation. Transients of interest include the one necessary to reach steady-state conditions (e.g., starting transient) and those that occur when something forces the operation away from steady-state conditions (e.g., a sudden change in load torque on a motor).

The discussion of air-gap magnetic fields was based on the simple model of a sinusoidal distribution of flux density in space and a sinusoidal variation of flux density with time. When the space distributions and time variations are not sinusoidal they can often be represented by Fourier series. Consequently, our discussion of air-gap magnetic fields can be applied to individual Fourier components to obtain insight into the interactions occurring in the machine.

In summary, our look at the smooth-air-gap machine in terms of simple models has a lot more generality than we at first might suspect. These simple models are building blocks with which we can build understanding of the behavior of complex machines.

4.1.6 Classification of Machine Types

The results of the steady-state analysis expressed as (4.1.18) for the configuration of Fig. 4.1.1 and as (4.1.30) for the machine of Fig. 4.1.7 are used to define conventional machine types. From one viewpoint rotating-machine theory boils down to the practical ways of satisfying the frequency conditions for average power conversion, given the available electrical and mechanical sources and loads and the desired machine characteristics.

In the following sections we indicate how the frequency condition is met and what the steady-state characteristics are for several conventional machine types.

4.1.6a Synchronous Machines

Consider the two-phase machine of Fig. 4.1.7 with direct current applied to the rotor ($\omega_r = 0$) and balanced, two-phase currents of frequency ω_s

applied to the stator. The condition of (4.1.30) indicates that the machine can convert average power only when the rotor is turning with the single value of speed

$$\omega_m = \omega_s. \quad (4.1.34)$$

These constraints yield a *synchronous* machine, so named because it can convert average power only at one mechanical speed—the synchronous speed ω_s determined by the stator excitation frequency.

In a similar way we can also make a synchronous machine by applying direct current to the stator and alternating current to the rotor. In this case the synchronous speed is determined by the frequency of the rotor currents.

The most common application of a smooth-air-gap, synchronous machine is as the generator, called alternator or turboalternator, that is driven by a steam turbine to generate power. In this machine the ac or *armature* windings are on the stator and the dc or *field* winding is on the rotor. The direct current for the field winding is usually fed through carbon or metal-graphite brushes that make contact with slip rings on the rotor. A large turboalternator with the end bell removed is shown in Fig. 4.1.10.

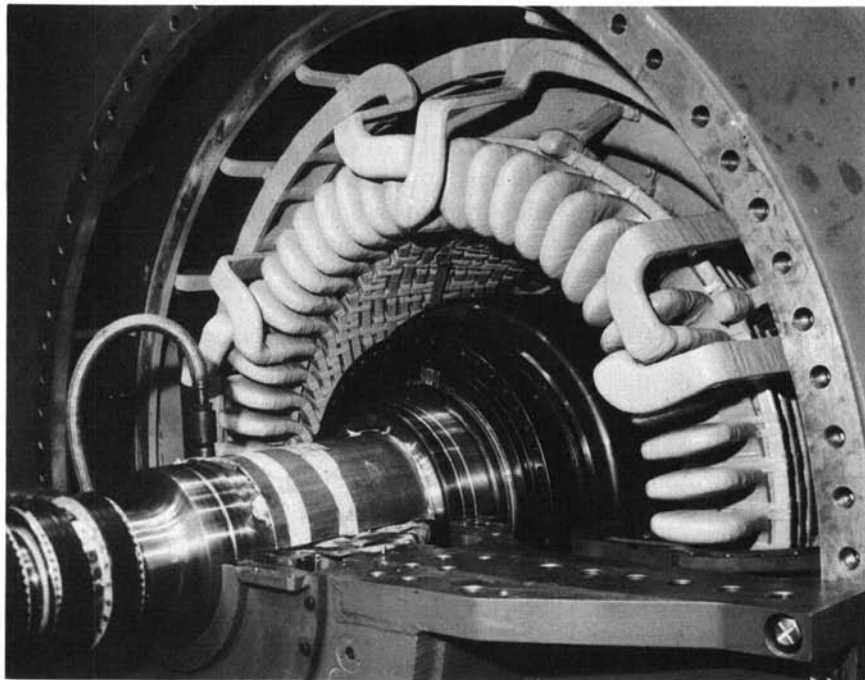


Fig. 4.1.10 Turboalternator partly assembled for test. This is a 3600-rpm, 192,000-kVA machine showing rotor in place with the upper half of the bearing and the end shield removed. The slip rings (or collector rings) for supplying current to the rotor (field) winding are in the foreground. (Courtesy of General Electric Company.)

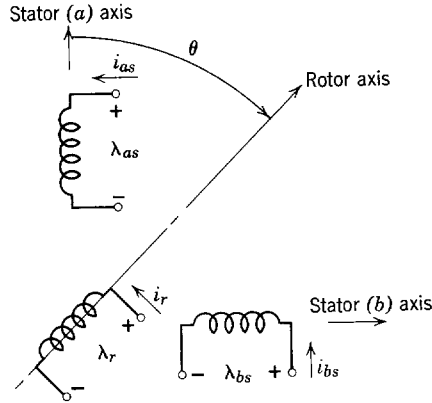


Fig. 4.1.11 Schematic representation of smooth-air-gap synchronous machine with field (dc) winding on the rotor and a balanced two-phase stator (armature) winding.

Some of the principal steady-state characteristics of synchronous machines can be illustrated by considering the machine shown schematically in Fig. 4.1.11. The electrical and mechanical terminal relations are

$$\lambda_{as} = L_s i_{as} + M i_r \cos \theta, \quad (4.1.35)$$

$$\lambda_{bs} = L_s i_{bs} + M i_r \sin \theta, \quad (4.1.36)$$

$$\lambda_r = L_r i_r + M(i_{as} \cos \theta + i_{bs} \sin \theta), \quad (4.1.37)$$

$$T^e = M i_r (i_{bs} \cos \theta - i_{as} \sin \theta). \quad (4.1.38)$$

[These are (4.1.19) to (4.1.21) and (4.1.23) with $i_{br} = 0$ and λ_{ar} and i_{ar} replaced by λ_r and i_r , respectively.]

Synchronous machines are normally operated with the stator windings excited by voltage sources. We express our results in these terms. We find it convenient, however, to constrain the stator winding currents analytically with the balanced, two-phase set

$$i_{as} = I_s \cos \omega t, \quad (4.1.39)$$

$$i_{bs} = I_s \sin \omega t, \quad (4.1.40)$$

and to consider I_s as an unknown in the analysis. We constrain the rotor current to be constant

$$i_r = I_r \quad (4.1.41)$$

and the angle θ with the position source

$$\theta = \omega t + \gamma. \quad (4.1.42)$$

We see that the condition for average power conversion (4.1.30) is automatically satisfied.

It can be verified by direct substitution of (4.1.39) to (4.1.42) into (4.1.37) that the rotor flux linkage λ_r is constant. Thus, under steady-state conditions, there is no voltage induced in the rotor circuit, and we could have applied a constant-voltage source, as we normally do in practice. In such a case the rotor current is constant and determined only by the applied voltage and rotor circuit resistance. Consequently, there is no loss in generality by specifying the rotor current as in (4.1.41). For transient conditions there is a difference between voltage and current excitation of the rotor.

Substitution of (4.1.39) to (4.1.42) into (4.1.38), yields for the instantaneous torque produced by the machine

$$T^e = -MI_r I_s \sin \gamma. \quad (4.1.43)$$

For the analysis of the energy conversion properties of large synchronous machines the mechanical (friction) losses are neglected because they are a small fraction of the power converted by the machine. Thus we follow this procedure and assume that the torque expressed by (4.1.43) is applied to the mechanical load (or source) on the shaft [T_m in Fig. 4.1.5].

To find the electrical terminal characteristics we need to evaluate the stator terminal voltages. Substitution of (4.1.39) to (4.1.42) into (4.1.35) and (4.1.36) yields

$$\lambda_{as} = L_s I_s \cos \omega t + MI_r \cos (\omega t + \gamma), \quad (4.1.44)$$

$$\lambda_{bs} = L_s I_s \sin \omega t + MI_r \sin (\omega t + \gamma). \quad (4.1.45)$$

These two flux linkages are sinusoidal functions of time with a single frequency ω . Moreover, λ_{bs} is the same as λ_{as} , except for a shift of 90° in time phase. The currents exciting these two windings are identical in amplitude and different in phase by the same 90° . Consequently, we expect the electrical behavior of the two stator windings to be the same except for this phase shift; thus we analyze only winding a . When considering power into or out of the stator, we multiply the result for one winding by two to account for the second winding.

As is the usual practice in the analysis of the energy conversion properties of large synchronous machines, we neglect winding resistances and express the terminal voltage of stator winding a as

$$v_{as} = \frac{d\lambda_{as}}{dt} = \frac{d}{dt} [L_s I_s \cos \omega t + MI_r \cos (\omega t + \gamma)]. \quad (4.1.46)$$

Because this expression involves sinusoidal functions of time with the single frequency ω , it is convenient to express the quantities in terms of complex functions and to use vector diagrams to illustrate electrical properties. Hence we write

$$i_{as} = \text{Re} (I_s e^{j\omega t}), \quad v_{as} = \text{Re} (\hat{V}_s e^{j\omega t}), \quad (4.1.47)$$

where I_s is real and $\hat{V}_s$ is complex. We substitute this expression for the voltage into (4.1.46) replace the time functions by their complex equivalents,

$$\cos \omega t = \text{Re} (e^{j\omega t}), \quad \cos (\omega t + \gamma) = \text{Re} (e^{j\gamma} e^{j\omega t}),$$

drop the Re, and cancel the $e^{j\omega t}$ terms to get

$$\hat{V}_s = j\omega L_s I_s + j\omega M I_r e^{j\gamma}. \quad (4.1.48)$$

The last term in this equation is a voltage source that depends on rotor current I_r and rotor phase angle γ . It is conventionally designated as

$$\hat{E}_f = j\omega M I_r e^{j\gamma}, \quad (4.1.49)$$

the subscript f denoting dependence on field (rotor) current. The use of (4.1.49) in (4.1.48) yields the expression

$$\hat{V}_s = j\omega L_s I_s + \hat{E}_f. \quad (4.1.50)$$

This steady-state stator (armature) terminal voltage is used to construct a simple steady-state equivalent circuit for one phase of a balanced two-phase synchronous machine with balanced excitation, as shown in Fig. 4.1.12.* Note from this figure that because $\hat{E}_f$ is independently adjustable the current I_s can be controlled in magnitude and phase relative to $\hat{V}_s$ and the synchronous machine can act as a motor or generator. The quantity $\omega_s L_s$ is conventionally called the *synchronous reactance* and is simply the reactance of the self-inductance of a stator winding.

The complex quantities in (4.1.50) and the equivalent circuit of Fig. 4.1.12 can be used to interpret the properties of a synchronous machine. To do this we sketch the complex quantities as vectors on the complex plane for two conditions in Fig. 4.1.13. To put our analysis in tune with convention we define the *torque angle* δ as measured from $\hat{V}_s$ to $\hat{E}_f$, as shown in Fig. 4.1.13. A simple geometrical construction illustrated in Fig. 4.1.13 shows that

$$\omega L_s I_s \sin \gamma = V_s \sin \delta, \quad (4.1.51)$$

where $V_s = |\hat{V}_s|$. Thus from (4.1.43) and (4.1.49)

$$T^e = -\frac{E_f V_s}{\omega^2 L_s} \sin \delta, \quad (4.1.52)$$

where $E_f = |\hat{E}_f|$. This is the expression for the torque normally used in the analysis of

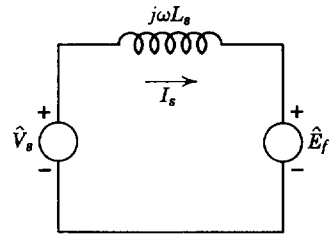


Fig. 4.1.12 Steady-state equivalent circuit for one phase of a balanced two-phase synchronous machine with balanced excitation; $\hat{E}_f$ is defined by (4.1.49).

* This same process, which uses an equivalent circuit for one phase, is also applied to machines with more than two phases.

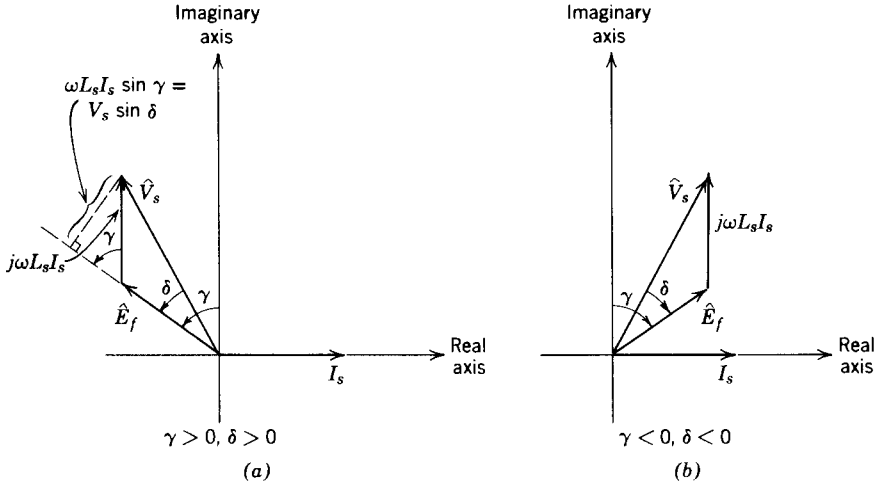


Fig. 4.1.13 Diagrams showing relations among variables in synchronous machine: (a) generator operation in which γ and δ (measured positive in the counterclockwise direction) are positive; (b) motor operation in which γ and δ are negative.

synchronous machines. It is expressed in terms of the magnitude of the stator voltage because this quantity is usually constant, as determined by the source supplying (or absorbing) electrical power.

The instantaneous power into the stator windings is

$$p_e = v_{as}i_{as} + v_{bs}i_{bs}. \quad (4.1.53)$$

It can be verified with some algebra and trigonometry that this power is equal to the mechanical power out of the shaft

$$p_e = p_m = \omega T^e = -\frac{E_f V_s}{\omega L_s} \sin \delta. \quad (4.1.54)$$

Thus the electromechanical power conversion occurs at a constant rate between the stator circuits and the mechanical system connected to the shaft. The rotor (field) circuit does not participate in the conversion process except to control the dependent source E_f . The power required to excite the field winding as a fraction of the stator (armature) power rating varies from 0.5 per cent in large turboalternators to a few per cent in synchronous motors. It is for this reason that the field winding is usually on the rotor, the result being that sliding contacts have to handle less power.

When a synchronous machine is operated with a constant voltage supply (V_s) to the stator and a constant field current (constant E_f), the torque-angle characteristic is the simple sinusoid shown in Fig. 4.1.14. The machine can

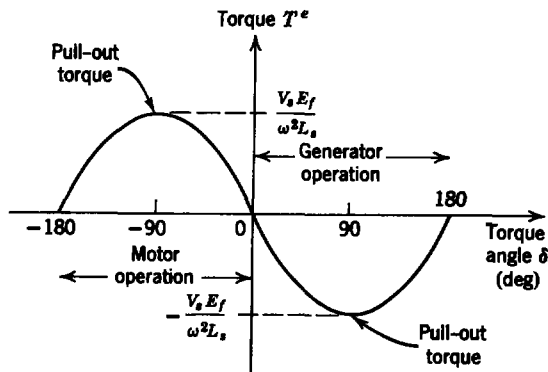


Fig. 4.1.14 Torque versus torque angle for a synchronous machine with constant stator voltage amplitude and constant rotor current.

supply any torque required by the shaft load (or supply) in the range

$$-\frac{V_s E_f}{\omega^2 L_s} < T^e < \frac{V_s E_f}{\omega^2 L_s}$$

and operate as a motor or generator. Any attempt to demand (or supply) a torque outside this range will surpass the ability of the machine and it will no longer run at synchronous speed and produce an average torque. This process of loss of synchronism is called *pulling out of step* and the maximum torque the machine can supply is called the *pull-out torque*.

We shall demonstrate one additional property of synchronous machines, that of adjusting the power factor (phase angle between stator terminal voltage and current) by varying the field (rotor) current. We shall illustrate the property by using motor operation; however, the general features of the analysis also hold for generator operation.

We assume motor operation with constant-amplitude, balanced, two-phase voltages applied to the stator windings of the machine in Fig. 4.1.11. A constant torque load T_m is applied to the machine, and the field (rotor) current I_f can be set to different values. We neglect stator winding resistance and friction losses. We consider three cases illustrated on the torque angle curves of Fig. 4.1.15a, the vector relations among the variables being shown in Figs. 4.1.15b, c, and d. Note two things in studying the vector diagrams of Fig. 4.1.15. First, as the field current I_f is increased from a low value I_{r1} to a high value I_{r3} , the magnitude of the stator current passes through a minimum. Second, for the same variation in I_f the phase angle between stator voltage and current reverses sign. An analysis of this type is used to produce a so-called *V-curve* like that shown in Fig. 4.1.15e, which is a plot of

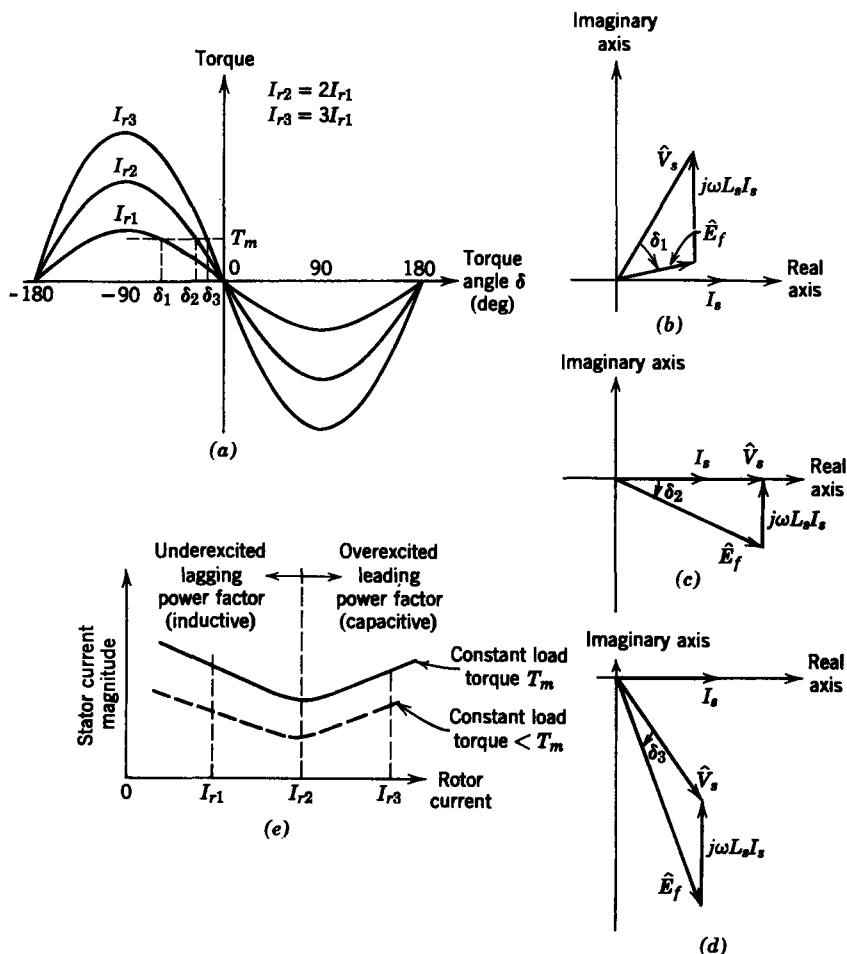


Fig. 4.1.15 Illustration of the performance of a synchronous motor as the field excitation is varied: (a) torque versus torque angle; (b) rotor current I_{r1} ; (c) rotor current I_{r2} ; (d) rotor current I_{r3} ; (e) synchronous motor V-curve.

stator current amplitude as a function of field current with *stator voltage amplitude and load torque held constant*. For small values of field current the machine is said to be underexcited; it appears inductive to the electrical sources and operates with a lagging power factor. For large values of field current the machine is said to be overexcited; it appears capacitive to the electrical sources and operates with a leading power factor.*

* Power factor is defined as $\cos \theta$, where θ is the phase-angle between the current and voltage. A power factor of unity indicates that the load is purely resistive. See, for example, R. M. Kerchner and G. F. Corcoran, *Alternating-Current Circuits*, Wiley, New York, 1955.

In many applications of synchronous motors the machines have adequate field winding capacity for overexcitation and the capacitive characteristics are used for power-factor correction. Special machines, called *synchronous condensers*, are actually synchronous motors that run with no mechanical load and are used as continuously variable capacitors to adjust power factor and regulate voltage on electric power transmission systems.

In this presentation we have discussed only a few of the principal features of synchronous machines. There are many others, depending on the application, and they are studied by the same techniques. Our treatment is intended to be only an introduction to synchronous-machine characteristics. A complete study would fill a book by itself.*

In our analysis we used a two-phase machine as an example. Virtually all synchronous machines manufactured are three-phase because the power supply is usually three-phase. Three-phase synchronous machines have the same energy conversion properties and steady-state characteristics as the two-phase machine of our examples. In fact, a standard procedure in the analysis of a three-phase machine is to transform the electrical variables to obtain the equations for an equivalent two-phase machine. This simplifies the mathematics in the analysis.†

This analysis was made with direct current applied to the rotor (field) winding. The rotor can, and sometimes is, replaced by a permanent-magnet rotor, an arrangement that has the two advantages of requiring no power to maintain the field and no sliding electrical contacts. This also has two primary disadvantages: (a) the amplitude of the magnetic field is fixed by the permanent magnet and cannot be controlled externally during operation, and (b) the magnetic flux densities obtainable with permanent magnet materials are considerably smaller than those obtainable with current-excited, high-permeability iron. As a result of the second disadvantage, permanent magnets are normally used in small synchronous machines.

4.1.6b Induction Machines

An *induction* machine is conventionally defined as one in which single-frequency alternating currents are fed into the stator circuits and the rotor circuits are all short circuited. Rotor currents are obtained by induction from the stator, hence the name.

To determine that an induction machine can convert average power, consider again the machine of Fig. 4.1.7 with the stator currents constrained by balanced two-phase sources.

$$i_{as} = I_s \cos \omega_s t, \quad (4.1.55)$$

$$i_{bs} = I_s \sin \omega_s t, \quad (4.1.56)$$

* See, for example, C. Concordia, *Synchronous Machines*, Wiley, New York, 1951.

† See, for example, White and Woodson, *op. cit.*, Chapter 9.

with the rotor circuits short-circuited

$$v_{ar} = v_{br} = 0, \quad (4.1.57)$$

and with the rotor constrained by the position source

$$\theta = \omega_m t + \gamma. \quad (4.1.58)$$

The terminal relations for the electromechanical coupling system are (4.1.19) to (4.1.23).

In the analysis that follows we neglect the resistance of the stator windings. This is standard practice when analyzing the energy conversion properties of a large induction machine. The primary effect of stator winding resistance is in heating the machine, and it therefore plays a major role in determining the machine's rating. For reasons that will become clear subsequently, we must retain the resistance of the rotor circuits in our analysis.

With the terminal constraints of (4.1.55) to (4.1.58) and with rotor circuit resistance denoted by R_r , we write the differential equations for the two rotor circuits:

$$\begin{aligned} 0 = R_r i_{ar} + L_r \frac{di_{ar}}{dt} + M I_s \frac{d}{dt} [\cos \omega_s t \cos (\omega_m t + \gamma) \\ + \sin \omega_s t \sin (\omega_m t + \gamma)] \end{aligned} \quad (4.1.59)$$

$$\begin{aligned} 0 = R_r i_{br} + L_r \frac{di_{br}}{dt} + M I_s \frac{d}{dt} [-\cos \omega_s t \sin (\omega_m t + \gamma) \\ + \sin \omega_s t \cos (\omega_m t + \gamma)]. \end{aligned} \quad (4.1.60)$$

The use of appropriate trigonometric identities* allows us to rewrite these equations in the forms

$$M I_s (\omega_s - \omega_m) \sin [(\omega_s - \omega_m)t - \gamma] = L_r \frac{di_{ar}}{dt} + R_r i_{ar}, \quad (4.1.61)$$

$$-M I_s (\omega_s - \omega_m) \cos [(\omega_s - \omega_m)t - \gamma] = L_r \frac{di_{br}}{dt} + R_r i_{br}. \quad (4.1.62)$$

The right sides are identical, linear, first-order differential operators with constant coefficients. The left sides are sinusoidal voltage drives of equal amplitudes, but 90 degrees phase difference (just like the stator currents). Thus we need to consider only one of these equations for a solution.

As indicated by (4.1.61) and (4.1.62), both rotor currents will have frequency $(\omega_s - \omega_m)$ which exactly satisfies the condition of (4.1.30). Thus the induction machine satisfies the condition for average power conversion at all mechanical speeds. With finite rotor resistance an induction machine can

* $\cos (x - y) = \cos x \cos y + \sin x \sin y$, $\sin (x - y) = \sin x \cos y - \cos x \sin y$.

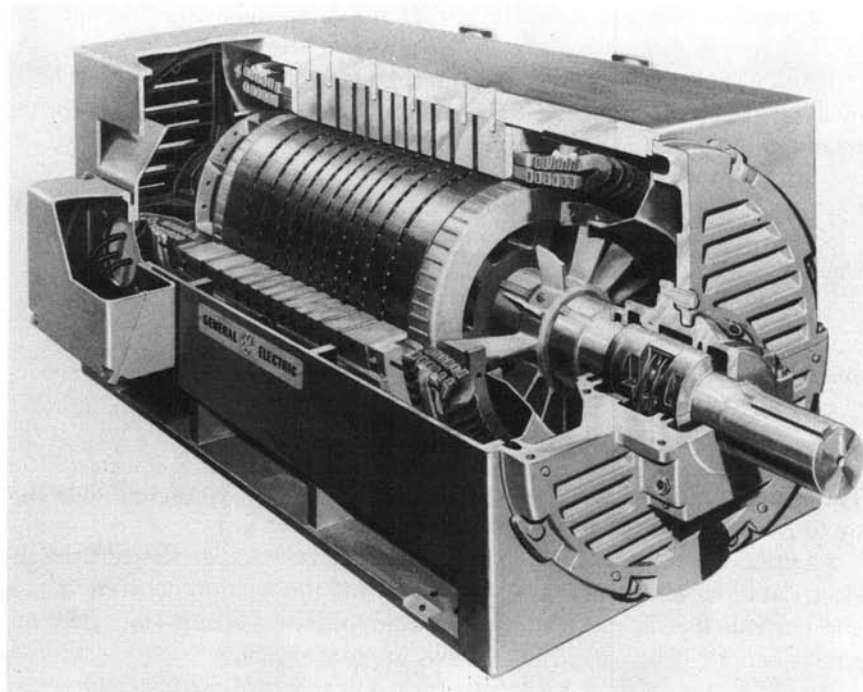


Fig. 4.1.16 Cutaway view of a squirrel-cage induction motor. (Courtesy of General Electric Company.)

Courtesy of General Electric Company. Used with permission.

convert average power at all speeds except synchronous ($\omega_m = \omega_s$), and its mode of operation (motor, generator, or brake) depends on the relative values of ω_m and ω_s , as we shall see subsequently.

An induction motor can also be obtained by exciting the rotor with ac and short-circuiting the stator circuits. This, however, requires that all the power be fed into the machine through sliding electrical contacts (brushes on slip rings), which is impractical in most cases in the light of the simple alternative.

Most induction machines have squirrel-cage rotors in which bare conductors are imbedded in slots in the rotor iron and are then all short-circuited together at the ends by conducting rings. A cutaway view of such a motor is shown in Fig. 4.1.16. The conductor assembly alone looks like a cage, hence the name. Some special-purpose induction machines have rotor circuits wound with insulated conductors with connections to the terminals made through brushes and slip rings. The rotor for a wound rotor induction machine was shown earlier in Fig. 4.1.3. Having access to the rotor circuits allows us to connect different sources, or loads, or to short circuit the rotor circuits externally and thereby obtain a variety of machine characteristics.

For reasons that are stated in Section 4.2.2, all induction machines are smooth-air-gap machines.

We shall now study the steady-state characteristics of induction machines by using the constraints of (4.1.55) to (4.1.58). We first solve (4.1.61) for the steady-state current in rotor circuit (a)*:

$$i_{ar} = \frac{(\omega_s - \omega_m)MI_s}{\sqrt{R_r^2 + (\omega_s - \omega_m)^2 L_r^2}} \cos [(\omega_s - \omega_m)t + \alpha], \quad (4.1.63)$$

where

$$\alpha = -\frac{\pi}{2} - \gamma - \beta$$

and

$$\beta = \tan^{-1} \frac{(\omega_s - \omega_m)L_r}{R_r}.$$

The current in rotor circuit b is identical except for a 90° phase shift indicated by (4.1.62) [The $\cos$ in (4.1.63) is replaced by $\sin$ for i_{br} .]

As usual, we want to know how the machine behaves, as viewed from the electrical input terminals; thus we wish to find the relation between voltage and current. It is helpful at the same time to draw a steady-state electrical equivalent circuit as we did for the synchronous machine.

It can be verified quite easily that for a balanced two-phase machine with balanced two-phase excitation, as we have here, we need to consider only one phase (stator circuit) because the behavior of the other circuit will be identical except for a 90° phase shift.

We use (4.1.19) with the definition of terminal voltage to write

$$v_{as} = \frac{d\lambda_{as}}{dt} = \frac{d}{dt}(L_s i_{as}) + \frac{d}{dt}(M i_{ar} \cos \theta - M i_{br} \sin \theta). \quad (4.1.64)$$

Substitution from (4.1.55), (4.1.58), and (4.1.63) into this expression yields

$$\begin{aligned} v_{as} = & \frac{d}{dt}(L_s I_s \cos \omega_s t) + \frac{d}{dt} \frac{(\omega_s - \omega_m)M^2 I_s}{\sqrt{R_r^2 + (\omega_s - \omega_m)^2 L_r^2}} \\ & \times \{ \cos [(\omega_s - \omega_m)t + \alpha] \cos (\omega_m t + \gamma) \\ & - \sin [(\omega_s - \omega_m)t + \alpha] \sin (\omega_m t + \gamma) \}. \end{aligned} \quad (4.1.65)$$

A trigonometric identity is used to simplify the second term; thus

$$v_{as} = \frac{d}{dt}(L_s I_s \cos \omega_s t) + \frac{d}{dt} \left[\frac{(\omega_s - \omega_m)M^2 I_s}{\sqrt{R_r^2 + (\omega_s - \omega_m)^2 L_r^2}} \cos \left(\omega_s t - \beta - \frac{\pi}{2} \right) \right]. \quad (4.1.66)$$

* A review of sinusoidal steady-state circuit analysis is given in Section 5.1.

It is now convenient to define the *slip* s as

$$s = \frac{\omega_s - \omega_m}{\omega_s}. \quad (4.1.67)$$

The slip is a measure of the difference between the actual mechanical speed (ω_m) and the synchronous speed (ω_s) expressed as a fraction of synchronous speed. When the mechanical speed is less than the synchronous speed, the slip is positive.

We now use the definition of slip s to rewrite (4.1.66) in the form

$$v_{as} = \frac{d}{dt}(L_s I_s \cos \omega_s t) + \frac{d}{dt} \left[\frac{\omega_s M^2 I_s}{\sqrt{(R_r/s)^2 + \omega_s^2 L_r^2}} \cos \left(\omega_s t - \beta - \frac{\pi}{2} \right) \right], \quad (4.1.68)$$

and we rewrite the definition of β in terms of slip s as

$$\beta = \tan^{-1} \frac{\omega_s L_r}{R_r/s}. \quad (4.1.69)$$

We now use complex notation by defining

$$i_{as} = \text{Re}(I_s e^{j\omega_s t}), \quad \cos \left(\omega_s t - \beta - \frac{\pi}{2} \right) = \text{Re}(-j e^{j\omega_s t} e^{-j\beta}),$$

$$v_{as} = \text{Re}(\hat{V}_s e^{j\omega_s t}),$$

and rewrite (4.1.68) as

$$\hat{V}_s = j\omega_s L_s I_s + \frac{\omega_s^2 M^2 I_s e^{-j\beta}}{\sqrt{(R_r/s)^2 + \omega_s^2 L_r^2}}. \quad (4.1.70)$$

This equation is conventionally represented by the steady-state equivalent circuit of Fig. 4.1.17 in which we have indicated a complex amplitude $\hat{I}_r$ that can be verified from the circuit to have the value

$$\hat{I}_r = \frac{-j\omega_s M I_s e^{-j\beta}}{\sqrt{(R_r/s)^2 + \omega_s^2 L_r^2}}. \quad (4.1.71)$$

Thus the second term in (4.1.70) is simply $j\omega_s M \hat{I}_r$, as it should be for the circuit in Fig. 4.1.17.

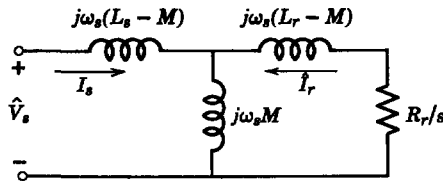


Fig. 4.1.17 Steady-state equivalent circuit for balanced two-phase induction machine with balanced excitation.

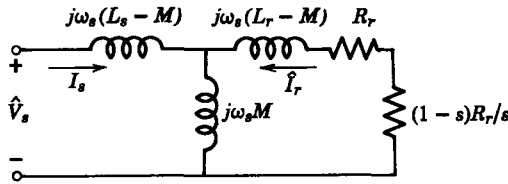


Fig. 4.1.18 Alternative form for steady-state equivalent circuit for balanced two-phase induction machine with balanced excitation. Here the resistance $(1-s)R_r/s$ gives rise to a power dissipation equal to the power converted to mechanical form.

By some simple manipulations it can be verified that the magnitude of $\hat{I}_r$ is the same as the magnitude of i_{ar} found in (4.1.63). The effect of the relative motion has been to change the frequency but not the magnitude of the rotor current as viewed from the stator winding and indicated in (4.1.64) and (4.1.66). Consequently, when the equivalent circuit of Fig. 4.1.17 is redrawn as in Fig. 4.1.18, the power loss calculated in R_r is the actual $I^2 R$ loss in one winding of the rotor. The power into the other resistance $(1-s)R_r/s$ represents power converted to mechanical form, as will be demonstrated.

The equivalent circuit of Fig. 4.1.17 or 4.1.18 can be used to study the steady-state electrical behavior of induction machines. Our use of i_{as} as having zero phase angle can be relaxed and I_s can be replaced by a complex amplitude. The equivalent circuit serves the important function of helping to determine the correct relative phase angles.

The b winding on the stator will behave like the a winding except for a 90° phase shift in all variables, as indicated by the excitations (4.1.55) and (4.1.56). This can be verified quite easily and is not done here.

To describe the behavior of the induction machine, as viewed from the mechanical terminal pair, we use (4.1.23) with (4.1.55), (4.1.56), (4.1.63) and the value of i_{br} , obtained by replacing the cosine in (4.1.63) with a sine, and obtain the expression for the torque

$$\begin{aligned}
 T^e &= \frac{(\omega_s - \omega_m)M^2 I_s^2}{\sqrt{R_r^2 + (\omega_s - \omega_m)^2 L_r^2}} \\
 &\times [\{\cos [(\omega_s - \omega_m)t + \alpha] \sin \omega_s t - \sin [(\omega_s - \omega_m)t + \alpha] \cos \omega_s t\} \\
 &\times \cos (\omega_m t + \gamma) - \{\cos [(\omega_s - \omega_m)t + \alpha] \cos \omega_s t \\
 &\quad + \sin [(\omega_s - \omega_m)t + \alpha] \sin \omega_s t\} \sin (\omega_m t + \gamma)]. \quad (4.1.72)
 \end{aligned}$$

The successive use of trigonometric identities* and the definition of angle β in (4.1.63) lead to the simplified result

$$T^e = \frac{(\omega_s - \omega_m)M^2 R_r I_s^2}{R_r^2 + (\omega_s - \omega_m)^2 L_r^2}. \quad (4.1.73)$$

* $\cos (x - y) = \cos x \cos y + \sin x \sin y$, $\sin (x - y) = \sin x \cos y - \cos x \sin y$.

The instantaneous torque T^e is constant. This is to be expected because balanced two-phase currents yield constant-amplitude rotating fields, as discussed in Section 4.1.4, and the interaction of these fields when the condition of (4.1.30) is satisfied produces a steady torque.

The mechanical power output of the machine is

$$p_m = \omega_m T^e = \left[\frac{\omega_s^2 M^2 I_s^2}{(R_r/s)^2 + \omega_s^2 L_r^2} \right] \left(\frac{1-s}{s} \right) R_r \quad (4.1.74)$$

where we have used the definition of slip s in (4.1.67) and manipulated the result to get this form. Note that the term in brackets is the square of the magnitude of $\hat{I}_r$, as given in (4.1.71); thus we have verified that the power absorbed by the resistance $(1-s)R_r/s$ in Fig. 4.1.18 is indeed the power converted to mechanical form when multiplied by two to account for both phases.

The total power input to the stator windings (excluding stator I^2R losses, which we have done) is defined as the *air-gap power* p_g . It is clear from Fig. 4.1.17 that

$$p_g = I_r^2 \frac{R_r}{s}, \quad (4.1.75)$$

where I_r is the magnitude of $\hat{I}_r$ given by (4.1.71) and (4.1.75) is twice the power input to one phase. As already indicated, the rotor I^2R losses p_r are given by

$$p_r = I_r^2 R_r = s p_g. \quad (4.1.76)$$

The power p_m converted to mechanical form is

$$p_m = I_r^2 \left(\frac{1-s}{s} \right) R_r = (1-s)p_g. \quad (4.1.77)$$

Thus the power into the stator equals the sum of rotor losses and converted power

$$p_g = p_r + p_m \quad (4.1.78)$$

and there is no rate of change of total energy stored in the machine. We knew this all the time, because for balanced excitation the air-gap magnetic fields have constant amplitudes.

We now use (4.1.75) to (4.1.78) to identify the three possible modes of operation of an induction machine as illustrated in Fig. 4.1.19. The arrow heads indicate the flow direction of power and the ranges of slip and speed are given in the figure. Note in particular that the rotor power p_r is always greater than zero as it must be because it is an I^2R loss. Note also that brake operation has power coming into the machine from both electrical and mechanical terminal pairs and all of this power is dissipated in the rotor resistance.

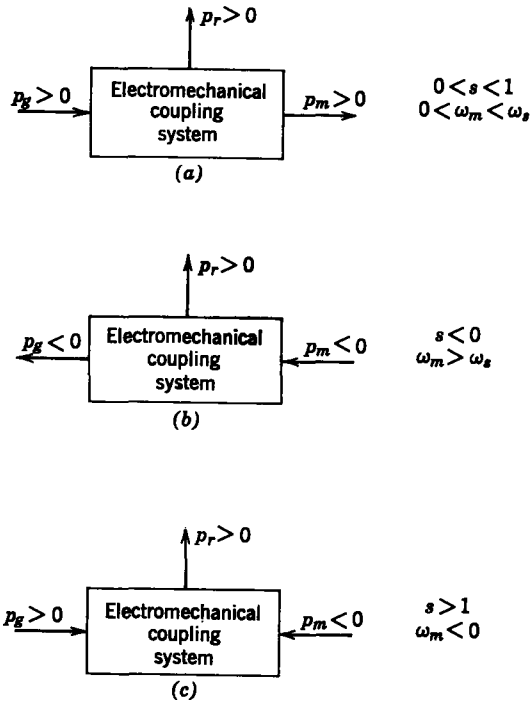


Fig. 4.1.19 The three modes of operation of an induction machine: (a) motor operation; (b) generator operation; (c) brake operation.

When operating as a motor, the machine efficiency η is defined as the mechanical power output divided by the electrical power input to the stator; thus

$$\eta = \frac{p_m}{p_g} = 1 - s. \quad (4.1.79)$$

As a consequence, large induction machines intended for the efficient production of mechanical power are designed to run at as small a slip (as close to synchronous speed) as possible.

Induction machines are normally excited by almost constant voltage sources. Consequently, the electromechanical coupling properties are of most interest for this condition and we need to express (4.1.73) in terms of the magnitude V_s of the terminal voltage $\hat{V}_s$. We use (4.1.70) to write

$$\hat{V}_s = j\omega_s L_s I_s + \frac{\omega_s^2 M^2 I_s (R_r/s)}{(R_r/s)^2 + \omega_s^2 L_r^2} - j \frac{\omega_s^3 M^2 L_r I_s}{(R_r/s)^2 + \omega_s^2 L_r^2}. \quad (4.1.80)$$

In obtaining this form we have used the definitions of angle β from (4.1.63)

and slip s from (4.1.67). The magnitude squared of $\hat{V}_s$ is then

$$V_s^2 = \left\{ \left[\omega_s L_s - \frac{\omega_s^3 M^2 L_r}{(R_r/s)^2 + \omega_s^2 L_r^2} \right]^2 + \left[\frac{\omega_s^2 M^2 (R_r/s)}{(R_r/s)^2 + \omega_s^2 L_r^2} \right]^2 \right\} I_s^2. \quad (4.1.81)$$

Solution of this equation for I_s^2 , substitution of that result in (4.1.73) and simplification lead to the result

$$T^e = \frac{(k^2/\omega_s)(L_r/L_s)(R_r/s)V_s^2}{[\omega_s(1 - k^2)L_r]^2 + (R_r/s)^2}. \quad (4.1.82)$$

We have used the square of the maximum coefficient of coupling

$$k^2 = \frac{M^2}{L_r L_s} \quad (4.1.83)$$

to simplify this expression. Note that this is the coefficient of coupling between the a windings on stator and rotor when rotor position θ is zero.

A curve of electromagnetic torque versus slip (and mechanical speed) typical of large squirrel-cage induction machines is shown in Fig. 4.1.20. The ranges over which the machine operates as a motor, generator, and brake are also indicated.

The torque given by (4.1.82) depends on rotor resistance R_r and slip s only through the ratio R_r/s . By differentiating (4.1.82) with respect to this ratio, setting the derivative equal to zero, and solving for the ratio we can determine that the torque has two maxima that occur when

$$\frac{R_r}{s} = \pm \omega_s(1 - k^2)L_r, \quad (4.1.84)$$

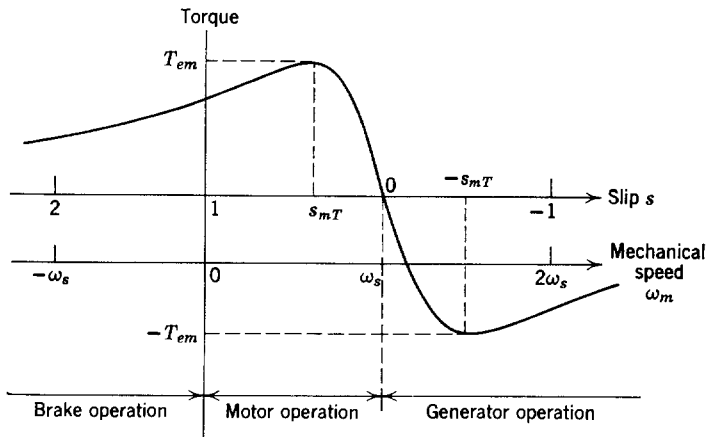


Fig. 4.1.20 Torque-slip curve of a two-phase induction machine with balanced excitation.

from which the slip at which maximum torque occurs is

$$s = \pm s_{mT} = \pm \frac{R_r}{\omega_s(1 - k^2)L_r}. \quad (4.1.85)$$

These two values of slip are indicated in Fig. 4.1.20. Substitution of (4.1.84) into (4.1.82) yields for the maximum torque

$$T^e = \pm T_{em} = \pm \frac{k^2 V_s^2}{2\omega_s^2 L_s(1 - k^2)}. \quad (4.1.86)$$

This maximum torque is indicated in Fig. 4.1.20.

The maximum torque given by (4.1.86) is independent of rotor resistance R_r . Thus for a wound rotor induction motor with which the rotor resistance can be set to any desired value we can get a set of steady-state torque-speed curves as sketched in Fig. 4.1.21. Note that as R_r increases the speed at which maximum torque occurs decreases but the maximum torque stays the same.

This fact is often used by introducing external resistance into the rotor circuit to achieve a high starting torque and then short-circuiting the rotor windings for normal running to get a small slip and therefore high efficiency.

The loading of an induction motor normally occurs in the region of negative slope near synchronous speed; for example, the torque-speed curve of an induction motor and a typical load (e.g., a fan) are shown in Fig. 4.1.22. The steady-state operating point of the system is indicated on the curves. If the fan load increases to the dashed curve, the new operating point occurs at a higher torque, lower speed, and higher slip. At the higher slip the motor produces more mechanical power but with less efficiency.

It is worthwhile to understand the reasons for the shape of the torque speed curve of an induction motor. First, in the normal operating range, which is the region of negative slope near synchronous speed in Fig. 4.1.20, the slip is

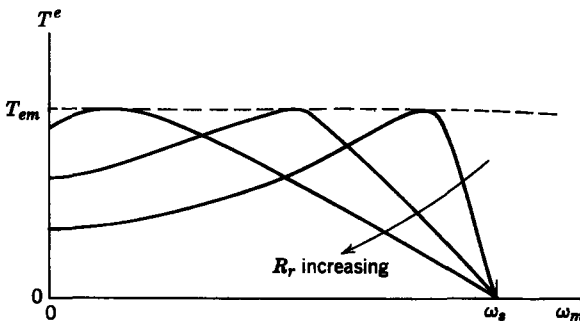


Fig. 4.1.21 Variation of torque-speed curves of induction motor with rotor resistance. Stator voltage held constant.

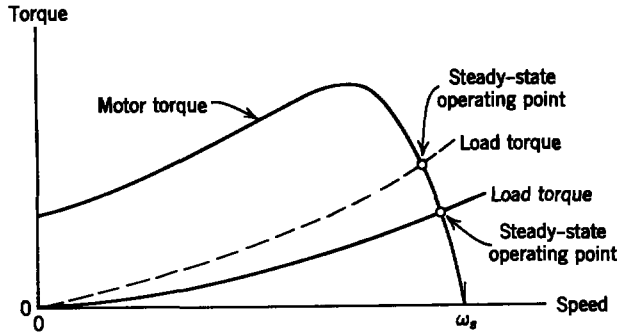


Fig. 4.1.22 Loading of an induction motor.

very small, hence the induced currents in the rotor are at very low frequency [see (4.1.63)]. Under this condition the resistance of the rotor windings has a much greater effect than the inductance. To express this mathematically, the resistive term in the denominator of (4.1.82) dominates such that

$$\left(\frac{R_r}{s}\right)^2 \gg [\omega_s(1 - k^2)L_r]^2$$

and the torque becomes

$$T^e \approx \frac{k^2 L_r s}{\omega_s L_s R_r} V_s^2. \quad (4.1.87)$$

This torque is a linear function of slip and therefore also of mechanical speed.

When the mechanical speed is far from synchronous speed (the slip is large), the frequency of the rotor currents (4.1.63) is high and inductance predominates over resistance. This region is defined from the denominator of (4.1.82) as the condition

$$[\omega_s(1 - k^2)L_r]^2 \gg \left(\frac{R_r}{s}\right)^2.$$

In this case the torque becomes

$$T^e \approx \frac{k^2}{\omega_s^3(1 - k^2)^2 L_r L_s} \frac{R_r}{s} V_s^2. \quad (4.1.88)$$

This expression varies inversely with slip.

The two asymptotes are sketched in Fig. 4.1.23 for positive slip. These two simplified models are useful for studying the behavior of the machine under particular conditions; for example, for the kind of torque-speed curve shown in Fig. 4.1.23, which is typical of large induction motors, the inductance-dominated model (4.1.88) is sufficient for starting conditions and the resistance-dominated model is adequate for normal running conditions. We have more to say about these kinds of approximations in the next chapter.

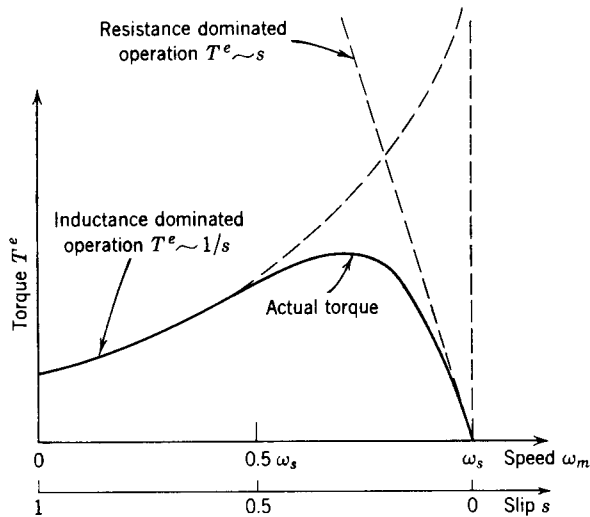


Fig. 4.1.23 Asymptotic behavior of an induction motor.

Our analysis of induction machines has been done with a two-phase model. Most large induction machines are actually excited by three-phase sources. Nonetheless, the standard technique for analysis is to transform a three-phase machine to an equivalent two-phase machine for a study of the energy conversion properties. This is true for any number of phases.* Consequently, our treatment is general and our conclusions are valid for all balanced polyphase machines with balanced polyphase excitation.

We have only highlighted the properties of polyphase induction machines with the idea of trying to establish some insight into the physical processes occurring. The subject of induction machines is complex and extensive enough to be the sole subject of books.†

All of our discussion so far has been relevant to polyphase induction machines. There are also single-phase induction machines which have some unique characteristics. A single-phase induction machine is constructed like the machine illustrated in Fig. 4.1.1. The stator is excited by a single-phase source and the rotor winding is short-circuited. (Actually, the rotor is almost always of squirrel-cage construction and therefore the equivalent of two windings, 90° apart in space.)

As we discovered in Section 4.1.4, single-phase excitation of a symmetrically distributed winding produces two equal-amplitude waves of flux density traveling in opposite directions in the air gap. A squirrel-cage rotor in this

* See, for example, White and Woodson, *op. cit.*, Chapter 8.

† See, for example, P. L. Alger, *The Nature of Polyphase Induction Machines*, Wiley, New York, 1951.

environment does not know which way to go and therefore will not start to rotate. Once started, however, the rotor will continue to run in the same direction. The torque-speed curve of a single-phase induction motor can be derived as the superposition of two machines operating with the traveling components of the air-gap flux, as illustrated in Fig. 4.1.24. Thus, with rotation in either direction, the flux wave traveling in that direction dominates and the machine continues to run in that direction. In the normal running range near synchronous speed the single-phase induction motor has properties similar to those of polyphase machines.

In view of the single-phase machine properties illustrated in Fig. 4.1.24, there is a problem in starting the machine. There is a variety of starting methods.* For moderate-size machines ($\frac{1}{10}$ to 5 hp, approximately) of the type installed in refrigerators, air conditioners, washing machines, and the like, an auxiliary winding is used. The auxiliary winding is wound with its magnetic axis displaced 90 degrees from that of the main winding. It is also excited from the single-phase source, but the phase angle of its current is different from that of the main winding, either because of a different L/R ratio or because a capacitor is added in series. This different phase angle of the current in the auxiliary winding causes an unbalance between the two rotating field waves; one of the waves dominates and starts the rotor turning. In most cases the auxiliary winding is disconnected by a centrifugal switch when the rotor reaches a predetermined speed during the acceleration.

For smaller, single-phase induction machines starting torque is provided by shading coils,† which are short-circuited turns on the stator that give the effect of making one flux wave dominate the other.

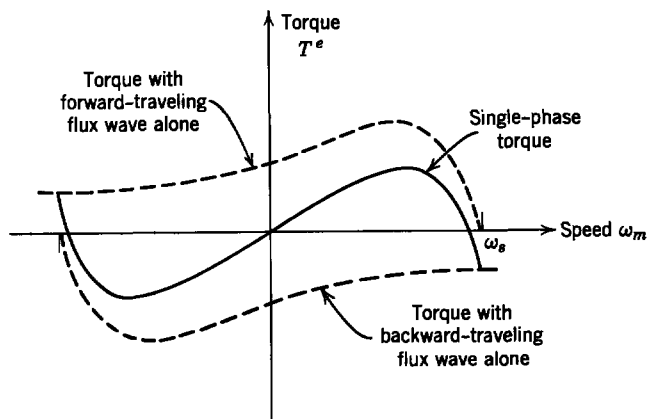


Fig. 4.1.24 The torque-speed curve of a single-phase induction motor.

* See, for example, Fitzgerald and Kingsley, *op. cit.*, Chapter 11.

† See, for example, Fitzgerald and Kingsley, *loc. cit.*

Because of the simple, rugged construction possible with a squirrel-cage rotor, the induction motor is the least expensive and most reliable means of converting electric energy to mechanical energy. As a result, induction motors are more numerous by far than any other type of motor.

Another important class of induction motor is the *two-phase servomotor*, which is essentially a two-phase induction motor with rotor resistance sufficiently high that maximum torque occurs at a slip of around 1.5 (see Fig. 4.1.21). The torque-speed curve then has a negative slope for all positive speeds and can therefore run stably at any speed between zero and synchronous speed. Such a motor is normally operated with full voltage applied to one winding and with variable voltage applied to the other to get smooth control of speed. Such operation is quite inefficient and thus servomotors are made in small sizes, mostly up to 20 W but sometimes up to 1000 W with auxiliary cooling for the rotors. They are, as their name implies, used mostly in servo systems for control applications. The analysis of servomotors is a straightforward application of the techniques we have introduced and is done quite well elsewhere.* Thus we do not discuss them further here.

4.1.6c Commutator Machines

The most widely used machine for control purposes is the *dc machine* which uses (or supplies) electrical power at zero frequency. It is evident from (4.1.18) that with zero-frequency rotor and stator currents it is impossible to satisfy the frequency condition with any nonzero mechanical speed. This problem is circumvented by the use of a *commutator* which can be viewed as a mechanical frequency changer. The stator circuit (field circuit) of the usual dc machine is excited by direct current and the rotor (armature) circuits are fed from direct-current sources through a commutator that provides the currents in the rotor conductors with components at ω_m . This frequency, with the zero stator-current frequency, satisfies the condition in (4.1.18) at all mechanical speeds.

A commutator is a mechanical switch whose state is determined by the rotor position θ . The simplest possible commutator is shown schematically for one rotor coil without the iron in Fig. 4.1.25a. When a constant current I is passed through the external terminals and the rotor carrying the coil and commutator is rotated about its axis with a mechanical velocity ω_m , the waveform of the current in the coil is as shown in Fig. 4.1.25b. It is clear from this waveform that the fundamental frequency of the coil current is ω_m , as stated above.

In practical machines commutators are made with many segments, and the many coils on the rotor are connected to one another and to the commutator

* See, for example, G. J. Thaler and M. L. Wilcox, *Electric Machines*, Wiley, New York, 1966, pp. 208–213.

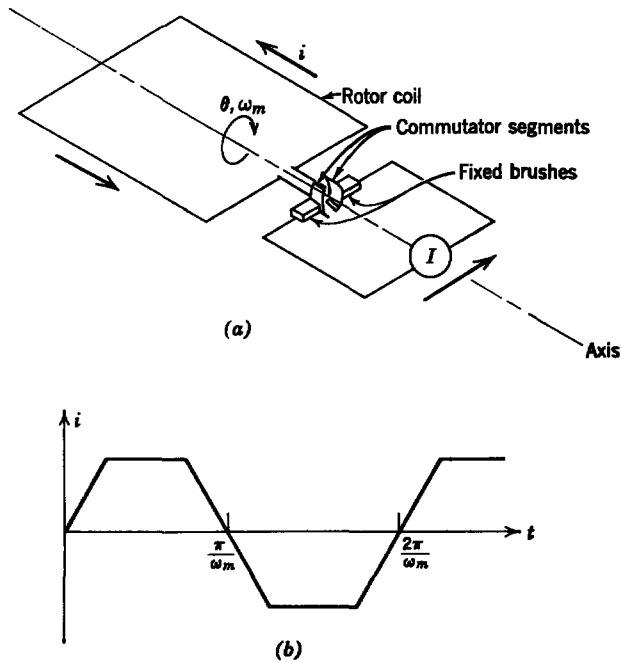


Fig. 4.1.25 Schematic representation of a simple commutator: (a) the physical arrangement; (b) coil current.

in one of two ways to obtain maximum utilization of the copper coils.* A dc machine rotor is shown in Fig. 4.1.26, and Fig. 4.1.27 illustrates how brushes are mounted to make contact with the commutator in a dc machine.

In spite of the apparent complications, the commutator can still be viewed as a mechanical frequency changer that is necessary to satisfy the frequency condition for average power conversion (4.1.18) when the electrical sources (or sinks) are at zero frequency (dc).

A variety of dc machine characteristics is possible, depending on whether the field (stator) circuit and the armature (rotor) circuit are connected in series, in parallel, or are excited separately (see Section 6.4.1).

The commutator has been described as having dc excitation, but it also acts as a frequency changer when alternating currents are fed into the brushes, the change in frequency being equal to the rotational speed of the commutator. Thus, if currents at frequency ω are fed into the stator circuits and into the rotor coils through a commutator, the rotor currents will contain components at frequency $\omega - \omega_m$ and the frequency condition of (4.1.18) is automatically satisfied at all mechanical speeds. This result gives rise to

* For details see Knowlton, *op. cit.*, Section 8.

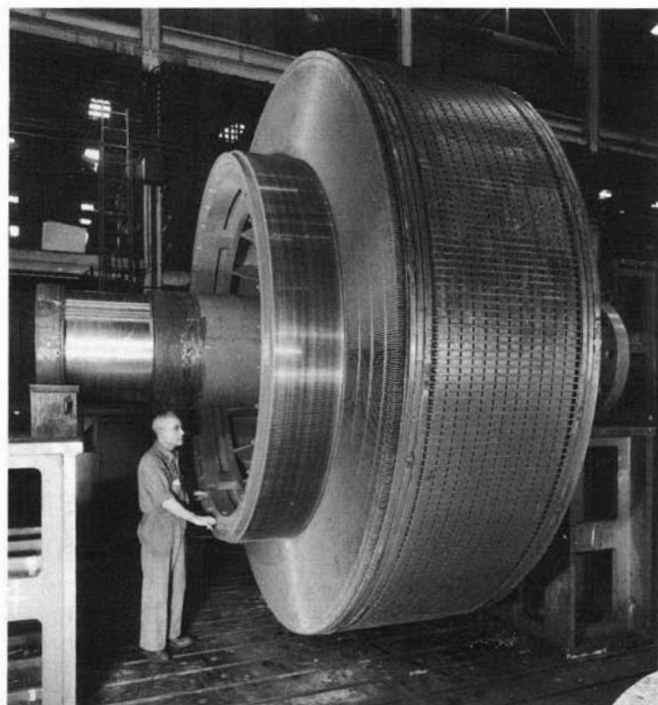


Fig. 4.1.26 A dc machine rotor (armature). Note the large number of slots and commutator bars. (Courtesy of Westinghouse Electric Corporation.)

many varieties of *ac commutator* machines*; the most common of which are used to drive vacuum cleaners, hand drills, electric egg beaters, and so forth.

Although we could develop the equations of motion and study the steady-state properties of commutator machines as we have done for synchronous and induction machines, it is more appropriate and meaningful to do so after we have developed some field theory for moving media. Thus we defer this treatment until we reach Chapter 6, Section 6.4.

4.1.7 Polyphase Machines

In our discussions so far machines have been considered with single-phase windings (Fig. 4.1.1) and two-phase windings (Fig. 4.1.7). In this section the definitions and configurations are given for machines with any number of phases, hence the name *polyphase*.

Polyphase electric power is generated and used for several reasons. It is economically optimum to generate and distribute three-phase power; the use

* For more detail see Knowlton, *op. cit.*, Section 7.

of polyphase power allows the operation of rotating machines to produce steady torque, even though the excitation is ac (see Section 4.1.3), and a polyphase machine for a particular application is smaller (and therefore less expensive) than a single-phase machine.

A set of balanced polyphase currents (or voltages) consists of a number of currents (or voltages) equal to the number of phases, each member of the set having the same amplitude and all members of the set being equally spaced in time phase; for example, a balanced set of three-phase currents (labeled by subscripts as phases a , b , and c) is specified as

$$\begin{aligned}i_a &= I \cos \omega t, \\i_b &= I \cos \left(\omega t - \frac{2\pi}{3} \right), \\i_c &= I \cos \left(\omega t - \frac{4\pi}{3} \right),\end{aligned}\tag{4.1.89}$$

where I is the amplitude and $2\pi/3$ is the phase difference between any two phases.

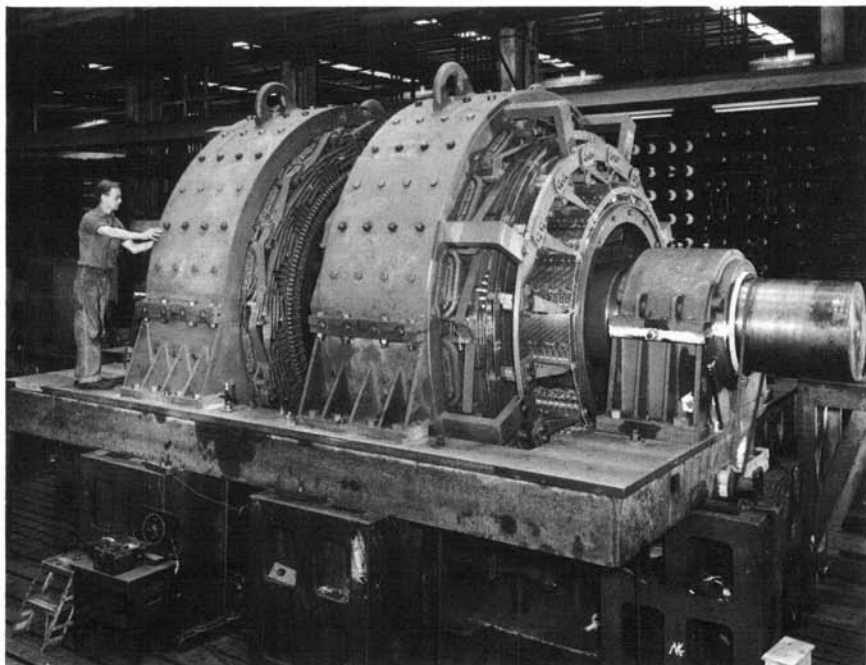


Fig. 4.1.27 A dc machine. Note how the brush rigging is assembled to hold the brushes in contact with the commutator. Note also the salient poles on the stator. (Courtesy of Westinghouse Electric Corporation.)

It is easy to extend this system to an arbitrary number of phases. Suppose there are n phases. Then a balanced set of voltages is written as

$$\begin{aligned}
 v_1 &= V \cos \omega t \\
 v_2 &= V \cos \left(\omega t - \frac{2\pi}{n} \right) \\
 v_3 &= V \cos \left(\omega t - \frac{4\pi}{n} \right) \\
 &\vdots \\
 v_n &= V \cos \left[\omega t - \frac{(n-1)2\pi}{n} \right].
 \end{aligned} \tag{4.1.90}$$

In these expressions V is the amplitude and the phase difference between any two adjacent phases is $(2\pi/n)$ rad.

In terms of this general scheme, a two-phase system like that in (4.1.24) and (4.1.25) is the special case of half a four-phase system. A four-phase set of currents is written as

$$\begin{aligned}
 i_a &= I \cos \omega t, \\
 i_b &= I \cos \left(\omega t - \frac{\pi}{2} \right), \\
 i_c &= I \cos (\omega t - \pi), \\
 i_d &= I \cos \left(\omega t - \frac{3\pi}{2} \right).
 \end{aligned} \tag{4.1.91}$$

A selection of the first two or the last two of this set will yield a set of two-phase currents with the same relative phase ($\pi/2$ rad) as that in (4.1.24) and (4.1.25).

Any set of phases will have a *phase sequence* defined as the order in which the phase variables reach a positive maximum (or any other convenient reference value). Thus the sequence of the three-phase system of (4.1.89) is a to b to c which is usually defined as *positive sequence*. A three-phase set with *negative sequence* (c to b to a) is

$$\begin{aligned}
 i_a &= I \cos \omega t, \\
 i_b &= I \cos \left(\omega t + \frac{2\pi}{3} \right), \\
 i_c &= I \cos \left(\omega t + \frac{4\pi}{3} \right).
 \end{aligned} \tag{4.1.92}$$

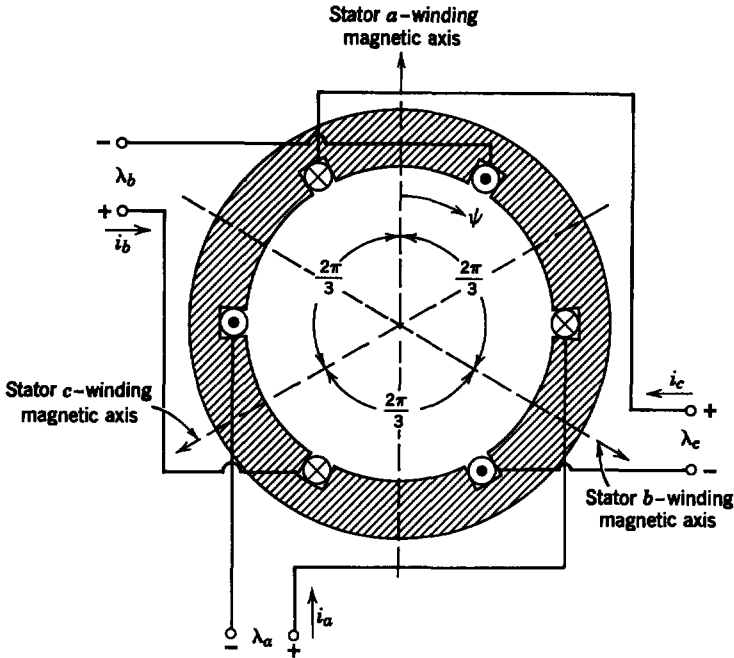


Fig. 4.1.28 A three-phase stator.

Note that the n -phase set of voltages of (4.1.90) and the four-phase set of currents in (4.1.91) are both positive sequence.

Now suppose that a rotating machine is to have its stator wound so that with balanced polyphase currents in its windings the rotating field due to stator currents will have constant amplitude and will rotate at constant speed around the periphery of the air gap. Such a system has already been discussed for a two-phase machine in Section 4.1.4. From that discussion it is clear that the number of windings must equal the number of phases and that the winding magnetic axes must be placed around the periphery in the same relative space positions that the currents are placed in relative time phase.

To illustrate this consider the three-phase windings on the stator of Fig. 4.1.28 in which the rotor is omitted for simplicity and the windings are shown lumped in single slots, although they would be distributed in an actual machine. When the positive-sequence, three-phase currents of (4.1.89) are applied to this machine, a field analysis similar to that of Section 4.1.4 will show that the air-gap flux density distribution will have constant amplitude and will rotate in the positive ψ -direction with the angular speed ω . Excitation of the stator of Fig. 4.1.28 with the negative sequence currents of (4.1.92) yields a constant-amplitude field pattern rotating in the negative ψ -direction with the angular speed ω .

This discussion, in which the example of a stator was used, applies equally well to a rotor. The general case is presented elsewhere.*

A machine can operate successfully with any number of phases on the stator and the same or any other number of phases on the rotor; for instance, a three-phase alternator (a synchronous machine used to generate electric power) usually has a three-phase stator (armature) winding and a single rotor (field) winding.

4.1.8 Number of Poles in a Machine

The number of poles in a machine is defined by the configuration of the magnetic field pattern that occurs; for example, consider the rotor of Fig. 4.1.29a with a single winding. When the instantaneous current is in the direction indicated by the dots and crosses, the resulting $\mathbf{B}$ field is as sketched in the figure. With the $\mathbf{B}$ field as shown, the rotor can be viewed as an electromagnet with north (N) and south (S) poles as indicated. In a trip around its periphery two poles are passed; therefore it is a two-pole rotor.

Consider now the rotor of Fig. 4.1.29b which has four slots carrying coils connected in series with the polarities indicated by dots and crosses. Once again this winding can be single-phase or it can be one phase of a polyphase winding. When the instantaneous winding current has the direction indicated, the resulting $\mathbf{B}$ field is as sketched in Fig. 4.1.29b and the rotor is effectively a four-pole electromagnet.

These ideas can be generalized to an arbitrary number of poles by stating

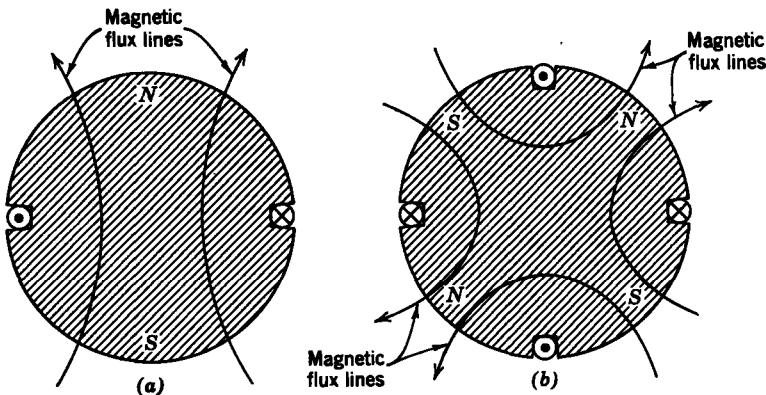


Fig. 4.1.29 Definition of number of poles in a machine: (a) two-pole rotor; (b) four-pole rotor.

* White and Woodson, *op. cit.*, Chapter 10.

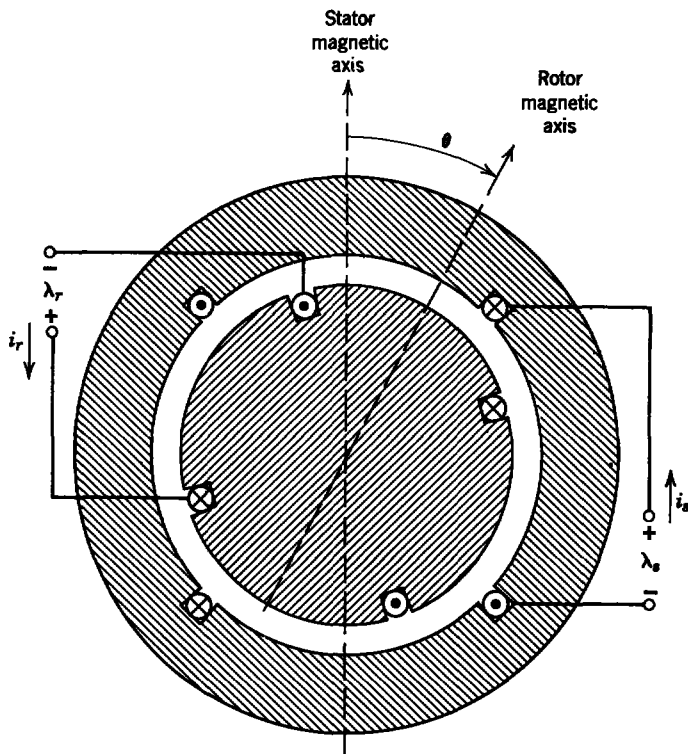


Fig. 4.1.30 Four-pole, single-phase machine.

that with current in one phase the number of poles (north and south) encountered in one turn around the periphery of the air gap defines the number of poles. It is clear that poles occur in pairs.

This discussion of the number of poles on a rotor applies equally well to stators.

Re-examination of the examples of earlier sections shows that they all concern two-pole machines. Some of those ideas are considered here for machines with more than two poles.

Consider the four-pole, single-phase machine illustrated in Fig. 4.1.30. The interconnections are not shown but current i_s is in all the stator coils in the directions shown and current i_r is in all the rotor coils in the directions shown. The slots are assumed to have negligible effects on the self-inductances (this is a smooth-air-gap machine) so that the self-inductances will be independent of rotor position θ . Because of the symmetries involved (see discussion in Section 4.1.1), the mutual inductance can be expressed as

$$L_{sr}(\theta) = M_1 \cos 2\theta + M_3 \cos 6\theta + M_5 \cos 10\theta + \cdots \quad (4.1.93)$$

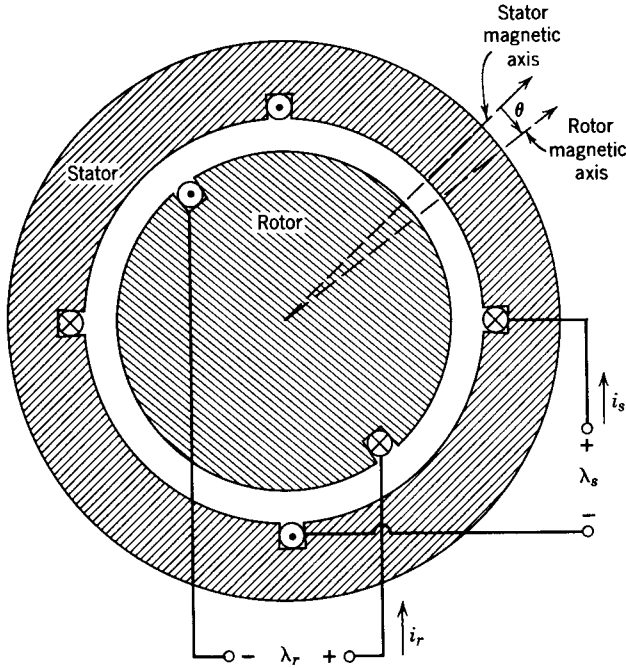


Fig. 4.1.31 Machine with four-pole stator and two-pole rotor. This configuration can produce no torque.

Compare this expression with (4.1.4). It can be generalized immediately by recognizing that for a p -pole-pair machine the mutual inductance between a stator winding and a rotor winding is expressible as

$$L_{sr}(\theta) = M_1 \cos p\theta + M_3 \cos 3p\theta + M_5 \cos 5p\theta + \cdots \quad (4.1.94)$$

To minimize the generation of harmonics multipole ac machines are designed to accentuate the lowest space harmonic of L_{sr} and to decrease as much as possible the higher space harmonics.

In the two examples in (4.1.93) and (4.1.94) it has been assumed that the rotor and stator have the same number of poles. This is necessary for successful operation of the machine as a power converter. If the rotor and stator had different numbers of poles, the mutual inductance between rotor and stator would be zero and, as evidenced by the terminal relations (4.1.1) to (4.1.3), the electromechanical coupling would disappear. To verify qualitatively that this is so, consider the machine in Fig. 4.1.31 which has a two-pole rotor and a four-pole stator. We can see that if the system has the usual type of symmetry and the stator is excited by direct current the result is that no net flux links the rotor circuit due to stator excitation for any rotor angle θ and the mutual inductance is indeed zero.

By carrying out a process similar to that used in Section 4.1.2, we find that the condition that must be satisfied by the frequencies and mechanical speed (4.1.18) for average power conversion must be generalized for a p -pole-pair machine to

$$\omega_m = \frac{1}{p} (\pm \omega_s \pm \omega_r). \quad (4.1.95)$$

Thus for given electrical frequencies the mechanical speed is reduced as the number of poles is increased; for example, for a synchronous machine operating on 60-Hz power,

$$\omega_s = 2\pi 60 \quad \text{and} \quad \omega_r = 0,$$

the mechanical speed is

$$\omega_m = \frac{2\pi 60}{p};$$

$$\text{for } p = 1 \quad \omega_m = 120\pi \text{ rad/sec} = 3600 \text{ rpm},$$

$$\text{for } p = 2 \quad \omega_m = 60\pi \text{ rad/sec} = 1800 \text{ rpm},$$

$$\text{for } p = 15 \quad \omega_m = 8\pi \text{ rad/sec} = 240 \text{ rpm}.$$

For a synchronous machine the maximum obtainable shaft speed is produced by a two-pole machine. The speed can be set at any submultiple of this maximum speed by setting the number of poles.

The freedom to set the number of poles allows for optimum design of systems; for instance, in the generation of electric power at 60 Hz generators for operation with steam turbines have two poles (a few have four) because steam turbines operate best at high speeds. On the other hand, generators for operation with hydraulic turbines (water wheels) usually have many poles, often as many as 40 or more, because hydraulic turbines operate best at low speeds.

Examination of Fig. 4.1.30 shows that the wire in the slots of the four-pole configuration could be reconnected at the end turns to yield a two-pole configuration. Thus a machine can be made to operate at two speeds by changing the number of poles. This is done frequently on induction machines for with a squirrel-cage rotor no rotor reconnections need to be made; for example, induction motors that drive automatic washing machines often operate with four-poles for the washing cycle and are reconnected as two-pole machines to run at approximately twice the speed for the spin-drying cycle.

It is clear from the foregoing analyses and discussions that a rotating machine is conceptually a simple device. It is simply a magnetic field-type, lumped-parameter, electromechanical device whose principal properties can be deduced by the straightforward techniques of Chapters 2 and 3. The many constructional variations (multipole and polyphase) and the wide variety of

possible excitations and characteristics lead to long and cumbersome, though necessary, mathematical analysis.* The amount of mathematics required should never be mistaken for conceptual complexity. Moreover, advantage should always be taken of the orderly mathematical procedures made possible by the symmetries that exist in machines.†

4.2 SALIENT-POLE MACHINES

The second geometrical configuration of rotating machines to be considered is that of the two-pole, single-phase, *salient-pole* machine illustrated in Fig. 4.2.1. This machine gets its name from the fact that one member (the rotor in Fig. 4.2.1) has protruding or *salient poles* and thus the air gap is not uniform around the periphery. The stator coil in Fig. 4.2.1 is shown lumped

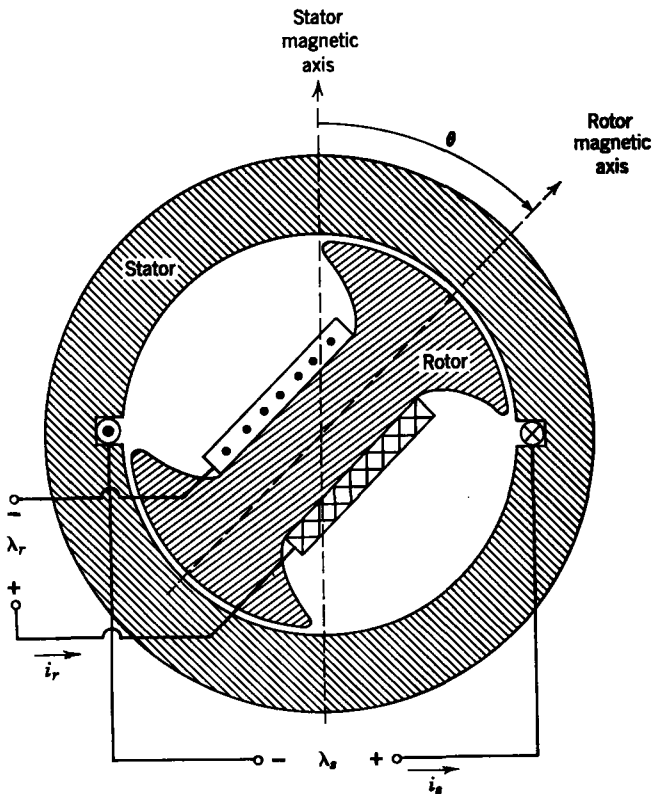


Fig. 4.2.1 Two-pole, single-phase, salient-pole machine with saliency on the rotor.

* See, for example, White and Woodson, *op. cit.*, Chapters 3, 4, and 7 to 11.

† White and Woodson, *op. cit.*, Chapter 4.

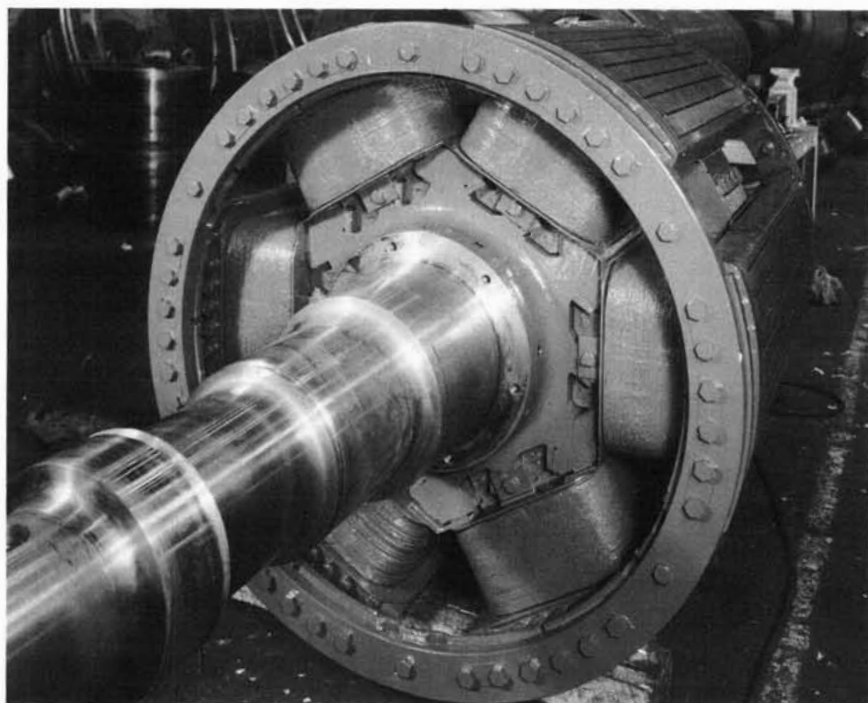


Fig. 4.2.2 Rotor of a salient-pole, synchronous motor. Note the method in which the coils are wound around the salient poles. (Courtesy of Westinghouse Electric Corporation.)

in two slots for simplicity. In practical machines the stator winding is distributed among several slots. The rotor winding in Fig. 4.2.1 is a fair representation of the method of winding salient rotors in practice, as indicated by the constructional details of the rotor for a salient-pole synchronous motor in Fig. 4.2.2. Another example of a salient-pole synchronous machine is shown in Fig. 4.2.3. This is a multipole generator driven by a hydraulic turbine. An example of a machine with saliency on the stator is the dc device in Fig. 4.1.27.

4.2.1 Differential Equations

Considering the system in Fig. 4.2.1 as an electrically linear, lumped-parameter, magnetic field-type, electromechanical device along the lines of Chapter 3, it is evident that the system is completely described when the inductances (electrical terminal relations) are known. Moreover, power conversion will occur only through inductances that depend on angular position θ ; thus to assess the effects of saliency on power conversion properties it is necessary only to investigate the inductances.

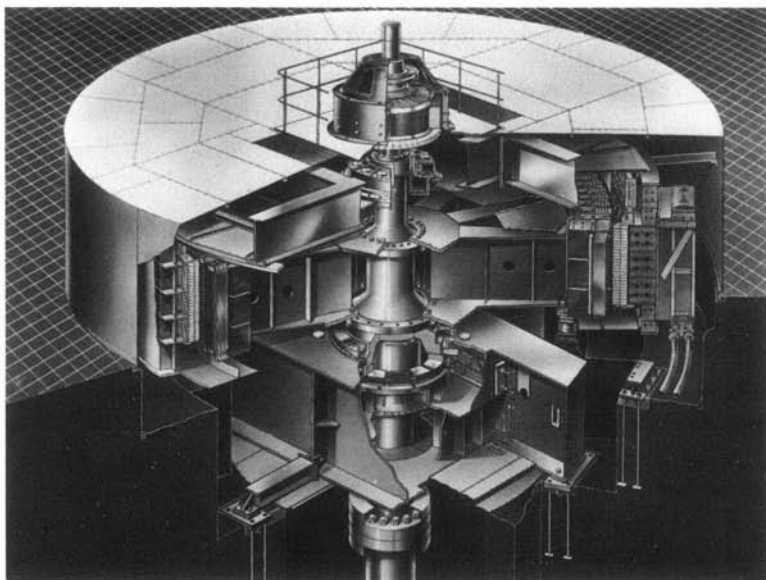


Fig. 4.2.3 Cutaway view of a hydroelectric generator. This is an example of a salient-pole, synchronous machine. (Courtesy of General Electric Company.)

Courtesy of General Electric Company. Used with permission.

First, when it is assumed that the slots that carry the stator winding in Fig. 4.2.1 cause negligible effects on the air-gap magnetic field, it is apparent that the self-inductance L_r of the rotor winding is independent of angle θ . It is also evident that the mutual inductance and the self-inductance of the stator winding depend on θ . Thus the electrical terminal relations can be written as

$$\lambda_s = L_s(\theta)i_s + L_{sr}(\theta)i_r, \quad (4.2.1)$$

$$\lambda_r = L_{sr}(\theta)i_s + L_r i_r. \quad (4.2.2)$$

Comparison of these expressions with the comparable ones for the smooth-air-gap machine (4.1.1) and (4.1.2) shows that the major difference introduced by saliency is the dependence of the stator self-inductance L_s on angular position θ , although an additional effect can occur in the form of the mutual inductance $L_{sr}(\theta)$.*

Consider first the stator self-inductance L_s . From the symmetry of the rotor structure in Fig. 4.2.1 it should be evident that the lowest space harmonic is the second because turning the rotor π rad in θ brings the inductance to its original value. This inductance is a maximum at $\theta = 0$ because the

* In polyphase machines saliency also affects the mutual inductances between windings on the nonsalient member. For an example of this and resulting forms see Section 4.2.2.

magnetic field encounters the smallest reluctance and a minimum at $\theta = \pi/2$ because the magnetic field encounters the maximum reluctance. Thus the stator self-inductance is expressible in general as

$$L_s(\theta) = L_0 + L_2 \cos 2\theta + L_4 \cos 4\theta + \cdots \quad (4.2.3)$$

In the design of ac salient-pole machines the salient-pole structure is shaped to accentuate the $\cos 2\theta$ term and to minimize all other space harmonics. Consequently, for the remainder of this treatment it is assumed that

$$L_s(\theta) = L_0 + L_2 \cos 2\theta. \quad (4.2.4)$$

This equation for inductance is often interpreted as representing the superposition of a smooth air gap (L_0) and a periodically varying air gap due to saliency ($L_2 \cos 2\theta$).

To ascertain the form of the mutual inductance $L_{sr}(\theta)$ we refer to Fig. 4.2.1 and reason physically. We recognize that reciprocity holds (Section 3.1.2c of Chapter 3) and consider flux linkages with stator windings due to rotor winding excitation by current of the direction indicated in Fig. 4.2.1. At the same time, we remember that the stator winding is actually distributed in many slots around the periphery to form a coil with the magnetic axis shown. Rotor position $\theta = 0$ results in maximum positive flux linking the stator winding, whereas rotor position $\theta = \pi$ yields maximum negative flux linkage. The symmetry indicates that these two maxima are of equal magnitude. As the rotor position θ is varied from 0 to π through positive angles, the flux linkage with the stator varies smoothly from the positive maximum to the negative maximum. Variation of rotor angle from 0 to π through negative angles gives exactly the same variation of flux linkages. Consequently, the mutual inductance is expressible as a Fourier series of odd space harmonics, exactly as it was in the smooth-air-gap machine in (4.1.4),

$$L_{sr}(\theta) = M_1 \cos \theta + M_3 \cos 3\theta + M_5 \cos 5\theta + \cdots \quad (4.2.5)$$

Although the forms of mutual inductance for the two machine types are the same, it should be clear that for a given frame size the coefficients in (4.1.4) will have different numerical values for the two cases.

For salient-pole ac machines the winding distribution on the member without salient poles (the stator in Fig. 4.2.1) is designed to maximize M_1 and minimize all other terms in (4.2.5), just as is done for smooth-air-gap machines. For the remainder of this treatment it is assumed that this design objective has been met and the mutual inductance is expressed as

$$L_{sr}(\theta) = M \cos \theta. \quad (4.2.6)$$

Substitution of (4.2.4) and (4.2.6) into (4.2.1) and (4.2.2) and calculation

of the mechanical torque of electric origin, using the techniques of Chapter 3 [see (g) of Table 3.1], yield

$$\lambda_s = (L_0 + L_2 \cos 2\theta)i_s + Mi_r \cos \theta, \quad (4.2.7)$$

$$\lambda_r = Mi_s \cos \theta + L_r i_r, \quad (4.2.8)$$

$$T^e = -i_s i_r M \sin \theta - i_s^2 L_2 \sin 2\theta. \quad (4.2.9)$$

These terminal relations for the electromechanical coupling system can be used with whatever external electrical and mechanical sources or loads are connected to the machine terminals to write the differential equations for the machine system. For example, if we specify the terminal constraints as those given in Fig. 4.1.5, which include the parameters normally associated with that machine alone, the differential equations derive from Kirchhoff's voltage law and Newton's second law:

$$v_s = R_s i_s + \frac{d\lambda_s}{dt}, \quad (4.2.10)$$

$$v_r = R_r i_r + \frac{d\lambda_r}{dt}, \quad (4.2.11)$$

$$T^e + T_m = J_r \frac{d^2\theta}{dt^2} + B_r \frac{d\theta}{dt} + T_{or} \frac{d\theta/dt}{|d\theta/dt|}, \quad (4.2.12)$$

where (4.2.7) to (4.2.9) are used to express λ_s , λ_r , and T^e . These equations have exactly the same form as (4.1.9) to (4.1.11) for the smooth-air-gap machine, as is to be expected. However, the terminal relations are different for the two machines. Compare (4.2.7) to (4.2.9) for the salient-pole machine with (4.1.6) to (4.1.8) for the smooth-air-gap machine. The sources in (4.2.10) to (4.2.12), v_s , v_r , and T_m , are completely general and can be independent or dependent on some variable.

4.2.2 Conditions for Conversion of Average Power

To establish conditions for average power conversion in a salient-pole machine we assume excitation of the electromechanical coupling system by ideal current and position sources as illustrated in Fig. 4.1.6. The sources are

$$i_s(t) = I_s \sin \omega_s t, \quad (4.2.13)$$

$$i_r(t) = I_r \sin \omega_r t, \quad (4.2.14)$$

$$\theta(t) = \omega_m t + \gamma, \quad (4.2.15)$$

where I_s , I_r , ω_s , ω_r , ω_m , and γ are positive constants. Note that these constraints are the same as those we used with the smooth-air-gap machine (4.1.12) to (4.1.14).

The first term in the torque expressed by (4.2.9) has the same form as (4.1.8) for the smooth-air-gap machine. Consequently, all the discussions of average power conversion in Section 4.1.2 apply equally well to the first term of (4.2.9). In order for the mutual inductance term of the torque in a salient-pole machine to participate in average power conversion, one of the four conditions of (4.1.18), which relates electrical excitation frequencies and mechanical velocity, must be satisfied. This condition is

$$\omega_m = \pm \omega_s \pm \omega_r. \quad (4.2.16)$$

The unique effect of salient poles on the power conversion process is represented by the second term in (4.2.9). With the terminal constraints of (4.2.13) to (4.2.15), the instantaneous mechanical power output of the coupling system due to the second term in (4.2.9) is

$$p_m = -\omega_m I_s^2 L_2 \sin^2 \omega_s t \sin (2\omega_m t + 2\gamma). \quad (4.2.17)$$

The use of trigonometric identities allows us to write this expression in the form

$$p_m = -\frac{\omega_m I_s^2 L_2}{4} \{2 \sin (2\omega_m t + 2\gamma) - \sin [2(\omega_m + \omega_s)t + 2\gamma] \\ - \sin [2(\omega_m - \omega_s)t + 2\gamma]\}. \quad (4.2.18)$$

A sinusoidal function of time has an average value only when the coefficient of t in its argument goes to zero. The first term in braces in (4.2.18) has an average value when $\omega_m = 0$, which is uninteresting because for this condition the power conversion is zero. The second term has an average value when

$$\omega_m + \omega_s = 0 \quad (4.2.19)$$

and the third term has an average value when

$$\omega_m - \omega_s = 0. \quad (4.2.20)$$

These two conditions are expressed in the compact form

$$\omega_m = \pm \omega_s, \quad (4.2.21)$$

and when either condition is satisfied the average power converted is

$$p_{m(av)} = \frac{\omega_m I_s^2 L_2}{4} \sin 2\gamma. \quad (4.2.22)$$

Sufficient conditions for nonzero average power conversion are (4.2.21) and $\sin 2\gamma \neq 0$.

It is worthwhile to interpret this result in terms of rotating fields along the

lines of Section 4.1.4. Recall that single-phase excitation of the stator winding with a current of frequency ω_s yields two component field patterns rotating in opposite directions with the angular speed ω_s . Thus each of the two conditions of (4.2.21) is interpreted as the condition under which the salient-pole structure is fixed in space with respect to one component of a stator rotating field. As a result, it is reasonable to expect that the use of balanced polyphase windings and balanced polyphase excitation, which produce a constant-amplitude rotating field (see Section 4.1.4), with saliency can produce a constant power conversion with no pulsating terms. Such is the case; in fact, saliency is used in machines (synchronous and direct current) in which such a result occurs. An example that illustrates how saliency affects the steady-state behavior of polyphase synchronous machines is given in Section 4.2.4.

4.2.3 Discussion of Saliency in Different Machine Types

The conditions expressed by (4.2.16) and (4.2.21) are now used to assess the usefulness of saliency in the principal machine types. We must recognize that within the framework of our general treatment there are possibilities for numerous unique machine types and many nonstandard machines are built for special applications. Most of these machines can be analyzed by using the general techniques developed here.

The simplest machine in which saliency is exploited is the *reluctance motor*. In the nomenclature of Fig. 4.2.1 a reluctance motor has a salient-pole rotor, one or two stator windings, but no rotor winding. The stator windings are excited by single-frequency alternating current. The only torque produced by this machine is that due to saliency or the *reluctance* torque given by the second term of (4.2.9) at a mechanical speed defined by (4.2.21). Thus the reluctance motor is a synchronous motor because it converts power at only one speed, ω_s . The steady-state energy conversion properties of a single-phase reluctance motor were studied in Example 3.1.2. Because of poor efficiency and power factor, reluctance motors are made in small sizes for such applications as clocks and phonograph turntables. Like any other synchronous machine, a reluctance motor has no starting torque and is usually started as an induction motor.

Saliency is most often exploited to improve the performance of machines that can operate successfully without it. To determine which machines can be helped by saliency, we must know when (4.2.16) (smooth-air-gap) and (4.2.21) (saliency) can be satisfied simultaneously with the same excitation. In one case we set $\omega_r = 0$ and $\omega_m = \pm\omega_s$, which yields a synchronous machine. We consider this subject in some detail in Section 4.2.4.

Saliency is also useful in dc machines in which the stator excitation is direct current ($\omega_s = 0$) and the commutator produces rotor currents of

fundamental frequency $\omega_r = \omega_m$ (see Section 4.1.6c). In this case the saliency is on the stator and (4.2.21) is replaced by

$$\omega_m = \pm \omega_r, \quad (4.2.23)$$

which is the same form that (4.2.16) assumes with $\omega_s = 0$. Thus saliency on the stator of a dc machine can enhance the power conversion capability. This is also true for commutator machines that operate on alternating current.

Although salient poles are sometimes used in small, single-phase induction motors to simplify the construction and make the machines less expensive, they are never used in large induction machines. As demonstrated in Section 4.1.6b, an induction machine has alternating current on both rotor and stator. Moreover, the machine converts power only when the rotor is *not* in synchronism with the stator-produced rotating field. Consequently, (4.2.16) and (4.2.21) cannot be satisfied simultaneously and saliency in an induction machine will produce only an oscillating power flow with no average value. The attendant noise and vibration make saliency undesirable in an induction machine.

4.2.4 Polyphase, Salient-Pole, Synchronous Machines

Salient poles are used in many synchronous machines; for example, all large synchronous motors, synchronous condensers, and hydro generators have them. It is therefore worthwhile to examine the effects of saliency on steady-state machine performance. The results achieved with saliency are compared with those obtained with a smooth-air-gap machine.

For the analysis we assume the balanced, two-phase, two-pole configuration shown schematically in Fig. 4.2.4. As usual we show the stator coils concentrated in two slots per phase for simplicity, but we realize that in an actual machine the stator windings are distributed in many slots around the periphery while maintaining the same relative symmetries with respect to magnetic axes.

We have already written the electrical terminal relations for a salient-pole machine with a single winding on both the rotor and stator in Section 4.2.1 [(4.2.7) and (4.2.8)]. These equations are still valid, except that saliency adds an angular-dependent mutual coupling term between the two stator windings. In a smooth-air-gap two-phase machine there is no mutual inductance between the two stator windings. [See (4.1.19) and (4.1.20).]

To obtain the form of this mutual inductance between stator windings we reason physically by using the configuration in Fig. 4.2.4. When the rotor magnetic axis is aligned with the magnetic axis of either stator coil ($\theta = 0, \pi/2, \pi, 3\pi/2$), the flux produced by either coil is symmetrical with respect to its magnetic axis and there is no net flux linking one stator coil

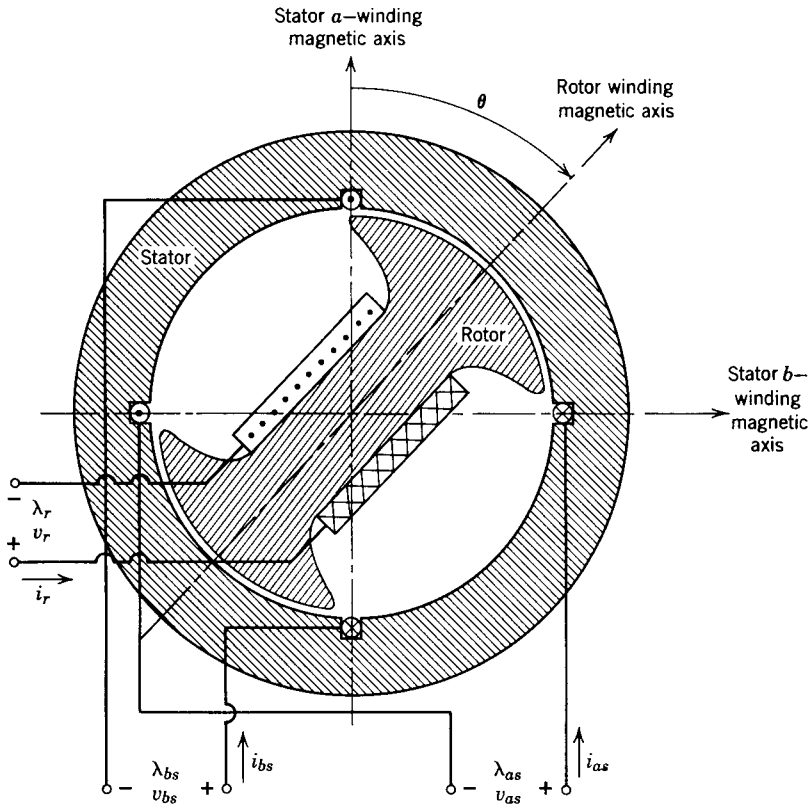


Fig. 4.2.4 Schematic representation of a salient-pole two-pole, balanced two-phase synchronous machine.

due to current in the other. Thus the stator mutual inductance L_{ss} is

$$L_{ss} = 0 \quad \text{for} \quad \theta = n \frac{\pi}{2}, \quad n = 0, \pm 1, \pm 2, \pm 3, \dots \quad (4.2.24)$$

Now assume a current in stator coil a of the polarity shown. When θ is in the range $(0 < \theta < \pi/2)$, the salient poles distort the flux pattern due to i_{as} and tend to concentrate it at the pole where the air gap is smallest. Thus for $(0 < \theta < \pi/2)$ the flux linkage with winding b on the stator is positive. A similar argument shows that for the range $(-\pi/2 < \theta < 0)$ the flux linkage with winding b is negative. Using these facts, recognizing the machine symmetries, and realizing that reciprocity applies, we write the stator-to-stator mutual inductance as the Fourier series

$$L_{ss}(\theta) = M_{s2} \sin 2\theta + M_{s6} \sin 6\theta + M_{s10} \sin 10\theta + \dots \quad (4.2.25)$$

This expression is justified by making use of Fig. 4.2.4 to determine the field patterns as functions of θ .

As discussed in Section 4.2.1, the winding distributions and salient-pole shape are adjusted in ac machine design to accentuate the lowest space harmonic and to minimize higher space harmonics in inductance. This is also true of stator mutual inductance. To be consistent with the assumptions made in Section 4.2.1 that this design objective has been met we simplify (4.2.25) to the form

$$L_{ss} = M_s \sin 2\theta. \quad (4.2.26)$$

This second harmonic variation in stator mutual inductance results from the same distortion of flux pattern that causes the $\cos 2\theta$ term in stator self-inductance indicated in (4.2.7). Furthermore, both stator windings have the same number of turns and consequently we assume

$$M_s = L_2. \quad (4.2.27)$$

This assumption is justified by a careful field analysis of the machine* and is borne out in practice.

We now use (4.2.7) and (4.2.8) with (4.2.26) and (4.2.27) to write the electrical terminal relations for the machine in Fig. 4.2.4:

$$\lambda_{as} = (L_0 + L_2 \cos 2\theta)i_{as} + L_2 i_{bs} \sin 2\theta + M_i r \cos \theta, \quad (4.2.28)$$

$$\lambda_{bs} = L_2 i_{as} \sin 2\theta + (L_0 - L_2 \cos 2\theta)i_{bs} + M_i r \sin \theta, \quad (4.2.29)$$

$$\lambda_r = M_i i_{as} \cos \theta + M_i i_{bs} \sin \theta + L_r i_r. \quad (4.2.30)$$

In writing (4.2.29) we have replaced θ with $(\theta - \pi/2)$ in (4.2.28) to obtain the self-inductance and stator-to-rotor mutual inductance terms. This change accounts for the angular difference of $\pi/2$ in the positions of the two stator coils. This equation (4.2.29) could have been obtained by reasoning physically with Fig. 4.2.4 and using the assumptions we have for design objectives.

The use of (4.2.28) to (4.2.30) with the techniques of Chapter 3 [see (g) of Table 3.1] leads to the mechanical terminal relation

$$T^e = M_i r (i_{bs} \cos \theta - i_{as} \sin \theta) - L_2 (i_{as}^2 - i_{bs}^2) \sin 2\theta + 2L_2 i_{as} i_{bs} \cos 2\theta. \quad (4.2.31)$$

Equations 4.2.28 to 4.2.31 should be compared with (4.2.7) to (4.2.9) for a single-phase, salient-pole machine to see the effects of adding the second phase and with (4.1.35) to (4.1.38) for a smooth-air-gap, two-phase machine to see the effects of adding saliency.

In our study of the steady-state characteristics of the salient-pole synchronous machine we neglect stator winding resistances and mechanical losses

* White and Woodson, *op. cit.*, pp. 180-190.

and use the same excitations we had in Section 4.1.6a, namely balanced two-phase currents on the stator,

$$i_{as} = I_s \cos \omega t, \quad (4.2.32)$$

$$i_{bs} = I_s \sin \omega t, \quad (4.2.33)$$

direct current on the rotor,

$$i_r = I_r, \quad (4.2.34)$$

and the position source

$$\theta = \omega t + \gamma. \quad (4.2.35)$$

Like the smooth-air-gap machine in Section 4.1.6a, it can be verified by direct substitution that there is constant flux linking the rotor winding, hence no induced voltage. Thus we could have excited the field winding with a constant-voltage source, as is usual in practice. It is convenient analytically, however, to use the constant current of (4.2.34), and there is no loss of generality in the steady-state analysis. For realistic transient analyses a rotor winding voltage source with a series resistance must be used.

Substitution of (4.2.32) to (4.2.35) into (4.2.31) yields for the steady-state instantaneous torque produced by the electromechanical coupling system,

$$\begin{aligned} T^e &= MI_r I_s [\sin \omega t \cos (\omega t + \gamma) - \cos \omega t \sin (\omega t + \gamma)] \\ &\quad - L_2 I_s^2 [\cos^2 \omega t - \sin^2 \omega t] \sin (2\omega t + 2\gamma) \\ &\quad + 2L_2 I_s^2 \cos \omega t \sin \omega t \cos (2\omega t + 2\gamma). \end{aligned} \quad (4.2.36)$$

The use of appropriate trigonometric* identities allows the simplification of this expression to the form

$$T^e = -MI_r I_s \sin \gamma - L_2 I_s^2 \sin 2\gamma. \quad (4.2.37)$$

This instantaneous torque is constant because the stator windings with balanced excitation produce a constant amplitude rotating flux wave and the salient-pole rotor is at an instantaneous position fixed with respect to this rotating field.

Comparison of (4.2.37) with (4.1.43) shows that saliency has added a term to the torque expression for a smooth-rotor machine.

Neglecting stator (armature) winding resistance, the terminal voltage of stator winding *a* is

$$v_{as} = \frac{d\lambda_{as}}{dt}. \quad (4.2.38)$$

* $\sin x \cos y - \cos x \sin y = \sin (x - y)$; $\cos 2x = \cos^2 x - \sin^2 x$; $2 \cos x \sin x = \sin 2x$.

Substitution of (4.2.32) to (4.2.35) into (4.2.28) and that result into (4.2.38) yields

$$v_{as} = I_s \frac{d}{dt} \{ [L_0 + L_2 \cos(2\omega t + 2\gamma)] \cos \omega t + L_2 \sin \omega t \sin(2\omega t + 2\gamma) \} \\ + MI_r \frac{d}{dt} [\cos(\omega t + \gamma)]. \quad (4.2.39)$$

The use of trigonometric identities reduces this equation to the form

$$v_{as} = \frac{d}{dt} [L_0 I_s \cos \omega t + L_2 I_s \cos(\omega t + 2\gamma) + MI_r \cos(\omega t + \gamma)]. \quad (4.2.40)$$

We define the complex quantities

$$v_{as} = \text{Re}(\hat{V}_s e^{j\omega t}), \quad i_{as} = \text{Re}(I_s e^{j\omega t}), \quad \cos(\omega t + \alpha) = \text{Re}(e^{j\omega t} e^{j\alpha})$$

and use the standard techniques of steady-state ac circuit theory to write

$$\hat{V}_s = j\omega L_0 I_s + j\omega L_2 I_s e^{j2\gamma} + j\omega MI_r e^{j\gamma}. \quad (4.2.41)$$

We define the complex voltage amplitude $\hat{E}_f$ generated by field (rotor) current, as we did for the smooth-air-gap machine in (4.1.19), as

$$\hat{E}_f = j\omega MI_r e^{j\gamma} \quad (4.2.42)$$

and rewrite (4.2.41) as

$$\hat{V}_s = j\omega L_0 I_s + j\omega L_2 I_s e^{j2\gamma} + \hat{E}_f. \quad (4.2.43)$$

This is the same form as (4.1.50) for the smooth-air-gap machine with the addition of the second term due to saliency.

Because of this term, it is not possible to draw a simple equivalent circuit for the salient-pole machine as we did in Fig. 4.1.12 for the smooth-air-gap machine. We can, however, draw vector diagrams to show the relations among variables as we did for the smooth-air-gap machine in Fig. 4.1.13. These diagrams, which illustrate generator and motor operation in the salient-pole machine, appear in Fig. 4.2.5. Note that the additional term due to saliency does not greatly change the over-all nature of the vector diagram.

When analyzing salient-pole synchronous machines with conventional nomenclature, the sum of the two reactance voltages ($j\omega L_0 I_s + j\omega L_2 I_s e^{j2\gamma}$) is normally decomposed into two components, one parallel to $\hat{E}_f$, called the *direct-axis* reactance voltage, and one perpendicular to $\hat{E}_f$, called the *quadrature axis* reactance voltage.*

To complete the description of the steady-state properties of salient-pole machines we need to assume stator excitation from constant-amplitude

* See, for example, Fitzgerald and Kingsley, *op. cit.*, Chapter 5.

voltage sources, as is done in practice, and to express the torque T^e in terms of the terminal voltage. It can be established quite easily from the vector diagram in Fig. 4.2.5a or 4.2.5b that

$$\omega(L_0 - L_2)I_s \sin \gamma = V_s \sin \delta \quad (4.2.44)$$

and

$$\omega(L_0 + L_2)I_s \cos \gamma = V_s \cos \delta - E_f, \quad (4.2.45)$$

where

$$V_s = |\hat{V}_s|,$$

$$E_f = |\hat{E}_f|;$$

δ is defined in Fig. 4.2.5 and is positive in the counterclockwise direction. The use of (4.2.44) and (4.2.45) to eliminate the angle γ and (4.2.42) for E_f in (4.2.37) lead to the desired form of the torque equation

$$T^e = -\frac{E_f V_s}{\omega^2(L_0 + L_2)} \sin \delta - \frac{L_2 V_s^2}{\omega^2(L_0^2 - L_2^2)} \sin 2\delta. \quad (4.2.46)$$

When we compare this result with (4.1.52) for the smooth-air-gap machine, we find that the first term of (4.2.46) is the same form as (4.1.52). The second term in (4.2.46) depends solely on the presence of saliency. When saliency is removed, $L_2 = 0$, the second term in (4.2.46) goes to zero, and the first term reduces to (4.1.52) for the smooth-air-gap machine.

The two terms of (4.2.46) are plotted separately with dashed lines and the

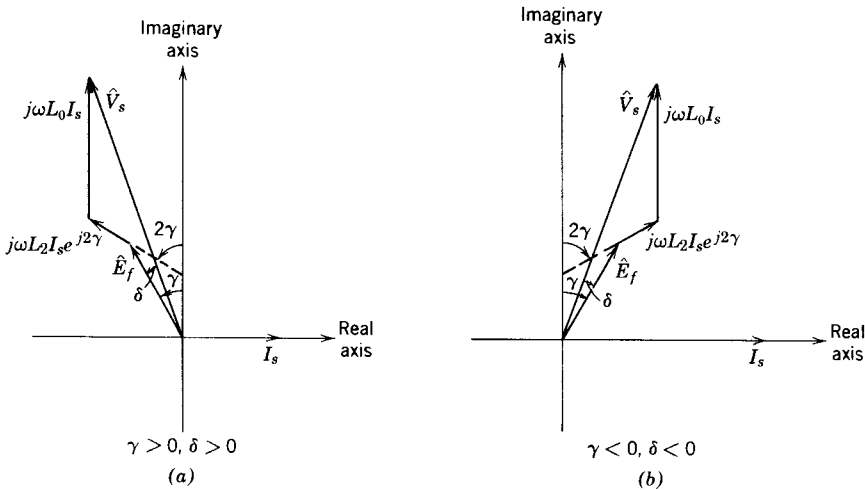


Fig. 4.2.5 Vector diagrams showing relations among variables in a salient-pole synchronous machine. Diagrams drawn for $L_0 = 3L_2$: (a) generator operation; (b) motor operation.

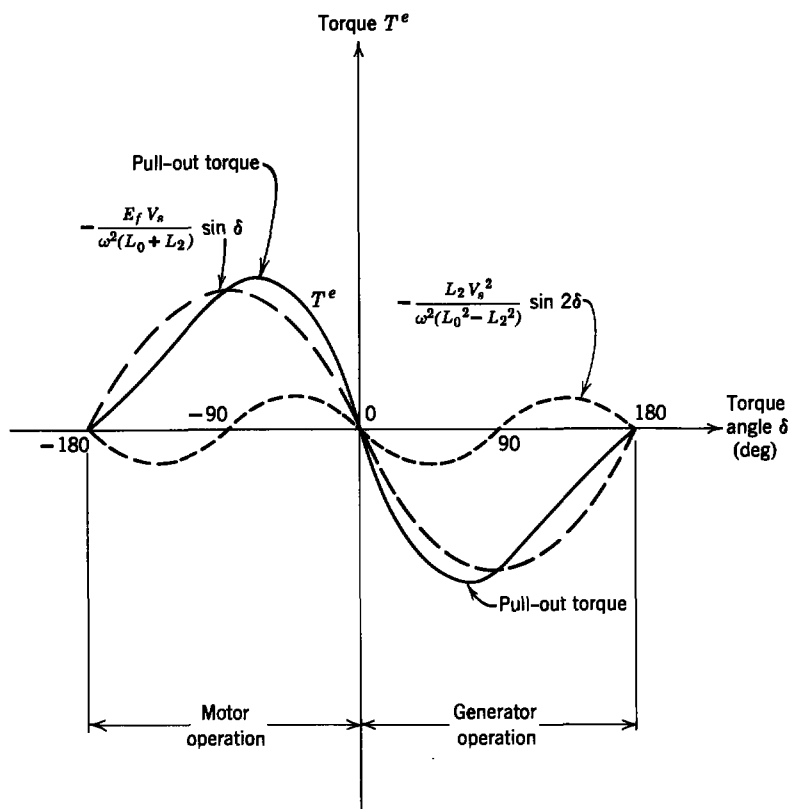


Fig. 4.2.6 Torque versus torque angle for a salient-pole synchronous machine. Curves plotted for $L_0 = 5L_2$ and $E_f = V_s$.

total torque in a solid line in Fig. 4.2.6. These curves are plotted for

$$L_0 = 5L_2,$$

which is typical for water-wheel generators,* and for

$$E_f = V_s.$$

Note from Fig. 4.2.6 that the presence of saliency has increased both the pull-out torque and the torque produced at small angles, which is quite important for transient behavior.

It should be clear from what we have done so far in this section that vector diagrams and V -curves can be drawn for salient-pole machines and that they will be similar to those for the smooth-air-gap machine shown in

* See, for example, Fitzgerald and Kingsley, *op. cit.*, Table 5-1, p. 237. In their nomenclature $X_d = \omega(L_0 + L_2)$ and $X_q = \omega(L_0 - L_2)$.

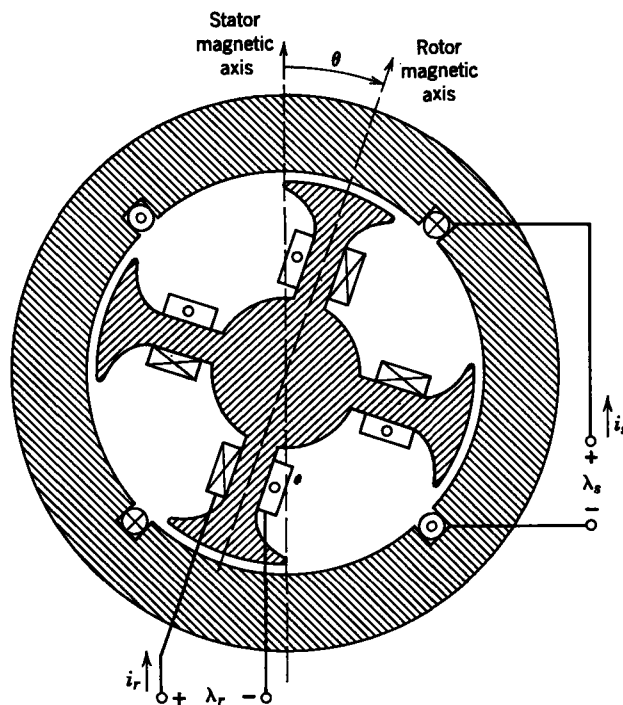


Fig. 4.2.7 Four-pole, single-phase, salient-pole machine with saliency on rotor.

Fig. 4.1.15. The process is straightforward and the interpretation is the same; consequently it is not repeated here.

All of the discussions of polyphase machines and excitations in Section 4.1.7 and of the number of poles in Section 4.1.8 apply equally well to salient pole machines, with the understanding that there is one polar projection per pole in a salient-pole machine; for example, Fig. 4.2.7 is a schematic drawing of a four-pole, single-phase machine.

In our discussions of synchronous machines in this section and in Section 4.1.6a we have made the point that a synchronous machine will produce a time-average torque and convert time-average power only at synchronous speed. Consequently, a synchronous machine alone can produce no starting torque. This is no problem with generators, but it is a problem with motors and synchronous condensers. A few machines are started by auxiliary starting motors, but the vast majority are started as induction machines. Conducting bars are mounted axially in the pole faces and shorted together at the ends to form a squirrel-cage winding, as shown for a motor in Fig. 4.2.2. Such a winding is conventionally called a *damp*er winding or *amortisseur winding* because, in addition to acting as an induction motor winding for

starting, it also damps out transients in torque angle. Operation as an induction motor brings the speed to near synchronous speed. The torque oscillations resulting from the interaction between the rotor field due to dc excitation and the rotating stator field occur at the slip frequency, which is quite low. This allows the oscillating torque ample time to accelerate the rotor inertia and pull it into step at synchronous speed during one half cycle. In a turbo-generator the solid steel rotor provides enough induction-motor action for adequate damping and no separate damper winding is used (see Fig. 4.1.10).

4.3 DISCUSSION

At this point it is worthwhile to re-emphasize several points made in this chapter.

First, although we have treated two geometrical configurations, the techniques are applicable to other rotating machines by simple extensions and modifications. Thus we should understand the basic concepts that are quite simple physically.

Second, we have considered in some detail the steady-state characteristics of some standard machine types for two purposes: to illustrate how the transition is actually made from basic concepts to practical descriptions of steady-state terminal behavior and to present the characteristics of some of the most important rotating machines.

Next, when the reader thinks back through the material presented in this chapter he will realize that the basic concepts of energy conversion in rotating machines are quite simple, though the mathematics sometimes becomes lengthy. As we indicated earlier, the symmetries that exist in rotating machines have led to orderly mathematical procedures for handling the manipulation. Thus rotating machine theory may appear formidable at first glance, but we, you and the authors, know that this is not so.

Finally, we want to state again that among all electromechanical devices, past, present, and foreseeable future, rotating machines occur in the greatest numbers and in the widest variety of sizes and types. Thus they form an important part of any study of electromechanics.

PROBLEMS

4.1. The object of this problem is to analyze a physical configuration that yields the electrical terminal relations of (4.1.6) and (4.1.7) almost exactly. The system of Fig. 4P.1 consists of two concentric cylinders of ferromagnetic material with infinite permeability and zero conductivity. Both cylinders have length l and are separated by the air gap g . As indicated in the figure, the rotor carries a winding of N_r turns distributed sinusoidally and

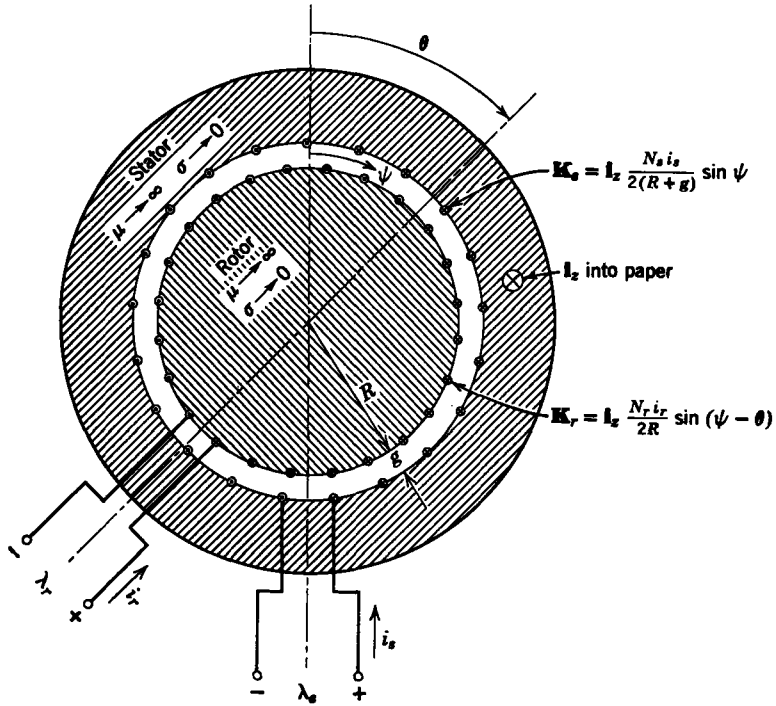


Fig. 4P.1

having negligible radial thickness. The stator carries a winding of N_s turns distributed sinusoidally and having negligible radial thickness. Current through these windings leads to sinusoidally distributed surface currents as indicated. In the analysis we neglect the effects of end turns and assume $g \ll R$ so that the radial variation of magnetic field can be neglected.

- Find the radial component of air-gap flux density due to stator current alone.
- Find the radial component of air-gap flux density due to rotor current alone.
- Use the flux densities found in parts (a) and (b) to find λ_s and λ_r in the form of (4.1.6) and (4.1.7). In particular, evaluate L_s , L_r , and M in terms of given data.

4.2. Rework Problem 4.1 with the more practical uniform winding distribution representable by surface current densities

$$\mathbf{K}_s = \begin{cases} i_s \frac{N_s i_s}{\pi(R+g)}, & \text{for } 0 < \psi < \pi, \\ -i_s \frac{N_s i_s}{\pi(R+g)}, & \text{for } \pi < \psi < 2\pi, \end{cases}$$

$$\mathbf{K}_r = \begin{cases} i_r \frac{N_r i_r}{\pi R}, & \text{for } 0 < (\psi - \theta) < \pi, \\ -i_r \frac{N_r i_r}{\pi R}, & \text{for } \pi < (\psi - \theta) < 2\pi. \end{cases}$$

In part (c) you will find the mutual inductance to be expressed as an infinite series like (4.1.4).

4.3. With reference to Problems 4.1 and 4.2, show that if either the rotor winding or the stator winding is sinusoidally distributed as in Problem 4.1, the mutual inductance contains only a space fundamental term, regardless of the winding distribution on the other member.

4.4. The machine represented schematically in Fig. 4P.4 has uniform winding distributions. As indicated by Problem 4.2, the electrical terminal relations are ideally

$$\lambda_s = L_s i_s + i_r \sum_{n \text{ odd}} \frac{M_o}{n^4} \cos n\theta,$$

$$\lambda_r = L_r i_r + i_s \sum_{n \text{ odd}} \frac{M_o}{n^4} \cos n\theta,$$

where L_s , L_r , and M_o are constants. We now constrain the machine as follows: $i_r = I = \text{constant}$; $\theta = \omega t$, $\omega = \text{constant}$, stator winding open-circuited $i_s = 0$.

- Find the instantaneous stator voltage $v_s(t)$.
- Find the ratio of the amplitude of the n th harmonic stator voltage to the amplitude of the fundamental component of stator voltage.
- Plot one complete cycle of $v_s(t)$ found in (a).

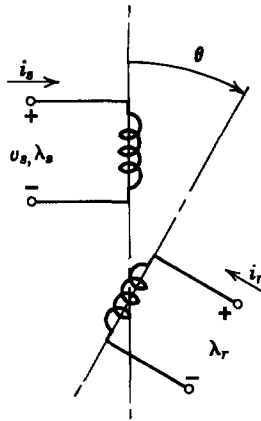


Fig. 4P.4

4.5. Calculate the electromagnetic torque T^e of (4.1.8) by using the electrical terminal relations (4.1.6) and (4.1.7) and the assumption that the coupling system is conservative.

4.6. A schematic representation of a rotating machine is shown in Fig. 4P.6. The rotor winding is superconducting and the rotor has moment of inertia J . The machine is constructed so that the electrical terminal relations are $\lambda_s = L_s i_s + M i_r \cos \theta$, $\lambda_r = M i_s \cos \theta + L_r i_r$. The machine is placed in operation as follows:

- With the rotor (r) terminals open-circuited and the rotor position at $\theta = 0$, the current i_s is raised to I_0 .
- The rotor (r) terminals are short circuited to conserve the flux λ_r , regardless of $\theta(t)$ and $i_s(t)$.
- The current i_s is constrained by the independent current source $i(t)$.

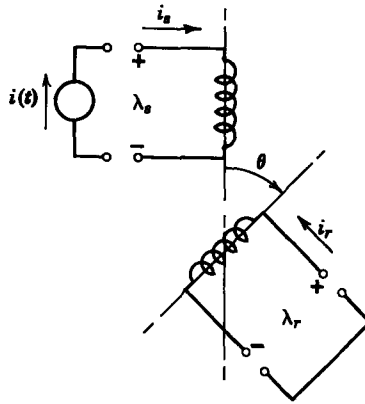


Fig. 4P.6

Write the equation of motion for the shaft with no external mechanical torque applied. Your answer should be one equation involving $\theta(t)$ as the only unknown. Damping may be ignored.

4.7. A smooth-air-gap machine with one winding on the rotor and one on the stator (see Fig. 4.1.1) has the electrical terminal relations of (4.1.1) and (4.1.2).

$$\lambda_s = L_s i_s + L_{sr}(\theta) i_r, \quad (4.1.1)$$

$$\lambda_r = L_{sr}(\theta) i_s + L_r i_r. \quad (4.1.2)$$

The mutual inductance $L_{sr}(\theta)$ contains two spatial harmonics, the fundamental and the third. Thus $L_{sr}(\theta) = M_1 \cos \theta + M_3 \cos 3\theta$, where M_1 and M_3 are constants.

- Find the torque of electric origin as a function of i_s , i_r , θ , M_1 , and M_3 .
- Constrain the machine with the current sources $i_s = I_s \sin \omega_s t$, $i_r = I_r \sin \omega_r t$ and the position source $\theta = \omega_m t + \gamma$, where I_s , I_r , ω_s , ω_r and γ are constants. Find the values of ω_m at which the machine can produce an average torque and find an expression for the average torque for each value of ω_m found.

4.8. The smooth-air-gap machine of Fig. 4.1.1 with the terminal relations given by (4.1.6) to (4.1.8) is constrained as follows: single-frequency rotor current, $i_r = I_r \sin \omega_r t$; stator current containing fundamental and third harmonic, $i_s = I_{s1} \sin \omega_s t + I_{s3} \sin 3\omega_s t$; and the position source $\theta = \omega_m t + \gamma$, where I_r , I_{s1} , I_{s3} , ω_r , ω_s , and γ are constants. Find the values of ω_m at which the machine can produce an average torque and give an expression for the average torque for each value of ω_m found.

4.9. Compute the torque T^e of (4.1.23) by using the electrical terminal relations of (4.1.19) to (4.1.22) and the assumption that the coupling system is conservative.

4.10. A smooth-air-gap machine has a two-phase set of stator windings, each with a total of N turns. The windings are distributed sinusoidally and currents in them produce surface current densities as indicated in Fig. 4P.10. When $g \ll R$, the radial flux density produced in the air gap by each winding (see Problem 4.1), is

$$B_{ra} = \frac{\mu_0 N i_a}{2g} \cos \psi,$$

$$B_{rb} = \frac{\mu_0 N i_b}{2g} \sin \psi.$$

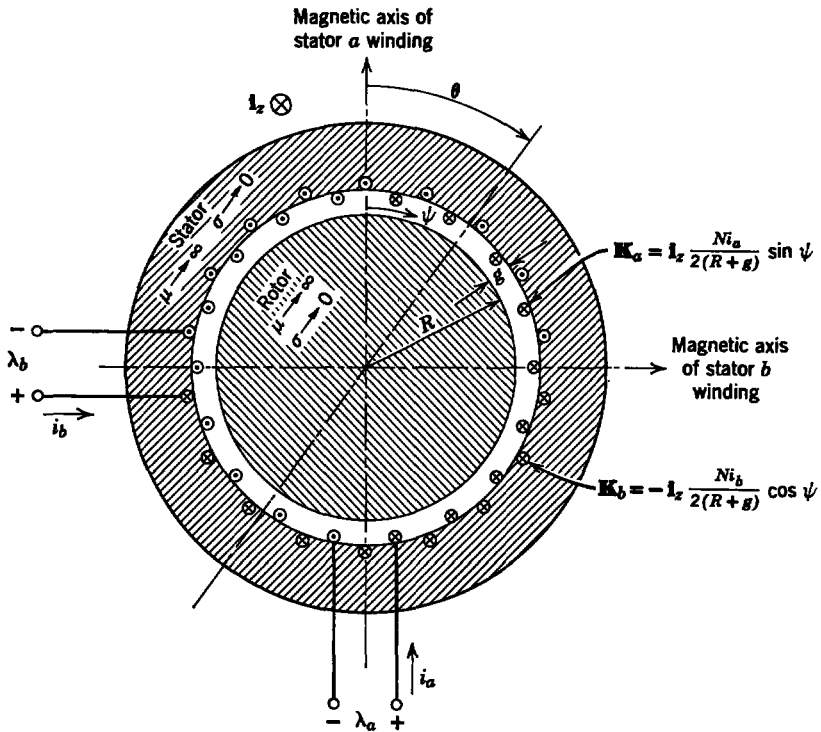


Fig. 4P.10

- (a) For the two-phase excitation $i_a = I_a \cos \omega t$, $i_b = I_b \sin \omega t$, which is unbalanced in amplitude, find the total radial flux density.
- (b) Express the answer to part (a) as a sum of two traveling waves. Identify the *forward* and *backward* components and show that their respective angular velocities are $\omega_f = \omega$ and $\omega_b = -\omega$.
- (c) Evaluate the ratio of the amplitudes of backward and forward waves. Show that the ratio $\rightarrow 0$ for a balanced excitation (i.e., consider the limit for $I_b \rightarrow I_a$).
- (d) Discuss how to achieve a constant amplitude backward wave only. This is the method used to reverse the direction of rotation of an ac machine.
- 4.11.** Rework Problem 4.10 and replace the excitation of part (a) with $i_a = I \cos \omega t$, $i_b = I \sin (\omega t + \beta)$. This is a two-phase set of currents, balanced in amplitude but unbalanced in phase. For part (c) balanced excitation occurs when $\beta \rightarrow 0$.
- 4.12.** Use (4.1.53) as the starting point to show that for steady-state operation the electrical power into a two-phase synchronous machine is equal to the mechanical power delivered, as expressed by (4.1.54).
- 4.13.** The two-phase equivalent of a large turbogenerator of the type now being used to generate power is as follows:
- 2-phase
60 Hz, 2-pole
Rated terminal voltage, 17,000 V rms

Rated terminal current, 21,300 A rms

Rating, 724×10^6 VA

Rated power factor, 0.85

Armature inductance, $L_s = 4.4 \times 10^{-3}$ H

Maximum value of armature-field mutual inductance, $M = 0.030$ H

Rated field current, $I_f = 6100$ A

Calculate and plot a family of V -curves for this generator. The V -curves are plots of armature current versus field current at constant power and constant terminal voltage (see Fig. 4.1.15). Your family of curves should be bounded by rated armature and field current, zero-power-factor, and 90° torque angle. Indicate which of these limits the curves. Also indicate on your plot the 0 and 0.85 power factors, both leading and lagging, and the unity power factor. Plot curves for 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and full rated load of 615 MW and for rated armature voltage. It will be convenient to normalize armature current to the rated value and field current to that value necessary to produce rated terminal voltage with the armature open-circuited.

4.14. It is customary to define the complex power produced by an alternator as $P + jQ$, where P is real power and Q is reactive power. For a two-phase machine with balanced currents and voltages and a phase angle ϕ

$$\begin{aligned} v_a &= \text{Re}(\hat{V}e^{j\omega t}), & i_a &= \text{Re}(\hat{I}e^{j\omega t}), \\ v_b &= \text{Re}(-j\hat{V}e^{j\omega t}), & i_b &= \text{Re}(-j\hat{I}e^{j\omega t}), \end{aligned}$$

where $\hat{V} = V$ and $\hat{I} = Ie^{-j\phi}$. The complex power supplied by both phases is $P + jQ = \hat{V}\hat{I}^* = VI \cos \phi + jVI \sin \phi$. By convention $Q > 0$ when $\hat{I}$ lags $\hat{V}$ (the load is inductive).

A capability curve for an alternator is a plot of P versus Q for constant armature voltage and for maximum allowable operating conditions defined by rated armature current, rated field current, or steady-state stability (torque angle δ approaching a critical value which we take to be 90°). Plot the capability curve for the alternator described in Problem 4.13 for operation at rated voltage. Indicate on your plot the limit that determines that part of the curve. It is useful to normalize both P and Q to the rating of the alternator.

4.15. An automobile speedometer consists of a permanent magnet mounted on a rotating shaft connected to the automobile transmission. An aluminum "drag cup" with a pointer mounted on it is placed around this rotating magnet. The cup is free to rotate through an angle ψ but is restrained by a torsion spring that provides a torque $T_s = -K\psi$. The angular position of the cup can be used to determine the angular velocity of the shaft connected to the magnet and therefore the speed of the automobile. The model to be used in analyzing the speedometer is illustrated in Fig. 4P.15. The permanent magnet is represented by a coil excited by a constant-current source. The drag cup is simulated by two coils shunted by resistances. These coils are attached to a rotatable frame, which in turn is restrained by the torsion spring. An appropriate electrical model of the coupling field is

$$\begin{aligned} \lambda_1 &= Mi_3 \cos(\phi - \psi) + Li_1, \\ \lambda_2 &= Mi_3 \sin(\phi - \psi) + Li_2, \\ \lambda_3 &= L_3 i_3 + Mi_1 \cos(\phi - \psi) + Mi_2 \sin(\phi - \psi). \end{aligned}$$

Assuming that the rotational velocity of the shaft is constant (i.e., the speed of the car is constant), find the deflection of the rotatable frame (of the speedometer pointer) as a function of the shaft rotational velocity ϕ . You may assume that the device is designed in such a way that

$$\left| L \frac{di}{dt} \right| \ll |Ri|.$$

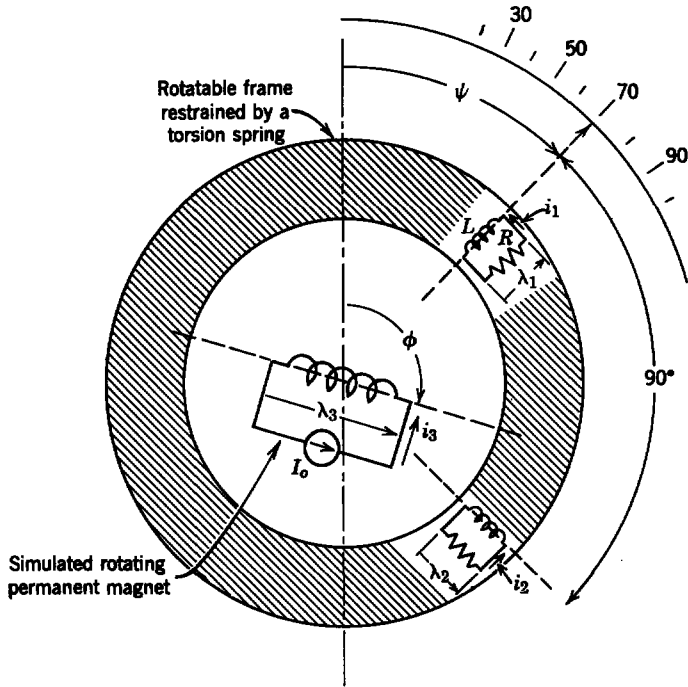


Fig. 4P.15

4.16. For nomenclature, refer to Fig. 4.1.17. The two-phase equivalent of a large, two-pole, polyphase, 60-Hz induction motor has the following parameters for operation at 60 Hz: $R_r = 0.100$ ohm, $\omega_s M = 4.50$ ohms, and $\omega_s(L_s - M) = \omega_s(L_r - M) = 0.300$ ohm. Neglect armature resistance. For operation at a constant amplitude of armature voltage $V_s = \sqrt{2} 500$ V peak, calculate and plot torque, armature current, volt-ampere input, electrical power input, and mechanical power output as functions of mechanical speed for the range $0 < \omega_m < \omega_s = 120\pi$ rad/sec.

4.17. The induction motor of Problem 4.16 is driving a fan load with the torque speed characteristic $T_m = -B\omega_m^3$, where $B = 7.50 \times 10^{-6}$ N-M sec³/rad³. Assume steady-state operation.

- For operation with balanced armature voltage of $V_s = \sqrt{2} 500$ V peak calculate the steady-state slip, mechanical power into the fan, electrical power input, and power factor.
- Calculate and plot the quantities of part (a) as functions of armature voltage for a range $\sqrt{2} 450 < V_s < \sqrt{2} 550$ V peak.

4.18. This problem is a version of the machine analysis in Problem 4.1 but with a three-phase winding on the stator. The geometry is illustrated in Fig. 4P.18; N_s is the total number of turns on each stator phase and N_r is the total number of turns in the rotor winding. The

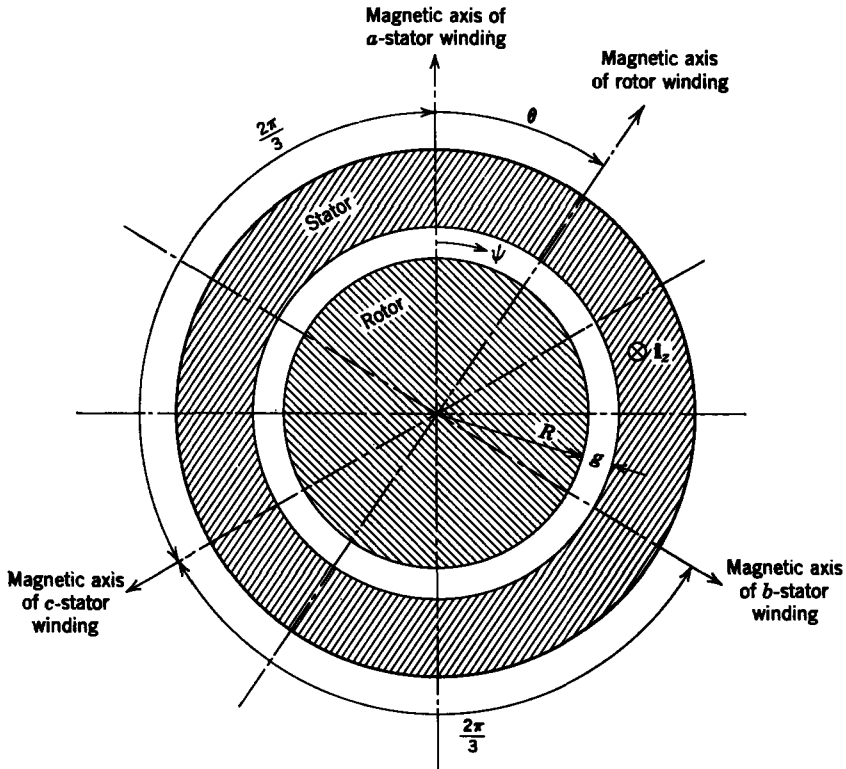


Fig. 4P.18

surface current densities produced by the three armature currents on the surface at $R + g$ are

$$\mathbf{K}_a = i_z \frac{N_s i_a}{2(R + g)} \sin \psi,$$

$$\mathbf{K}_b = i_z \frac{N_s i_b}{2(R + g)} \sin \left(\psi - \frac{2\pi}{3} \right),$$

$$\mathbf{K}_c = i_z \frac{N_s i_c}{2(R + g)} \sin \left(\psi - \frac{4\pi}{3} \right).$$

The surface current density due to rotor current on the surface at R is

$$\mathbf{K}_r = i_z \frac{N_r i_r}{2R} \sin (\psi - \theta).$$

Assume $g \ll R$ so that there is no appreciable variation in the radial component of magnetic field across the air gap.

- Find the radial flux density due to current in each winding.
- Find the mutual inductance between the a and b windings on the stator.
- Write the electrical terminal relations for the machine.
- Find the torque T^e of electrical origin.

4.19. Consider the machine in Problem 4.18 with the stator excitations

$$i_a = I_a \cos \omega t,$$

$$i_b = I_b \cos \left(\omega t - \frac{2\pi}{3} \right),$$

$$i_c = I_c \cos \left(\omega t - \frac{4\pi}{3} \right).$$

- (a) Show that the radial component of air-gap flux density is expressible as a combination of two constant-amplitude waves, one rotating in the positive θ -direction with the speed ω and the other rotating in the negative θ -direction with speed ω .
 (b) Show that when $I_a = I_b = I_c$ the amplitude of the wave traveling in the negative θ -direction goes to zero.

4.20. A four-pole smooth-air-gap machine has a two-phase set of stator windings, each with a total of N turns. The windings are distributed sinusoidally and currents in them produce surface current densities as indicated in Fig. 4P.20. When $g \ll R$, the radial flux density produced in the air gap by each winding is (see Problems 4.1 and 4.10)

$$B_{ra} = \frac{\mu_0 N i_a}{2g} \cos 2\psi,$$

$$B_{rb} = \frac{\mu_0 N i_b}{2g} \sin 2\psi.$$

- (a) For the two-phase excitation, $i_a = I_a \cos \omega t$, $i_b = I_b \sin \omega t$, which is unbalanced in amplitude, find the total radial flux density.
 (b) Express the answer to (a) as a sum of two constant-amplitude traveling waves.

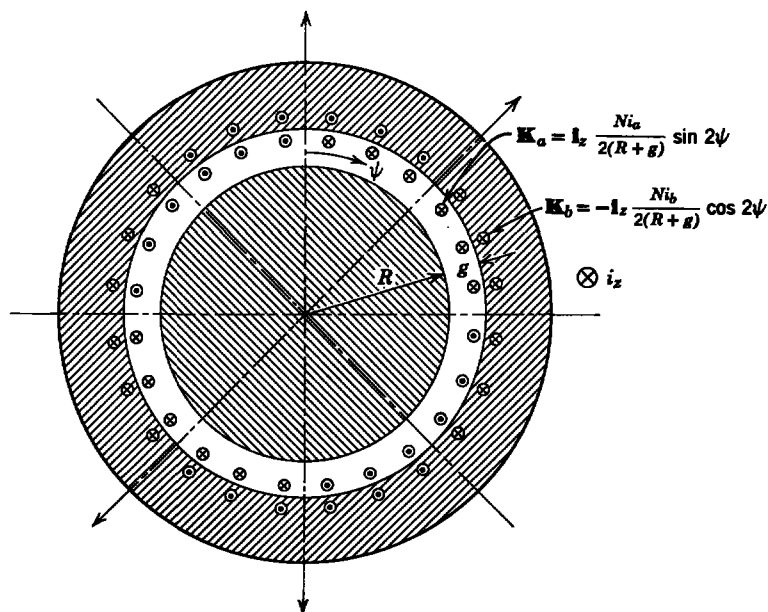


Fig. 4P.20

Identify the forward and backward components and show that their respective angular velocities are $\omega_f = \omega/2$ and $\omega_b = -\omega/2$.

- (c) Show that the amplitude of the backward wave goes to zero when $I_b = I_a$ and that the amplitude of the forward wave goes to zero when $I_b = -I_a$.

4.21. Rework Problem 4.20 for a p -pole-pair machine for which the component radial air-gap flux densities are

$$B_{ra} = \frac{\mu_0 N i_a}{2g} \cos p\psi,$$

$$B_{rb} = \frac{\mu_0 N i_b}{2g} \sin p\psi.$$

Assume the same excitation as in part (a) of Problem 4.20. In part (b) the forward and backward waves have angular velocities $\omega_f = \omega/p$ and $\omega_b = -\omega/p$.

4.22. Derive the electromagnetic torque of (4.2.9), starting with the electrical terminal relations of (4.2.7) and (4.2.8) and the assumption that the coupling system is conservative.

4.23. The salient-pole, synchronous machine of Fig. 4P.23 is electrically linear and lossless and has a terminal inductance expressed as

$$L = \frac{L_o}{(1 - 0.25 \cos 4\theta - 0.25 \cos 8\theta)},$$

where L_o is a positive constant. This is an alternative mathematical representation to the form given by (4.2.3).

- Describe briefly why the dependence of this inductance on θ is physically reasonable.
- Find the torque of electric origin T^e as a function of flux linkage λ , angle θ , and the constants of the system.
- As shown in Fig. 4P.23, the terminals are excited by a sinusoidal voltage source such that the flux λ is given by $\lambda(t) = \Lambda_o \cos \omega t$, where Λ_o and ω are positive constants. The rotor is driven by a constant-angular-velocity source such that $\theta(t) = \Omega t + \delta$, where Ω and δ are constants. Find the values of Ω , in terms of the electrical frequency ω , at which time-average power can be converted by the machine between the electrical and mechanical systems.

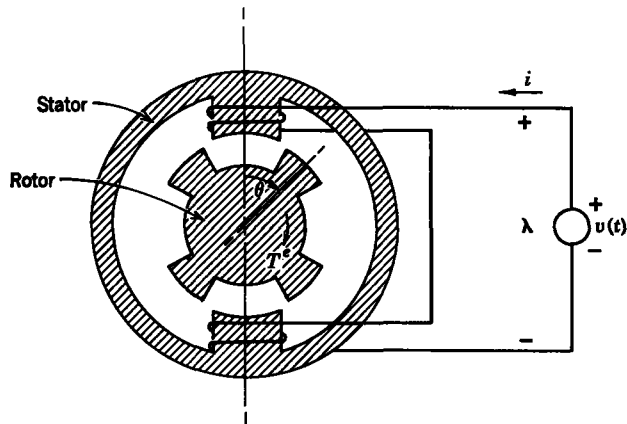


Fig. 4P.23

4.24. The two-phase equivalent of a salient-pole, synchronous motor has the following parameter values and ratings [see (4.2.28) to (4.2.30) for definitions]

2-phase	60 Hz
Rated output power,	6000 hp
Power factor,	0.8 leading
Rated armature voltage,	3000 V rms
Voltage coefficient,	$\omega M = 350 \text{ V/A}$
Direct axis reactance,	$\omega(L_0 + L_2) = 4 \text{ ohms}$
Quadrature axis reactance,	$\omega(L_0 - L_2) = 2.2 \text{ ohms}$

- Find the field current necessary to give maximum rated conditions at rated voltage. This is rated field current.
- Calculate and plot a family of V -curves for loads of 6000, 3000, and zero hp and rated voltage; V -curves are plots of armature current as a function of field current for constant load power (see Problem 4.13). Indicate the factor that limits the extent of the plot: rated armature current, rated field current, or steady-state stability (pull-out torque is approached).

4.25. As discussed at the end of Section 4.1.6a, synchronous condensers are essentially synchronous machines operating with no shaft torque. They are used for power-factor correction and they are conventionally of the salient-pole type of construction. Start with (4.2.41), assume zero shaft torque [$\gamma = 0$ from (4.2.37)] and operation at constant armature voltage amplitude, and construct vector diagrams to show the machine appearing capacitive and inductive.

4.26. This is a problem that involves the use of a synchronous condenser to correct power factor in a power system. The correction is actually achieved by using the synchronous condenser to regulate voltage. We consider one phase of a balanced two-phase system. In Fig. 4P.26a a power system feeds a steady-state load which has admittance $Y e^{-j\phi}$ as shown.

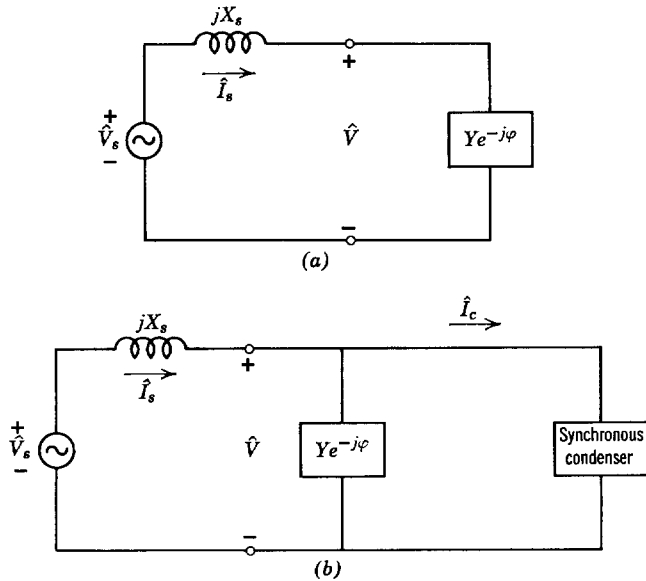


Fig. 4P.26

The Thevenin equivalent circuit of the system, as viewed from the load, is the source $\hat{V}_s$ in series with the inductive reactance jX_s . To fix ideas assume the following parameters and excitations: $V_s = \sqrt{2} 100,000$ V peak, $X_s = 10$ ohms, $Y = 0.01$ mho.

- Find the ratio of the magnitudes of the load voltage V and the source voltage V_s for $\phi = 0$ and $\phi = 45$ degrees.
- Now a synchronous condenser is connected across the load as shown in Fig. 4P.26b and draws current $\hat{I}_c$. Find the volt-ampere rating required for the synchronous condenser to make the ratio $|\hat{V}|/|\hat{V}_s|$ equal to unity for each case in part (a). Compare each with the real power drawn by the load.

4.27. A two-phase, 60-Hz, salient-pole, 2-pole, synchronous motor has the following ratings and constants:

Rated output power,	1000 hp
Rated armature volts,	$\sqrt{2} 1000$ V peak
Rated power factor,	unity
Direct axis reactance,	$\omega(L_0 + L_2) = 3.0$ ohms
Quadrature axis reactance,	$\omega(L_0 - L_2) = 2.0$ ohms
Speed voltage coefficient,	$\omega M = 150$ V/A

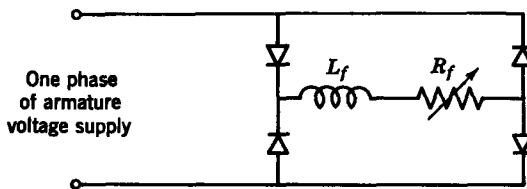


Fig. 4P.27

- The field winding of the motor is supplied from one phase of the supply by a full-wave bridge rectifier as shown in Fig. 4P.27. The field winding inductance is large enough that only the dc component of field voltage need be considered. Calculate the total field circuit resistance R_f necessary to achieve unity-power-factor operation at rated voltage with 1000 hp load.
- Calculate and plot the torque angle δ as a function of armature supply voltage from 10 per cent above rating down to the value at which the motor can no longer carry the load.

4.28. The two-phase equivalent of a large, salient-pole, 72-pole, water-wheel generator of the type now being used has the following constants and ratings:

Rating,	200×10^6 V-A
Frequency,	60 Hz
Power factor,	0.85 lagging
Rated terminal voltage,	10,000 V rms
Rated armature current,	10,000 A rms
Armature inductance,	$L_0 = 2.65 \times 10^{-3}$ H
	$L_2 = 0.53 \times 10^{-3}$ H
Maximum armature-field mutual inductance,	$M = 0.125$ H

- (a) Calculate the field current necessary to achieve rated conditions of armature voltage, current, and power factor.
- (b) Plot a capability curve for this generator. See Problem 4.14 for a description of a capability curve. In this case the stability limit of maximum steady-state torque will occur for $\delta < 90^\circ$ (see Fig. 4.2.6).

4.29. Figure 4P.29 shows a pair of grounded conductors that form the rotor of a proposed rotating device. Two pairs of fixed conductors form the stator; one pair is at the potential

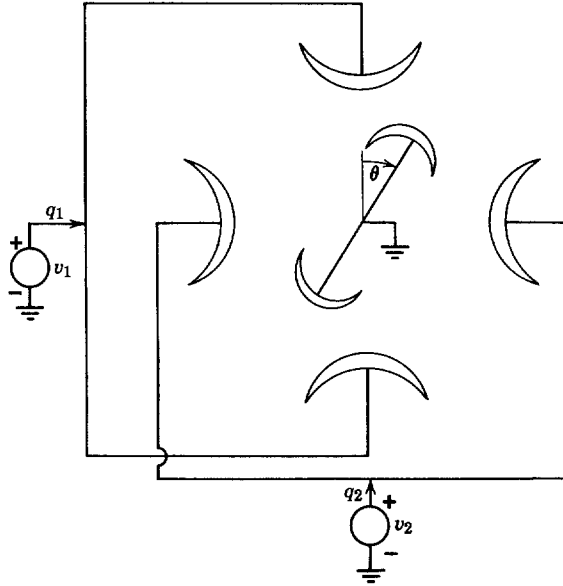


Fig. 4P.29

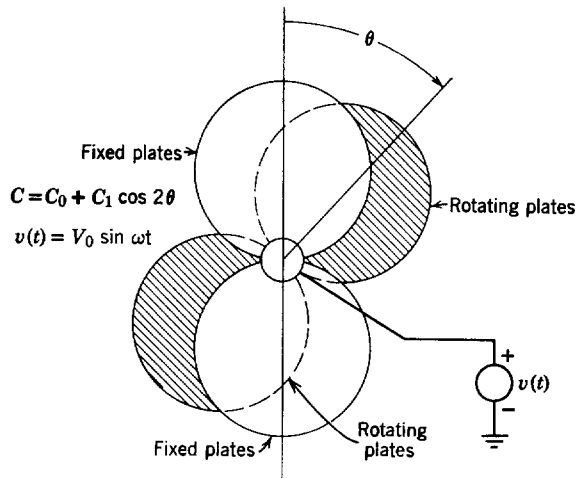


Fig. 4P.30

v_1 and supports a total charge q_1 ; the other is at the potential v_2 and supports the total charge q_2 . Given that $q_1 = C_o(1 + \cos 2\theta)v_1$, $q_2 = C_o(1 + \sin 2\theta)v_2$, where C_o is a given positive constant,

- (a) what is the electrical torque exerted on the rotor in the θ direction?
- (b) The voltages v_1 and v_2 are now constrained to be $v_1 = V_o \cos \omega t$, $v_2 = V_o \sin \omega t$. Under what condition(s) will the device produce a time-average torque?
- (c) Under the condition(s) of (b), what is the time-average torque?

4.30. A pair of capacitor plates is attached to a rotating shaft in such a way that when θ is zero they are directly opposite a pair of fixed plates. It is assumed that the variation in capacitance can be approximately described by the relation $C = C_o + C_1 \cos 2\theta$. If a potential difference $v(t) = V_o \sin \omega t$ is applied to the plates through a slip ring, what are the shaft rotational velocities at which the device can behave like a motor?

Chapter 5

LUMPED-PARAMETER ELECTROMECHANICAL DYNAMICS

5.0 INTRODUCTION

The representation of lumped-parameter electromechanical systems by means of mathematical models has been the subject of the preceding chapters. Our objective in this chapter is to study their dynamical behavior. Mathematically, we are interested in the solution of differential equations of motion for given initial conditions and with given driving sources. Physically, we are interested in important phenomena that occur in electromechanical systems.

It is clear from previous examples that the differential equations that describe electromechanical systems are in most cases nonlinear. Consequently, it is impossible to develop a concise and complete mathematical theory, as is done for linear circuit theory. We shall find many systems for which we can assume “small-signal” behavior and linearize the differential equations. In these cases we have available to us the complete mathematical analysis developed for linear systems. If exact solutions are required for nonlinear differential equations, each situation must be considered separately. Machine computation is often the only efficient way of obtaining theoretical predictions. Some simple cases however, are amenable to direct integration. The physical aspects of a given problem often motivate simplifications of the mathematical model and lead to meaningful but tractable descriptions. Hence in this chapter we are as much concerned with illustrating approximations that have been found useful as with reviewing and expanding fundamental analytical techniques.

Lumped-parameter systems are described by ordinary differential equations. The partial differential equations of continuous or distributed systems are often solved by a reduction to one or more ordinary differential equations. Hence many concepts used here will prove useful in the chapters that follow.

Similarly, the physical behavior of a distributed system is sometimes most easily understood in terms of lumped parameter concepts. Examples discussed in this chapter are in many cases motivated by the physical background that they provide for more complicated interactions to be considered later.

Because the mathematics of linear systems is comparatively simple, we begin our study of the dynamic behavior of lumped-parameter electromechanical systems by considering the several types of system for which a linear model provides an adequate description. We shall then consider the types of system that are basically nonlinear and for which the differential equations can be integrated directly.

5.1 LINEAR SYSTEMS

We have stated that electromechanical systems are not usually described by linear differential equations. Many devices, however, called incremental-motion transducers, are designed to operate *approximately* as linear systems. Moreover, meaningful descriptions of the basic properties of nonlinear systems can often be obtained by making small-signal linear analyses. In the following sections we develop and illustrate linearization techniques, linearized models, and the dynamical behavior of typical systems.

5.1.1 Linear Differential Equations

First, we should recall the definition of a linear ordinary differential equation.* An n th-order equation has the form

$$\frac{d^n x}{dt^n} + A_1(t) \frac{d^{n-1} x}{dt^{n-1}} + \cdots + A_n(t)x = f(t), \quad (5.1.1)$$

where the order is determined by the highest derivative. Note that the coefficients $A_i(t)$ can in general be functions of the *independent* variable t . If, however, the coefficients were functions of the *dependent* (unknown) variable $x(t)$, the equation would be nonlinear. The “driving function” $f(t)$ is a known function of time.

The “homogeneous” form of (5.1.1) is provided by making $f(t) = 0$. There are n independent solutions $x_n(t)$ to the homogeneous equation. The general solution to (5.1.1) is a linear combination of these homogeneous solutions, plus a particular solution $x_p(t)$ to the complete equation:

$$x(t) = c_1 x_1(t) + \cdots + c_n x_n(t) + x_p(t). \quad (5.1.2)$$

Although (5.1.1) is linear, it has coefficients that are functions of the

* A review of differential equations can be found in such texts as L. R. Ford, *Differential Equations*, 2nd Ed., McGraw-Hill, New York, 1955.

independent variable and this can cause complications; for example, if $f(t)$ is a steady-state sinusoid of a given frequency, the solution may contain all harmonics of the driving frequency. Alternatively, if $f(t)$ is an impulse, the response varies with the time at which the impulse is applied. These complications are necessary in some cases; most of our linear systems, however, are described by differential equations with constant coefficients. For now we limit ourselves to the case in which the coefficients $A_i = a_i = \text{constant}$, and (5.1.1) becomes

$$\frac{d^n x(t)}{dt^n} + a_1 \frac{d^{n-1} x(t)}{dt^{n-1}} + \cdots + a_n x(t) = f(t). \quad (5.1.3)$$

The solution to equations having this form is the central theme of circuit theory.* The solutions $x_i(t)$ to the homogeneous equation, when the coefficients are constant, are exponentials $e^{s_i t}$, where s can in general be complex; that is, if we let

$$x(t) = \sum_{i=1}^n c_i e^{s_i t} \quad (5.1.4)$$

and substitute it in the homogeneous equation, we obtain

$$(s_i^n + a_1 s_i^{n-1} + \cdots + a_n) \sum_{i=1}^n c_i e^{s_i t} = 0 \quad (5.1.5)$$

and (5.1.4) is a solution, provided that the complex frequencies satisfy the condition

$$s_i^n + a_1 s_i^{n-1} + \cdots + a_n = 0. \quad (5.1.6)$$

Here we have an n th-order polynomial in s , hence a condition that defines the n possible values of s required in (5.1.4). The frequencies s_i that satisfy (5.1.6) are called the *natural frequencies* of the system and (5.1.6) is sometimes called the *characteristic equation*.†

Many commonly used devices are driven in the sinusoidal steady state. In this case the driving function $f(t)$ has the form

$$f(t) = \text{Re} [\hat{F} e^{j\omega t}]. \quad (5.1.7)$$

Here $\hat{F}$ is in general complex and determines the phase of the driving signal; for example, if $\hat{F} = 1$, $f(t) = \cos \omega t$, but, if $\hat{F} = -j$, $f(t) = \sin \omega t$. To find

* See, for example, E. A. Guillemin, *Theory of Linear Physical Systems*, Wiley, New York, 1963 (especially Chapter 7).

† If the characteristic equation has repeated roots, the solution must be modified slightly, see, for example, M. F. Gardner and J. L. Barnes, *Transients in Linear Systems*, Wiley, New York, 1942, pp. 159–163.

the particular solution with this drive we assume

$$x_p(t) = \text{Re} [\hat{X}e^{j\omega t}] \quad (5.1.8)$$

and substitute into (5.1.3) to obtain

$$\text{Re} \{e^{j\omega t} [\hat{X}((j\omega)^n + a_1(j\omega)^{n-1} + \cdots + a_n) - \hat{F}]\} = 0. \quad (5.1.9)$$

It follows that (5.1.8) is the particular solution if

$$\hat{X} = \frac{\hat{F}}{(j\omega)^n + a_1(j\omega)^{n-1} + \cdots + a_n}. \quad (5.1.10)$$

Note that the natural frequencies (5.1.6) are the values of $j\omega$ in (5.1.10) which lead to the possibility of a finite response $\hat{X}$ when $\hat{F} = 0$; thus the term natural frequency.

The general solution is the sum of the homogeneous solution and the driven solution (5.1.4) and (5.1.8):

$$x(t) = \sum_{i=1}^n c_i e^{s_i t} + \text{Re} \left[\frac{\hat{F} e^{j\omega t}}{(j\omega)^n + a_1(j\omega)^{n-1} + \cdots + a_n} \right]. \quad (5.1.11)$$

Given n initial conditions [e.g., $x(0)$, $(dx/dt)(0)$, $\dots$, $(d^{n-1}x/dt^{n-1})(0)$], the constants c_i can be evaluated. The first term in (5.1.11) is the transient part of the solution and the second term is the driven or steady-state part. If the system is stable (i.e., if all the s_i have negative real parts), the transient term in (5.1.11) will damp out. After a long enough time the first term will become small enough to be neglected. Then the system is said to be operating in the sinusoidal steady state and the response is given by the second term alone. When we wish to calculate the sinusoidal steady-state response, we find only the particular solution.

5.1.2 Equilibrium, Linearization, and Stability

We have already stated that useful information can be obtained about many electromechanical systems by making small-signal linear analyses in the vicinity of equilibrium points. In this section we introduce the concept of equilibrium and illustrate how to obtain small-signal, linear equations. In the process we shall study the nature of the small-signal behavior and define two basic types of instability that can occur in the vicinity of an equilibrium point.

5.1.2a Static Equilibrium and Static Instability

In general, the term *equilibrium* is used in connection with a dynamical system to indicate that the motion takes on a particularly simple form; for example, a mass M , constrained to move in the x -direction and subject

to a force $f(x)$ will have a position $x(t)$ predicted by the equation [see (2.2.10) of Chapter 2]

$$M \frac{d^2x}{dt^2} = f(x). \quad (5.1.12)$$

We say that the mass is in equilibrium at any point $x = X$ such that $f(X) = 0$. Physically, we simply mean that at the point $x = X$ there is no external force to accelerate the mass, hence it is possible for the mass to retain a static position (or be in equilibrium) at this point.

The word equilibrium is used to refer not only to cases in which the dependent variables (x) take on static values that satisfy the equations of motion but also to situations, such as uniform motion, in which the general (nonlinear) equations of motion are satisfied by the equilibrium solution. Equilibria of this type were of primary interest in Chapter 4, in which the steady-state behavior of rotating magnetic field devices was considered.

Small perturbations from the equilibrium positions are predicted approximately by linearized equations of motion, which are found by assuming that the dependent variables have the form

$$x(t) = X + x'(t), \quad (5.1.13)$$

where X is the equilibrium position and $x'(t)$ is the small perturbation. It is then possible to expand nonlinear terms in a Taylor series* about the equilibrium values; for example, $f(x)$ in (5.1.12) can be expanded in the series

$$f(x) = f(X) + x' \frac{df}{dx}(X) + \frac{1}{2} x'^2 \frac{d^2f}{dx^2}(X) + \cdots \quad (5.1.14)$$

Now, if x' is small enough, it is likely that the first two terms will make the most significant contributions to the series, hence the remaining terms can be ignored. Recall that by definition $f(X) = 0$ and 5.1.12 has the form,

$$\frac{d^2x'}{dt^2} + \omega_o^2 x' = 0, \quad (5.1.15)$$

where

$$\omega_o^2 = - \frac{1}{M} \frac{df}{dx}(X).$$

The resulting equation is linear and can be solved as described in Section 5.1.1. The solution has the form

$$x' = c_1 e^{j\omega_o t} + c_2 e^{-j\omega_o t}. \quad (5.1.16)$$

* F. B. Hildebrand, *Advanced Calculus for Engineers*, Prentice-Hall, New York, 1949, p. 125.

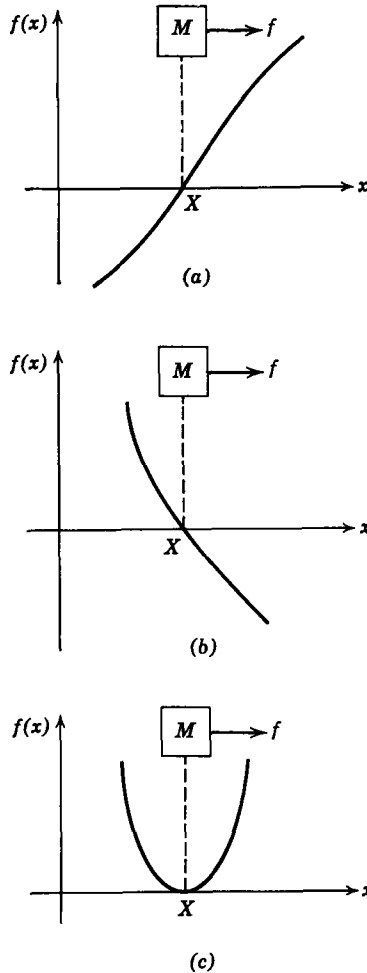


Fig. 5.1.1 Graphical representation of a force $f(x)$ which acts on the mass M with a static equilibrium at $x = X$: (a) unstable; (b) stable; (c) nonlinear.

We see that if $(df/dx)(X)$ is positive ω_o is imaginary and any small displacement of the mass (as will inevitably be supplied by noise) will lead to a motion that is unbounded. In this case we say that the equilibrium position X is unstable, and we call this type of pure exponential instability a *static* instability. We can interpret this situation physically by reference to Fig. 5.1.1a which shows a plot of $f(x)$ with a positive slope at $x = X$. If the mass moves a small distance to the right of the equilibrium point, the force f becomes positive and tends to increase the displacement still further. Thus a static experiment will reveal the presence of the instability. Although our

solution cannot be trusted for long after the start of a transient, the small-signal instability provides the essential information that the mass will not remain at the equilibrium point. If df/dx is negative at $x = X$, ω_o is real and the mass will execute a sinusoidal motion about the equilibrium point with the angular frequency ω_o , as can be seen from (5.1.15) and (5.1.16). This result is also easily understood physically. Figure 5.1.1*b* shows a plot of force f having a negative slope at the equilibrium point. If the mass moves slightly to the right, the force becomes negative and tends to return the mass to $x = X$. When the mass reaches the equilibrium point it has finite velocity and overshoots. In the absence of further external disturbances the mass will oscillate sinusoidally about the equilibrium point with constant amplitude. We call this motion stable because the response is bounded. If there were damping in the system, the amplitude of the oscillation would decay until the mass came to rest at the equilibrium point.

Once a solution has been found to the linearized equations of motion, it is possible to check the accuracy of the prediction by considering the significance of the terms that were dropped in (5.1.14), compared with the second term. An extreme case in which linearized equations would not adequately describe the motion is illustrated in Fig. 5.1.1*c*, where the slope of $f(x)$ is also zero at the equilibrium point. In this case the lowest order, nonzero term in (5.1.14) must be retained.

Example 5.1.1. To illustrate some of these ideas we consider again the magnetic field transducer shown schematically in Fig. 5.1.2. The electric terminal relations were calculated in Example 2.1.1 and the equations of motion were written in Example 3.2.1.

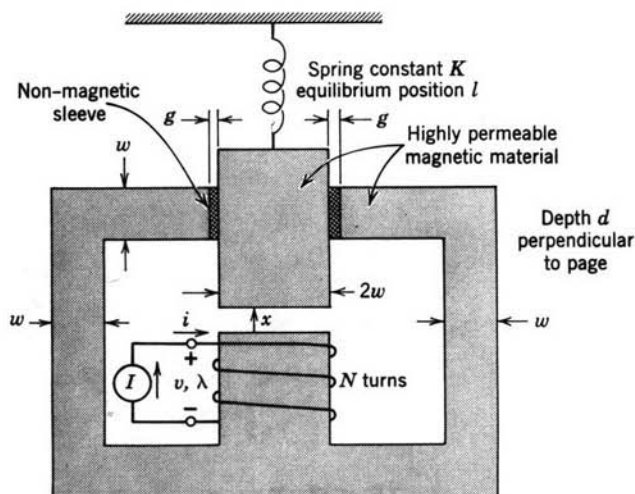


Fig. 5.1.2 Magnetic field transducer used to demonstrate linearization techniques in Example 5.1.1.

The electric terminal relation, as described by (d) and (e) of Example 2.1.1, is

$$\lambda = \frac{L_o i}{1 + x/g} \quad (a)$$

where

$$L_o = \frac{\mu_0 N^2 (2wd)}{g} \quad (b)$$

is the inductance with the air gap closed ($x = 0$). The force of electric origin is given by (d) of Example 3.2.1 and is

$$f^e = -\frac{L_o i^2}{2g(1 + x/g)^2}. \quad (c)$$

Because the electrical excitation is a current source, the equation for the electrical part of the system is not of interest. In the mechanical part we neglect damping; consequently, the equation of motion for the mechanical node (x) is [see (e) of Example 3.2.1]

$$M \frac{d^2 x}{dt^2} = -\frac{L_o I^2}{2g(1 + x/g)^2} - K(x - l). \quad (d)$$

This equation has the form of (5.1.12):

$$f(x) = -\frac{L_o I^2}{2g(1 + x/g)^2} - K(x - l) = f^e(x) + f^s(x). \quad (e)$$

In static equilibrium at $x = X$, (d) becomes

$$f(X) = -\frac{L_o I^2}{2g(1 + X/g)^2} - K(X - l) = 0. \quad (f)$$

This is a cubic equation in X which cannot be solved easily. Its properties, however, can be investigated by sketching the two terms as shown in Fig. 5.1.3. In this figure the negative of f^e has been plotted as a function of X on the same scale as a plot of f^s . Hence the intersections represent solutions to (f); that is, the points X_1 and X_2 are positions at which the

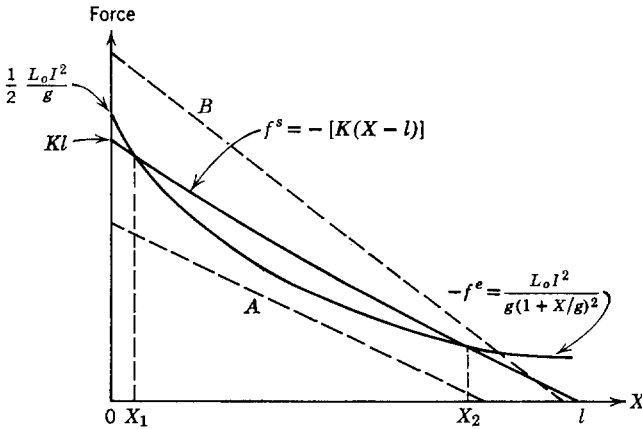


Fig. 5.1.3 Sketch for determining equilibrium points for system of Fig. 5.1.2.

plunger experiences no external forces, hence it can be in static equilibrium. There are, of course, three roots to the cubic equation, but one is not physical, since it requires that X be less than zero or that the plunger extend into the magnetic yoke. The relative values of the parameters can be such that there are no possible equilibrium points, as illustrated by the dashed curve A in Fig. 5.1.3, or there may be only one equilibrium point, as indicated by the dashed curve B . Note that all equilibrium points are such that $X < l$. Physically, this is expected, since the force f^e always tends to pull the plunger into the yoke, hence to extend the length of the spring.

We now assume that the conditions represented by the solid curves of Fig. 5.1.3 have been established and consider the dynamics for small excursions from the equilibrium points; for example, about X_1 ,

$$x(t) = X_1 + x'(t)$$

and (d) becomes (see 5.1.15)

$$\frac{d^2x'}{dt^2} + \omega_o^2 x' = 0, \quad (g)$$

where

$$\omega_o^2 = -\frac{1}{M} \frac{df}{dx}(X_1) = \frac{1}{M} \left[K - \frac{L_o I^2}{g^2(1 + X_1/g)^3} \right]. \quad (h)$$

It should be clear from (5.1.15) and (5.1.16), and the associated discussion, that the relative magnitudes of the two terms in brackets determines whether this system is stable at the operating point X_1 .

Although we could proceed in a formal mathematical way to study the stability at the equilibrium points, we shall pursue the subject with some qualitative study of the curves in Fig. 5.1.3.

At the point X_1 the magnitude of df^e/dX is larger than the magnitude of df^s/dX . Hence the derivative df/dX at X_1 has the sign of df^e/dX ; that is,

$$\frac{df}{dX}(X_1) > 0, \quad (i)$$

and we conclude that this equilibrium point is unstable. We have found mathematically that a small excursion of the mass to the right of $X = X_1$ (Fig. 5.1.3) subjects the plunger to a force dominated by the spring force, which tends to force the plunger further to the right. (Remember that f^s is defined as a force that acts in the $+x$ -direction if it is positive.)

Similarly, at X_2 the slope df^s/dX has a larger magnitude than df^e/dX , hence the sign of df/dX is negative at X_2 ,

$$\frac{df}{dX}(X_2) < 0 \quad (j)$$

and the equilibrium point X_2 is stable.

The process of using the first two terms in a Taylor series expansion to make a linear approximation has been described for the case in which there is a single dependent variable (x). It can be generalized to an arbitrary number of dependent variables. Suppose there are M variables $x_1, x_2, \dots, x_M$, in terms of which a general function is expressed as $f(x_1, x_2, \dots, x_M)$. If there is an equilibrium point $(X_1, X_2, X_3, \dots, X_M)$ about which we wish to obtain a linear approximation to f by using a Taylor series expansion, we express each variable as

$$x_k = X_k + x'_k.$$

Then we write the fixed and linearly varying parts of the Taylor series about the fixed point as

$$f(x_1, x_2, \dots, x_M) \approx f(X_1, X_2, \dots, X_M) + \sum_{k=1}^M \frac{\partial f}{\partial x_k}(X_1, X_2, \dots, X_M) x'_k. \quad (5.1.17)$$

The range of x'_k over which this approximation is valid within specified limits of error must be evaluated by using higher order terms in the Taylor series.* In Section 5.2.1 we consider the errors that result from using a linear approximation.

5.1.2b Dynamic or Steady-State Equilibrium

In the preceding section we studied small-signal operation about a static equilibrium. In many cases it is desirable to analyze devices as they are perturbed from a steady-state dynamic condition. This often occurs in rotating devices that are commonly designed to operate with constant angular velocity (as discussed in Chapter 4). Changes in external constraints (excitation or load) produce changes in the angular velocity that can be described as perturbations from the steady-state condition.

The linearization techniques that are the subject of this section are also applicable to many situations containing continuous media. The steady-state equilibrium may involve a moving medium such as a fluid, which has a constant velocity at a given point in space. The dynamics that result from perturbations from this steady flow could be described in a way similar to that developed here and illustrated in Chapter 10.

In the following example a synchronous magnetic field machine is used to illustrate the ideas involved in studying linearized motions about a dynamic equilibrium. The steady-state behavior of this type of device was studied in Section 4.1.6a and the example picks up the equations of motion developed there.

Example 5.1.2. A synchronous machine is modeled by the system of three coils shown schematically in Fig. 5.1.4. The physical arrangement of these coils may be as shown in Fig. 4.1.10.

The magnetic torque on the rotor, as a function of the stator currents i_{as} and i_{bs} , the rotor current i_r , and the rotor angle θ , is (4.1.38)

$$T^e = Mi_r(i_{bs} \cos \theta - i_{as} \sin \theta). \quad (a)$$

* For a discussion of this expansion see, for example, F. B. Hildebrand, *Advanced Calculus for Engineers*, Prentice-Hall, New York, 1949, p. 353.

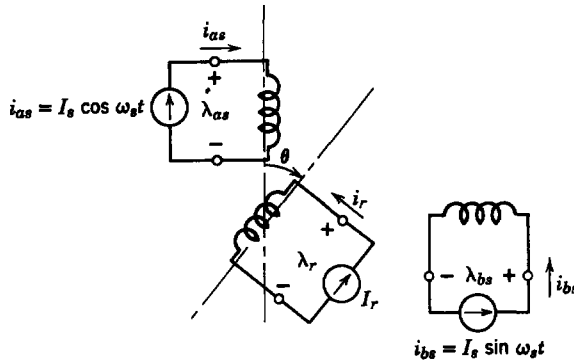


Fig. 5.1.4 Schematic representation of synchronous machine showing two fixed (stator) and one rotatable (rotor) coils.

To obtain steady-state synchronous conditions the stator and rotor terminals are excited by the current sources

$$i_r = I_r, \quad (b)$$

$$i_{as} = I_s \cos \omega_s t, \quad (c)$$

$$i_{bs} = I_s \sin \omega_s t, \quad (d)$$

where it is helpful for purposes of discussion to consider I_r , I_s , and ω_s as positive constants. In addition to the torque T^e , the shaft is subject to an inertial torque, a friction torque, and a load torque. We represent the total moment of inertia about the axis of rotation as J [see (2.2.27)], the friction torque as linear with coefficient B [see (2.2.6)], and the load torque as a driving function $T_m(t)$. Thus we can write the mechanical equation of motion for the angular deflection θ of the rotor as

$$J \frac{d^2 \theta}{dt^2} + B \frac{d\theta}{dt} + MI_r I_s (\cos \omega_s t \sin \theta - \sin \omega_s t \cos \theta) = T_m(t). \quad (e)$$

To establish a dynamic equilibrium we assume $T_m = 0$ and constant angular velocity

$$\frac{d\theta}{dt} = \Omega = \text{constant} \quad (f)$$

and write

$$\theta = \Omega t + \gamma_o, \quad (g)$$

where γ_o is a constant to be determined. We substitute (g) into (e) with $T_m = 0$ to obtain the equilibrium equation

$$B\Omega + MI_r I_s [\cos \omega_s t \sin (\Omega t + \gamma_o) - \sin \omega_s t \cos (\Omega t + \gamma_o)] = 0. \quad (h)$$

The use of a trigonometric identity to simplify the term in brackets yields

$$B\Omega = MI_r I_s \sin [(\omega_s - \Omega)t - \gamma_o]. \quad (i)$$

The left side of this equation is constant; consequently, the equation can be satisfied for all t only if

$$\Omega = \omega_s. \quad (j)$$

This is the synchronous speed at which the rotor can run in dynamic equilibrium. Using

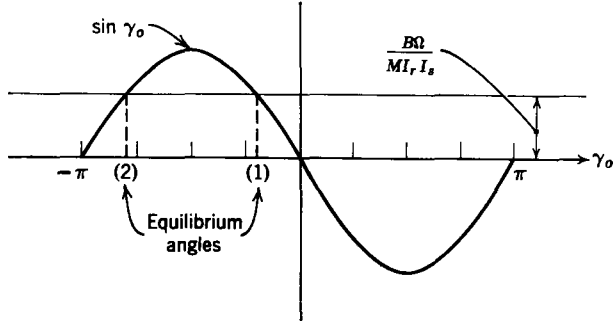


Fig. 5.1.5 The equilibrium values of angle γ_o .

(j) in (i), we obtain an expression for γ_o :

$$\sin \gamma_o = \frac{-B\Omega}{MI_t I_s}. \quad (k)$$

Dynamic equilibrium is defined for the system of Fig. 5.1.4 with the electrical excitations of (b) to (d) by (g), (j), and (k). Note that there is a limited range of parameters over which a dynamic equilibrium is possible because

$$-1 < \sin \gamma_o < 1.$$

Also, for any value of

$$\left| \frac{B\Omega}{MI_t I_s} \right| < 1$$

there are basically two different solutions for γ_o , as indicated in Fig. 5.1.5.

We now assume operation in dynamic equilibrium and describe perturbations from this equilibrium by $\gamma'(t)$; thus

$$\theta(t) = \Omega t + \gamma_o + \gamma'(t). \quad (l)$$

Substitution of this expression into (e), use of appropriate trigonometric identities, and retention of only linear terms yields after some simplification

$$J \frac{d^2 \gamma'}{dt^2} + B \frac{d\gamma'}{dt} + K\gamma' = T_m(t), \quad (m)$$

where $K = MI_t I_s \cos \gamma_o$ is the effective spring constant of the magnetic torque. In writing this equation we have subtracted out the equilibrium equation represented by (k).

To study the dynamic behavior we assume that the load torque T_m is a small step occurring at $t = 0$:

$$T_m(t) = Tu_{-1}(t), \quad (n)$$

where T is a constant and $u_{-1}(t)$ is the unit step occurring at $t = 0$.

The method of solution reviewed in Section 5.1.1 can be used here. A particular solution is

$$\gamma' = \frac{T}{K}. \quad (o)$$

The initial conditions are $\gamma'(0) = 0$ and $(d\gamma'/dt)(0) = 0$. The complete solution therefore is

$$\gamma'(t) = \frac{T}{K} \left(1 + \frac{s_2}{s_1 - s_2} e^{s_1 t} + \frac{s_1}{s_2 - s_1} e^{s_2 t} \right), \quad (p)$$

where

$$s_1 = -\frac{B}{2J} + \left[\left(\frac{B}{2J} \right)^2 - \frac{K}{J} \right]^{1/2},$$

$$s_2 = -\frac{B}{2J} - \left[\left(\frac{B}{2J} \right)^2 - \frac{K}{J} \right]^{1/2}$$

are the natural frequencies that satisfy the characteristic equation

$$s^2 + \frac{B}{J}s + \frac{K}{J} = 0. \quad (q)$$

We refer now to the two equilibrium points illustrated in Fig. 5.1.5. At equilibrium point (2) the effective spring constant K is negative because $\cos \gamma_o$ is negative. Consequently, s_1 is positive, the response is unbounded, and the machine has a static instability at equilibrium point (2).

At equilibrium point (1) in Fig. 5.1.5 $\cos \gamma_o$ is positive, K is positive, and the real parts of s_1 and s_2 are both negative. Consequently, this equilibrium point is stable. When we assume the system to be underdamped,

$$\frac{K}{J} > \left(\frac{B}{2J} \right)^2,$$

we can write (p) in the form

$$\gamma'(t) = \frac{T}{K} \left[1 - e^{-\alpha t} \left(\cos \omega t + \frac{\alpha}{\omega} \sin \omega t \right) \right], \quad (r)$$

where

$$\alpha = \frac{B}{2J},$$

$$\omega = \left[\frac{K}{J} - \left(\frac{B}{2J} \right)^2 \right]^{1/2}.$$

The response of (r) is plotted as a function of time for two values of α/ω in Fig. 5.1.6. Two

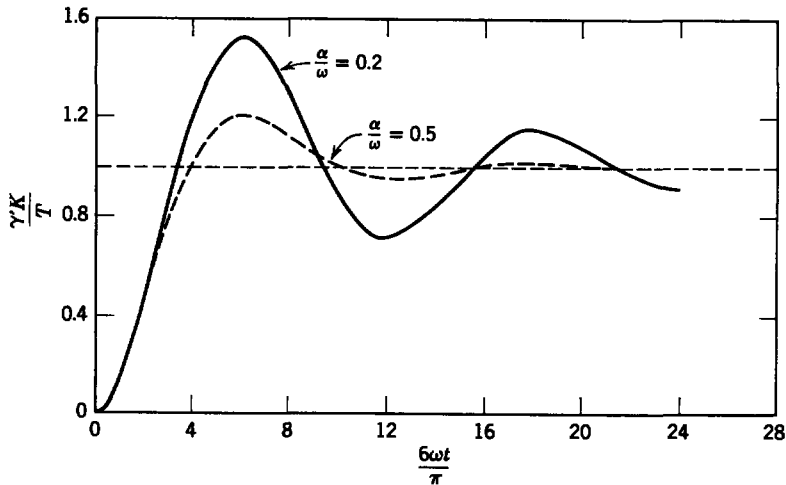


Fig. 5.1.6 Hunting transient of a synchronous machine.

general features of this “hunting” transient should be noted. First the initial parts of both transients are the same because, as (e) indicates, the initial part of the transient is dominated by the angular acceleration term $J(d^2\theta/dt^2)$, which is the same for both cases. Second, the damping retards the phase as indicated by the second term in parentheses in (x). The normalization for the time axis is different in the two cases because ω differs by a small amount.

5.1.2c Overstability or Dynamic Instability

In Section 5.1.2a we described a static instability characterized by a pure exponential growth in time. There is a second basic type of instability, called *overstability* or *dynamic instability*, which we now illustrate.

The equilibrium points of the system described by (5.1.12) are not changed if the mass is subject to an additional force proportional to velocity. The equation of motion has the form

$$M \frac{d^2x}{dt^2} + B \frac{dx}{dt} = f(x), \quad (5.1.18)$$

and in static equilibrium the additional term makes no contribution. The linearized equation, however, is

$$\frac{d^2x'}{dt^2} + \frac{B}{M} \frac{dx'}{dt} + \omega_o^2 x' = 0 \quad (5.1.19)$$

and the natural frequencies of the system are

$$s_{(1)} = \frac{-B}{2M} \pm \left(\frac{B^2}{4M^2} - \omega_o^2 \right)^{1/2}. \quad (5.1.20)$$

If the term $B(dx/dt)$ is due to viscous damping, B will be positive (see Section 2.2.1b). In this case, if $\omega_o^2 < 0$, one of the natural frequencies will be positive, and the result will be a static instability that can be detected by the static experiment described in Section 5.1.2a. On the other hand, if $\omega_o^2 > 0$, the system will be stable, regardless of the magnitude of ω_o^2 .

As we show in Example 5.1.3, feedback can be used to make $B < 0$ in (5.1.18). If $\omega_o^2 < 0$, one of the natural frequencies (5.1.20) will be real and positive and will result in a pure exponential growth that is a static instability detectable with a static experiment. Alternatively, if

$$0 < \omega_o^2 < \frac{B^2}{4M^2}, \quad B < 0, \quad (5.1.21)$$

the natural frequencies will be real and both will be positive, thus indicating an instability that cannot be detected by our static argument. If

$$0 < \frac{B^2}{4M^2} < \omega_o^2, \quad B < 0, \quad (5.1.22)$$

the radical in (5.1.20) will be imaginary and the natural frequencies will be

complex with positive real parts. In this case the transient is an exponentially growing sinusoid. This is *overstability** or *dynamic instability*.

Note that under conditions of overstability $\omega_o^2 > 0$ and our static experiment of a small displacement will result in a force that tends to return the mass toward the equilibrium point; but this force is reinforced by the negative damping and the system overshoots the equilibrium point and reaches a larger displacement in the opposite direction.

It is clear that the static argument we used in Section 5.1.2a to detect a static instability will not detect overstability and one type of exponential growth. Thus we must regard the static argument as a sufficient, but not a necessary, condition for instability.

We have discussed stability in Section 5.1.2a and in this section in terms of a system describable by a second-order differential equation. Many systems have differential equations of an order higher than 2. Whenever a system goes unstable, however, the instability is usually caused by one or two natural frequencies. Because the instability dominates the dynamical behavior, the system differential equation can sometimes be approximated by a first- or second-order differential equation for studying the instability.

It is worthwhile to establish the physical significance of the three modes of instability. For this purpose, suppose that the mass is given an initial position $x' = x_o$ with no initial velocity. Then the appropriate solution to (5.1.19) is [in terms of the roots s_1 and s_2 defined by (5.1.20)],

$$x'(t) = \frac{x_o}{s_2 - s_1} (s_2 e^{s_1 t} - s_1 e^{s_2 t}). \quad (5.1.23)$$

This solution is shown in Fig. 5.1.7 for the three cases of instability that have been illustrated. Further insight is provided by the following example.

Example 5.1.3. In many situations it is desirable to support an object with a magnetic field; for example, in a wind tunnel effects of the mechanical structure (stinger) supporting the model under study introduce errors in drag and lift measurements. One solution to this problem is to use a magnetic field.† Then, if the fluid is an ordinary nonconducting gas, the magnetic field will not interfere with the flow. To support a large mass it is desirable to use a ferromagnetic core in the model so that magnetic forces will be of a useful magnitude. It is familiar to anyone who has held a piece of magnetic material near a magnet, that any static equilibrium achieved with these forces is unstable. The example undertaken here shows how feedback can be used to stabilize an inherently unstable equilibrium. Without feedback the equilibrium exhibits a static instability. This is obviated by the introduction of feedback, but then dynamic instability comes into play. This second type of instability is removed by additional feedback.

* For additional discussion of the terminology we use for describing instabilities see S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Oxford University Press, London, 1961, pp. 1-3.

† J. E. Chrisinger et al., "Magnetic Suspension and Balance System for Wind Tunnel Application," *J. Roy. Aeron. Soc.*, 67, 717-724 (1963).

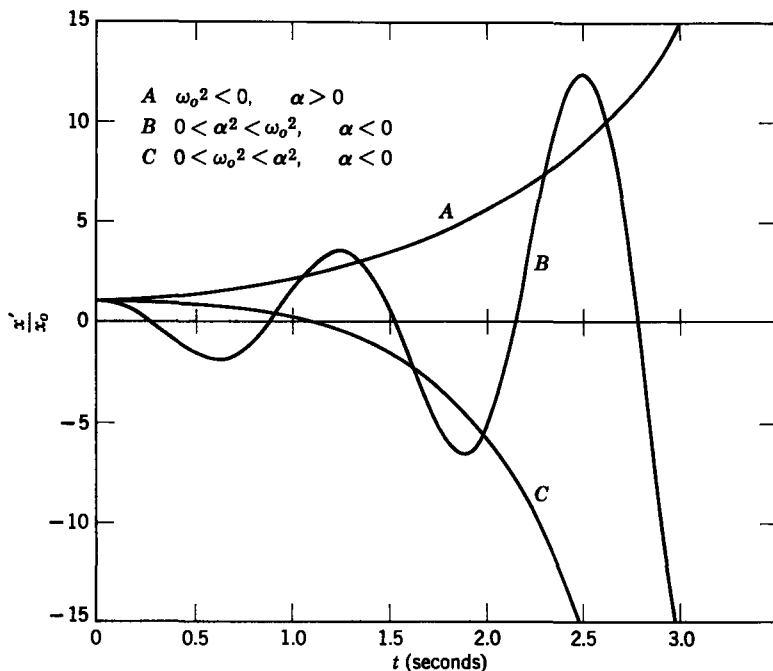


Fig. 5.1.7 Three modes of instability for a second-order system. The normalized damping α is negative for cases *B* and *C* ($\alpha = B/2M$).

The simple example to be considered here is shown schematically in Fig. 5.1.8*a* and an operating system of this type is illustrated in Fig. 5.1.8*b*. This system has the basic ingredients of systems constructed to levitate a mass M to be used in gyroscopes and accelerometers. The spherical particle is magnetic and is therefore attracted upward by the magnetic field induced by I . Hence there is a position $x = d$ at which the mass is supported against gravity by the magnetic field. With no feedback ($i' = 0$) the equilibrium is unstable, for, as the ferromagnetic mass approaches the inductor, the upward force increases, whereas the gravitational force remains constant. Feedback is introduced by using the optical system to detect the position of the mass. The photomultiplier and amplifier are adjusted to give a current i' as nearly as possible proportional to the deflection x' from the equilibrium position $x = d$. Hence with feedback there is an addition to the magnetic force proportional to the deflection x' . By adjustment of the loop gain it is possible to make the effective spring constant introduced by the feedback large enough so that the equilibrium will appear to be stable on the basis of a static experiment of displacing the sphere from equilibrium and finding a restoring force. The amplifier, however, is not an ideal current source, and the effect of the coil inductance with finite amplifier output impedance causes the equilibrium to be dynamically unstable. A feedback signal proportional to sphere velocity is then added to stabilize the equilibrium completely. This discussion characterizes the design process used with the system shown in Fig. 5.1.8*b*.*

* The analysis was used in a student laboratory project to achieve the stable suspension of the sphere shown.

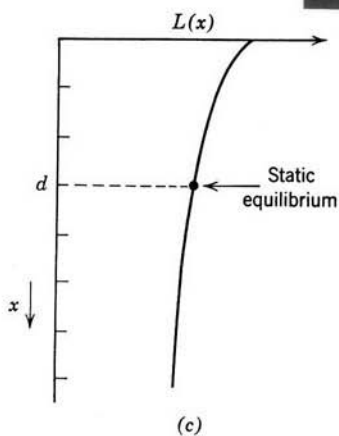
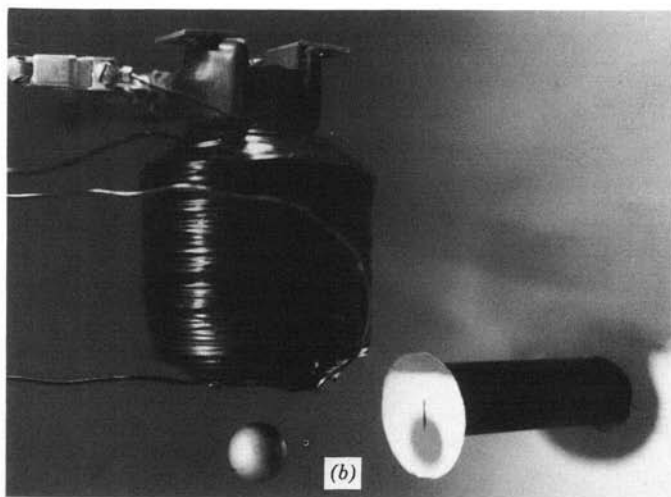
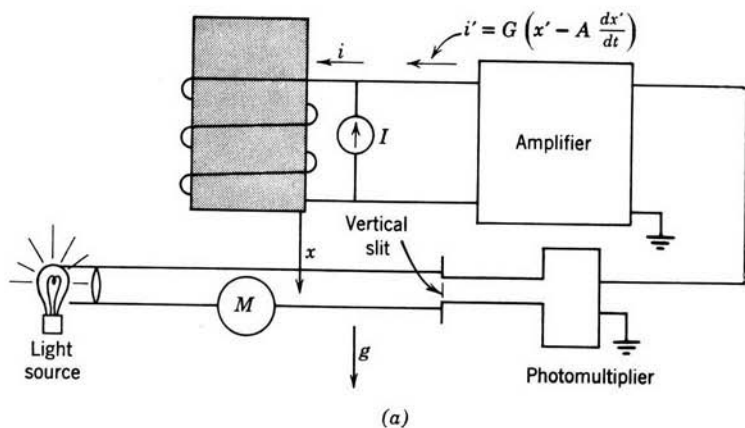


Fig. 5.1.8 (a) Mass M levitated by a magnetic field. The optical system provides a signal that is fed back to stabilize the static equilibrium of the mass at $x = d$; (b) view of laboratory project shown schematically by (a); (c) dependence of coil inductance on the mass displacement.

Measurement of the inductance L as a function of the position of the mass M would produce a curve like that shown in Fig. 5.1.8c. The inductance has its largest value when the ferromagnetic sphere is next to the coil and decreases to a constant as the sphere is removed to $x = \infty$. For the present purposes we take this dependence as

$$L(x) = L_1 + \frac{L_o}{1 + x/a}, \quad (a)$$

where L_1 , L_o , and a are positive constants. Then the methods of Chapter 3 provide the force on the mass M ; that is, the coenergy (since the electrical terminal relation is linear) is

$$W'(i, x) = \frac{1}{2} \left(L_1 + \frac{L_o}{1 + x/a} \right) i^2 \quad (b)$$

and the force of electrical origin follows as

$$f = \frac{\partial W'}{\partial x} = -\frac{1}{2a} \frac{L_o}{(1 + x/a)^2} i^2. \quad (c)$$

When the sphere is in static equilibrium, the gravitational force is balanced by this force:

$$Mg = \frac{1}{2a} \frac{L_o}{(1 + d/a)^2} I^2. \quad (d)$$

Given the current I , the equilibrium position d is determined. Perturbations x' from the equilibrium lead to a perturbation current i' ; that is,

$$\begin{aligned} x &= d + x', \\ i &= I + i'. \end{aligned} \quad (e)$$

To linear terms in the perturbation quantities the force of (c) becomes

$$f = -\frac{L_o}{2a} \left[\frac{I^2}{(1 + d/a)^2} - \frac{2x'I^2}{a(1 + d/a)^3} + \frac{2Ii'}{(1 + d/a)^2} \right]. \quad (f)$$

It follows that the incremental equation of motion is

$$M \frac{d^2 x'}{dt^2} - \frac{L_o I^2}{a^2 (1 + d/a)^3} x' + \frac{L_o I i'}{a(1 + d/a)^2} = 0, \quad (g)$$

where use has been made of (d) to cancel out the constant part of the force equation.

In the absence of feedback ($i' = 0$) it is clear from (g) that the equilibrium is statically unstable. To consider first the effect of ideal feedback, assume that the output voltage of the photomultiplier is linear with x' and that the amplifier is a perfect current source feeding the coil. In this case

$$i' = Gx', \quad (h)$$

where G is a constant, including the amplifier gain. Substitution of this expression into (g) yields

$$M \frac{d^2 x'}{dt^2} + \left[\frac{GL_o I}{a(1 + d/a)^2} - \frac{L_o I^2}{a^2 (1 + d/a)^3} \right] x' = 0. \quad (i)$$

This system has natural frequencies determined from [see (5.1.15)]

$$\omega_o^2 = \frac{L_o I}{aM(1 + d/a)^2} \left(G - \frac{I}{a + d} \right). \quad (j)$$

Thus the response is bounded and the system is stable when the feedback gain is raised high enough to make

$$G > \frac{I}{a + d}. \quad (k)$$

No amplifier is absolutely ideal; consequently, we consider next the system performance in which the amplifier has a large but finite output impedance. In this case the amplifier produces a voltage proportional to sphere displacement x' and this voltage is applied to the series RL circuit illustrated in Fig. 5.1.9 in which the resistance R includes the internal resistance of the amplifier. The relation between x' and i' is now determined from the equation

$$Gx' = \frac{1}{R} \frac{d(Li')}{dt} + i'. \quad (l)$$

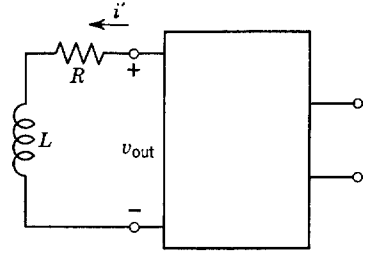


Fig. 5.1.9 The driving current of Fig. 5.1.8a is produced by a voltage amplifier connected in series with a large resistance R . The output voltage is proportional to the displacement x' . The inductance L is the equilibrium inductance of the coil shown in Fig. 5.1.8a.

Expansion of the derivative and retention of only linear terms yields

$$Gx' = \frac{1}{R} \left(L_1 + \frac{L_o}{1 + d/a} \right) \frac{di'}{dt} - \frac{L_o I}{Ra(1 + d/a)^2} \frac{dx'}{dt} + i'. \quad (m)$$

Using the exponential forms

$$x' = \hat{x}e^{st} \quad \text{and} \quad i' = \hat{i}e^{st}, \quad (n)$$

we rewrite (m) in the form

$$\hat{i} = G\hat{x} \left[\frac{1 + \frac{L_o I}{GRa(1 + d/a)^2} s}{1 + \frac{1}{R} \left(L_1 + \frac{L_o}{1 + d/a} \right) s} \right]. \quad (o)$$

For relatively high gain G and relatively high amplifier output impedance we assume

$$\frac{L_o I}{GRa(1 + d/a)^2} s \ll 1,$$

$$\frac{1}{R} \left(L_1 + \frac{L_o}{1 + d/a} \right) s \ll 1,$$

and approximate (o) by the form

$$\hat{i} = G\hat{x}(1 - As), \quad (p)$$

where

$$A = \frac{1}{R} \left[\left(L_1 + \frac{L_o}{1 + d/a} \right) - \frac{L_o I}{Ga(1 + d/a)^2} \right]. \quad (q)$$

It is evident that when the inequality k is satisfied $A > 0$.

We rewrite (p) in differential form as

$$i' = Gx' - GA \frac{dx'}{dt} \quad (r)$$

and substitute this result in (g) to obtain

$$M \frac{d^2 x'}{dt^2} - \frac{GAL_o I}{a(1 + d/a)^2} \frac{dx'}{dt} + \frac{L_o I}{a(1 + d/a)^2} \left(G - \frac{I}{a + d} \right) x' = 0. \quad (s)$$

This has the form of (5.1.19) with $B < 0$; thus the system is overstable or dynamically unstable when the nonideal nature of the amplifier is included. The response of this system to a small disturbance will be like curve *B* or curve *C* of Fig. 5.1.7, depending on the relative sizes of the coefficients in (s) [see (5.1.19) to (5.1.22)].

We can interpret the three curves of Fig. 5.1.7 physically with reference to this example. For curve *A* the electrical force due to the bias current *I* exceeds the restoring force induced by the feedback. Thus, when the mass is released from rest with a small initial displacement, the displacement increases exponentially. This occurs regardless of whether the damping is negative or positive, the only effect of the damping being to change the *rate* of exponential growth.

Curve *B* of Fig. 5.1.7 represents the situation in which the feedback force dominates the force due to bias current *I* to provide a restoring force but the derivative term due to electrical feedback is negative. Also, the relative parameter values are such that the system is oscillatory. When the mass is released from rest with an initial displacement, the feedback force immediately accelerates the mass back toward equilibrium. The negative damping force adds to this feedback force to cause the position to overshoot equilibrium by more than the magnitude of the initial displacement. The process repeats periodically as the amplitude of the oscillation grows exponentially.

Curve *C* of Fig. 5.1.7 represents the situation in which the feedback force dominates the electrical force due to the bias current *I* to provide a static restoring force. The negative damping due to the electrical feedback is large enough to make the system's natural frequencies *s* real. When the mass is released from rest with an initial displacement, the feedback force accelerates the mass back toward equilibrium. As the mass starts moving, however, the negative damping adds a force to accelerate the mass further toward equilibrium. As the mass passes through equilibrium, the negative damping force dominates to accelerate the mass further along a rising exponential in a direction opposite that of the initial displacement.

To stabilize the equilibrium with a nonideal amplifier it is necessary to modify the amplifier signal so that its output current contains a component proportional to (dx'/dt) with the proper sign. This process is called compensation.* The simplest method of compensation is achieved by using the *RC* circuit of Fig. 5.1.10 between the photomultiplier output and the amplifier input. Implicit in what follows is the assumption that the internal impedance of the photomultiplier is very low and the amplifier input impedance is very high.

In terms of complex exponentials

$$v_2 = \hat{v}_2 e^{st} \quad \text{and} \quad v_1 = \hat{v}_1 e^{st},$$

* Compensating networks and their use in automatic control systems are discussed in such texts as J. J. D'Azzo and C. H. Houpis, "Feedback Control System Analysis and Synthesis." McGraw-Hill, New York, 2nd ed., 1966 p. 158.

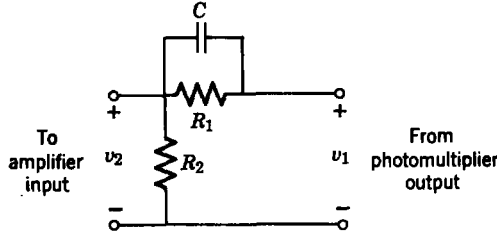


Fig. 5.1.10 Compensating network to provide stabilization of dynamic instability caused by finite amplifier output impedance.

the transfer function of the compensating network is

$$\frac{\hat{v}_2}{\hat{v}_1} = \frac{R_2}{R_1 + R_2} \left[\frac{R_1 Cs + 1}{[R_1 R_2 / (R_1 + R_2)] Cs + 1} \right]. \quad (t)$$

For proper operation of the network it is conventional to set $R_2 \ll R_1$; thus

$$\frac{R_1 R_2}{R_1 + R_2} Cs \ll R_1 Cs$$

and for low frequencies we can approximate (t) by the form

$$\hat{v}_2 = \frac{R_2}{R_1 + R_2} (R_1 Cs + 1) \hat{v}_1$$

or in differential form as

$$v_2 = \frac{R_2}{R_1 + R_2} \left(R_1 C \frac{dv_1}{dt} + v_1 \right) \quad (u)$$

Because v_1 is the output voltage of the photomultiplier and is therefore proportional to x' , we now rewrite (m) as

$$G_1 x' + G_1 R_1 C \frac{dx'}{dt} = \frac{1}{R} \left(L_1 + \frac{L_o}{1 + d/a} \right) \frac{di'}{dt} - \frac{L_o I}{Ra(1 + d/a)^2} \frac{dx'}{dt} + i', \quad (v)$$

where

$$G_1 = \frac{R_2}{R_1 + R_2} G.$$

Using the exponential forms of (n), we solve for i to obtain

$$i = G_1 \hat{x} \left[\frac{1 + R_1 Cs + \frac{L_o I}{G_1 Ra(1 + d/a)^2} s}{1 + \frac{1}{R} \left(L_1 + \frac{L_o}{1 + d/a} \right) s} \right] \quad (w)$$

Making the assumption that parameter values and frequency s are such that we need retain

only first-order terms in s , (w) becomes

$$t = G_1 \dot{x} [1 + (R_1 C - A)s], \quad (x)$$

where A is defined in (q) and $A > 0$ when inequality k is satisfied. Writing (x) in differential form, we have

$$i' = G_1 x' + G_1 (R_1 C - A) \frac{dx'}{dt}. \quad (y)$$

Substitution of this expression into (g) yields

$$M \frac{d^2 x'}{dt^2} + \frac{L_o I G_1 (R_1 C - A)}{a(1 + d/a)^2} \frac{dx'}{dt} + \frac{L_o I}{a(1 + d/a)^2} \left(G_1 - \frac{I}{a + d} \right) x' = 0. \quad (z)$$

It is clear from this equation that the compensation circuit has added positive damping to the system and that the system is completely stable when

$$G_1 > \frac{I}{a + d} \quad \text{and} \quad R_1 C > A.$$

Note that because $G_1 = [R_2 / (R_1 + R_2)]G$ and $R_2 \ll R_1$ for proper compensation the amplifier gain must be greater when compensation is used. This is a principal consequence of compensation—that amplifier gain can be traded for a change in dynamic system behavior.

5.1.2d Steady-State Sinusoidal Response

Many incremental-motion transducers, such as speakers, microphones, and electromechanical filters, are designed to operate approximately as linear systems. One of the most important design factors is the driven response. A transducer may be used to convert a mechanical signal (pressure for example) to an electrical form, in which case the electrical signal is the response to a driving force. It may also convert the output of an amplifier to an acoustic signal, and thus the pressure or velocity response to a driving voltage is of interest. The most commonly used and convenient driven response for a linear system with constant coefficients is the sinusoidal steady state. Many systems operate largely in a sinusoidal steady-state condition, but even for those that do not the techniques of Fourier transforms and Fourier series are available. The response to an arbitrary signal can be synthesized from the response to sinusoidal driving signals ranging over the frequency spectrum of interest.*

In this section we show, by means of an example, how the techniques of Section 5.1.1 can be used to find the steady-state response to a sinusoidal excitation. The example also serves to illustrate characteristic dynamical behavior and impedance levels in an electric field system.

* See, for example, S. J. Mason and H. J. Zimmermann, *Electronic Circuits, Signals, and Systems*, Wiley, New York, 1960, Chapter 7.

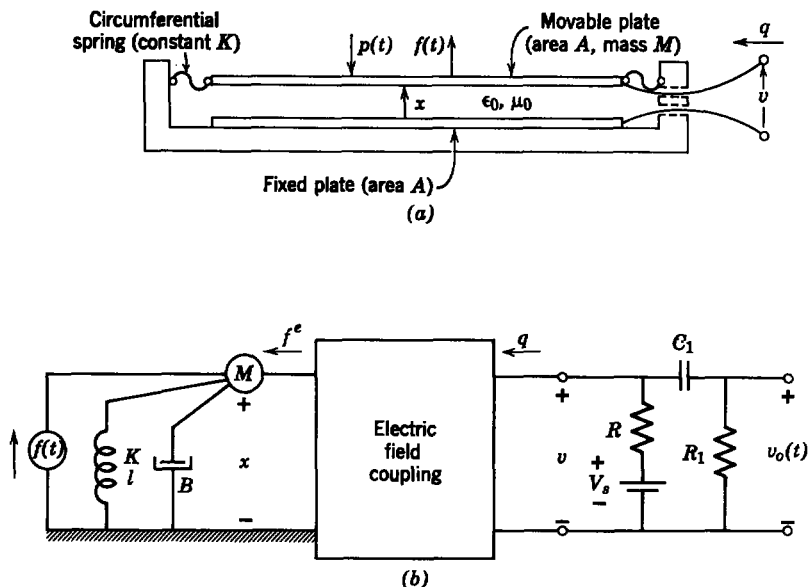


Fig. 5.1.11 Electric field transducer that can be used as a pressure sensor (microphone) or pressure source (speaker): (a) cross section of device (side view); it would appear circular from above; (b) equivalent electromechanical circuit.

Example 5.1.4. The simple variable capacitor shown schematically in Fig. 5.1.11 illustrates the basic construction of an electric field transducer that can be used as a microphone or as a speaker.* A circular movable plate with a total mass M is mounted on a peripheral bellows spring which has an equivalent constant K . The plate moves against a damping force (primarily caused by the surrounding air), which we assume is linear, with a coefficient B . When the device operates as a microphone, the differential pressure $p(t)$ acting over the area of the moving plate exerts a force $f(t)$ (defined as positive in the positive x -direction) which moves the plate and changes the capacitance, as seen from the electrical terminals. When the electrical terminals are biased with a constant voltage V_s (see Fig. 5.1.11b), a change in capacitance due to the motion induces a current dq/dt in the circuit. This current, at least in part, flows through resistance R_1 and produces a voltage $v_o(t)$ at the output terminals.

In this example we wish to analyze the behavior of the device of Fig. 5.1.11 for steady-state sinusoidal operation as a microphone. We assume that the driving force (pressure) is a sinusoidal function of time with constant amplitude and ask for the steady-state response of the output voltage. In the process we make approximations appropriate for the analysis of devices of this type.

Because the general properties of the coupling network should be described without taking into account the external elements, we begin with the electric field coupling network. We neglect fringing fields at the edges of the plates and describe the capacitance of this

* L. L. Beranek, *Acoustic Measurements*, Wiley, New York, 1949, pp. 173–176 and pp. 211–224.

electrically linear system as

$$C(x) = \frac{\epsilon_0 A}{x}. \quad (a)$$

Thus the electrical terminal relation for the coupling network is

$$q(v, x) = C(x)v = \frac{\epsilon_0 A v}{x}. \quad (b)$$

The system is conservative, hence we obtain the force f^e as

$$f^e = \frac{\partial W'_e(v, x)}{\partial x}. \quad (c)$$

Because the system is electrically linear, we express the electric coenergy as

$$W'_e = \frac{1}{2} C(x) v^2 = \frac{\epsilon_0 A v^2}{2x}. \quad (d)$$

We use this expression with (c) to evaluate the force

$$f^e = -\frac{\epsilon_0 A v^2}{2x^2}. \quad (e)$$

The equation of motion for the mechanical node (x) is written by referring to the circuit of Fig. 5.1.11b:

$$M \frac{d^2 x}{dt^2} + B \frac{dx}{dt} + K(x - l) = f^e + f(t). \quad (f)$$

We use (e) in this expression to rewrite (f) as

$$M \frac{d^2 x}{dt^2} + B \frac{dx}{dt} + K(x - l) + \frac{\epsilon_0 A v^2}{2x^2} = f(t). \quad (g)$$

Before equations for the electric circuit shown in Fig. 5.1.11b are written, we shall make some appropriate approximations to simplify the problem. The circuit comprised of capacitance C_1 and resistance R_1 is used to isolate the output terminals from the bias voltage V_s and is not intended to affect the dynamic behavior of the system in normal operation. Consequently, with a driving frequency ω we assume that the values of R_1 and C_1 satisfy the inequalities

$$\frac{1}{\omega C_1} \ll R_1 \quad \text{and} \quad R_1 \gg R \quad (h)$$

over the frequency range of interest. Thus the output voltage $v_o(t)$ is essentially the time-varying component of v and the current through R_1 can be neglected compared with the current through R .

Using the inequality (h) we now write the node equation for the electric circuit as

$$\frac{dq}{dt} = \frac{V_s - v}{R} = \frac{d}{dt} \left(\frac{\epsilon_0 A v}{x} \right) \quad (i)$$

and describe the output voltage as

$$-v_o(t) = R \frac{dq}{dt} = R \frac{d}{dt} \left(\frac{\epsilon_0 A v}{x} \right); \quad (j)$$

(g), (i), and (j) are the general equations from which we can calculate the output voltage $v_o(t)$ once the driving force $f(t)$ is specified. Note that these equations are nonlinear.

This capacitor microphone is representative of a class of devices constructed and operated purposely to behave as linear devices. Nonlinear effects cause distortion and loss of fidelity. Linearization techniques are especially meaningful because they are appropriate under conditions that must be fulfilled in construction and operation to achieve linearity.

We now use the technique presented in Section 5.1.2 to linearize (g), (i), and (j) for small-signal operation about a static equilibrium. We define the static equilibrium by requiring that all time derivatives and $f(t)$ be zero. Thus from (j) the equilibrium value for $v_o(t)$ is zero. Defining the equilibrium values of x and v as X and V , respectively, we find the relations from (g) and (i):

$$K(X - l) + \frac{\epsilon_0 A V^2}{2X^2} = 0, \quad (k)$$

$$V_s - V = 0. \quad (l)$$

It is clear that the equilibrium value of the terminal voltage is the bias voltage V_s and that the equilibrium position X is determined from the cubic equation. This equation and the properties of the equilibria are similar to those studied in Example 5.1.1. For our purposes here it suffices to state that we select the solution of (k) that represents a stable equilibrium position X .

We now assume the two variables x and v to be perturbed from their equilibrium values by small time-dependent functions $x'(t)$ and $v'(t)$. Thus

$$x(t) = X + x'(t),$$

$$v(t) = V_s + v'(t).$$

We substitute these variables into (g), (i), and (j), subtract out equilibrium terms, and retain only linear terms in the perturbation variables to obtain

$$M \frac{d^2 x'}{dt^2} + B \frac{dx'}{dt} + K_o x' + C_o E_o v' = f(t), \quad (m)$$

$$C_o \frac{dv'}{dt} - C_o E_o \frac{dx'}{dt} = -\frac{v'}{R}, \quad (n)$$

$$v_o(t) = v'(t), \quad (o)$$

where we have defined the following constants:

$$C_o = \frac{\epsilon_0 A}{X} \quad \text{is the capacitance at equilibrium,}$$

$$E_o = \frac{V_s}{X} \quad \text{is the magnitude of the electric field intensity between the plates at equilibrium,}$$

$$K_o = K - C_o E_o^2 \quad \text{is the effective (net) spring constant and is positive for the stable equilibrium we are using.}$$

Our interest here is in the steady-state response of the system to a sinusoidal driving force. Thus (see Section 5.1.1 and Fig. 5.1.11) we assume that

$$f(t) = -A p(t) = \operatorname{Re} (F e^{j\omega t}) = F \cos \omega t, \quad (p)$$

where F and ω are positive real constants. The system equations are linear with constant coefficients; thus we assume solutions of the form

$$x'(t) = \text{Re} (\hat{X}e^{j\omega t}),$$

$$v_o(t) = v'(t) = \text{Re} (\hat{V}e^{j\omega t}),$$

where $\hat{X}$ and $\hat{V}$ are complex amplitudes. We substitute these forms of the dependent variables into (m) and (n), cancel the $e^{j\omega t}$, and drop the Re to obtain the algebraic equations.

$$[(j\omega)^2 M + j\omega B + K_o]\hat{X} + C_o E_o \hat{V} = F, \quad (q)$$

$$\left(j\omega C_o + \frac{1}{R}\right)\hat{V} = j\omega C_o E_o \hat{X}. \quad (r)$$

We solve these two equations to find the complex amplitude of the output voltage $\hat{V}$ as a function of the amplitude of the driving force F :

$$\hat{V} = \left[\frac{j\omega C_o R E_o}{(K_o - \omega^2 M + j\omega B)(j\omega C_o R + 1) + j\omega (C_o E_o)^2 R} \right] F. \quad (s)$$

This expression could be used to determine the time response $v_o(t)$ for any set of parameters, value of frequency, and amplitude of drive. It is customary, however, to describe the steady-state sinusoidal response by plotting the magnitude (and often the phase) of the complex amplitude as a function of frequency for a constant input amplitude. Such a plot could be made for (s), but the denominator is quite complex and in its general form obscures the fact that different physical phenomena predominate in different frequency ranges.

We plot the amplitude of the transfer function $\hat{V}/F$ as a function of frequency by making approximations to simplify the expression in three frequency ranges. Our approximations are those made to achieve good microphone design.*

We consider first the low frequency behavior of the microphone and set the limit of (s) as $\omega \rightarrow 0$ to obtain

$$\left(\frac{\hat{V}}{F}\right)_{\text{lf}} = \frac{j\omega C_o R E_o}{K_o}. \quad (t)$$

The operation at very low frequencies can be interpreted as follows: first, the velocity and acceleration are so small that the inertia and damping forces are much smaller than the spring force in (g) and can be neglected. Next, the perturbation voltage $\hat{V}$ has negligible effect in the force equation and the term $C_o E_o \hat{V}$ can be neglected in (q). These two assumptions lead to the result in (t) if we recognize that the first term is small compared to the second on the left hand side of (r).

The fact that we can neglect the $\hat{V}$ term in (q) indicates that from a mechanical viewpoint the microphone capacitance is constrained to constant voltage. Thus the spring constant K_o includes the electrical spring constant due to the constant (bias) voltage [see (m)].

In summary, in the low-frequency limit the microphone capacitance operates at constant voltage and the mechanical system behaves as a spring, which includes electrical "spring" effects.

The low-frequency approximation breaks down when the frequency becomes large enough that

$$\omega C_o R \approx 1.$$

* *Ibid.*, pp. 211–218 and especially Fig. 5.30a.

This occurs in conventional microphones around a frequency of 10 Hz.* In the region of transition between the low-frequency and the mid-frequency ranges the transfer function is

$$\frac{\hat{V}}{\hat{F}} = \frac{j\omega C_o R E_o}{K_o(j\omega C_o R + 1) + j\omega(C_o E_o)^2 R}. \quad (u)$$

We next define the mid-frequency range as starting at a frequency such that

$$\omega \gg \frac{1}{RC_o}$$

and continuing until inertia and damping forces become appreciable. This results in a mid-frequency transfer function

$$\left(\frac{\hat{V}}{\hat{F}}\right)_{\text{mf}} = \frac{E_o}{K}. \quad (v)$$

Note that the denominator is the spring constant K alone, without the electrical spring force [see (m)]. This is an indication that the microphone capacitance is operating at constant charge because a parallel plate capacitor with negligible fringing fields and constant charge will have a force that is independent of plate spacing. The constraint of constant charge results because the resistance R is so large ($R \gg 1/\omega C_o$) that appreciable charge cannot flow on or off the plates. We can obtain this mid-frequency solution by neglecting the $1/R$ term in (r) and using that result to eliminate the term involving $\hat{V}$ in (q).

The mid-frequency range is the normal operating range of the microphone. The amplitude and phase of the transfer function are constant over this range; thus the output voltage is an exact replica of the input force and high fidelity is obtained.

The approximate transfer function for the mid-frequency range (v) breaks down when inertia and damping forces become appreciable. In practice, the mechanical system is lightly damped and a resonance occurs. In fact, elaborate means are used to provide additional mechanical damping to reduce the size of the resonance peak.† With a resonance, the mid-frequency transfer function breaks down when the frequency becomes high enough so that

$$\omega^2 M \approx K.$$

The frequency corresponding to this transition is usually around 10,000 Hz.‡

In the transition between the mid-frequency and high-frequency ranges the transfer function is

$$\frac{\hat{V}}{\hat{F}} = \frac{E_o}{K - \omega^2 M + j\omega B}. \quad (w)$$

Note that in this region the microphone capacitance is still operating at constant charge, as evidenced by the presence of only the mechanical spring constant K in the denominator.

The high-frequency range is defined by the condition

$$\omega^2 M \gg K \quad \text{and} \quad \omega^2 M \gg \omega B.$$

Thus the high-frequency transfer function is

$$\left(\frac{\hat{V}}{\hat{F}}\right)_{\text{hf}} = -\frac{E_o}{\omega^2 M}. \quad (x)$$

* *Ibid.*

† *Ibid.*, pp. 217 and 220.

‡ *Ibid.*, p. 220.

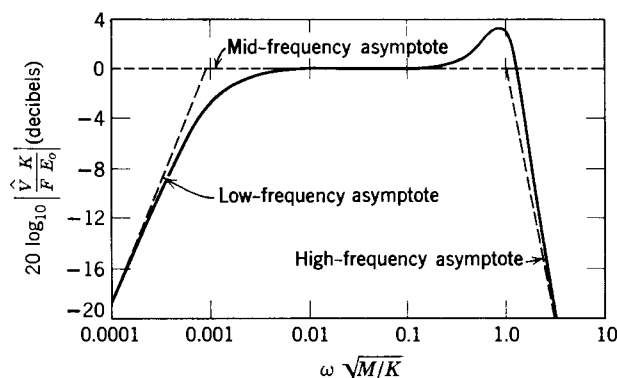


Fig. 5.1.12 Frequency response of capacitor microphone.

In this case the microphone capacitance is still operating at constant charge and the mechanical dynamics are determined completely by the mass M .

The amplitude of the transfer function $\dot{V}/F$ is plotted as a function of frequency in Fig. 5.1.12. The approximate solutions, commonly called asymptotes, given by (t), (v), and (x) are shown as dashed lines. The more exact solutions in the transition regions (u) and (w) are shown as solid lines. The curves are plotted for the following relations among parameters:

$$\frac{B}{\sqrt{MK}} = 0.7; \quad \frac{K_o}{K} = 0.9; \quad C_o R \left(\frac{K}{M} \right)^{1/2} = 1000.$$

Note that in Fig. 5.1.12 both the amplitude and frequency scales are logarithmic. This is a Bode plot, used for plotting frequency-response data.*

5.1.3 Physical Approximations

There are two indices by which the usefulness of an engineering model can be measured. First, there is the degree to which it represents the essential features of the physical situation. Second, there is the amount of effort required to use it for an analytical study. Obviously, these two considerations are in conflict and the choice of a model represents a compromise.

The selection of an appropriate model demands an awareness of the interplay between physical approximations and mathematical techniques; for example, if dissipation mechanisms are not significant in a given situation, it may be a simple matter to describe the nonlinear dynamics. In the opposite extreme, if dissipation dominates the dynamics, it may also be possible to include nonlinear effects. In the intermediate case of moderate damping nonlinear effects may be included only with a great deal of effort.

In this section a simple example is used to illustrate how the mathematical model can be simplified by recognizing the important physical effects at the

* F. E. Nixon, *Principles of Automatic Controls*, Prentice-Hall, Englewood Cliffs, N.J., 1953, pp. 165–174.

outset. This is often done in circuit theory. Suppose that the voltage v in the LR circuit of Fig. 5.1.13 is given and the current i is to be computed. We can, of course, solve this problem with little trouble. This allows us to see that if interest is confined to the current at very low frequencies

$$v \simeq iR; \quad (5.1.24)$$

that is at low frequencies virtually all of the voltage drop is across the resistance. By contrast, at high frequencies the inductive reactance greatly exceeds the resistance and virtually all the voltage drop is across the inductance. In this limit

$$v \simeq L \frac{di}{dt}. \quad (5.1.25)$$

The frequency is considered to be low or high, depending on the relationship between the period of excitation $2\pi/\omega$ and the time constant L/R of the circuit. This is normally expressed as a ratio of inductive reactance and resistance. Thus, when $\omega L/R \ll 1$, (5.1.24) can be used; and, when $\omega L/R \gg 1$, (5.1.25) can be used. When $\omega L/R \approx 1$, neither approximation is appropriate.

By contrast with the circuit of Fig. 5.1.13, most electromechanical problems are represented by nonlinear equations unless the dynamics are limited to incremental motions. In these situations approximations analogous to those represented by (5.1.24) and (5.1.25) are useful. The electromechanical approximation, however, is more subtle because the frequency or characteristic time constant of the system is often not known until after the problem has been solved. In Fig. 5.1.13 we knew at the outset that the current i had the same frequency as the driving voltage. With the circuit coupled to a mechanical system and natural or free motions of the system under consideration (not the sinusoidal steady state resulting from a given driving function), the temporal behavior of the system is at least in part determined by mechanical effects. Hence the characteristic frequencies of the response to initial conditions can be low or high, compared with the natural frequencies of the electrical system.

The pair of coils shown in Fig. 5.1.14 provides a concrete example of the physical consequences of making electrical approximations in a magnetic field system. A pair of fixed coils is driven by the constant current $i_2 = I$ and arranged to give a uniform magnetic flux density B_0 in the region of a pivoted coil with the angular deflection θ . The rotatable coil is short-circuited but has a

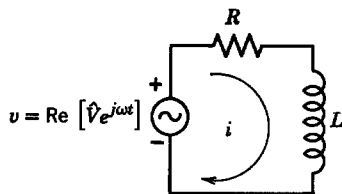


Fig. 5.1.13 L - R circuit to illustrate appropriate electrical approximations when the period of excitation is extreme compared with the time constant L/R .

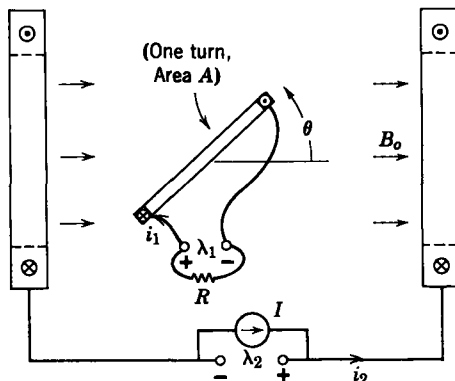


Fig. 5.1.14 A pivoted coil is free to rotate with the angular deflection θ in a uniform magnetic field produced by i_2 .

resistance R , which is represented by a resistance connected to the terminals. We wish to study the mechanical response of the coil when it is given an initial angular velocity $d\theta/dt = \Omega$ at the angular position $\theta = 0$.

The equations of motion are found by first writing the electrical terminal relations as

$$\lambda_1 = L_1 i_1 + M i_2 \sin \theta, \quad (5.1.26)$$

$$\lambda_2 = M i_1 \sin \theta + L_2 i_2, \quad (5.1.27)$$

where L_1 , M , and L_2 are constants. The dependence of the mutual inductance on θ should be evident from Fig. 5.1.14, and in the absence of magnetic materials the self-inductances are independent of θ . For this electrically linear system the coenergy follows from (5.1.26) and (5.1.27) as

$$W' = \frac{1}{2} L_1 i_1^2 + M i_1 i_2 \sin \theta + \frac{1}{2} L_2 i_2^2. \quad (5.1.28)$$

Hence the electrical torque is

$$T^e = \frac{\partial W'}{\partial \theta} = M i_1 i_2 \cos \theta. \quad (5.1.29)$$

The rotatable coil has a moment of inertia J , so that if mechanical damping is ignored the mechanical equation of motion is ($i_2 M = IM = AB_o$).

$$J \frac{d^2 \theta}{dt^2} = AB_o i_1 \cos \theta. \quad (5.1.30)$$

The electrical equation requires that $-i_1 R = d\lambda_1/dt$, which, in view of (5.1.26) and the fact that $M i_2 = AB_o$, is

$$-i_1 R - L_1 \frac{di_1}{dt} = AB_o \cos \theta \frac{d\theta}{dt}. \quad (5.1.31)$$

The voltage on the right-hand side of this equation is induced by the motion of the coil through the magnetic induction B_o . The equation expresses the fact that this "speed voltage" is absorbed by the self-inductance of the coil and by the resistance R . Note that the relative magnitudes of these terms on the left are determined by the same considerations discussed in connection with Fig. 5.1.13. Now, however, the current i_1 has a temporal behavior that depends on the mechanical deflection of the coil. From (5.1.31) it is clear that the moment of inertia plays a part in determining whether the inductive reactance or the resistance (or both) are significant.

We approach the problem here by assuming at the outset that one or the other of the terms on the left in (5.1.31) dominates, investigating the analytical consequences, and returning to check the validity of the initial assumption by using the predicted response. Suppose first that

$$|i_1 R| \gg L_1 \left| \frac{di_1}{dt} \right|. \quad (5.1.32)$$

Then, i_1 can be found explicitly from (5.1.31) and substituted into (5.1.30). The result, after some trigonometric manipulation, has the form

$$\frac{d}{dt} \left[J \frac{d\theta}{dt} + \frac{(AB_o)^2}{2R} (\cos \theta \sin \theta + \theta) \right] = 0. \quad (5.1.33)$$

Here one derivative has been factored to show that the quantity in brackets is constant. The initial conditions that $d\theta/dt = \Omega$ when $\theta = 0$ fix this constant so that (5.1.33) can be integrated.

$$J \frac{d\theta}{dt} + \frac{(AB_o)^2}{2R} (\cos \theta \sin \theta + \theta) = J\Omega. \quad (5.1.34)$$

This first-order equation can be integrated to find $\theta(t)$ without approximations concerning the amplitude of the angular deflection θ . This approach to nonlinear dynamics is the subject of Section 5.2.2. It serves our purpose here to establish the physical significance of the approximation by limiting consideration to small amplitude (linearized) deflections about $\theta = 0$, in which case (5.1.34) is approximated by

$$\frac{d\theta}{dt} + \frac{\theta}{\tau_o} = \Omega, \quad (5.1.35)$$

where

$$\tau_o = \frac{RJ}{(AB_o)^2}.$$

In view of the initial conditions, this linear equation has the solution

$$\theta = \Omega \tau_o (1 - e^{-t/\tau_o}), \quad (5.1.36)$$

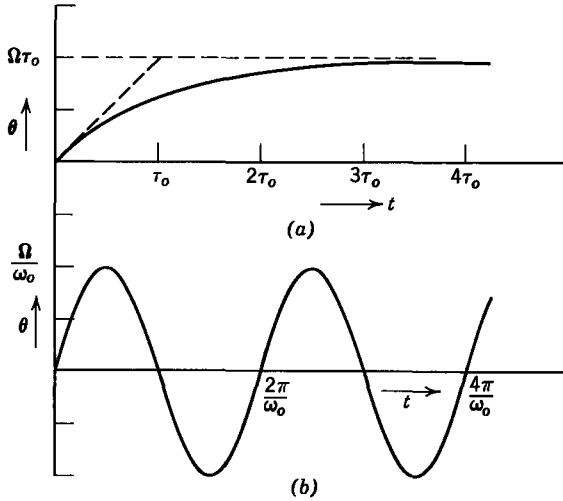


Fig. 5.1.15 Response in angular deflection of the rotatable coil of Fig. 5.1.14 to an initial angular velocity Ω at $\theta = 0$: (a) motion dominated by electrical resistance ($\tau_o \gg L_1/R$); (b) flux λ_1 conserved at zero ($2\pi/\omega_o \ll L_1/R$) as it would be if the resistance R were very small.

which is sketched in Fig. 5.1.15a. Remember that this expression is valid only if the inequality of (5.1.32) is satisfied. Use of (5.1.36) shows that the inequality requires

$$\tau_o \gg \frac{L_1}{R}. \quad (5.1.37)$$

Note that the electromechanical time constant τ_o is proportional to the moment of inertia J . When τ_o is large enough to satisfy (5.1.37), it simply means that the inertial effect slows the motion to the point at which the inductive reactance (which depends on the rate of change of i_1) is of negligible influence.

This approximation is typical of those used in the analysis of large, magnetic-field type devices such as rotating machines. Mechanical and electromechanical time constants are so long compared with electrical time constants that mechanical and electromechanical transients are assumed to occur with the electrical system always operating in the steady state [$L_1(di_1/dt)$ neglected in the example just completed]. Conversely, electrical transients are so fast that they are assumed to occur with the mechanical system operating at constant speed.

It is important to see that an equation of motion in the form of (5.1.35) would be obtained if the magnetic induction B_o were absent but the coil rotated in a viscous fluid. In the limit in which the reactance of the coil can be ignored the magnetic field and short-circuited coil combine to act as a mechanical damper. This is the limit used in synchronous machines when

short-circuited damper windings are added to the rotor to assist in damping electromechanical oscillations (see Section 4.1.6a and Example 5.1.2).

It is evident from the solution (5.1.36) that there are no oscillations (as would be expected from “springlike” torques). The reactance represents the contribution of self-currents to the total magnetic field. When $L_1 di_1/dt$ is ignored in (5.1.31), it means that we are ignoring the magnetic field induced by the current i_1 . This kind of physical approximation is useful in dealing with continuum interactions (see Chapter 10). We shall also find that in the limit in which electrical dissipation dominates, media tend to “ooze” rather than “bounce.”

In both lumped parameter and continuum electromechanics it is often meaningful to model a conducting medium as “perfectly” conducting. This model is illustrated here by taking an extreme (to that so far considered) in which characteristic times of the electromechanical system are short enough to warrant neglecting the drop across the resistance compared with that across the reactance; that is, the first term in (5.1.31) is ignored compared with the second. That expression can then be integrated to give

$$i_1 = -\frac{AB_o}{L_1} \sin \theta. \quad (5.1.38)$$

Here, we assume that when the motion is initiated at $t = 0$, not only do $\theta = 0$ and $d\theta/dt = \Omega$ but $i_1 = 0$. Note that (5.1.38) requires that the flux λ_1 linking the rotatable coil be conserved. The initial conditions require that this flux be conserved at $\lambda_1 = 0$.

It follows from (5.1.38) and (5.1.30) that the equation of motion is

$$J \frac{d^2\theta}{dt^2} = -\frac{(AB_o)^2}{L_1} \sin \theta \cos \theta. \quad (5.1.39)$$

This nonlinear expression can be integrated without further approximations. For now, we delay this nonlinear problem until Section 5.2.1 and illustrate the physical consequences of the approximation by considering small amplitude deflections about $\theta = 0$. Then (5.1.39) becomes

$$\frac{d^2\theta}{dt^2} + \omega_o^2 \theta = 0, \quad (5.1.40)$$

where

$$\omega_o = \frac{AB_o}{\sqrt{JL_1}}.$$

The solution to this equation, which satisfies the initial conditions, is

$$\theta = \frac{\Omega}{\omega_o} \sin \omega_o t. \quad (5.1.41)$$

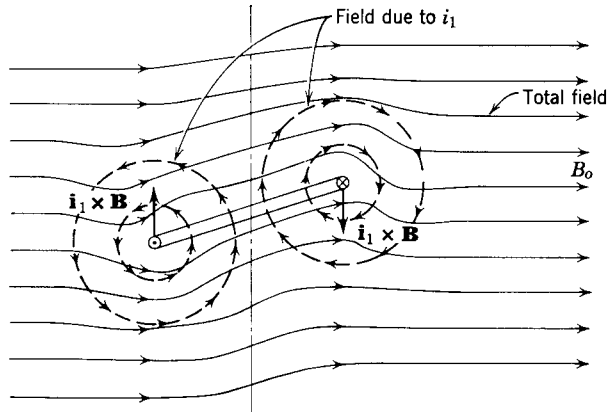


Fig. 5.1.16 Distortion of the initially uniform magnetic field caused by angular deflection of coil shown in Fig. 5.1.14. The flux of the moveable coil is constrained to be zero.

This expression is plotted in Fig. 5.1.15*b*, where it can be compared with the loss-dominated case.

When the electrical dissipation can be ignored, the magnetic torque has the same effect on the motion as a torsional spring (in the nonlinear case, a nonlinear torsional spring). The reason for this can be seen physically with the help of Fig. 5.1.16. Remember that in this limit the total flux λ_1 through the rotatable coil is constrained to be zero. With the angular deflection shown in Fig. 5.1.16, the flux density B_0 links the coil, thus contributing to λ_1 . This flux must be canceled by a flux induced by the current i_1 . The deflection shown in Fig. 5.1.16 is accompanied by the currents, as indicated, which induce a flux that cancels that from B_0 . The total magnetic field is distorted to remain tangential to the plane of the coil. Note that the magnetic force $\mathbf{i}_1 \times \mathbf{B}$ tends to restore the coil to the angle $\theta = 0$. Because the induced current is proportional to the angular deflection (and not to its rate of change), the magnetic torque is similar to that of a spring.

The assumption that the inductive reactance is of primary importance to the dynamics is equivalent to recognizing that the effect of the magnetic field induced by the motion is on the same order as that of the imposed magnetic field. More is said on this point in Section 7.0, in which the same physical arguments appear in the context of a distributed interaction. In the context of lumped parameters perfectly conducting media (in a magnetic field system) behave in a “springlike” or “bouncing” fashion. In continuous media (e.g., a “perfectly” conducting fluid) the same approximation leads to the possibility of wavelike motions, as illustrated in Section 12.2.3.

Our remarks in this section have been limited to electrical approximations that are appropriate in magnetic field systems. We could further illustrate the

role of electrical dissipation in electric field systems. This was done implicitly however, in Example 5.1.4, in which it was shown that the electrical variables of a capacitor microphone were essentially constrained to constant potential and constant charge in the low and high-frequency ranges, respectively; that is, if the capacitor plate in Fig. 5.1.11 responded at a low frequency, the $R dq/dt$ drop across the resistance R could be ignored and the potential on the plate taken as the constant V_0 . In the opposite extreme rapid variations in the capacitance of the microphone meant that there was little chance of charge leaking off through the resistance R . The result was an essentially constant charge on the movable plate. The critical parameter that determined which approximation was valid was ωRC_0 , or again essentially the ratio of the electrical time constant and the period of the mechanical response. The simplifying feature of Example 5.1.4 was the known response frequency ω . We could, however, easily envision a situation like the one considered in this section, in which the characteristic dynamic time would not be known until after the problem had been solved.

5.2 NONLINEAR SYSTEMS

As we have seen, most lumped-parameter electromechanical devices are described in general by nonlinear differential equations. Section 5.1 was devoted to showing that for many purposes these equations can be approximated by linearized equations. There are cases in which the nonlinear dynamics are essential, and indeed nonlinear interactions represent possibilities for engineering applications that are not available within the framework of linear systems. Unfortunately, there is no general mathematical theory to cover the solution of all types of nonlinear differential equation. This is not surprising, since nonlinear equations include all types that are not linear, as defined in Section 5.1.1.

The most direct way of obtaining numerical answers to nonlinear problems is to use machine computation, either analog or digital. In some simple cases it is possible to integrate the equations of motion. In the following two sections we illustrate two classes of these simple systems and the analytical techniques that are useful in obtaining solutions. Our objective is not only to study techniques for describing nonlinear systems but to gain a deeper physical insight into electromechanical dynamics.

5.2.1 Conservative Systems

In mechanics, if the energy remains constant throughout the motion of a system, the system is said to be conservative; that is, although the velocity and position of a mass change with time, the total energy is conserved at its

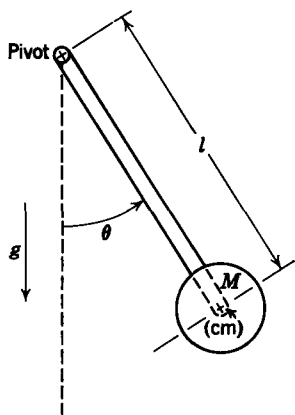


Fig. 5.2.1 Simple pendulum with an angular deflection $\theta(t)$ in the gravitational field g .

initial value. As we show in this section, this provides a basis for finding the motions of many electromechanical systems.

A simple mechanical system that illustrates the approach while allowing considerable physical insight is the simple pendulum of Fig. 5.2.1. It consists of a mass M whose center of mass is connected by a rigid, weightless rod of length l to a frictionless pivot. We consider the motion in which there is no externally applied torque except that due to gravity, which acts downward as shown.

The torque equation is therefore

$$J \frac{d^2\theta}{dt^2} = T_g, \quad (5.2.1)$$

where the moment of inertia $J = Ml^2$ and T_g , the torque due to gravity, can be written as

$$T_g = -\frac{\partial V}{\partial \theta}; \quad V = -Mgl \cos \theta. \quad (5.2.2)$$

It is useful to write the torque as the derivative of the potential V because (5.2.1) can then be written as

$$\frac{d}{dt} \left[\frac{J}{2} \left(\frac{d\theta}{dt} \right)^2 + V \right] = 0. \quad (5.2.3)$$

The best way to see that this is true is to take the first time derivative in (5.2.3) and see that (5.2.1) is recovered. It follows from (5.2.3) that the quantity in brackets is constant, or conserved. If we call this constant E ,

$$\frac{J}{2} \left(\frac{d\theta}{dt} \right)^2 = E - V(\theta). \quad (5.2.4)$$

To understand the physical significance of this equation it is helpful to think in terms of the potential plot shown in Fig. 5.2.2. The constant E , which is the sum of the kinetic and potential energies, is independent of θ . According to (5.2.4), the kinetic energy, hence the square of the angular velocity, is proportional to the difference between E and $V(\theta)$. This is shown graphically in Fig. 5.2.2. It is apparent from the diagram that at points (a) and (b), at which $|\theta| = \theta_m$, the angular velocity is zero, whereas at $\theta = 0$ the magnitude of the angular velocity has its largest value. The kinetic energy $J(d\theta/dt)^2/2$ is always positive. Hence, given the value of E , we can picture the angular

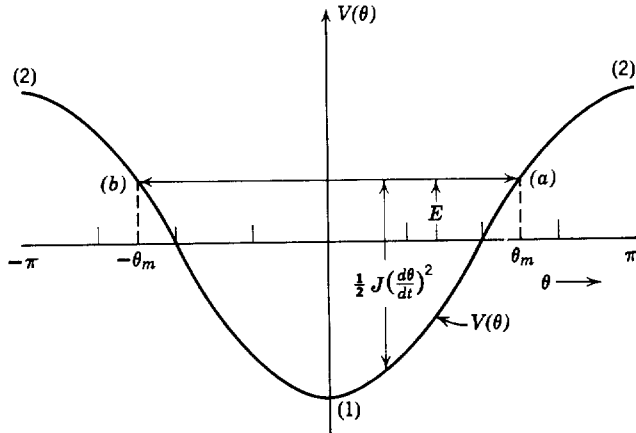


Fig. 5.2.2 Potential well $V(\theta)$ for the pendulum of Fig. 5.2.1 and for the rotatable coil of Fig. 5.2.4. If the pendulum is released from a stationary state at $\theta = \theta_m$, it will have an excursion between the points (a) and (b). The square of the angular velocity at any given position is proportional to the vertical distance between the constant E line and the potential $V(\theta)$.

deflection as limited to those regions of the potential plot in which the constant E line is above the potential $V(\theta)$. For the value of E shown in Fig. 5.2.2 the pendulum oscillates between the angles $\theta = \theta_m$ and $\theta = -\theta_m$.

The invariant E is established by the initial conditions. Suppose that at $t = 0$, $\theta = \theta_m$ and $d\theta/dt = 0$; that is, the pendulum is released from an initial static deflection $\theta = \theta_m$. Then from (5.2.4) $E = V(\theta_m)$ and

$$\frac{J}{2} \left(\frac{d\theta}{dt} \right)^2 = V(\theta_m) - V(\theta). \quad (5.2.5)$$

Similarly, the pendulum could be given an initial velocity $(d\theta/dt)_m$ at $\theta = 0$, and it follows from (5.2.4) that $E = (J/2)(d\theta/dt)_m^2 + V(0)$. If the initial kinetic energy exceeds $V(\pi) - V(0)$, the line of constant E in Fig. 5.2.2 never intersects the potential curve and the pendulum deflection increases monotonically. This simply means that, given a large enough initial energy, the pendulum rotates continuously on its pivot, moving rapidly at $\theta = 0$, 2π , 4π , ... and slowing down at $\theta = \pi$, 3π , ...

Note that at the angles $\theta = 0, \pi, \dots$ the pendulum can be in static equilibrium, for at these points $\partial V / \partial \theta = 0$ ($T_g = 0$). The question whether these equilibria are stable can be answered in terms of the potential plot. Suppose that the pendulum is given an initial static position $\theta = 0$. This establishes the constant E line in Fig. 5.2.2 as passing through point (1). To see if the equilibrium is stable, the pendulum is given a slight kinetic energy, which

raises the constant E line just above point (1) and shows that small amplitude oscillations result. In the sense defined in Section 5.1.2a we say that this equilibrium is stable. Similarly, if the pendulum is placed in static equilibrium at (2), the constant E line just grazes the peaks of the potential curve. To test for stability the pendulum is given a slight additional kinetic energy (greater E). It is clear that the pendulum does not tend to return to position (2), which is a point of unstable static equilibrium. This is not surprising to someone who has tried to balance a broom on his finger. The pendulum is upside down at $\theta = \pi$.

As a consequence of our deductions concerning Fig. 5.2.2, we can interpret the system behavior as if the moment of inertia were a particle that slides without friction on a physical "hill" with the shape $V(\theta)$. Motions within a *potential well* are bounded and therefore stable.

It is worthwhile to place the linear stability theory of Section 5.1.2a in perspective by relating it to the potential well. The torque T_θ can be expanded about a point of static equilibrium $\theta = \Theta$:

$$T_\theta = -\frac{\partial V}{\partial \theta}(\Theta) - (\theta - \Theta)\frac{\partial^2 V}{\partial \theta^2}(\Theta). \quad (5.2.6)$$

Because the equilibrium is static, the first term on the right is zero. Hence (5.2.1) has been linearized and is

$$J \frac{d^2 \theta}{dt^2} + \left[\frac{\partial^2 V}{\partial \theta^2}(\Theta) \right] (\theta - \Theta) = 0. \quad (5.2.7)$$

From this constant coefficient linear equation it follows that the solutions will be oscillatory, hence stable, at points of zero slope on the potential plot at which

$$\frac{\partial^2 V}{\partial \theta^2} > 0. \quad (5.2.8)$$

Our linearized techniques of Section 5.1.2a tell us that small perturbations about a point in the bottom of a potential well (1) are stable, whereas those from the peaks (2) are unstable. We shall see in the examples that follow cases in which even though small amplitude deflections are unstable large deflections remain bounded.

We return to this example of the mechanical pendulum to show how (5.2.4) can be integrated to determine the detailed temporal behavior. Before doing so, however, it would be more to the point to see how these ideas can be extended into the context of electromechanics.

In Fig. 5.2.3 an electromechanical system is represented schematically by a mechanical system connected to an electromechanical coupling network.

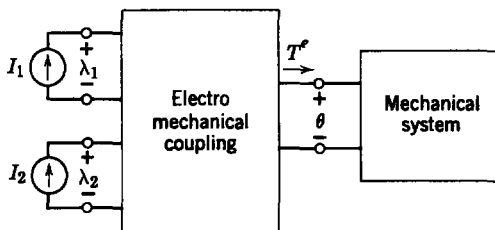


Fig. 5.2.3 Electromechanical system in which conservation of an energy function can be used to determine the dynamics. The mechanical system is composed of lossless elements and one of each of the electrical terminal variables is held constant.

This network, as defined in Chapter 3, does not contain elements that dissipate energy, but, of course, it in turn couples to an electrical system that does contain such elements. If, however, the electrical terminals are constrained so that a variable at each of the terminal pairs is held fixed, we expect to find a constant of the motion for the total electromechanical system. This is true because any additions to the total electromechanical energy of the system must be made through the electrical terminals. If a terminal variable is fixed, there is either no flow of energy or no flow of coenergy through that terminal pair; for example, in the case shown in Fig. 5.2.3 an increment of coenergy at the (i_1, λ_1) terminal pair is $\lambda_1 di_1$, which is constrained to zero as long as $I_1 = \text{constant}$. Now we know from Chapter 3 that the torque of electrical origin T^e can be found as the derivative of an “energy function” written as a function of θ and one of each of the electrical terminal variables (the “independent” variables): for example, in Fig. 5.2.4

$$T^e = \frac{\partial W'}{\partial \theta}(i_1, i_2, \theta), \quad (5.2.9)$$

where W' is the coenergy function (see, for example, Section 3.1.2b). If i_1 and i_2 are constrained to be constant, then T^e is the derivative of a known function of θ :

$$T^e = \frac{\partial W'}{\partial \theta}(I_1, I_2, \theta). \quad (5.2.10)$$

Because the mechanical system is composed of elements that do not dissipate energy, the mechanical torque can also be written as the derivative with respect to θ of an

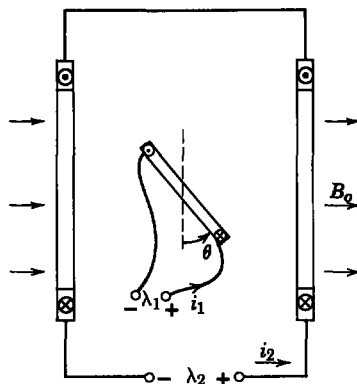


Fig. 5.2.4 A pair of fixed coils is excited in series by the current i_2 . A pivoted coil has the angular deflection θ and the electrical terminal variables (i_1, λ_1) . When both coils are driven by constant current sources, the potential well is as shown in Fig. 5.2.2.

energy function, which we might call $U(\theta)$

$$T^m = - \frac{\partial U}{\partial \theta}. \quad (5.2.11)$$

It follows that the torque equation can be written as

$$J \frac{d^2 \theta}{dt^2} + \frac{\partial V}{\partial \theta} = 0, \quad (5.2.12)$$

where

$$V(\theta) = U(\theta) - W'(I_1, I_2, \theta), \quad (5.2.13)$$

which now takes on the same form as (5.2.3).

Example 5.2.1 is based on this generalization of the potential well to include the electromechanical energy function.

Example 5.2.1. The electromechanical system shown in Fig. 5.2.4 has the schematic description of Fig. 5.2.3. The pivoted coil has the terminal variables (i_1, λ_1) , whereas the fixed coils are connected in series, with the terminal variables (i_2, λ_2) . There is no magnetic material in the problem, so that self-inductances remain constant and the electrical terminal relations can be written as

$$\lambda_1 = L_1 i_1 + M(\theta) i_2, \quad (a)$$

$$\lambda_2 = M(\theta) i_1 + L_2 i_2, \quad (b)$$

where for this particular case $M(\theta) = M_o \cos \theta$, L_1 , L_2 , and M_o are constants. From the terminal relations the coenergy $W'(i_1, i_2, \theta)$ follows as

$$W' = \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 + M(\theta) i_1 i_2. \quad (c)$$

The first two terms in this expression are constant and can be absorbed in the constant of the motion E . Hence from (5.2.13) we have

$$V = -I_1 I_2 M(\theta). \quad (d)$$

For the particular case of Fig. 5.2.4

$$V = -I_1 I_2 M_o \cos \theta. \quad (e)$$

This potential has the same form as the mechanical pendulum of Fig. 5.2.1; hence the developments relevant to dynamics of the pendulum are equally applicable here. For quantitative purposes $Mg l$ is replaced by $I_1 I_2 M_o$.

Note that the state of stable static equilibrium at (1) in Fig. 5.2.2 now corresponds to the situation in which the magnetic field generated by i_1 at the center of the pivoted coil is aligned with the field produced by i_2 . If either I_1 or I_2 is made negative, the potential well in Fig. 5.2.2 is inverted, with $\theta = 0$ becoming an unstable static equilibrium and point (2) becoming a stable equilibrium.

Now that we have discussed the basic considerations involved in using potential-well techniques for describing electromechanical problems it would be interesting to embark on examples that illustrate characteristic dynamic behavior. It is not often that an engineer is concerned with the detailed temporal behavior of a system. Example 5.2.2 illustrates how a knowledge of the

electrical terminal relations can be used to establish the significant features of dynamic behavior under a variety of electrical constraints. This approach is extremely useful because the relevant features of the electrical terminal relations can be found quantitatively by simple measurements or qualitatively by sketching the electric or magnetic fields.

Example 5.2.2. A coil (mass M) is mounted on a massless pendulum of length R , as shown in Fig. 5.2.5, to allow it to swing through the magnetic field generated by a pair of series-connected fixed coils. We wish to study the dynamical consequences of energizing the fixed coils with a current source I_2 and constraining the electrical terminals of the movable coil in two different ways. First, the pendulum motions that result when $i_1 = I_1$ or the current through the moving coil is constant provide us with an opportunity to illustrate how a combination of electrical and mechanical potentials is handled, a situation that can then be contrasted with the second case to be considered in which the terminals (i_1, λ_1) are constrained to zero flux linkage. This is the physical result if the terminals of the moving coil are short-circuited and the resistance of the coil is "small." This limit is discussed in Section 5.1.3, in which it is shown that the flux can be considered essentially constant if the current i_1 is limited by the self-reactance of the coil and not by the resistance. This demands that the characteristic time constant of the motion be short compared with the L/R time constant of the moving coil.

The mutual inductance between the moving and fixed coils is shown in Fig. 5.2.5c. The

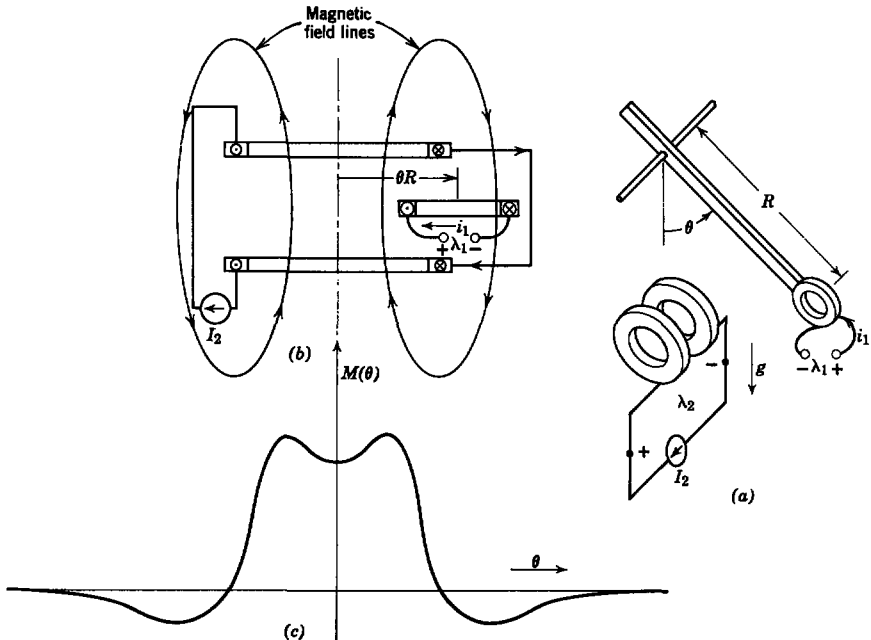


Fig. 5.2.5 (a) A coil is attached to a pendulum in such a way that when $\theta = 0$ it is directly between a pair of fixed coils energized by the current I_2 ; (b) top view of (a) showing the magnetic field produced by the fixed coils; (c) mutual inductance between the fixed and movable coils of (b).

dependence on θ can be understood qualitatively by considering Fig. 5.2.5*b*. When θR is very large, it is clear that the magnetic field generated by the fixed coils does not link the movable coil. As the movable coil comes into the vicinity of the fixed coils, it links a magnetic field having the opposite direction to that linked when it is directly between the coils. Hence the mutual inductance first becomes negative and then positive. If the diameter of the fixed coils is large compared with their spacing, the field, which tends to concentrate near the currents, will be smaller at the center ($\theta = 0$) than at off-center angles where the movable coil is adjacent to the fixed windings. Hence the mutual inductance is shown with a dip in the vicinity of $\theta = 0$.

This inductance could be measured by exciting the fixed coil with a sinusoidal current and measuring the induced voltage in the movable coil as a function of position. Therefore we can regard $M(\theta)$ as determined either qualitatively or quantitatively. The electric terminal relations have the form of (a) and (b) in Example 5.2.1.

CONSTANT CURRENT CONSTRAINTS

First, consider the consequences of driving both fixed and movable coils by constant-current sources. The total potential is then the sum of a gravitational potential (5.2.2) and a potential due to the magnetic field [(d) of Example 5.2.1]:

$$V = -Mgl \cos \theta - I_1 I_2 M(\theta). \quad (a)$$

There are two possibilities. Either $I_1 I_2 > 0$, in which case the total potential appears as shown in Fig. 5.2.6*a*, or $I_1 I_2 < 0$, and $V(\theta)$ is as shown in Fig. 5.2.6*b*.

Consider first the case in which both currents are positive. If the pendulum holding the coil is released from an initial static position at $\theta = \theta_m$, the line of constant E appears as shown in Fig. 5.2.6*a*. The pendulum swings completely through the region of the fixed coils. The effect of the negative slope of the mutual inductance is not sufficient to decelerate it as it approaches them. Once the movable coil is between the fixed coils, the $\mathbf{I} \times \mathbf{B}$ force tends to accelerate it toward the center, except very near $\theta = 0$. The effect of the constant current constraints with both currents positive is to make the potential well centered on $\theta = 0$ even deeper than it would be without the field.

A similar experiment with one of the currents reversed results in motions characterized by the constant E line shown in Fig. 5.2.6*b*. For this case the pendulum released from the initial angle θ_m is reflected by the magnetic interaction with the fixed coil. This is expected, since the $\mathbf{I} \times \mathbf{B}$ force on the moving coil as it nears the origin is now in the direction required to retard the motion. Of course, given enough initial energy, the pendulum will pass on through the interaction region. Note that the pendulum could be trapped in a region near the origin.

A significant feature of the constant current dynamics is its dependence on the sign of the excitation current. If one of the currents is reversed, that part of the potential due to the magnetic field is turned upside down; for example, in Fig. 5.2.6*b*, in which one of the currents is reversed, the static equilibrium at the origin (*a*) is stable, and there are two additional points (*d* and *e*) at which the pendulum can be in stable static equilibrium. The equilibria (*b* and *c*) that were stable in Fig. 5.2.6*a* are replaced by unstable equilibria (*b* and *c*) in Fig. 5.2.6*b*.

CONSTANT-CURRENT CONSTANT-FLUX CONSTRAINTS

We now embark on describing the motions when the pendulum coil of Fig. 5.2.5 is constrained to constant (zero) flux. As is evident from the development, the electro-mechanical coupling must now be represented by a hybrid energy function, for neither all of the currents nor all of the fluxes are constrained to be constant.

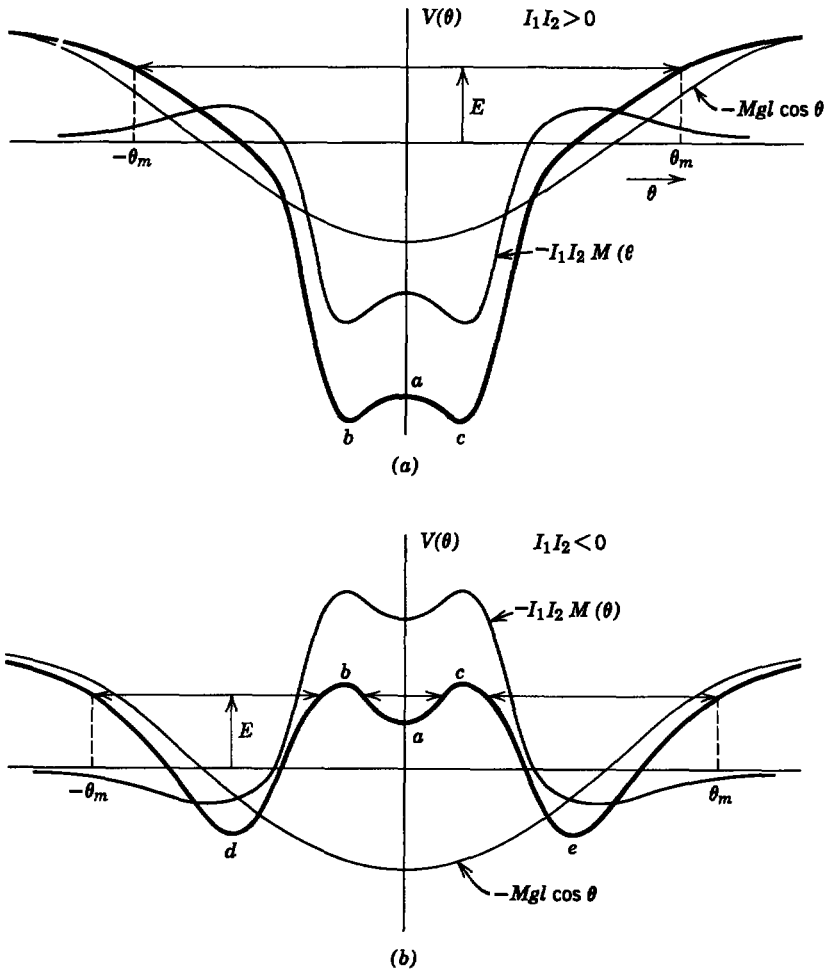


Fig. 5.2.6 Potential plots for the system of Fig. 5.2.5 constrained to constant current: (a) both coil currents positive; (b) one coil current positive and one negative.

A schematic representation of the terminal constraints is shown in Fig. 5.2.7. To use the potential techniques, we require a function $W''(\theta)$ from which we can find the electrical torque by taking a derivative:

$$T^e = \frac{\partial W''}{\partial \theta}. \quad (b)$$

This function, with the terminations of Fig. 5.2.3, was simply the coenergy W' , since in the coenergy function the currents are used as independent variables. For our present purposes it is helpful to recall that any of the energy functions are derived from a statement of conservation of energy for the electromechanical coupling.

$$i_1 d\lambda_1 + i_2 d\lambda_2 = dW + T^e d\theta; \quad (c)$$

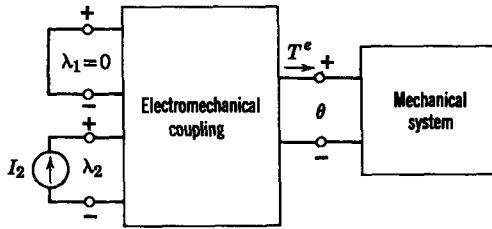


Fig. 5.2.7 Schematic representation of the system shown in Fig. 5.2.5a when the pendulum coil is constrained to zero flux.

for example, in the case in which the currents were held constant it was appropriate to transform this expression to one involving the currents i_1 and i_2 as independent variables (as discussed in Section 3.1.2b). For the present situation λ_1 and i_2 are fixed and therefore should be used as independent variables. Hence the second term in (c) is rewritten as

$$i_2 d\lambda_2 = d(i_2\lambda_2) - \lambda_2 di_2 \quad (d)$$

so that (c) becomes

$$\lambda_2 di_2 - i_1 d\lambda_1 = dW'' - T^e d\theta, \quad (e)$$

where

$$W'' = i_2\lambda_2 - W.$$

From (e) it follows that the electrical torque is given by (b). As for the energy and coenergy functions, W'' is found by integrating a form of the energy equation, which is now (e). In carrying out this integration, it must be remembered that i_2 and λ_1 are independent variables and must be used to express λ_2 and i_1 in (e). Thus (a) and (b) of Example 5.2.1 are written as

$$i_1 = \frac{\lambda_1}{L_1} - \frac{M(\theta)}{L_1} i_2, \quad (f)$$

$$\lambda_2 = \left(L_2 - \frac{M^2(\theta)}{L_1} \right) i_2 + \frac{M(\theta)}{L_1} \lambda_1. \quad (g)$$

Now, if the integration of (e) is carried out in the usual way (Section 3.1.1), we obtain

$$W'' = \frac{1}{2} \left(L_2 - \frac{M^2}{L_1} \right) i_2^2 - \frac{1}{2} \frac{\lambda_1^2}{L_1} + \frac{M}{L_1} i_2 \lambda_1. \quad (h)$$

The terminal constraints require that $i_2 = I_2$ and that $\lambda_1 = 0$. (The initial conditions determine the constant λ_1 the short-circuited pendulum coil will retain.) For the present purposes we assume that the coil is initially outside the magnetic field, where a short circuit establishes the flux $\lambda_1 = 0$. We have established the function W'' to be used in (b):

$$W'' = \frac{1}{2} \left(L_2 - \frac{M^2}{L_1} \right) I_2^2. \quad (i)$$

The potential V , which includes both the effects of the magnetic field and gravity, is [from (5.2.2)]

$$V = -Mgl \cos \theta + \frac{I_2^2}{2L_1} M^2(\theta). \quad (j)$$

Here, a constant has been absorbed in the constant of the motion E [which appears in (5.2.4)].

The potential well for the flux-constrained coil is shown in Fig. 5.2.8 as the superposition of the same gravitational potential used before and a magnetic potential that is proportional to the square of $M(\theta)$, as given in Fig. 5.2.5c. At the outset two observations are of physical significance; the potential well is unaltered by reversing the direction of the field due to the fixed coils [I_2 is squared in (j)] and the interaction does not depend on the sign of the mutual inductance. This is by contrast with the current constrained situation in which the potential well could take on the alternative forms shown in Fig. 5.2.6. These observations reflect the fact that the current i_1 is *induced* in a direction that cancels any flux due to i_2 linking the pendulum coil. This type of dynamics is familiar from Section 5.1.3 and is evident here from the potential diagram.

The field generated by the fixed coils represents a magnetic barrier to the pendulum coil; for example, if the pendulum is given the initial velocity required to establish the constant of the motion as E_1 in Fig. 5.2.8, the moving coil will bounce off the potential barrier set up by the magnetic field. In fact, at an energy E_2 , the pendulum coil oscillates between a maximum deflection magnitude determined by gravity and a minimum determined by the magnetic interaction.

At an energy E_3 it is possible to trap the coil in a well created solely by the imposed magnetic field. This magnetic trapping is a lumped-parameter illustration of how a magnetic field can be shaped to "bottle up" a highly conducting continuum such as a plasma. We can think of the pendulum coil as replaced by a highly conducting "blob" of material, which in turn can be modeled by many arbitrarily oriented, perfectly conducting loops. Each of these loops tends to behave as described here.

The effects of a finite coil resistance were considered in Section 5.1.3 to place the dynamics as found here in perspective. The zero flux constraint is a meaningful model for the actual

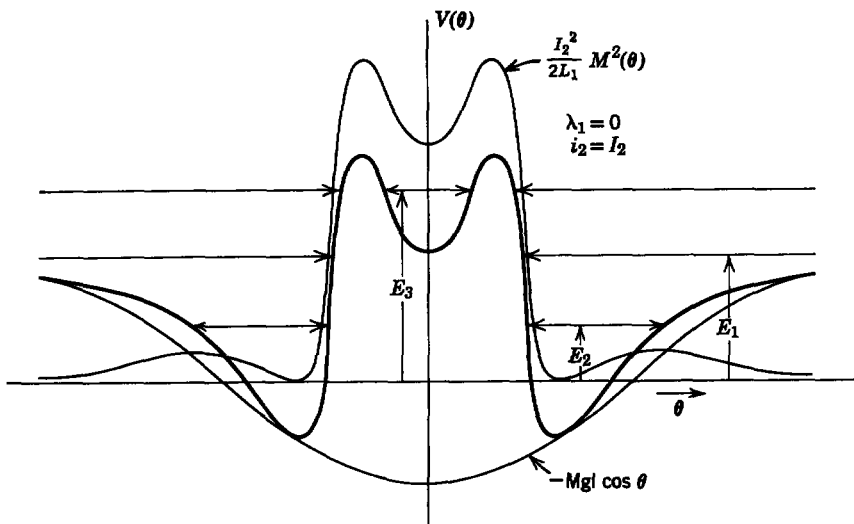


Fig. 5.2.8 Potential well for the system of Fig. 5.2.5a when the pendulum coil is constrained to zero flux. The potential has this shape regardless of the sign of the current I_2 .

physical situation only as long as characteristic times for the motion are short compared with the L/R time constant of the pendulum coil.

So far in this discussion of nonlinear conservative systems we have not concerned ourselves with detailed temporal behavior. Sometimes nonlinear periods of oscillation or deflections as a function of time are required and (5.2.4) must be integrated. For this purpose it is written as

$$\frac{d\theta}{dt} = \pm \left\{ \frac{2}{J} [E - V(\theta)] \right\}^{1/4}. \quad (5.2.14)$$

The plus and minus signs indicate that the pendulum can be moving in either direction at a given angle θ .

If we specify that when $t = 0$ the angle $\theta = \theta_m$, (5.2.14) can be integrated.

$$t = \pm \int_{\theta_m}^{\theta} \frac{d\theta'}{\left\{ \frac{2}{J} [E - V(\theta')] \right\}^{1/4}}. \quad (5.2.15)$$

The parameter θ' is a running variable of integration. Whether (5.2.15) can be integrated analytically depends on the form of $V(\theta)$. In any case, given $V(\theta)$, numerical integration is a straightforward matter.

In the case of the mechanical pendulum of Fig. 5.2.1, $V(\theta)$ is given by (5.2.2) and the integration of (5.2.15) can be carried out. Suppose that the pendulum is released from rest at $\theta = \theta_m$. Then from (5.2.4) $E = V(\theta_m)$ and the integral of (5.2.15) takes the standard form of an elliptic integral* for which solutions are tabulated. One fourth of a cycle of oscillation is shown as a function of time in Fig. 5.2.9, where for (a) the pendulum is released from an initial "small" angle $\theta_m = 20^\circ$ and for (b) the initial amplitude is 90° . For these plots the time is normalized to the frequency $\omega = \sqrt{g/L}$. This is the frequency of oscillation for small amplitudes, as can be seen by combining and linearizing (5.2.1) and (5.2.2) to obtain

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0. \quad (5.2.16)$$

For the initial conditions considered in Fig. 5.2.9 the solution to this equation is

$$\theta = \theta_m \cos \omega t. \quad (5.2.17)$$

This response, predicted by the linearized equation of motion, is shown as dashed curves in Fig. 5.2.9. For an amplitude of $\theta = 20^\circ$ the results from the linear and nonlinear models are almost identical, although there is some

* P. Franklin, *Methods of Advanced Calculus*, McGraw-Hill, New York, 1944, Chapter VII.

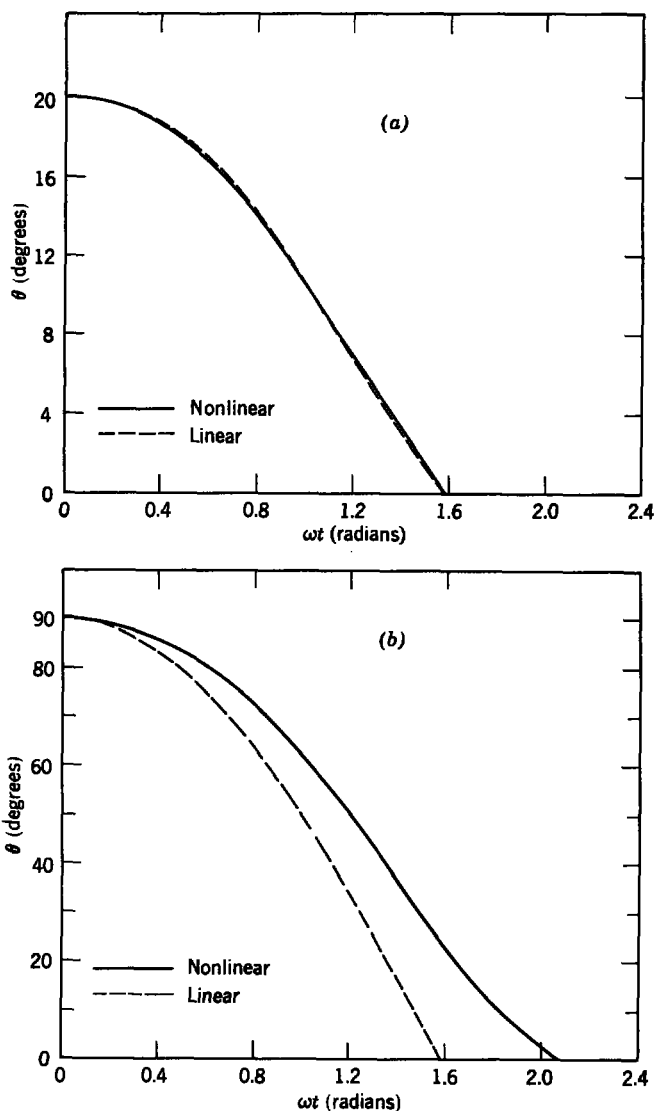


Fig. 5.2.9 Oscillation of a simple pendulum: (a) small amplitude; (b) large amplitude.

discrepancy between the predictions for $\theta = 90^\circ$. This comparison between the exact and linearized solutions should help to place the methods of Section 5.1 in perspective. Of course, the adequacy of a linearized model will depend greatly on the nature of the nonlinearity. The following example is one in which a linearized model would be difficult to make and would be of doubtful usefulness.

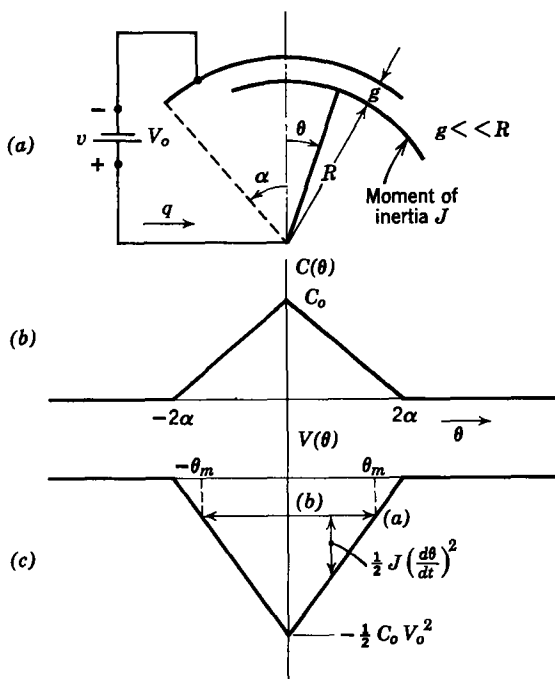


Fig. 5.2.10 (a) Capacitor plates just overlap when $\theta = 0$ and are constrained to constant potential V_o ; (b) capacitance as a function of θ ; (c) potential for the system of (a).

Example 5.2.3. The electric field system shown in Fig. 5.2.10 illustrates how the period of oscillation, computed from (5.2.15), can have nonlinear behavior. A capacitor is constructed from fixed and rotatable plates with the shape of sections from coaxial cylinders. At $\theta = 0$ the plates are aligned and the capacitance has the maximum value C_o . At $\theta = \pm 2\alpha$ there is no overlap between the plates and the capacitance is essentially zero. Hence $C(\theta)$ depends on θ essentially as shown in Fig. 5.2.10b. With the plates constrained to have the constant potential difference V_o , we expect that the rotatable plate can be at rest at $\theta = 0$, for the induced charges will tend to make the plates attract one another. We wish to determine the period of oscillation that results when the plate is deflected from this equilibrium.

The terminals are constrained to constant potential; hence it is appropriate to write the electric torque in terms of the coenergy $W'(v, \theta)$ (see Section 3.1.2b):

$$T^e = \frac{\partial W'}{\partial \theta}(V_o, \theta). \quad (a)$$

Hence the potential V is [from (5.2.1) and (5.2.2)]

$$V = -W'(V_o, \theta), \quad (b)$$

which for this example is

$$V = -\frac{1}{2} V_o^2 C(\theta), \quad (c)$$

with $C(\theta)$ as shown in Fig. 5.2.10b. It follows that the potential produced by the electric field appears as shown in Fig. 5.2.10c. From this diagram it is clear that if the movable

plate (with the moment of inertia J but with no effect from gravity) is given a kinetic energy greater than

$$\frac{1}{2}J\left(\frac{d\theta}{dt}\right)^2 = \frac{1}{2}C_oV_o^2 \quad (d)$$

at $\theta = 0$ the motions will not be oscillatory. Instead, the plate will rotate continuously with constant velocity when the plates do not overlap and with a maximum velocity at $\theta = 0$.

We can compute the period of oscillation T from (5.2.15) when the initial conditions are such that the motions are oscillatory. For this purpose we call the peak deflection θ_m so that the line of constant energy is as shown in Fig. 5.2.10c. In terms of that figure the rotor moves from (a) to (b) in $T/4$ sec. Over this interval the potential can be written as

$$V(\theta) = \frac{C_o}{2\alpha} (\theta - 2\alpha) \frac{V_o^2}{2}. \quad (e)$$

For our purposes (5.2.15) becomes

$$\frac{T}{4} = - \int_{\theta_m}^0 \frac{d\theta'}{\left\{ \frac{2}{J} [V(\theta_m) - V(\theta')] \right\}^{1/2}}, \quad (f)$$

where from (e)

$$V(\theta_m) - V(\theta) = \frac{C_o}{2\alpha} (\theta_m - \theta) \frac{V_o^2}{2}. \quad (g)$$

Substitution of (g) into (f), followed by integration, gives the required period of oscillation:

$$T = 8 \left(\frac{2J\alpha}{C_oV_o^2} \right)^{1/2} \sqrt{\theta_m}; \quad |\theta_m| < 2\alpha. \quad (h)$$

Of course, this result is limited to a range of θ_m in which the plates overlap. Beyond this range the motions are not bounded (oscillatory) and

$$T \rightarrow \infty; \quad |\theta_m| > 2\alpha. \quad (i)$$

The dependence of the oscillation period on θ_m is shown in Fig. 5.2.11. In a linear system the period of oscillation is independent of amplitude. Hence the plot emphasizes the nonlinear character of the motion.

It should be recognized that the approximate function $C(\theta)$ is valid only if the plate spacing g is small compared with deflections θR of interest. In a more exact model the functional dependence of C would be smoothed in the region near $\theta = 0$ and $\theta = \pm 2\alpha$ in Fig. 5.2.10. This is true because the fringing fields would extend beyond the edge of the overlapping plates a distance on the order of g . Hence we cannot expect the period of oscillation given by (h) to be correct unless $\theta_m R \gg g$.

5.2.2 Loss-Dominated Systems

The approach to the analysis of nonlinear problems in Section 5.2.1 took advantage of the small mechanical and electrical energy dissipations. In this section we wish to illustrate how simplifying assumptions can be made valid when there is a *large* effect from damping mechanisms. We can illustrate briefly the notion involved by returning to the example of the pendulum used in Section 5.2.1 (Fig. 5.2.1). If there is viscous damping (with coefficient B),

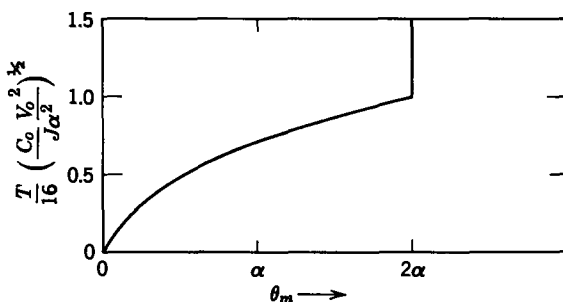


Fig. 5.2.11 Normalized period of oscillation for the electric pendulum of Fig. 5.2.10a as a function of peak amplitude.

(5.2.1) for the deflection of the pendulum is replaced by (see Section 2.2.1b for a discussion of torsional viscous dampers)

$$J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} = -Mgl \sin \theta. \quad (5.2.18)$$

In the analysis presented in Section 5.2.1 the viscous term was implicitly assumed small compared with the other terms in the equation. In the opposite extreme the viscous damping is so large that the inertial effects of the first term are ignorable. This would be the result of immersing the pendulum in heavy oil. Then we can approximate (5.2.18) by

$$B \frac{d\theta}{dt} = -Mgl \sin \theta, \quad (5.2.19)$$

which is a simple nonlinear expression to integrate. Rearranging, we have

$$\frac{d\theta}{\sin \theta} = -\frac{Mgl}{B} dt. \quad (5.2.20)$$

If, when $t = 0$, $\theta = \theta_m$, we can integrate this equation between θ_m and $\theta(t)$

$$\int_{\theta_m}^{\theta(t)} \frac{d\theta}{\sin \theta} = -\frac{Mgl}{B} \int_0^t dt. \quad (5.2.21)$$

We carry out this integration to obtain*

$$t \left(\frac{Mgl}{B} \right) = -\ln \left[\frac{\tan(\theta/2)}{\tan(\theta_m/2)} \right]. \quad (5.2.22)$$

This expression for θ is plotted in Fig. 5.2.12.

We now consider an example of a nonlinear, lossy system, for which we can make simplifying assumptions to allow analytical integration of nonlinear equations, and analyze a time-delay relay in which the dynamic

* C. D. Hodgman, *Mathematical Tables from Handbook of Chemistry and Physics*, 9th ed., Chemical Rubber Publishing Co., Cleveland, Ohio.

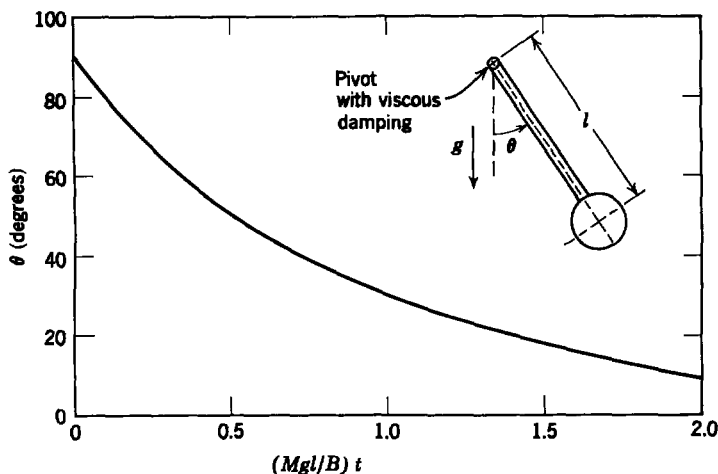


Fig. 5.2.12 Angular deflection θ of a pendulum heavily damped and released from $\theta_m = 90^\circ$.

behavior is controlled by mechanical losses. Although we treat a time-delay relay in this example, the approximations and analytical techniques are applicable to many other systems.

Example 5.2.4. In Fig. 5.2.13 we show schematically the basic actuator for a mechanically damped time-delay relay.* The basic operation to be analyzed is as follows: with switch S open, the spring pulls the plunger against a mechanical stop at $x = x_0$; when switch S is closed, current in the coil causes a magnetic force that pulls the plunger against the stop at $x = 0$. This displacement opens or closes relay contacts. The motion of the plunger when driven by the magnetic force is dominated by the mechanical damper. The damper, or dashpot, normally used in this application consists of a piston with a small orifice, moving in a cylinder filled with oil (see Fig. 2.2.10a). As discussed in Section 2.2.1b, a damper of this type is represented quite well by a damping force proportional to the square of the velocity [see (2.2.8)].

The electromechanical coupling of the configuration in Fig. 5.2.13 has been analyzed in Examples 2.1.1, 3.2.1, and 5.1.1. Neglecting fringing fields and assuming infinitely permeable magnetic material, we obtain the electrical and mechanical terminal relations from (a) and (c) of Example 5.1.1.

$$\lambda = \frac{L_o i}{1 + x/g} \quad (a)$$

$$f^e = - \frac{L_o i^2}{2g(1 + x/g)^2}, \quad (b)$$

where

$$L_o = \frac{\mu_0 N^2 (2wd)}{g}.$$

* A. E. Knowlton, ed., *Standard Handbook for Electrical Engineers*, 9th ed., McGraw-Hill, New York, 1957, Section 5-150.

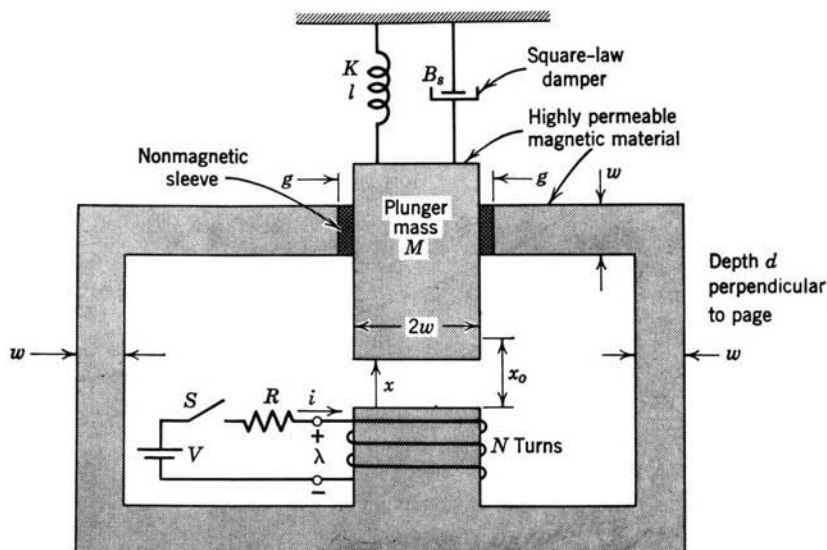


Fig. 5.2.13 Basic configuration of actuator for mechanically damped time-delay relay.

Thus, with switch S closed and the plunger between the stops

$$0 < x < x_o,$$

the equations of motion are

$$V = Ri + \frac{L_o}{1 + x/g} \frac{di}{dt} - \frac{L_o i}{g(1 + x/g)^2} \frac{dx}{dt}, \quad (c)$$

$$M \frac{d^2 x}{dt^2} \pm B_s \left(\frac{dx}{dt} \right)^2 + K(x - l) = - \frac{L_o i^2}{2g(1 + x/g)^2}. \quad (d)$$

The $+$ or $-$ sign on the damping force is chosen to make the damping force oppose the motion.

As already stated, we wish to analyze the transient that occurs when the switch S is closed at $t = 0$ with the initial conditions at $t = 0$

$$x = x_o, \quad i = 0. \quad (e)$$

The mechanical damper dominates the mechanical motion and slows down the motion of the plunger so that the closing time can be of the order of seconds to minutes. Consequently, the speed voltage, the last term in (c), can be neglected throughout the analysis, and the transient in current is complete before the position x changes significantly from the initial position x_o . Thus we find the current from the simplified equation

$$V = Ri + \frac{L_o}{1 + x_o/g} \frac{di}{dt}, \quad (f)$$

which has constant coefficients. The current transient is

$$i = \frac{V}{R} (1 - e^{-t/\tau_e}), \quad (g)$$

where

$$\tau_e = \frac{L_0}{R(1 + x_0/g)}.$$

The time constant τ_e is the electrical time constant with the gap fully open.

With the purely electrical transient completed before the mechanical motion starts and with the neglect of the speed voltage in (c) the mechanical motion occurs with the current essentially constant. Thus from the viewpoint of the mechanical motion the voltage source V and resistance R form an effective current source. This situation, in which electrical time constants are much shorter than mechanical time constants (see Section 5.1.3), occurs often in electromechanical transducers and is the source of significant simplification.

As stated before, the mechanical motion is dominated by the damper. This means that except for the very short time during which the mass initially accelerates, the damping force, the second term on the left of (d), is much greater than the acceleration force and the spring force. Consequently, when our interest is in the time required for the air gap to close, we can simplify (d) to

$$-B_s \left(\frac{dx}{dt} \right)^2 = -\frac{L_0(V/R)^2}{2g(1 + x/g)^2}, \quad (h)$$

where we have chosen the minus sign because $B_s(dx/dt)^2$ must act to retard the motion of the plunger in the $-x$ -direction. Solution of this expression for the velocity yields

$$\frac{dx}{dt} = -\frac{\sqrt{L_0/2gB_s} V/R}{1 + x/g}. \quad (i)$$

We have specified a minus sign because we know from the initial conditions and the physical nature of the problem that x is decreasing.

With the initial condition as specified by (e), we integrate (i) to find the time t at which the plunger is at position x as

$$\int_{x_0}^x \left(1 + \frac{x'}{g} \right) dx' = -\int_0^t \left(\frac{L_0}{2gB_s} \right)^{1/2} \frac{V}{R} dt'. \quad (j)$$

Integration of this expression yields

$$(x_0 - x) + \frac{(x_0^2 - x^2)}{2g} = \left(\frac{L_0}{2gB_s} \right)^{1/2} \frac{V}{R} t. \quad (k)$$

We find x as a function of time by solving this quadratic equation. The result is

$$x = -g + \left[(g + x_0)^2 - \left(\frac{2L_0g}{B_s} \right)^{1/2} \frac{V}{R} t \right]^{1/2} \quad (l)$$

Note that this expression satisfies the initial condition that at $t = 0$, $x = x_0$. Because the other root of (h) does not satisfy this condition, it is extraneous.

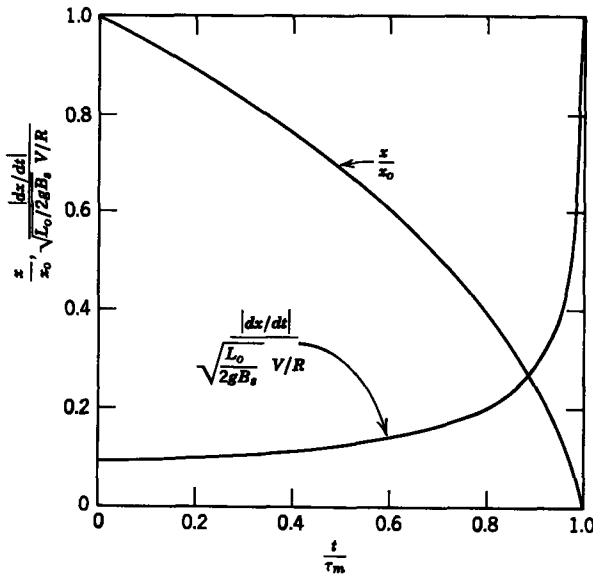


Fig. 5.2.14 Dynamic response of system of Fig. 5.2.13.

Alternatively, if we wish to evaluate the time τ_m necessary for the airgap to close, we set $x = 0$ in (k) and obtain

$$\tau_m = \left(\frac{2gB_s}{L_0} \right)^{1/2} \frac{R}{V} x_0 \left(1 + \frac{x_0}{2g} \right). \quad (m)$$

It is clear from this expression how the closing time can be controlled by the damping constant B_s or by the current V/R .

The response of the position as a function of time (l) is plotted in Fig. 5.2.14. For this purpose (l) is normalized in the following way:

$$\frac{x}{x_0} = -\frac{g}{x_0} + \left[\left(\frac{g}{x_0} + 1 \right)^2 - \left(\frac{2g}{x_0} + 1 \right) \frac{t}{\tau_m} \right]^{1/2}, \quad (n)$$

and we assume the ratio

$$\frac{g}{x_0} = 0.1.$$

Also plotted in Fig. 5.2.14 is the magnitude of the velocity, normalized, by using (i) and (n) in the form

$$\frac{|dx/dt|}{(L_0/2gB_s)^{1/2} V/R} = \left[\left(1 + \frac{x_0}{g} \right)^2 - \frac{x_0}{g} \left(2 + \frac{x_0}{g} \right) \frac{t}{\tau_m} \right]^{-1/2}. \quad (o)$$

We note from the curves of Fig. 5.2.14 that the plunger moves at almost constant velocity over most of the travel and then accelerates markedly. This results because the magnetic force (b) increases rapidly as $x \rightarrow 0$. A characteristic of this type is desirable in a time-delay relay to ensure that the contacts will close rapidly to avoid arcing. In any given situation,

however, the validity of ignoring the acceleration and speed voltage effects in this last region should be examined.

The curve of velocity in Fig. 5.2.14 indicates that this system behaves much like a linear statically unstable system (see curve *A* of Fig. 5.1.7). In some cases it may be desirable to approximate the equations of motion by a linear set with a static instability. This is especially true if the nonlinear expressions cannot be integrated analytically.

Although we have confined our attention in this section to discussing examples in which the damping is mechanical in nature, electrical damping can also dominate the dynamics. An example in which this is the case was discussed in Section 5.1.3. There the dynamic behavior of a coil rotating in a magnetic field was discussed in the limit at which the inductive reactance could be ignored [condition of (5.1.32)]. This made it possible to reduce the solution of the nonlinear motions to the problem of integrating (5.1.34), a procedure that is analogous to integrating (5.2.19).

5.3 DISCUSSION

In this chapter there have been two objectives. For the first, important types of dynamical behavior have been illustrated in which attention has been given to the relation of basic electromechanical interactions to mathematical models. For the second objective we have formed a basis on which to build an understanding of continuum interactions. In this regard both the mathematical techniques and physical approximations of this chapter are important in the chapters that follow.

PROBLEMS

5.1. Two parallel, perfectly-conducting plates are constrained as shown in Figure 5P.1 in such a way that the bottom plate is fixed and the top one is free to move only in the x -direction. A field is applied between the plates by the voltage source $v(t)$. When $x = 0$, the spring is in its equilibrium position.

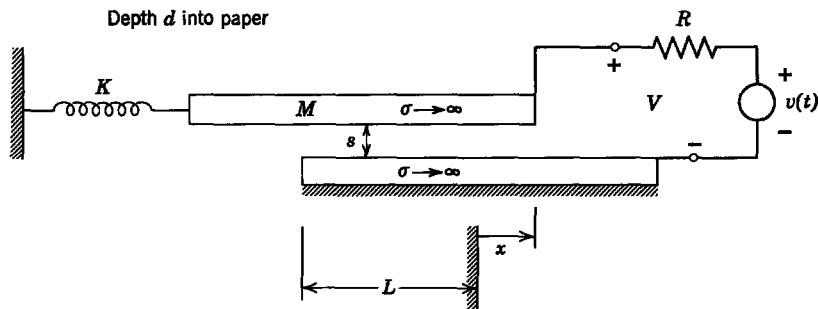


Fig. 5P.1

- What is the force of electric origin exerted on the upper plate?
- Write the complete equations of motion for this system.
- If $R = 0$ and $v(t) = (V_o \sin \omega t)u_{-1}(t)$, where $u_{-1}(t)$ is the unit step function, what is $x(t)$? Assume that the system is in static equilibrium when $t < 0$.

5.2. The system illustrated in Fig. 5P.2 is a schematic model of a differential transformer, which is a device for measuring small changes in mechanical position electrically. The movable core is constrained by bearings (not shown) to move in the x -direction. The two excitation windings, each having N_1 turns, are connected in series with relative polarity such that, when the movable core is centered as shown, there is no coupling between the excitation circuit and the signal winding. When the core moves from the center position in either direction, a voltage is induced in the signal winding. In the analysis neglect fringing fields and assume that the magnetic material is infinitely permeable.

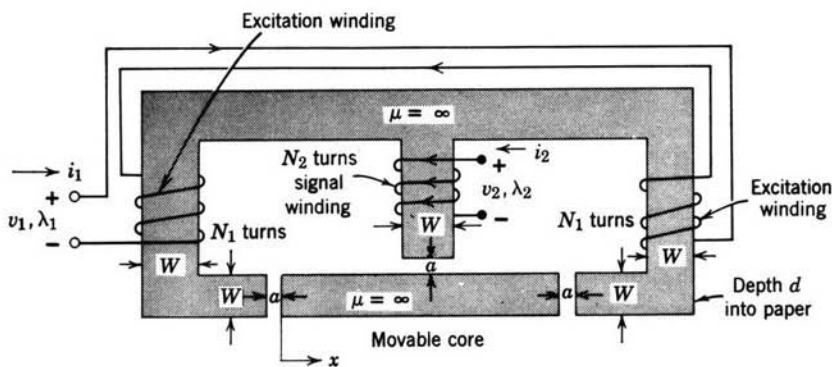


Fig. 5P.2

- Calculate the lumped-parameter equations of state, $\lambda_1(i_1, i_2, x)$, $\lambda_2(i_1, i_2, x)$, for this system.
- Terminal pair 1 is constrained by a current source $i_1 = I_o \cos \omega t$ and the system operates in the steady state. The open-circuit voltage v_2 is measured. Calculate the amplitude and phase angle of v_2 as a function of displacement x for the range $-a < x < a$.

5.3. A pair of highly conducting plates is mounted on insulating sheets, as shown in Fig. 5P.3a. The bottom sheet is immobile and hinged to the top sheet along the axis A . The top sheet is therefore free to rotate through an angle ψ . A torsion spring tends to make $\psi = \psi_o$ so that there is a spring torque in the $+\psi$ -direction $T^s = K(\psi_o - \psi)$. A source of charge $Q(t)$ is shunted by a conductance G and connected by flexible leads between the conducting plates. We wish to describe mathematically the motion of the upper plate [e.g., find $\psi(t)$]. To do this complete the following steps:

- Find the static electric field $\mathbf{E}$ between the two flat, perfectly conducting plates shown in Fig. 5P.3b. Assume that each of the plates extends to infinity in the r - and z -directions.
- If the angle ψ is small [$\psi a \ll D$, $\psi a \ll (b - a)$] so that fringing fields are not important, the electric field of part (a) can be used to approximate $\mathbf{E}$ between the metal plates of Fig. 5P.3a. Under this assumption find the charge q on the upper metal plate. Your answer should be in the form of $q = q(V, \psi)$ and is the electrical terminal relation for the block diagram of Fig. 5P.3c.

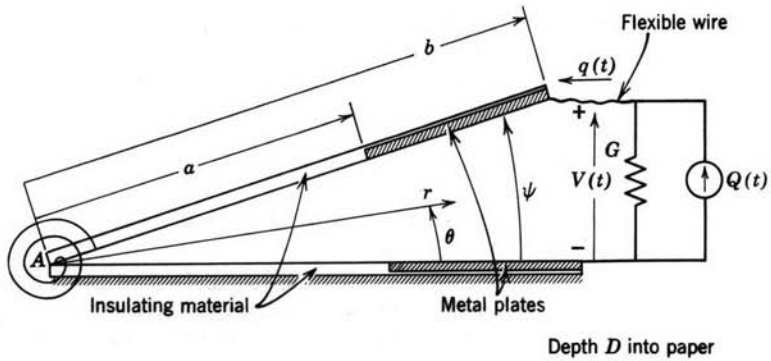


Fig. 5P.3a

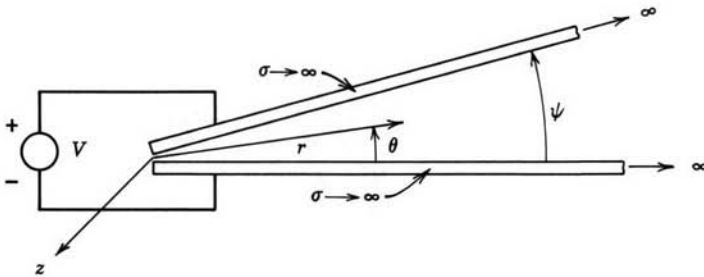


Fig. 5P.3b

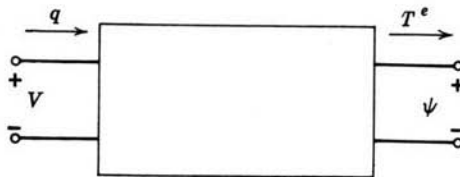


Fig. 5P.3c

- (c) Use part (b) to find the electrical energy stored between the plates $W = W(q, \psi)$ and, using W , find the torque of electrical origin T^e exerted by the field on the movable plate.
- (d) Write two differential equations that, with initial conditions, define the motion of the top plate. These equations should be written in terms of the two dependent variables $\psi(t)$ and $q(t)$. The driving function $Q(t)$ is known and the movable plate has a moment of inertia J .
- (e) Use the equations of part (d) to find the sinusoidal steady-state deflection $\psi(t)$ if $G = 0$ and $Q(t) = Q_o \cos \omega t$. You may wish to define ψ as $\psi = \psi_1 + \psi'(t)$, where ψ_1 is a part of the deflection which is independent of time. Identify the steady-state frequency at which the plate vibrates and give a physical reason why this answer would be expected.

5.4. This is a continuation of Problem 3.4, in which the equations of motion for the system shown in Fig. 3P.4 were developed.

- The resistance R is made large enough to be ignorable and the current $I(t) = I_o$, where I_o is a constant. Write the equation of motion for $x(t)$.
- Use a force diagram (as in Example 5.1.1) to determine the position $x = x_o$, where the mass can be in static equilibrium, and show whether this equilibrium is stable.
- With the mass M initially in static equilibrium, $x(0) = x_o$, it is given an initial velocity v_o . Find $x(t)$ for $x \approx x_o$.

5.5. Two small spheres are attached to an insulating rod, and a third sphere is free to slide between them. Each of the outside spheres has a charge Q_1 , whereas the inside sphere has a charge Q_o and a mass M . Hence the equation of motion for the inside sphere is

$$M \frac{d^2x}{dt^2} = \frac{Q_o Q_1}{4\pi\epsilon(d+x)^2} - \frac{Q_1 Q_o}{4\pi\epsilon(d-x)^2}.$$

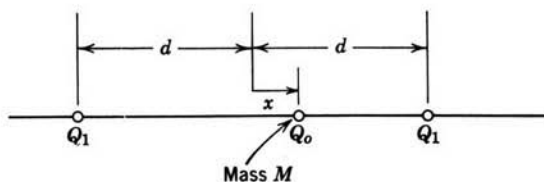


Fig. 5P.5

- For what values of Q_o and Q_1 will the movable sphere have a stable static equilibrium at $x = 0$? Show your reasoning.
- Under the conditions of (a), what will be the response of the sphere to an initial small static deflection $x = x_o$? $x_o \ll d$. (When $t = 0$, $x = x_o$, $dx/dt = 0$.)

5.6. Figure 5P.6 shows a sphere of magnetic material in the magnetic field of a coil. The coenergy of the coil is

$$W'(i, x) = \frac{L_o}{2} \left[1 - \left(\frac{x}{b} \right)^4 \right] i^2,$$

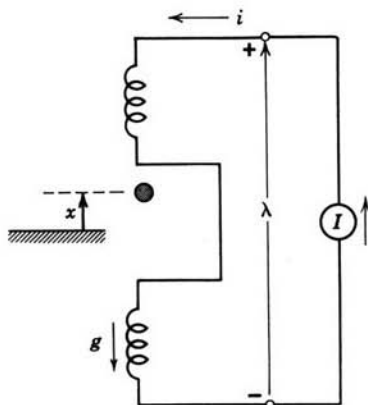


Fig. 5P.6

where L_o is positive. The sphere has a mass M and is subject to a gravitational force Mg (as shown).

- Write the differential equation that determines $x(t)$.
- Find the equilibrium position(s).
- Show whether this (these) equilibrium position(s) is (are) stable.

5.7. In Problem 3.15 we developed the equation of motion for a magnetic wedge, and it was shown that, with a constant current $i = I_o$ applied, the wedge could be in static equilibrium at $\theta = 0$. Under what conditions is this equilibrium stable?

5.8. The system shown in Fig. 5P.8 is one third of a system (governing vertical motion only) for suspending an airfoil or other test vehicle in a wind tunnel without mechanical support. The mass M_o , which represents the airfoil, contains magnetizable material and is constrained by means not shown to move in the vertical direction only. The system is designed so that the main supporting field is generated by current i_1 and the stabilizing field is generated by current i_2 . Over the range of positions (x) of interest, the electrical terminal relations may be expressed as:

$$\lambda_1(i_1, i_2, x) = \frac{L_1 i_1}{(1 + x/a)^3} + \frac{M i_2}{(1 + x/a)^3},$$

$$\lambda_2(i_1, i_2, x) = \frac{M i_1}{(1 + x/a)^3} + \frac{L_2 i_2}{(1 + x/a)^3},$$

where a , L_1 , L_2 , and M are positive constants and $M^2 < L_1 L_2$.

- Find the force of electric origin $f^e(i_1, i_2, x)$ acting on mass M_o .
- Set $i_1 = I$, a constant current, and set $i_2 = 0$. Find the equilibrium position X_o where f^e is just sufficient to balance the gravitational force on the mass M_o .
- With the currents as specified in part (b), write the linear incremental differential equation that describes the motion of mass M_o for small excursions $x'(t)$ from the equilibrium X_o . If an external force $f(t) = I_o u_o(t)$ (an impulse) is applied to the mass in the positive x -direction with the mass initially at rest, find the response $x'(t)$.
- For stabilization of the equilibrium at X_o a feedback system, which uses a light source, photoelectric sensor, and amplifiers, supplies a current i_2 such that $i_2(t) = \alpha x'(t)$, where α is a real constant. Keeping $i_1 = I$, write the equation of motion for $x'(t)$. For what range of α is the impulse response $x'(t)$ bounded?

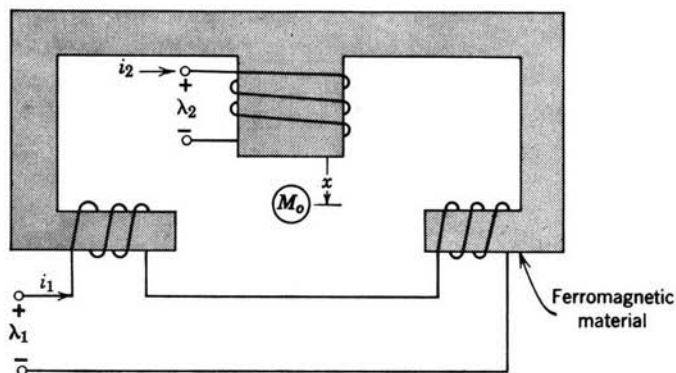


Fig. 5P.8

- (e) To make the impulse response tend to zero as $t \rightarrow \infty$, the signal from the photoelectric sensors is operated on electronically to produce a current i_2 such that

$$i_2 = \alpha x'(t) + \beta \frac{dx'}{dt},$$

where α and β are real constants. Again, write the equation of motion for $x'(t)$.

For what ranges of α and β does the impulse response $x'(t)$ tend to zero as $t \rightarrow \infty$?

5.9. A conservative magnetic field transducer for which variables are defined in Fig. 5P.9 has the electrical equation of state $\lambda = Ax^3i$, for $x > 0$, where A is a positive constant. The system is loaded at its mechanical terminals by a spring, whose spring constant is K

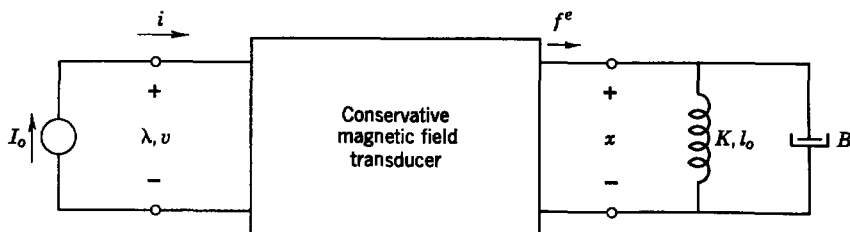


Fig. 5P.9

and whose force is zero when $x = l_o$, and a mechanical damper with the constant B . The electrical terminals are excited by a direct-current source I_o with the value

$$I_o = \left(\frac{K}{8Al_o} \right)^{1/4}.$$

- Write the mechanical equation in terms of f^e .
- Find f^e in terms of data given above.
- Find by *algebraic* techniques the possible equilibrium positions for the system and show whether each equilibrium point is stable.
- Check the results of part (c) by using *graphical* techniques to investigate the stability of the equilibrium points.

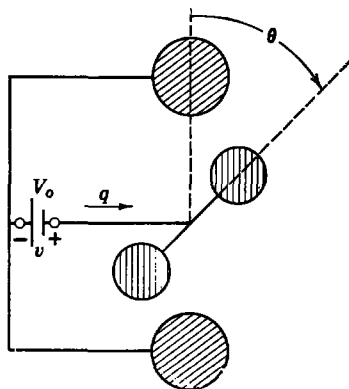


Fig. 5P.10

5.10. An electric field system has a single electrical terminal pair and one mechanical degree of freedom θ (Fig. 5P.10). The electrical terminal variables are related by $q = C_o(1 + \cos 2\theta)v$, where θ is the angular position of a shaft. The only torques acting on this shaft are of electrical origin. The voltage $v = -V_o$, where V_o is a constant.

- At what angles θ can the shaft be in static equilibrium?
- Which of these cases represents a stable equilibrium? Show your reasoning.

5.11. Figure 5P.11 shows a diagrammatic cross section of a two-phase, salient-pole synchronous machine. The windings in an actual machine are distributed in many slots

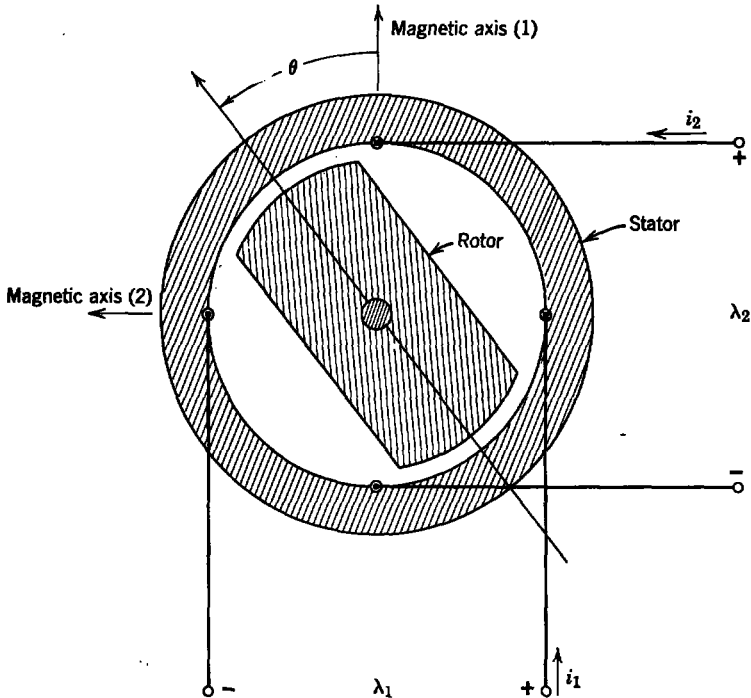


Fig. 5P.11

along the periphery of the stator, rather than as shown. The rotor is made of magnetically soft iron which has no residual permanent magnetism. The electrical terminal relations are given by

$$\lambda_1 = (L_o + M \cos 2\theta)i_1 + M \sin 2\theta i_2,$$

$$\lambda_2 = M \sin 2\theta i_1 + (L_o - M \cos 2\theta)i_2.$$

- Determine the torque of electrical origin $T^e(i_1, i_2, \theta)$.
- Assume that the machine is excited by sources such that $i_1 = I \cos \omega_s t$, $i_2 = I \sin \omega_s t$, and the rotor has the constant angular velocity ω_m such that $\theta = \omega_m t + \gamma$. Evaluate the instantaneous torque T^e . Under what conditions is it constant?

- (c) The rotor is subjected to a mechanical torque (acting on it in the $+\theta$ -direction): $T = T_o + T'(t)$, where T_o is a constant. The time-varying part of the torque perturbs the steady rotation of (b) so that $\theta = \omega_m t + \gamma_o + \gamma'(t)$. Assume that the rotor has a moment of inertia J but that there is no damping. Find the possible equilibrium angles γ_o between the rotor and the stator field. Then write a differential equation for $\gamma'(t)$, with $T'(t)$ as a driving function.
- (d) Consider small perturbations of the rotation $\gamma'(t)$, so that the equation of motion found in (c) can be linearized. Find the response to an impulse of torque $T'(t) = I_o \mu_o(t)$, assuming that before the impulse in torque the rotation velocity is constant.
- (e) Which of the equilibrium phase angles γ_o found in (c) is stable?

5.12. An electromechanical model for a magnetic transducer is shown in Fig. 5P.12. A force $f(t)$ is to be transduced into a signal $v_o(t)$ which appears across the resistance R . The

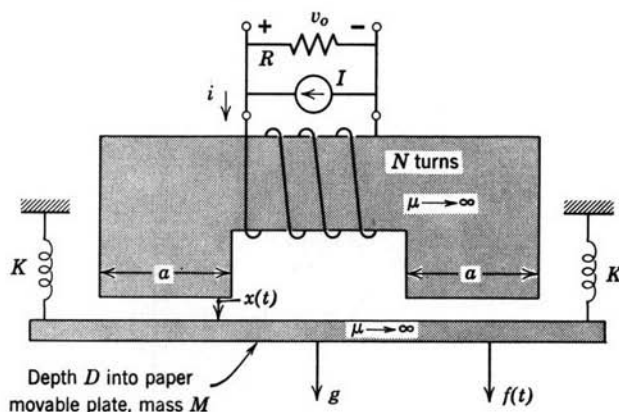
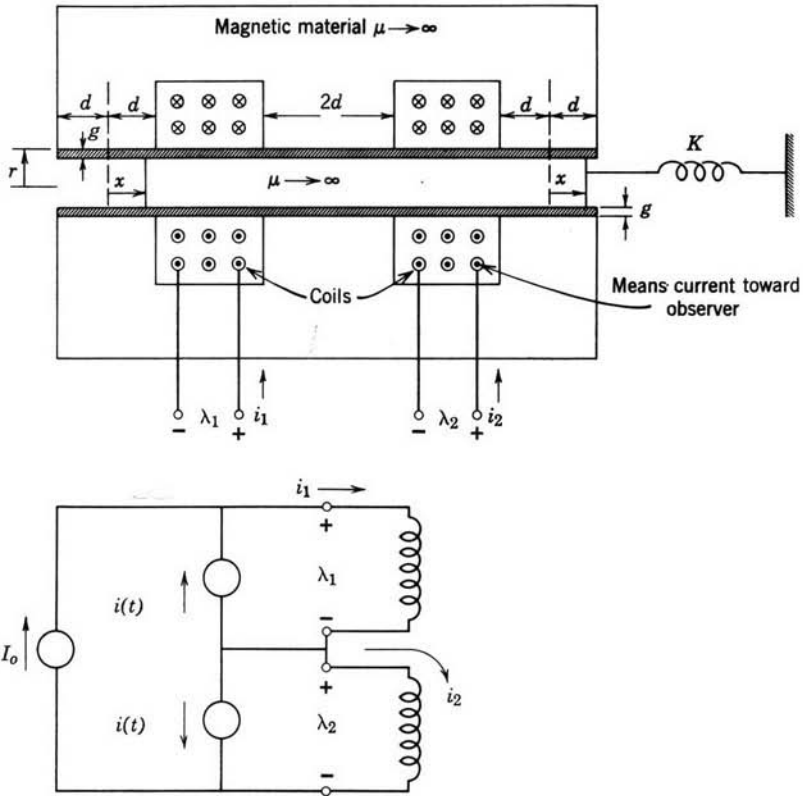


Fig. 5P.12

system is designed to provide linear operation about an equilibrium where the coil is excited by a constant current I . The plate is constrained at each end by springs that exert no force when $x = 0$.

- (a) Find the force of electrical origin $f^e(x, i)$ on the plate in the x -direction.
- (b) Write the equations of motion for the system. These should be two equations in the dependent variables (i, x) .
- (c) The static equilibrium is established with $f(t) = 0$, $x = X$, and $i = I$. Write the equilibrium force equation that determines X . Use a graphical sketch to indicate the equilibrium position X at which the system is stable. Assume in the following that the system is perturbed from this stable static equilibrium.
- (d) The resistance R is made large enough so that the voltage drop across the resistance is much larger than that across the self-inductance of the coil. Use this fact as the basis for an approximation in the electrical equation of motion. Assume also that perturbations from the equilibrium conditions of (c) are small enough to justify linearization of the equations. Given that $f(t) = \text{Re} [\hat{f} e^{j\omega t}]$, $v_o(t) = \text{Re} [\hat{v}_o e^{j\omega t}]$, solve for the frequency response $\hat{v}_o / \hat{f}$.

5.13. The cross-section of a cylindrical solenoid used to position the valve mechanism of a hydraulic control system is shown in Fig. 5P.13. When the currents i_1 and i_2 are equal, the plunger is centered horizontally ($x = 0$). When the coil currents are unbalanced, the plunger



Note. When $x = 0$, the spring force is zero!

Fig. 5P.13

moves a distance x . The nonmagnetic sleeves keep the plunger centered radially. The mass of the plunger is M , the spring constant K , and the viscous friction coefficient is B . The displacement x is limited to the range $-d < x < d$. You are given the terminal conditions

$$\begin{bmatrix} \lambda_1 = L_{11}i_1 + L_{12}i_2 \\ \lambda_2 = L_{12}i_1 + L_{22}i_2 \end{bmatrix}$$

where

$$L_{11} = L_o \left(3 - 2 \frac{x}{d} - \frac{x^2}{d^2} \right),$$

$$L_{22} = L_o \left(3 + 2 \frac{x}{d} - \frac{x^2}{d^2} \right),$$

$$L_{12} = L_o \left(1 - \frac{x^2}{d^2} \right).$$

- Write the mechanical equation of motion.
- Assume that the system is excited by the bias current I_0 and the two signal current sources $i(t)$ in the circuit of Fig. 5P.13 with the restriction that $|i(t)| \ll I_0$. Linearize the mechanical equation of motion obtained in part (a) for this excitation.
- Is the system stable for all values of I_0 ?
- The system is under damped. Find the response $x(t)$ to a step of signal current $i(t) = I u_{-1}(t)$.
- Find the steady-state response $x(t)$ to a signal current $i(t) = I \sin \omega t$.

5.14. A plane rectangular coil of wire can be rotated about its axis as shown in Fig. 5P.14. This coil is excited electrically through sliding contacts and the switch S by the constant-current source I in parallel with the conductance G . A second coil, not shown, produces a

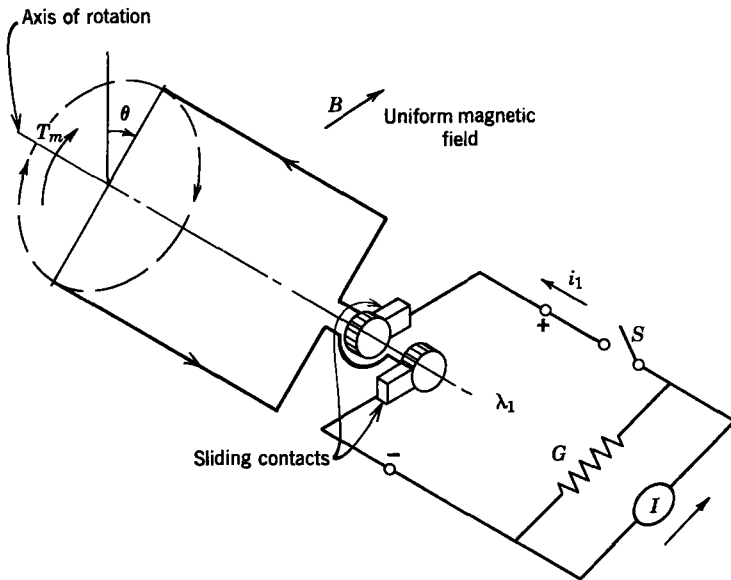


Fig. 5P.14

uniform magnetic field perpendicular to the plane of the rectangular coil when $\theta = 0$. Assume that the terminals of the second coil are described by the variables i_2, λ_2 , so that we can write the electrical terminal relations as $\lambda_1 = L_1 i_1 + i_2 M \cos \theta$, $\lambda_2 = i_1 M \cos \theta + L_2 i_2$, where L_1, M , and L_2 (the self- and maximum mutual inductances of the coils) are constants. Concentrate attention on the electrical variables of the rotating coil by assuming that $i_2 = I_2 = \text{constant}$. This is the excitation that provides the uniform constant magnetic field. In addition, assume that the mechanical position is constrained by the source $\theta = \Omega t$.

- Write the electrical equation for the coil. This equation, together with initial conditions, should determine $i_1(t)$.

- (b) Assume that the switch S is closed at $t = 0$; that is, the initial conditions are, when $t = 0$, $\theta = 0$ and $i_1 = 0$. Find the current $i_1(t)$.
- (c) Find the flux $\lambda_1(t)$ that links the rotating coil.
- (d) Consider the limiting case of (b) and (c) in which the current i_1 can be considered as constrained by the current source.

$$\Omega GL_1 \ll 1 \quad \text{and} \quad \Omega GL_1 \ll \frac{L_1 I}{M I_2}.$$

Sketch $i_1(t)$ and $\lambda_1(t)$.

- (e) Consider the limiting case in which the electrical terminals can be considered to be constrained to constant flux $\Omega GL_1 \gg 1$ and sketch $i_1(t)$ and $\lambda_1(t)$.
- (f) Compute the instantaneous torque of electrical origin on the rotating coil.
- (g) Find the average power required to rotate the coil. Sketch this power as a function of the normalized conductance ΩGL_1 . For what value of ΩGL_1 does G absorb the maximum power?

5.15. In the system illustrated in Fig. 5P.15 the lower capacitor plate is fixed and the upper plate is constrained to move only in the x -direction. The spring force is zero when $x = l$, and the damping with coefficient B is so large that we can neglect the mass of the movable

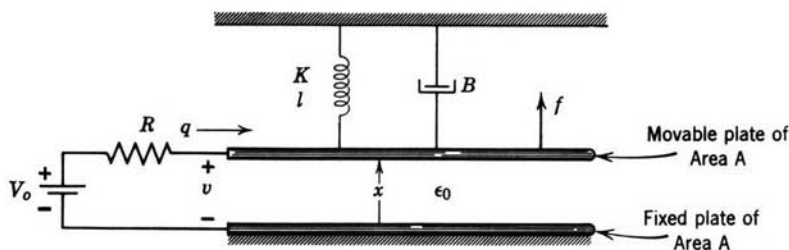


Fig. 5P.15

plate. The capacitor plates are excited by a constant voltage source in series with a resistance R . Neglect fringing fields. The voltage V_0 is adjusted so that a stable, static equilibrium occurs at $X_0 = 0.7l$. With the system at rest at this equilibrium position, a small step of mechanical force f is applied: $f = Fu_{-1}(t)$. In all of your analyses assume that the perturbations from equilibrium are small enough to allow use of linear incremental differential equations.

- (a) Calculate the resulting transient in position x .
- (b) Specify the condition that must be satisfied by the parameters in order that the mechanical transient may occur essentially at constant voltage. Sketch and label the transient under this condition.
- (c) Specify the condition that must be satisfied by the parameters in order that the initial part of the mechanical transient may occur essentially at constant charge. Sketch and label the transient under this condition.

5.16. A mass M has the position $x(t)$. It is subjected to forces f_1 and f_2 which have the dependence on x shown in Fig. 5P.16. The mass is released at $x = 0$ with the velocity v_0 . In terms of F_0 and K , what is the largest value of v_0 that will lead to bounded displacements of M ?

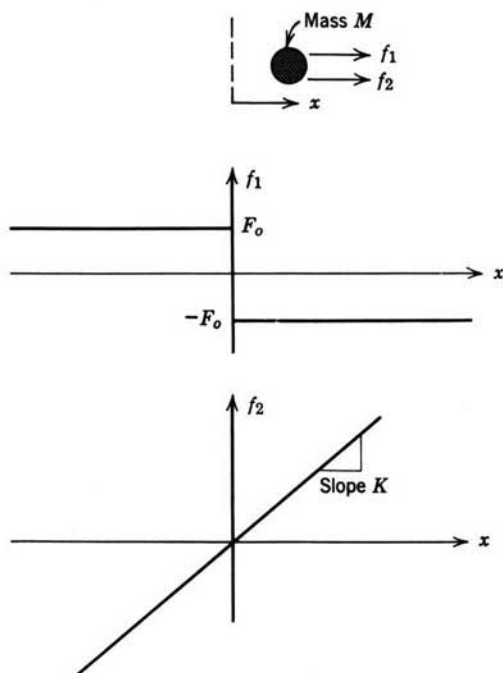


Fig. 5P.16

5.17. The electric field transducer shown in Fig. 5P.17 has two electrical terminal pairs and a single mechanical terminal pair. Both plates and movable elements can be regarded as perfectly conducting.

- Find the electrical terminal relations $q_1 = q_1(v_1, v_2, x)$, $q_2 = q_2(v_1, v_2, x)$.
- Now the terminals are constrained so that $v_2 = V_o = \text{constant}$ and $q_1 = 0$. Find the energy function $U(x)$ such that the force of electrical origin acting in the x -direction on the movable element is

$$f^e = -\frac{\partial U}{\partial x}.$$

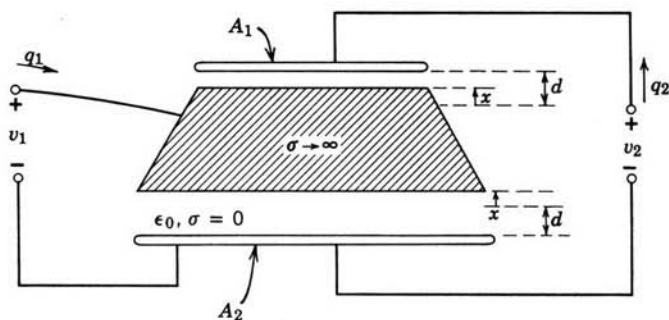


Fig. 5P.17

5.18. The central plate of a three-plate capacitor system (Fig. 5P.18) has one mechanical degree of freedom, x . The springs are relaxed in the equilibrium position $x = 0$ and fringing can be neglected. For a long time the system is maintained with the central plate fixed at $x = 0$ and the switch closed. At $t = 0$ the switch is opened and the center plate is released simultaneously.

- (a) Find, in terms of given parameters, a hybrid energy function W' such that $f^e = -\partial W'/\partial x$ for $t > 0$.

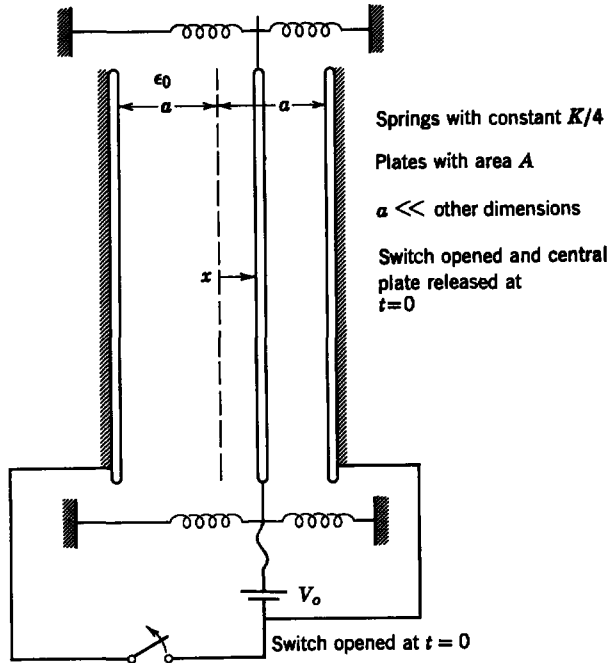


Fig. 5P.18

- (b) Determine the criterion that the central plate be in stable equilibrium at $x = 0$.
 (c) In the case in which the criterion of part (b) is satisfied, sketch a potential well diagram for $-a \leq x \leq a$, indicating all static equilibrium points, and whether they are stable or unstable.

5.19. An electromechanical system with one electrical and one mechanical terminal pair is shown in Fig. 5P.19. The electrical terminal relation is

$$\lambda = \frac{L_0 i}{(1 - x/a)^4},$$

where L_0 and a are given constants. The system is driven by a voltage $V_0 + v(t)$, where V_0 is constant. The mass of the plunger can be ignored. Gravity acts on M as shown.

- (a) Write the complete equations of motion for the system. There should be two equations in the unknowns i and x .

- (a) Write the equation of motion in the form

$$J \frac{d^2\theta}{dt^2} + \frac{\partial V}{\partial \theta} = 0.$$

(see Section 5.2.1) and sketch the potential well.

- (b) Indicate on the potential-well sketch the angular positions at which the rotor can be in stable static equilibrium and in unstable static equilibrium.
- (c) With the rotor initially at rest at $\theta = 0$, how much kinetic energy must be imparted to the rotor to make it rotate continuously?

5.22. The system of Fig. 5P.22 contains a simple pendulum with mass M and length l . The pivot has viscous (linear) friction of coefficient B . The mass is made of ferromagnetic material. It causes a variation of coil inductance with angle θ that can be represented

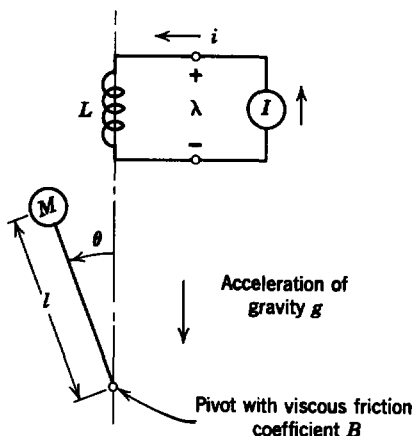


Fig. 5P.22

approximately by the expression $L = L_0(1 + 0.2 \cos \theta + 0.05 \cos 2\theta)$, where L_0 is a positive constant. The coil is excited by a constant-current source I at a value such that $I^2 L_0 = 6Mg/l$ with no externally applied forces other than gravity g .

- (a) Write the mechanical equation of motion for the system.
- (b) Find all of the possible static equilibria and show whether or not each one is stable.

5.23. The one-turn inductor shown in Fig. 5P.23 is made from plane parallel plates with a spacing w and depth (into the paper) D . The plates are short-circuited by a sliding plate in the position $x(t)$. This movable plate is constrained by a spring (constant K) and has a mass M .

- (a) Find the equation of motion for the plate, assuming that the electrical terminals are constrained to constant flux $\lambda = \Lambda = \text{constant}$.
- (b) Find the position(s) $x = X_0$ at which the plate can be in static equilibrium. Determine if each point represents a stable equilibrium. Can you assign an equivalent spring constant to the magnetic field for small-signal (linear) motions?

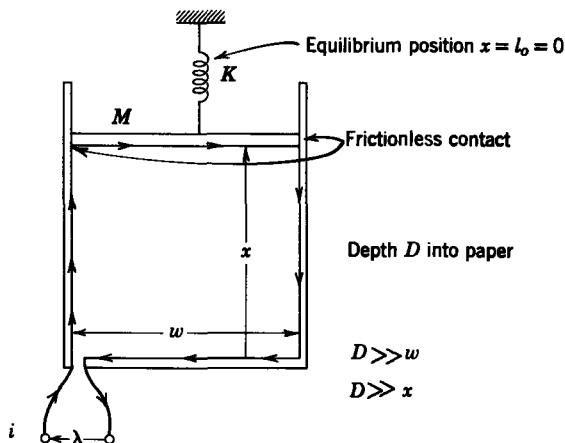


Fig. 5P.23

- (c) Use a potential-well argument to describe the nonlinear motions of the plate. Include in your discussion how you would use the initial conditions to establish the constant of the motion E .
- (d) Briefly describe the motions of the plate constrained so that $i = I = \text{constant}$.
- 5.24.** The terminals of the device shown in Problem 4.23 are now constrained to a constant value of λ : $\lambda = \Lambda_0 = \text{constant}$. The rotor has a moment of inertia J and is free of damping.
- When $\theta = 0$, the angular velocity $d\theta/dt = \Omega$. Find an analytical expression for $d\theta/dt$ at each value of θ . (Given the angle θ , this expression should provide the angular velocity.)
 - What is the minimum initial angular velocity required to make the rotor rotate continuously in one direction?
 - For what values of θ can the rotor be in static equilibrium? Which of these equilibria is stable?
 - Describe quantitatively the angular excursion of the rotor when it is given an initial angular velocity less than that found in (b).
- 5.25.** A mass M_1 attached to a weightless string rotates in a circle of radius r on a fixed frictionless surface as illustrated in Fig. 5P.25. The other end of the string is passed through a frictionless hole in the surface and is attached to a movable capacitor plate of mass M_2 . The other capacitor plate is fixed and the capacitor is excited by a voltage source $v(t)$. The necessary dimensions are defined in the figure. The length of the string is such that when $x = 0$, $r = 0$. You may assume that $a \gg x$ and ignore the effects of gravity and electrical resistance. With $v(t) = V_0 = \text{constant}$ and $r = l$, the mass M_1 is given an angular velocity ω_m necessary for equilibrium.
- Find the force of electromagnetic origin exerted on the capacitor plate.
 - Determine the equilibrium value of ω_m .
 - Show that the angular momentum $M_1 r^2 d\theta/dt$ is constant, even if $r = r(t)$ and $\theta = \theta(t)$. [See Problem 2.8 for writing force equations in (r, θ) coordinates.]
 - Use the result of (c) to write the equation of motion for $r(t)$. Write this equation in a form such that potential well arguments can be used to deduce the dynamics.
 - Is the equilibrium found in (b) stable?

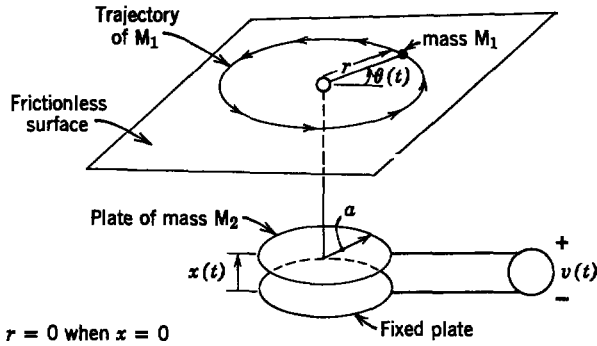


Fig. 5P.25

5.26. As an example of a lossy nonlinear system, consider the basic actuator for an electrically damped time-delay relay* illustrated in Fig. 5P.26. The transducer is designed to operate as follows. With switch S open, the spring holds the plunger against a mechanical stop at $x = x_o$. When switch S is closed, the magnetic field is excited, but the winding that is short-circuited through resistance R_2 limits the rate of buildup of flux to a low value. As the flux builds up slowly, the magnetic force increases. When the magnetic force equals the spring force, the plunger starts to move and close the air gap. The velocity of the plunger is so low that inertia and friction forces can be neglected; thus, when the plunger is moving, the spring force is at all times balanced by the magnetic force (see Section 5.2.2).

- Write the electrical circuit equations. The magnetic flux Φ is defined such that $\lambda_1 = N_1\Phi$ and $\lambda_2 = N_2\Phi$. Use these equations to find a single equation involving (Φ, x) with V as a driving function.
- Define two constants: the flux Φ_o linking the coils with the air gap closed ($x = 0$), $\Phi_o = 2\mu_0 wd N_1 V / g R_1$, and the time constant τ_o for flux buildup when the air gap is closed,

$$\tau_o = \frac{2\mu_0 wd}{g} \left(\frac{N_1^2}{R_1} + \frac{N_2^2}{R_2} \right).$$

Show that the result of (a) can be written in the form

$$\Phi_o = \left(1 + \frac{x}{g} \right) \Phi + \tau_o \frac{d\Phi}{dt}.$$

The transient behavior of this device can be divided into three intervals:

- The switch S is closed with the plunger at $x = x_o$ and with zero initial flux Φ . The flux builds up to a value necessary to provide a magnetic force equal to the spring force that is holding the plunger against the stop at $x = x_o$.
- The plunger moves from the stop at $x = x_o$ to the stop at $x = 0$. During this motion the spring force is the only appreciable mechanical force and is balanced by the magnetic force.
- The plunger is held against the stop at $x = 0$ by the magnetic force, whereas the flux Φ continues to build up to Φ_o .

* *Standard Handbook for Electrical Engineers*, 9th ed., McGraw-Hill, New York, 1957, Sections 5-150 and 5-168.

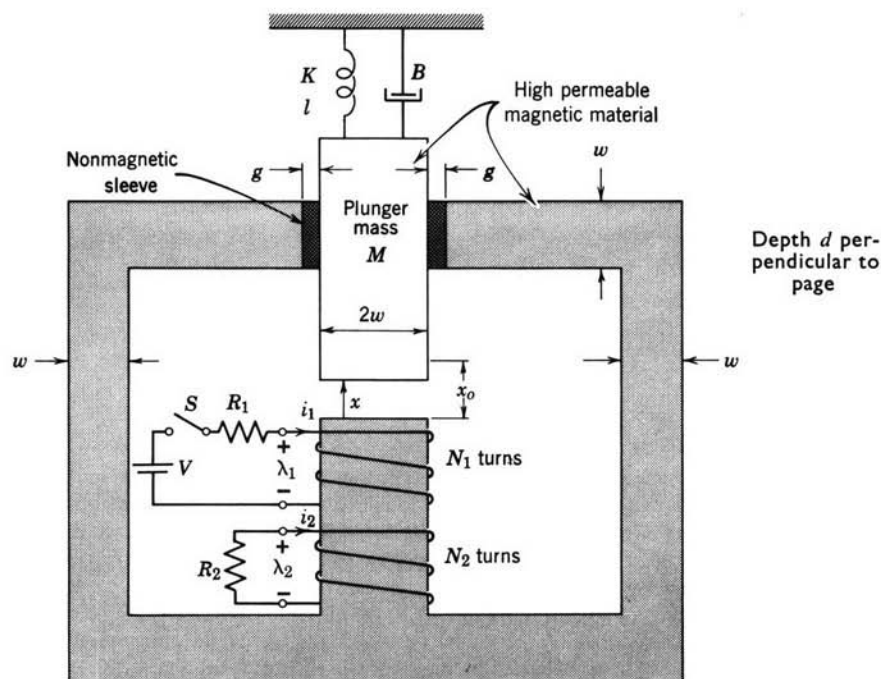


Fig. 5P.26

- (c) Determine the transient in Φ during interval (1). Write an equation of force equilibrium for the plunger to determine the flux $\Phi = \Phi_1$ when interval (1) ends.
- (d) Write an equation for Φ during interval (2). Assume the parameters

$$\frac{l}{x_o} = 2, \quad \frac{x_o}{g} = 4, \quad \frac{\Phi_o}{\Phi_1} = 10$$

and integrate the equation resulting from (c) to find $\Phi(t)$ in interval (2).

- (e) Find the transient in Φ during interval (3).
- (f) Sketch Φ and x as functions of time throughout the three intervals.

CHAPTER 6

FIELDS AND MOVING MEDIA

6.0 INTRODUCTION

In Chapter 1 we reviewed the basic postulates and definitions of electromagnetic theory. We defined the quasi-static electromagnetic field theory suitable for efficient analysis of low-frequency, low-velocity electromechanical systems. In Chapter 2 we used the quasi-static electromagnetic equations to calculate lumped parameters for important classes of electromechanical systems. The effect of mechanical motion on the electric or magnetic fields was accounted for by allowing the electrical lumped parameters to have a dependence on the mechanical displacements. In Chapter 3 this lumped-parameter model was used to determine the electric or magnetic forces on the mechanical system. We were then prepared for the study in Chapters 4 and 5 of the dynamics of lumped-parameter electromechanical systems.

In this chapter we return to the field description introduced in Chapter 1. This is necessary if we are to extend the class of electromechanical situations with which we can deal beyond the lumped-parameter systems of Chapters 2 to 5. In subsequent chapters we shall be treating continuum systems, that is, those in which both the electrical and mechanical parts of the system are described, at least in part, by partial differential equations. This necessitates a field description of the electromechanical coupling.

Even if we are concerned only with lumped-parameter systems, a field description of the electromechanical interaction provides a useful alternative to the lumped-parameter models of Chapters 2 to 5. In many cases forces of electrical origin can be most easily deduced from the fields themselves, thus bypassing the computation of lumped parameters and an energy function. Similarly, the effects of material motion on the electrical system can be deduced from field considerations. On the other hand, an understanding of lumped-parameter systems, based on the viewpoint of Chapters 2 to 5,

provides considerable insight into what is required of a field description that includes effects of moving media. By the time this chapter is completed, the reader should be acquainted with Appendix B, in which the subject of quasi-statics is reviewed.

A simple example explains how the following sections lead to a generalization of electromagnetic theory to include the effects of material motion. A disk of copper is shown in Fig. 6.0.1*a* as it moves with velocity $\mathbf{v}$ into a region of flux density $\mathbf{B}$ imposed by a magnet. (We could do this experiment by attaching a handle to the disk so that we could wield it through the magnetic field.) Our lumped-parameter model provides a qualitative description of what happens. Suppose we model the disk as a one-turn, perfectly conducting loop shorted through a resistance, as shown in Fig. 6.0.1*b*. Then, as the loop enters the magnetic field region, it links an increasing flux λ from the imposed magnetic field $\mathbf{B}$, and an induced current i flows in a direction that induces a magnetic field which tends to cancel the flux of the imposed field $\mathbf{B}$. As the loop enters the field, this current i , interacting with $\mathbf{B}$, gives rise to a force tending to retard the motion. As we know from Section 5.1.3, the nature of this force depends on the resistance R of the loop. For a copper disk of reasonable size, as we move it through the magnetic field, it is likely that there would be an impression of passing it through a viscous liquid. The important point is that there is a magnetic force on the disk, hence a current within the disk.

Suppose that we are to analyze this problem in terms of fields. As discussed in Chapter 1, we are concerned with solutions to field equations for a quasi-static magnetic field system (equations summarized in Table 1.2).

$$\nabla \times \mathbf{H} = \mathbf{J}_f, \quad (6.0.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (6.0.2)$$

$$\nabla \cdot \mathbf{J}_f = 0, \quad (6.0.3)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (6.0.4)$$

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \quad (\mathbf{M} \simeq 0 \text{ for copper}). \quad (6.0.5)$$

In addition, there are boundary conditions on the surfaces of the disk and magnet. Also, we need the continuum equivalent of Ohm's law, for that was used in the lumped parameter model to explain the experiment. The necessary constituent relation was introduced in Chapter 1 as (1.1.9) and is

$$\mathbf{J}_f = \sigma \mathbf{E} \quad (6.0.6)$$

where σ is the electrical conductivity of the disk material. One approach to solving the obviously difficult problem at hand is to guess a reasonable

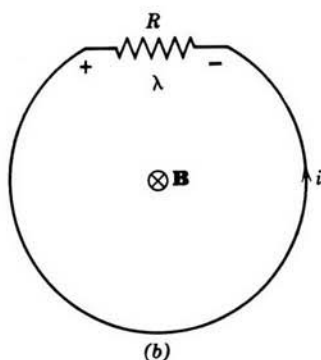
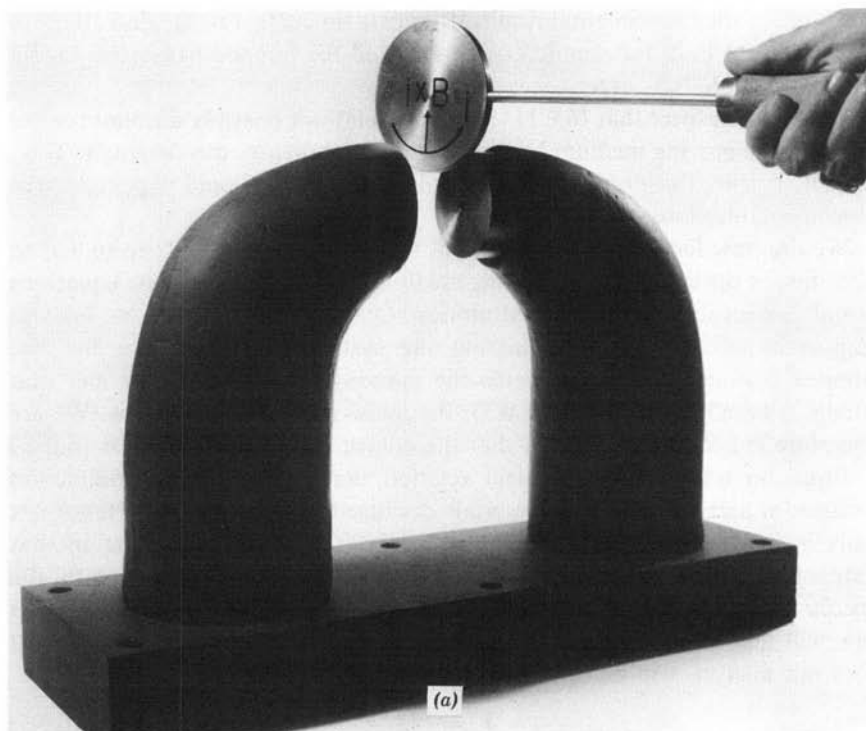


Fig. 6.0.1 (a) As the copper disk moves into the field $\mathbf{B}$, there is an induced current i which interacts with $\mathbf{B}$ to retard the motion; (b) conducting loop equivalent to the disk of copper.

solution and show that it satisfies all of the equations and boundary conditions. In the absence of the moving conductor we have the fields $\mathbf{H} = \mathbf{H}_s$, $\mathbf{E} = 0$, $\mathbf{B} = \mu_0 \mathbf{H}_s$ and $\mathbf{J}_f = 0$, where $\mathbf{H}_s$ is a function of position but is independent of time. By definition this solution satisfies (6.0.1) to (6.0.6). In fact, it satisfies these equations and boundary conditions even as the disk passes through the magnetic field!

What we have found is that our field equations in the form given do not

account for the experimental result. If there is no current in the disk, there is no force. Yet both the simple experiment and the lumped parameter model show that there is a force.

We might suspect that (6.0.1) to (6.0.6) could not possibly account for the effect of the moving medium because they do not involve the velocity $\mathbf{v}$. This, in fact, is true, but note that the boundary conditions could depend on the mechanical motions.

We are now faced with the question, how do we alter (6.0.1) to (6.0.6) to account for the effects of the moving medium? At least one of these equations is not correct if there is material motion. One possibility is that we left out important effects by approximating the system as quasi-static; but the lumped parameter model explains the induced force on the disk and that model is based on (6.0.1) to (6.0.5), the quasi-static field equations. We are therefore led to the conclusion that the culprit in our description is (6.0.6.)

Equation 6.0.6 is a constituent relation that represents the conduction process in a certain class of materials discussed in Section 1.1.1. Hence not only is it a law deduced from experiments [(6.0.1) to (6.0.6) are in that category] but it is found to hold for certain media in a particular state: the media are at rest. To analyze our experiment we must know what form this law will take when the material is in motion. Suffice it to say at this point that our analysis will be correct when we rewrite (6.0.6) as

$$\mathbf{J}'_r = \sigma \mathbf{E}', \quad (6.0.7)$$

where $\mathbf{J}'_r$ and $\mathbf{E}'$ are the current density and field intensity that would be measured by an observer moving with the material. Of course, we wish to formulate the problem in the laboratory reference frame in which the current density and electric field intensity are $\mathbf{J}_r$ and $\mathbf{E}$, respectively. Hence our first objective is to relate the field variables measured in a frame of reference moving with a constant velocity to the field variables measured in the laboratory frame. We then discuss constituent relations for moving media.

We confine our attention to quasi-static electric and magnetic field systems. As discussed in the following sections, this means that we consider Galilean transformations that are appropriate also for Newtonian mechanics. This approach is the logical extension of a division of electromechanics into electric and magnetic field systems. The relationship of Galilean and Lorentz transformations and the relevance of Einstein relativity are discussed at the appropriate points.

Our treatment of relative motion is based on two postulates: (a) the equations of motion, including Maxwell's equations, are always written for an inertial coordinate system, that is, a coordinate system that is traveling with a velocity of constant magnitude and fixed direction; and (b) the laws of physics (e.g., Newton's laws and Maxwell's equations) are the same in every

inertial coordinate system. These postulates are normally associated with Einstein relativity but they are also valid for Galilean systems.*

In Sections 6.3 and 6.4 examples that demonstrate how field transformations, boundary conditions, and constituent relations (the subjects of Sections 6.1 to 6.3) are used in the analysis of practical problems are considered. Emphasis is given to the magnetic field system; Section 6.4 is devoted to a class of rotating machines we were not prepared to discuss in detail in Chapter 4. These commutator-type machines, which are also of considerable practical significance, illustrate the fundamental point of this chapter.

6.1 FIELD TRANSFORMATIONS

In this treatment we are interested in field phenomena that occur in systems with material media in relative motion. The Lorentz force was introduced as the definition of the $\mathbf{E}$ and $\mathbf{B}$ fields in (1.1.28) of Chapter 1:

$$\mathbf{f} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}. \quad (6.1.1)$$

This expression states that a charge q in motion with velocity $\mathbf{v}$ with respect to an observer will experience the force $\mathbf{f}$ when subjected to the fields $\mathbf{E}$ and $\mathbf{B}$. Because Newton's laws must be the same in all inertial reference systems, another observer in a different reference frame will measure the same force on the charge but the charge will have a different velocity. It should be clear then that the two observers will measure different values of electric field intensity and magnetic flux density. The object of this section is to find the relations between electromagnetic quantities that are measured by two observers in uniform relative motion.

We have already stated that when an observer defines electromagnetic quantities he does so with the understanding that they are defined in his reference frame (coordinate system). Hence they are related by Maxwell's equations written in his coordinate system. It is a postulate of special relativity that physical laws, such as Maxwell's equations, must be the same in all inertial coordinate systems. We use this postulate to determine the relations between electromagnetic quantities measured in different inertial coordinate systems.

We define two inertial coordinate systems $\mathbf{r}$ and $\mathbf{r}'$ which are moving with a constant relative velocity $\mathbf{v}'$. The times t and t' measured by observers in the two coordinate systems are assumed to be the same

$$t = t'. \quad (6.1.2)$$

* For discussions of the postulates and consequences of the special theory of relativity see, for example, J. D. Jackson, *Classical Electrodynamics*, Wiley, New York, 1962, Chapters 11 and 12; L. Landau and E. Lifshitz, *The Classical Theory of Fields*, Addison-Wesley, Reading, Mass., 1951, Chapters 1 and 2.

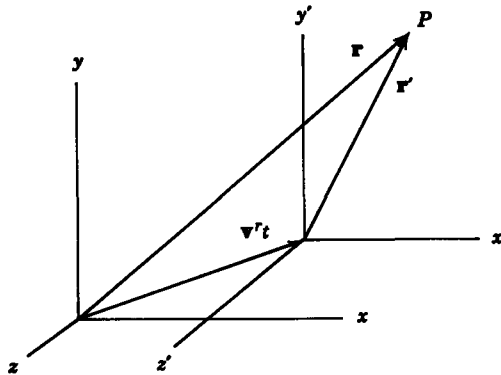


Fig. 6.1.1 Two inertial coordinate systems in relative motion.

We select the origins of the coordinate systems to coincide at $t = 0$. The relative geometry at a time t is illustrated in Fig. 6.1.1, from which we can obtain the relation between $\mathbf{r}$ and $\mathbf{r}'$, the instantaneous position vectors of point P , as measured in the two coordinate systems

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t. \quad (6.1.3)$$

Equations 6.1.2 and 6.1.3 define a *Galilean coordinate transformation* between inertial coordinate systems. We now show that this transformation is consistent with the quasi-static Maxwell equations. An analogous procedure can be used to show that the Galilean transformation is also consistent with all mechanical equations of motion introduced in Chapter 2 or in later chapters.

It is worthwhile to interject at this point that the general form of Maxwell's equations cannot be transformed consistently by means of the Galilean transformation. This, in fact, is the basis for a relativistic treatment, which demands that the transformation be consistent (that the equations be invariant) and results in the Lorentz transformation.* Because the relativistic terms usually make no significant contribution to the electromechanics, it is most convenient to work with the quasi-static equations from the outset, as is done here. This avoids our having to discuss effects that we would end up neglecting in a practical context.

We obtain our transformations for field variables from the differential equations; consequently, before we derive the transformations, we need to consider the differential operators in the two coordinate systems and how they are related. The space differential operator ∇ for cartesian coordinates

* For the relativistic treatment see, for example, J. A. Stratton, *Electromagnetic Theory*, McGraw-Hill, New York, 1941, pp. 59–82.

x, y, z of the unprimed coordinate system $\mathbf{r}$ is

$$\nabla = \left(\mathbf{i}_x \frac{\partial}{\partial x} + \mathbf{i}_y \frac{\partial}{\partial y} + \mathbf{i}_z \frac{\partial}{\partial z} \right). \quad (6.1.4)$$

The space differential operator ∇' for coordinates x', y', z' of the primed coordinate system $\mathbf{r}'$ is

$$\nabla' = \left(\mathbf{i}_x \frac{\partial}{\partial x'} + \mathbf{i}_y \frac{\partial}{\partial y'} + \mathbf{i}_z \frac{\partial}{\partial z'} \right). \quad (6.1.5)$$

To determine the relation between ∇ and ∇' we first write out the three coordinates of the vector equation (6.1.3).

$$x' = x - v_x^* t, \quad (6.1.6a)$$

$$y' = y - v_y^* t, \quad (6.1.6b)$$

$$z' = z - v_z^* t. \quad (6.1.6c)$$

Consider a function $f'(x, y, z, t)$ which can also be written as $f'(x', y', z', t')$ by making substitutions from (6.1.2) and (6.1.6). The gradient of this function in the primed coordinate system is

$$\nabla' f' = \mathbf{i}_x \frac{\partial f'}{\partial x'} + \mathbf{i}_y \frac{\partial f'}{\partial y'} + \mathbf{i}_z \frac{\partial f'}{\partial z'}. \quad (6.1.7)$$

The chain rule of differentiation* is used to write

$$\frac{\partial f'}{\partial x} = \frac{\partial f'}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial f'}{\partial y'} \frac{\partial y'}{\partial x} + \frac{\partial f'}{\partial z'} \frac{\partial z'}{\partial x} + \frac{\partial f'}{\partial t'} \frac{\partial t'}{\partial x}. \quad (6.1.8)$$

It is evident from (6.1.2) and (6.1.6) that

$$\frac{\partial x'}{\partial x} = 1; \quad \frac{\partial y'}{\partial x} = \frac{\partial z'}{\partial x} = \frac{\partial t'}{\partial x} = 0, \quad (6.1.9a)$$

$$\frac{\partial y'}{\partial y} = 1; \quad \frac{\partial x'}{\partial y} = \frac{\partial z'}{\partial y} = \frac{\partial t'}{\partial y} = 0, \quad (6.1.9b)$$

$$\frac{\partial z'}{\partial z} = 1; \quad \frac{\partial x'}{\partial z} = \frac{\partial y'}{\partial z} = \frac{\partial t'}{\partial z} = 0. \quad (6.1.9c)$$

We now use (6.1.8) and (6.1.9) with (6.1.7) to establish that

$$\nabla' f' = \nabla f'. \quad (6.1.10)$$

The scalar function f' may be a component of a vector; therefore we can use

* P. Franklin, *Methods of Advanced Calculus*, McGraw-Hill, New York, 1944, Chapter 2.

this same formalism to establish that for any vector $\mathbf{A}'(x', y', z', t')$ the space derivatives can be written as

$$\nabla' \cdot \mathbf{A}' = \nabla \cdot \mathbf{A}', \quad (6.1.11)$$

$$\nabla' \times \mathbf{A}' = \nabla \times \mathbf{A}'. \quad (6.1.12)$$

The same mathematical techniques are used to establish the relation between time derivatives. We again assume a function $f'(x', y', z', t')$ and write the time derivative in the unprimed system as

$$\frac{\partial f'}{\partial t} = \frac{\partial f'}{\partial t'} \frac{\partial t'}{\partial t} + \frac{\partial f'}{\partial x'} \frac{\partial x'}{\partial t} + \frac{\partial f'}{\partial y'} \frac{\partial y'}{\partial t} + \frac{\partial f'}{\partial z'} \frac{\partial z'}{\partial t}. \quad (6.1.13)$$

From (6.1.2) and (6.1.6) it follows that

$$\begin{aligned} \frac{\partial t'}{\partial t} &= 1; & \frac{\partial x'}{\partial t} &= -v_x^r, \\ \frac{\partial y'}{\partial t} &= -v_y^r; & \frac{\partial z'}{\partial t} &= -v_z^r. \end{aligned} \quad (6.1.14)$$

Substitution of these results into (6.1.13) yields

$$\frac{\partial f'}{\partial t} = \frac{\partial f'}{\partial t'} - \left(v_x^r \frac{\partial}{\partial x'} + v_y^r \frac{\partial}{\partial y'} + v_z^r \frac{\partial}{\partial z'} \right) f'. \quad (6.1.15)$$

The term in parentheses can be written as $\mathbf{v}^r \cdot \nabla'$; thus (6.1.15) is written in the form

$$\frac{\partial f'}{\partial t} = \frac{\partial f'}{\partial t'} - (\mathbf{v}^r \cdot \nabla') f'. \quad (6.1.16)$$

We use (6.1.10) to write this result in the alternative form

$$\frac{\partial f'}{\partial t'} = \frac{\partial f'}{\partial t} + (\mathbf{v}^r \cdot \nabla) f'. \quad (6.1.17)$$

The function f' can be a component of a vector; thus, if we define a vector $\mathbf{A}'(x', y', z', t')$, the same mathematical process leads to

$$\frac{\partial \mathbf{A}'}{\partial t'} = \frac{\partial \mathbf{A}'}{\partial t} + (\mathbf{v}^r \cdot \nabla) \mathbf{A}'. \quad (6.1.18)$$

Suppose that the unprimed frame is the fixed or laboratory frame. Then, from the left-hand side of (6.1.18) it is clear that the right-hand side is *the rate of change with respect to time of $\mathbf{A}'$ for an observer moving with velocity $\mathbf{v}^r$* . This derivative, written in terms of the coordinates (x, y, z, t) of the fixed

frame, is used not only in this chapter but in many of the chapters that follow. Hence it is designated

$$\frac{DA'}{Dt} \equiv \frac{\partial A'}{\partial t} + (\mathbf{v}^r \cdot \nabla) A' \quad (6.1.19)$$

and is called the *substantial* or *convective derivative*. An example will help to clarify the significance of this derivative.

Example 6.1.1. To illustrate the significance of (6.1.19) consider an example in which A' is the displacement of a surface from the x - z plane given by

$$A' = \xi(x, t) \mathbf{i}_y, \quad (a)$$

as shown in Fig. 6.1.2a. The function ξ gives the y -coordinate of the surface. At a given position on the surface this y -coordinate has the same value, no matter whether it is viewed from the fixed frame or from a (primed) frame moving in the x -direction with velocity $\mathbf{v}^r = V \mathbf{i}_x$; that is, $A' = A$ and $\xi' = \xi$ for this particular case. If we evaluate (6.1.19), using (a), it follows that

$$\frac{DA'}{Dt} = \left(\frac{\partial \xi}{\partial t} + V \frac{\partial \xi}{\partial x} \right) \mathbf{i}_y. \quad (b)$$

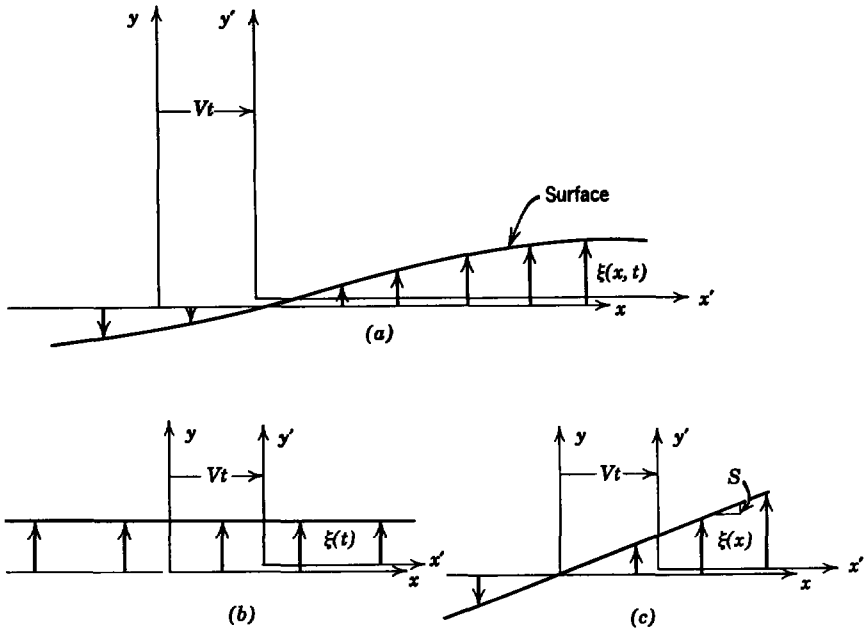


Fig. 6.1.2 (a) A surface described by $y = \xi(x, t)$ has an elevation above the x - z plane which is the same whether viewed from the moving (primed) frame or the fixed frame ($\xi' = \xi$); (b) ξ is independent of position so that only the first term in (6.1.19) makes a contribution to D/Dt ; (c) ξ is independent of time and only the second term in (6.1.19) makes a contribution.

The significance of the two terms in this expression can be understood by considering limiting cases in which they, individually, make the sole contribution to the substantial derivative.

Figure 6.1.2*b* shows a surface ξ that is independent of position (x or x') but has a displacement that varies with time. Then the second term in (b) is zero and the rate of change for an observer moving with velocity V is the same as for a stationary observer. This we would have known from Fig. 6.1.2*b* without recourse to a mathematical equation.

In Fig. 6.1.2*c* the surface elevation is independent of time, since $\xi = \xi(x)$ and the rate of change of ξ with respect to time for the fixed observer is zero. By contrast the time rate of change for an observer moving with the velocity V is

$$\frac{DA'}{Dt} = V \frac{\partial \xi}{\partial x} i_y. \quad (c)$$

This result is not surprising either, because an observer in the moving frame travels to the right with a velocity V and sees a deflection ξ that increases in proportion to the slope of the surface $\partial \xi / \partial x$ and in proportion to the velocity V . In particular, if

$$\xi = Sx, \quad (d)$$

then from (c)

$$\frac{DA'}{Dt} = VS i_y. \quad (e)$$

This result could be obtained by inspection of Fig. 6.1.2*c*.

We shall find it useful later to write (6.1.18) in a different form. Because $\mathbf{v}^r$ is constant, a vector identity* makes it possible to write (6.1.18) in the form

$$\frac{\partial \mathbf{A}'}{\partial t'} = \frac{\partial \mathbf{A}'}{\partial t} + \mathbf{v}^r(\nabla \cdot \mathbf{A}') - \nabla \times (\mathbf{v}^r \times \mathbf{A}'). \quad (6.1.20)$$

We are now in a position to obtain transformations for electromagnetic quantities from the field equations for magnetic and electric field systems.

6.1.1 Transformations for Magnetic Field Systems

The differential equations that define the relations of the field quantities to sources in quasi-static magnetic field systems were given in Section 6.0 (6.0.1) to (6.0.5) and are repeated here for convenience:

$$\nabla \times \mathbf{H} = \mathbf{J}_f, \quad (6.1.21)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (6.1.22)$$

$$\nabla \cdot \mathbf{J}_f = 0, \quad (6.1.23)$$

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}, \quad (6.1.24)$$

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}). \quad (6.1.25)$$

* $\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a})$.

These equations describe the field quantities measured by an observer who is fixed in the unprimed inertial coordinate system of Fig. 6.1.1.

It is a postulate of special relativity that physical laws must be the same in any inertial coordinate system. Consequently, we write the equations to describe the field quantities measured by an observer who is fixed in the primed inertial coordinate system of Fig. 6.1.1 as

$$\nabla' \times \mathbf{H}' = \mathbf{J}'_f, \quad (6.1.26)$$

$$\nabla' \cdot \mathbf{B}' = 0, \quad (6.1.27)$$

$$\nabla' \cdot \mathbf{J}'_f = 0, \quad (6.1.28)$$

$$\nabla' \times \mathbf{E}' = -\frac{\partial \mathbf{B}'}{\partial t'}, \quad (6.1.29)$$

$$\mathbf{B}' = \mu_0(\mathbf{H}' + \mathbf{M}'). \quad (6.1.30)$$

Use is now made of (6.1.11), (6.1.12), and (6.1.20) to express (6.1.26) to (6.1.29) in the equivalent forms

$$\nabla \times \mathbf{H}' = \mathbf{J}'_f, \quad (6.1.31)$$

$$\nabla \cdot \mathbf{B}' = 0, \quad (6.1.32)$$

$$\nabla \cdot \mathbf{J}'_f = 0, \quad (6.1.33)$$

$$\nabla \times (\mathbf{E}' - \mathbf{v}' \times \mathbf{B}') = -\frac{\partial \mathbf{B}'}{\partial t}. \quad (6.1.34)$$

We have made use of (6.1.32) to simplify the form of (6.1.34).

It has been postulated that (6.1.31) to (6.1.34) describe the same physical laws as (6.1.21) to (6.1.24). A comparison of the two sets of equations shows that a consistent set of transformations which satisfies this requirement is

$$\mathbf{H}' = \mathbf{H}, \quad (6.1.35)$$

$$\mathbf{J}'_f = \mathbf{J}_f, \quad (6.1.36)$$

$$\mathbf{B}' = \mathbf{B}, \quad (6.1.37)$$

$$\mathbf{E}' = \mathbf{E} + \mathbf{v}' \times \mathbf{B}. \quad (6.1.38)$$

We also use (6.1.35) and (6.1.37) with (6.1.25) and (6.1.30) to obtain the transformation for magnetization density

$$\mathbf{M}' = \mathbf{M}. \quad (6.1.39)$$

The transformations of (6.1.35) to (6.1.39) relate the values of electromagnetic quantities in a quasi-static, magnetic field system that would be

measured by two observers in relative motion with constant relative velocity $\mathbf{v}^r$ at a particular point in space at a given instant of time. Note that there is no contradiction or inconsistency among these transformations as there would have been had we kept terms such as the displacement current in Ampère's law. Note also that the transformation for free current density (6.1.36) indicates that current flow by the convection of net free charge is consistently neglected in a magnetic field system. We can now return to the integral form of the magnetic system equations (Table 1.2) to see that the definition of $\mathbf{E}'$ postulated there is consistent with what we have found here. Still another derivation of the integral form of Ampère's law for deforming contours of integration is given in Section B.4.1.

It is interesting to interpret (6.1.38) in terms of the Lorentz force (6.1.1). Consider a charge q at rest in the primed coordinate system. The force measured by an observer in that system is simply

$$\mathbf{f}' = q\mathbf{E}'.$$

An observer in the unprimed system who measures fields $\mathbf{E}$ and $\mathbf{B}$ will see the charge moving with a velocity $\mathbf{v}^r$ and will therefore describe the force as

$$\mathbf{f} = q\mathbf{E} + q\mathbf{v}^r \times \mathbf{B}.$$

The transformation of (6.1.38) is just the relation between $\mathbf{E}$ and $\mathbf{E}'$ that must exist if the force on the charge is to be independent of the coordinate system in which it is expressed. Some writers actually use the Lorentz force to obtain the transformation for the electric field rather than the differential equations as we have.* Although this can be done, it is important to see that there is a close connection between the field transformations and the field equations. The field equations for the magnetic field systems do not include the displacement current, and it would be *inconsistent* to use field transformations based on equations that did not include this same approximation. For this reason it is not surprising that in the next section a *different* set of field transformations is found for the electric field systems.

Example 6.1.2. The most interesting of the field transformations introduced in this section is given by (6.1.38) and it is important to understand the close connection between this expression for $\mathbf{E}'$ in terms of the fields in the fixed frame and the lumped parameter models of preceding chapters. For this purpose consider the idealized problem shown in Fig. 6.1.3, in which a pair of perfectly conducting plates are shorted by a conducting bar. The bar moves to the right with the velocity V and there is a uniform magnetic flux density $\mathbf{B}$ imposed in the z -direction by an external source. We assume that the plates are terminated at the left in an essentially open circuit so that no currents flow to make additions to the field $\mathbf{B}$.

* R. M. Fano, L. J. Chu, and R. B. Adler, *Electromagnetic Fields, Energy, and Forces*, Wiley, New York, 1960, p. 390.

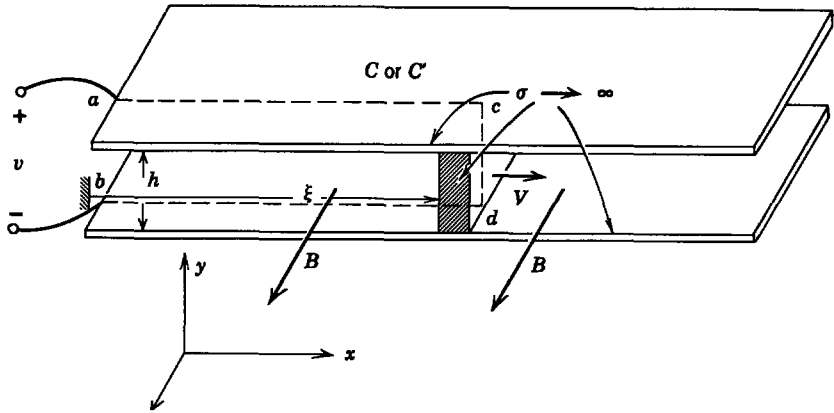


Fig. 6.1.3 A pair of parallel perfectly conducting plates are short-circuited by a moving perfectly conducting bar. Because of the magnetic field B , a voltage v is induced which can be computed either by integrating the induction equation around the fixed loop C' that passes through the bar or by integrating the induction equation around a loop C that expands in area as the bar moves to the right. The field transformation of (6.1.38) guarantees that both integrations will give the same result.

First recall how the voltage v is computed in Chapter 2. A contour C , as shown in Fig. 6.1.3, passes through the perfectly conducting bar. Then the induction equation is written in the form (2.1.6) and (2.1.7)

$$\oint_C \mathbf{E}' \cdot d\mathbf{l} = - \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da, \quad (a)$$

where $\mathbf{E}'$ is the electric field in the frame of the conductor. Hence the integral of $\mathbf{E}'$ along the contour a - c - d - b makes no contribution and (a) reduces to the familiar form

$$v = \frac{d\lambda}{dt}, \quad (b)$$

where

$$\lambda = \int_S \mathbf{B} \cdot \mathbf{n} \, da = -\xi h B. \quad (c)$$

In the viewpoint represented by this derivation the voltage v arises because the contour C is expanding, thus enclosing more magnetic flux. In particular (b) and (c) give

$$v = -hBV. \quad (d)$$

The field transformations make it possible to take an alternative approach to this problem. The integral form of the induction equation can also be written for a contour that is fixed in space

$$\oint_{C'} \mathbf{E} \cdot d\mathbf{l} = - \frac{d}{dt} \int_{S'} \mathbf{B} \cdot \mathbf{n} \, da \quad (e)$$

This expression has the same form as (a), but now C' and S' are fixed and $\mathbf{E}$ is the electric field intensity evaluated in the fixed frame. In the present example we can consider the

contour C' shown in Fig. 6.1.3, but even though this contour has the same instantaneous position as before it is now fixed in space rather than moving. As a direct consequence the right-hand side of (e) vanishes (remember, we assume that B is constant). If we further recognize that the integral of $\mathbf{E}$ through the perfectly conducting plates from a - c and d - b makes no contribution, (e) reduces to

$$\int_b^a \mathbf{E} \cdot d\mathbf{l} + \int_c^d \mathbf{E} \cdot d\mathbf{l} = 0. \quad (f)$$

In the region in which the terminals are located we assume (as in the preceding approach) that there is no time-varying magnetic field so that $\mathbf{E} = -\nabla\phi$ and

$$\int_b^a \mathbf{E} \cdot d\mathbf{l} = -(\phi_a - \phi_b) = -v. \quad (g)$$

Hence (f) reduces to

$$v = \int_c^d \mathbf{E} \cdot d\mathbf{l}. \quad (h)$$

The remaining integration from c - d must provide the voltage v . Note that this "speed voltage" is given by the term on the right in (a), but is now accounted for by the term on the left in (e). This term can be evaluated by recognizing that because $\mathbf{E}' = 0$ in the bar (6.1.38)

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} = V B \mathbf{i}_y. \quad (i)$$

This result can be incorporated into (h) to give

$$v = -hBV, \quad (j)$$

which will be recognized as the same result obtained with the deforming contour of integration (d).

6.1.2 Transformations for Electric Field Systems

The differential equations that define the fields and their relations to sources in quasi-static, electric field systems were given in Table 1.2.

$$\nabla \times \mathbf{E} = 0, \quad (6.1.40)$$

$$\nabla \cdot \mathbf{D} = \rho_f, \quad (6.1.41)$$

$$\nabla \cdot \mathbf{J}_f = -\frac{\partial \rho_f}{\partial t}, \quad (6.1.42)$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}, \quad (6.1.43)$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}. \quad (6.1.44)$$

These equations describe the field quantities measured by an observer who is fixed in the unprimed inertial coordinate system of Fig. 6.1.1.

Our procedure here is analogous to that of the preceding section. We

recognize, by postulate, that these physical laws must be the same in any other inertial coordinate system. We write them for the primed inertial coordinate system of Fig. 6.1.1 as

$$\nabla' \times \mathbf{E}' = 0, \quad (6.1.45)$$

$$\nabla' \cdot \mathbf{D}' = \rho'_f, \quad (6.1.46)$$

$$\nabla' \cdot \mathbf{J}'_f = -\frac{\partial \rho'_f}{\partial t'}, \quad (6.1.47)$$

$$\nabla' \times \mathbf{H}' = \mathbf{J}'_f + \frac{\partial \mathbf{D}'}{\partial t'}, \quad (6.1.48)$$

$$\mathbf{D}' = \epsilon_0 \mathbf{E}' + \mathbf{P}'. \quad (6.1.49)$$

We now use (6.1.11), (6.1.12), (6.1.17)*, and (6.1.20) to express (6.1.45) to (6.1.48) in the forms

$$\nabla \times \mathbf{E}' = 0, \quad (6.1.50)$$

$$\nabla \cdot \mathbf{D}' = \rho'_f, \quad (6.1.51)$$

$$\nabla \cdot (\mathbf{J}'_f + \rho'_f \mathbf{v}^r) = -\frac{\partial \rho'_f}{\partial t}, \quad (6.1.52)$$

$$\nabla \times (\mathbf{H}' + \mathbf{v}^r \times \mathbf{D}') = \mathbf{J}'_f + \rho'_f \mathbf{v}^r + \frac{\partial \mathbf{D}'}{\partial t}. \quad (6.1.53)$$

We have used (6.1.51) to obtain (6.1.53).

Using the postulate that (6.1.40) to (6.1.43) express the same physical laws as (6.1.50) to (6.1.53) we obtain the following consistent set of transformations:

$$\mathbf{E}' = \mathbf{E}, \quad (6.1.54)$$

$$\mathbf{D}' = \mathbf{D}, \quad (6.1.55)$$

$$\rho'_f = \rho_f, \quad (6.1.56)$$

$$\mathbf{H}' = \mathbf{H} - \mathbf{v}^r \times \mathbf{D}, \quad (6.1.57)$$

$$\mathbf{J}'_f = \mathbf{J}_f - \rho_f \mathbf{v}^r. \quad (6.1.58)$$

We use (6.1.54) and (6.1.55) with (6.1.44) and (6.1.49) to obtain the transformation for polarization density

$$\mathbf{P}' = \mathbf{P}. \quad (6.1.59)$$

Note that these transformations are consistent with those postulated in

* Remember $\mathbf{v}^r$ is constant, so $(\mathbf{v}^r \cdot \nabla)f' = \nabla \cdot (\mathbf{v}^r f)$.

Section 1.1.2*b* to express the integral form of the equations for an electric field system. Yet another derivation of these integral laws is given in Section B.4.2.

Example 6.1.3. The simple significance of the field transformations for the electric field systems can be illustrated by means of the parallel-plate capacitor shown in Fig. 6.1.4. Here a battery is used to induce surface charges on the plates, as shown. Hence in the laboratory (unprimed) frame there is an electric field intensity between the plates related to the surface charge density by

$$\mathbf{E} = \frac{\sigma_f}{\epsilon_0} \mathbf{i}_y, \quad (a)$$

whereas there is no magnetic field $\mathbf{H}$. (We assume here that there are no external currents that would induce a magnetic field in the laboratory frame.) For the purpose of the example consider that the plates have infinite extent in the x -direction. Then, according to the electric field transformations, an observer in the moving frame of Fig. 6.1.4*a* would measure the magnetic field intensity (6.1.57)

$$\mathbf{H}' = -V\epsilon_0 E_y \mathbf{i}_z = -V\sigma_f \mathbf{i}_z. \quad (b)$$

This magnetic field is present in the moving frame because in that frame of reference the surface charges give rise to surface currents. These currents induce the field $\mathbf{H}'$. To see this,

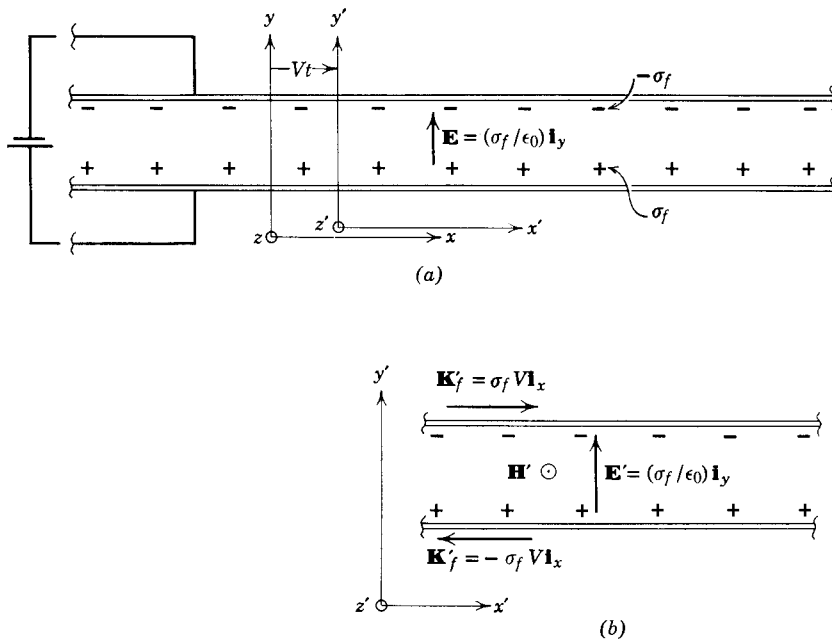


Fig. 6.1.4 (a) A parallel-plate capacitor is biased by a voltage source so that surface charges of opposite polarity are induced; (b) the fields in the moving frame can be found by computing the magnetic field induced by the convection of the surface charges or by using the field transformation of (6.1.57).

consider that the plates are being viewed from the moving frame of reference, as shown in Fig. 6.1.4*b*. In this frame surface currents flow in the x -direction, thus giving rise to the magnetic field of (b). Note that implicit to this reasoning is the transformation for the free current density (6.1.58).

In this and the preceding section we have obtained transformations that describe the relations between the field quantities measured by two observers in relative motion with a constant relative velocity. These transformations have been obtained for quasi-static systems and are valid only for such systems. We have stated that these transformations are consistent. By this we mean that we can use our transformation relations to transform field quantities repeatedly back and forth between two inertial reference frames without generating inconsistencies. A summary of transformations is given in Table 6.1.

6.2 BOUNDARY CONDITIONS

It is often found that electrical properties change significantly over distances that are infinitesimal with respect to significant dimensions of an electromechanical system. Such changes occur at the surface of a medium or at an interface between two media. In such cases we can represent the abrupt changes mathematically as spatial discontinuities in the electromagnetic variables. It is these discontinuities that provide boundary conditions on the electromagnetic variables.

The conventional treatment in electromagnetic theory considers conditions at stationary boundaries.* Because we are interested here in electromechanics we require boundary conditions at moving boundaries. The conditions derived are correct only for quasi-static systems.

First, we define the surface Σ , illustrated in Fig. 6.2.1, which separates medium a from medium b . Media a and b move with velocities $\mathbf{v}^a$ and $\mathbf{v}^b$ with respect to the inertial coordinate system $\mathbf{r}$ in which all field and source quantities ($\mathbf{E}$, $\mathbf{B}$, $\mathbf{P}$, $\mathbf{M}$, $\mathbf{J}_f$, ρ_f) are defined. Superscripts a and b indicate the medium in which a quantity exists. The normal vector $\mathbf{n}$ is defined as normal to the surface Σ and has a positive direction from medium b to medium a , as shown.

In order that the surface Σ may be a well-defined boundary between the two media, the normal components of the two velocities $\mathbf{v}^a$ and $\mathbf{v}^b$ must be the same at the surface; thus

$$\mathbf{n} \cdot (\mathbf{v}^a - \mathbf{v}^b) = 0. \quad (6.2.1)$$

If this condition is not satisfied, the two materials are diffusing through each other or moving apart, leaving a vacuum between them. In either case a

* Fano et al., *op. cit.*, pp. 86–89.

Table 6.1 Differential Equations, Transformations, and Boundary Conditions for Quasi-static Electromagnetic Systems with Moving Media

	Differential Equations		Transformations		Boundary Conditions	
Magnetic field systems	$\nabla \times \mathbf{H} = \mathbf{J}_f$	(1.1.1)	$\mathbf{H}' = \mathbf{H}$	(6.1.35)	$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K}_f$	(6.2.14)
	$\nabla \cdot \mathbf{B} = 0$	(1.1.2)	$\mathbf{B}' = \mathbf{B}$	(6.1.37)	$\mathbf{n} \cdot (\mathbf{B}^a - \mathbf{B}^b) = 0$	(6.2.7)
	$\nabla \cdot \mathbf{J}_f = 0$	(1.1.3)	$\mathbf{J}'_f = \mathbf{J}_f$	(6.1.36)	$\mathbf{n} \cdot (\mathbf{J}_f^a - \mathbf{J}_f^b) + \nabla_\Sigma \cdot \mathbf{K}_f = 0$	(6.2.9)
	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	(1.1.5)	$\mathbf{E}' = \mathbf{E} + \mathbf{v}^r \times \mathbf{B}$	(6.1.38)	$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = v_n(\mathbf{B}^a - \mathbf{B}^b)$	(6.2.22)
	$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$	(1.1.4)	$\mathbf{M}' = \mathbf{M}$	(6.1.39)		
Electric field systems	$\nabla \times \mathbf{E} = 0$	(1.1.11)	$\mathbf{E}' = \mathbf{E}$	(6.1.54)	$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = 0$	(6.2.31)
	$\nabla \cdot \mathbf{D} = \rho_f$	(1.1.12)	$\mathbf{D}' = \mathbf{D}$	(6.1.55)	$\mathbf{n} \cdot (\mathbf{D}^a - \mathbf{D}^b) = \sigma_f$	(6.2.33)
			$\rho'_f = \rho_f$	(6.1.56)		
	$\nabla \cdot \mathbf{J}_f = -\frac{\partial \rho_f}{\partial t}$	(1.1.14)	$\mathbf{J}'_f = \mathbf{J}_f - \rho_f \mathbf{v}^r$	(6.1.58)	$\mathbf{n} \cdot (\mathbf{J}_f^a - \mathbf{J}_f^b) + \nabla_\Sigma \cdot \mathbf{K}_f = v_n(\rho_f^a - \rho_f^b) - \frac{\partial \sigma_f}{\partial t}$	(6.2.36)
	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$	(1.1.15)	$\mathbf{H}' = \mathbf{H} - \mathbf{v}^r \times \mathbf{D}$	(6.1.57)	$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K}_f + v_n \mathbf{n} \times [\mathbf{n} \times (\mathbf{D}^a - \mathbf{D}^b)]$	(6.2.38)
	$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$	(1.1.13)	$\mathbf{P}' = \mathbf{P}$	(6.1.59)		

well-defined boundary of the type postulated does not exist. From the electromagnetic theory viewpoint no requirement is necessary on the tangential velocities of the media at the boundary. Consequently, the media can slide past each other at the boundary with no restrictions on the tangential components of the velocities. We shall find that the boundary conditions depend on the normal component of the velocity of the boundary. One or both media may be vacuum.

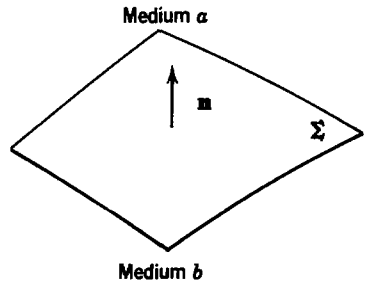


Fig. 6.2.1 Surface separating two media.

In general, the surface Σ is not plane and it is moving and deforming. When a mathematical description of the surface is given, we must be able to evaluate the normal vector $\mathbf{n}$. In the coordinate system $\mathbf{r}$ a surface can be described by the general functional form

$$f(\mathbf{r}, t) = 0. \quad (6.2.2)$$

The normal vector $\mathbf{n}$ can then be evaluated as

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|}. \quad (6.2.3)$$

This statement is familiar from electromagnetic field theory; that is, if we assume that (6.2.2) defines one equipotential surface of the set

$$f(\mathbf{r}, t) = \phi,$$

where ϕ is the potential, the electric field is the negative gradient of the potential and is normal to an equipotential surface. Hence we can think of the normal vector $\mathbf{n}$, defined by (6.2.3) as the negative of the normalized electric field, evaluated at the zero-potential surface.

Example 6.2.1. To illustrate the manner in which a surface is represented by an expression like that in (6.2.2), consider the surface defined in Fig. 6.2.2 in which the height of the surface above the x - z plane is given by

$$y = A \sin \omega t \cos \frac{2\pi x}{l} + B. \quad (a)$$

where A , B , ω , and l are positive constants. This represents a surface whose position is independent of z and whose height varies as the cosine function with x . The amplitude of the variation in height at a fixed position x is a sinusoidal function of time.

To obtain the equation for the surface in the form of (6.2.2) we write

$$f(x, y, t) = y - A \sin \omega t \cos \frac{2\pi x}{l} - B = 0. \quad (b)$$

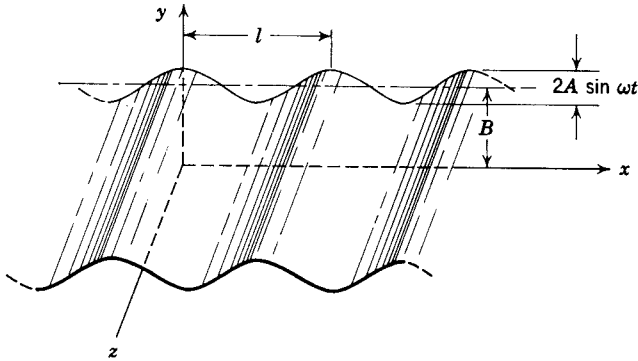


Fig. 6.2.2 Example of surface that varies in space and time.

Taking the gradient of this expression yields

$$\nabla f = \mathbf{i}_x \frac{2\pi A}{l} \sin \omega t \sin \frac{2\pi x}{l} + \mathbf{i}_y.$$

The magnitude of this gradient is

$$|\nabla f| = \left(1 + \frac{4\pi^2 A^2}{l^2} \sin^2 \omega t \sin^2 \frac{2\pi x}{l} \right)^{1/2}$$

and the normal vector is

$$\mathbf{n} = \frac{\mathbf{i}_x (2\pi A/l) \sin \omega t \sin (2\pi x/l) + \mathbf{i}_y}{\sqrt{1 + (4\pi^2 A^2/l^2) \sin^2 \omega t \sin^2 (2\pi x/l)}} \quad (c)$$

This normal vector becomes $\mathbf{i}_y$ at

$$\frac{2\pi x}{l} = \pm n\pi; \quad n = 0, 1, 2, \dots;$$

that is, on the crests and in the troughs of the corrugations the normal vector is vertical. At other values of x the normal vector is not vertical and its direction can be determined from (c). The direction of the normal can be reversed by defining f as the negative of (b). Consequently, we can label media a and b and make sure that the definition of f yields a normal vector as described in Fig. 6.2.1 or we can define f and label the materials after the direction of the normal has been determined.

6.2.1 Boundary Conditions for Magnetic Field Systems

For studying boundary conditions in a quasi-static, magnetic field system we assume that the surface Σ carries a free surface current density $\mathbf{K}_f$ (amperes per meter) and a free surface charge density σ_f (coulombs per square meter). The free surface current density $\mathbf{K}_f$ is part of the primary excitation, whereas the surface charge density is a quantity that can be determined from auxiliary relations after the fields have been determined.

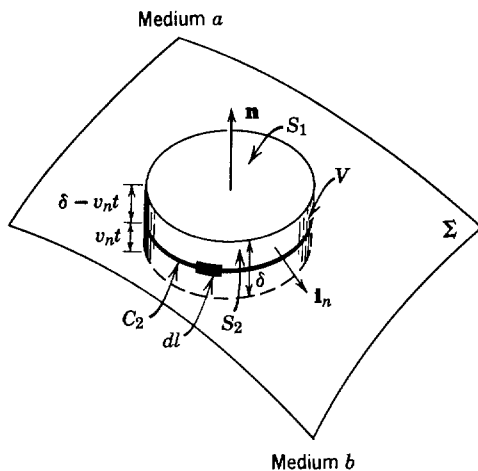


Fig. 6.2.3 Geometry for calculating discontinuities in normal components of field vectors.

Consider first the equations in integral form that determine how the sources excite the fields. They were given in (1.1.20) and (1.1.21) and are

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot \mathbf{i}_n da, \quad (6.2.4)$$

$$\oint_S \mathbf{B} \cdot \mathbf{i}_n d\mathbf{a} = 0. \quad (6.2.5)$$

Here we have introduced $\mathbf{i}_n$ as the unit vector perpendicular to the area of integration so that it can be distinguished from the vector $\mathbf{n}$ normal to the surface Σ . To find the boundary condition imposed by (6.2.5) we define a small, right-circular cylindrical volume V , enclosed by a surface S consisting of the top and bottom surfaces of areas S_1 and a lateral surface of height δ and area S_2 , as shown in Fig. 6.2.3. The volume V is fixed in the inertial coordinate system $\mathbf{r}$ and is so oriented that it intersects the boundary Σ as shown in Fig. 6.2.3. The surface S_1 is small enough that the boundary Σ can be assumed plane in its vicinity and the top and bottom surfaces S_1 are parallel to the boundary Σ . Hence the vector $\mathbf{n}$ is normal to both Σ and S_1 . We assume that the height δ of the pillbox is so small that the lateral area S_2 is much smaller than the area of surface S_1 . When we integrate (6.2.5) over the surface S and assume that S_1 is so small that $\mathbf{B}$ does not change appreciably over S_1 , we obtain

$$(\mathbf{B}^a \cdot \mathbf{n})S_1 - (\mathbf{B}^b \cdot \mathbf{n})S_1 = 0. \quad (6.2.6)$$

We cancel S_1 from this expression to obtain

$$\mathbf{n} \cdot (\mathbf{B}^a - \mathbf{B}^b) = 0. \quad (6.2.7)$$

Equation 6.2.7 states that the normal component of $\mathbf{B}$ must be continuous at the boundary. This is the same result obtained in the electromagnetic theory of stationary systems; thus the motion has not affected this boundary condition.

Like the flux density the free-current density $\mathbf{J}_f$ in a quasi-static magnetic field system has no divergence [see (1.1.3) of Table 1.2]. However, when deriving the boundary condition on $\mathbf{J}_f$, the integral expression of (1.1.22) of Table 1.2 must be used with due regard for surface currents at the discontinuity. The current density $\mathbf{J}_f$, unlike $\mathbf{B}$, can be singular. If we use (1.1.22) ($\oint \mathbf{J}_f \cdot \mathbf{i}_n da = 0$) with the pillbox in Fig. 6.2.3 and neglect the contribution of volume current density $\mathbf{J}_f$ over the lateral surface S_2 , we obtain

$$(\mathbf{J}_f^a \cdot \mathbf{n})S_1 - (\mathbf{J}_f^b \cdot \mathbf{n})S_1 + \oint_{C_2} \mathbf{K}_f \cdot \mathbf{i}_n dl = 0. \quad (6.2.8)$$

This is simply an expression of the fact that current into the pillbox from the two media must equal surface current across the contour C_2 because no appreciable free charge density (volume or surface) can exist in a magnetic field system. Dividing (6.2.8) by S_1 and taking the limit as $S_1 \rightarrow 0$ yields the desired boundary condition

$$\mathbf{n} \cdot (\mathbf{J}_f^a - \mathbf{J}_f^b) + \nabla_\Sigma \cdot \mathbf{K}_f = 0, \quad (6.2.9)$$

where

$$\nabla_\Sigma \cdot \mathbf{K}_f = \lim_{S_1 \rightarrow 0} \frac{\oint_{C_2} \mathbf{K}_f \cdot \mathbf{i}_n dl}{S_1} \quad (6.2.10)$$

is the surface (two-dimensional) divergence of $\mathbf{K}_f$ applied in the plane of the surface Σ at the point in question.*

We consider next the boundary condition imposed by (6.2.4). For this purpose we use the contour C which encloses the open surface S and is fixed in the coordinate system $\mathbf{r}$. The contour instantaneously intersects the boundary, as illustrated in Fig. 6.2.4. The surface S is a plane rectangle and is small enough for the boundary Σ to be assumed plane in its vicinity. The surface S is perpendicular to Σ and the height δ of the contour is much smaller than the length L .

$$\delta \ll L$$

* The two-dimensional divergence is simply the sum of the derivatives of the two orthogonal components of a vector in the surface with respect to the distance in the component directions; for example, assume a surface Σ with normal vector $\mathbf{n} = \mathbf{i}_z$. A vector $\mathbf{A}$ lying in the surface Σ will have only x - and y -components $\mathbf{A} = A_x \mathbf{i}_x + A_y \mathbf{i}_y$ and the surface divergence of this vector is $\nabla_\Sigma \cdot \mathbf{A} = [\mathbf{i}_x(\partial/\partial x) + \mathbf{i}_y(\partial/\partial y)] \cdot \mathbf{A} = \partial A_x/\partial x + \partial A_y/\partial y$. Note that the surface divergence in this case is just two terms of the volume divergence.

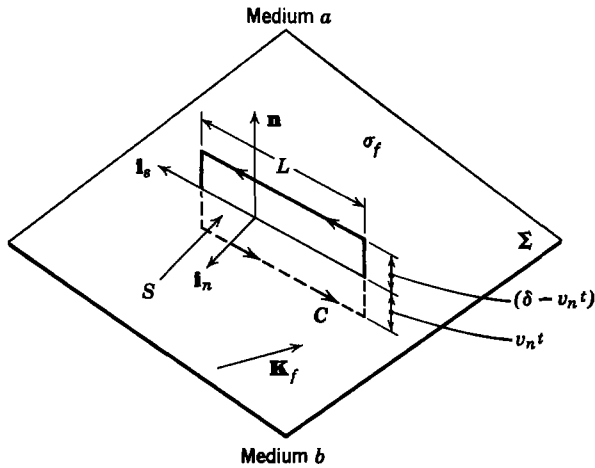


Fig. 6.2.4 Contour and surface for determining discontinuities in tangential components of field vectors.

The three unit vectors $\mathbf{n}$, $\mathbf{i}_n$, and $\mathbf{i}_s$, shown in Fig. 6.2.4, are mutually orthogonal. We assume that the contour C is small enough that fields do not vary appreciably over its length L . With these assumptions, and ignoring contributions from the ends (δ), we integrate (6.2.4) to obtain (after canceling out the length L)

$$(\mathbf{H}^a - \mathbf{H}^b) \cdot \mathbf{i}_s = \mathbf{K}_f \cdot \mathbf{i}_n. \quad (6.2.11)$$

This expression states that the discontinuity in the tangential component of $\mathbf{H}$ in the direction of $\mathbf{i}_s$ is equal to the component of $\mathbf{K}_f$ perpendicular to $\mathbf{i}_s$.

Equation 6.2.11 can be put into a more useful form in the following way. We substitute

$$\mathbf{i}_s = \mathbf{i}_n \times \mathbf{n} \quad (6.2.12)$$

into (6.2.11) and use a vector identity* to obtain

$$[\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b)] \cdot \mathbf{i}_n = \mathbf{K}_f \cdot \mathbf{i}_n. \quad (6.2.13)$$

By definition, the vector $\mathbf{K}_f$ lies in the boundary Σ . The vector $[\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b)]$ also lies in the boundary Σ . The vector $\mathbf{i}_n$ has an arbitrary direction except that it also lies in the boundary Σ . Therefore from (6.2.13) we obtain the result

$$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K}_f. \quad (6.2.14)$$

Note once again that this is the same boundary condition obtained for stationary systems (it is independent of the boundary velocity).

* $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$.

We now derive the boundary condition for the electric field by starting with (1.1.23) written for a fixed contour C :

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = - \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{i}_n da. \quad (6.2.15)$$

We integrate (6.2.15) by using the contour C and surface S defined in Fig. 6.2.4. The restrictions on orientation and size in the derivation of (6.2.14) also apply here. By using the unit vectors and dimensions defined in Fig. 6.2.4 we obtain the contour integral

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = (\mathbf{E}^a - \mathbf{E}^b) \cdot \mathbf{i}_s L. \quad (6.2.16)$$

In this expression we have neglected the contribution to this integral from the sides perpendicular to Σ because we require that $\delta \ll L$.^{*} The time origin is defined as the instant in which the surface Σ coincides with the lower edge of the contour. Thus, defining the normal component of velocity of the boundary as

$$v_n = \mathbf{n} \cdot \mathbf{v}^a = \mathbf{n} \cdot \mathbf{v}^b, \quad (6.2.17)$$

we write the surface integral in (6.2.15) as

$$\int_S \mathbf{B} \cdot \mathbf{i}_n da = [\mathbf{B}^a(\delta - v_n t) + \mathbf{B}^b(v_n t)] \cdot \mathbf{i}_n L. \quad (6.2.18)$$

In this expression we have assumed that δ is small enough that v_n does not change appreciably as the boundary Σ passes the contour C . Note that this does not require that v_n be a constant, for we shall shortly take the limit in which $\delta \rightarrow 0$.

We take the time derivative of (6.2.18)

$$\begin{aligned} \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{i}_n da &= -v_n(\mathbf{B}^a - \mathbf{B}^b) \cdot \mathbf{i}_n L \\ &+ \left[\frac{\partial v_n}{\partial t} t(\mathbf{B}^b - \mathbf{B}^a) + \frac{\partial \mathbf{B}^a}{\partial t} (\delta - v_n t) + \frac{\partial \mathbf{B}^b}{\partial t} (v_n t) \right] \cdot \mathbf{i}_n L. \end{aligned} \quad (6.2.19)$$

The time derivatives of the fields are finite in the two media: consequently, in the limit as $\delta \rightarrow 0$ the term in brackets on the right of (6.2.19) vanishes. (Note that by definition $\delta \geq v_n t$ so that the interval of time during which the surface is within the volume V is the largest time t with which we are concerned. Hence, as $\delta \rightarrow 0$, so also does the largest value of t .) We equate the negative of (6.2.19) to (6.2.16) and cancel the length L to obtain

$$(\mathbf{E}^a - \mathbf{E}^b) \cdot \mathbf{i}_s = v_n(\mathbf{B}^a - \mathbf{B}^b) \cdot \mathbf{i}_n. \quad (6.2.20)$$

^{*} There is the implicit assumption here that although $\mathbf{E}$ can be discontinuous at the boundary it must be finite. Otherwise we could not ignore the contribution to the integral along the sides of length δ .

By using the relation

$$\mathbf{i}_s = \mathbf{i}_n \times \mathbf{n}$$

and a vector identity,* we put (6.2.20) in the form

$$[\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b)] \cdot \mathbf{i}_n = v_n(\mathbf{B}^a - \mathbf{B}^b) \cdot \mathbf{i}_n. \quad (6.2.21)$$

The vector $\mathbf{i}_n$ lies in the boundary Σ but otherwise it has an arbitrary direction. The vector $\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b)$ also lies in the boundary Σ . The normal component of $\mathbf{B}$ is continuous [see (6.2.7)]; consequently, the vector $(\mathbf{B}^a - \mathbf{B}^b)$ lies in the boundary Σ . Therefore we conclude from (6.2.21) that

$$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = v_n(\mathbf{B}^a - \mathbf{B}^b). \quad (6.2.22)$$

This is the desired boundary condition on the electric field.

We indicate an alternative method of deriving (6.2.22) by putting it in a different form. We define the velocity $\mathbf{v}$ as

$$\mathbf{v} = \mathbf{n}v_n, \quad (6.2.23)$$

which is simply the normal velocity of the boundary Σ . We now write (6.2.22) in the form

$$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = (\mathbf{n} \cdot \mathbf{v})(\mathbf{B}^a - \mathbf{B}^b) \quad (6.2.24)$$

and use a vector identity† with the boundary condition on the normal component of $\mathbf{B}$ (6.2.7) to write (6.2.24) as

$$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = -\mathbf{n} \times [\mathbf{v} \times (\mathbf{B}^a - \mathbf{B}^b)]. \quad (6.2.25)$$

When we define $\mathbf{E}^{a'}$ and $\mathbf{E}^{b'}$ as

$$\mathbf{E}^{a'} = \mathbf{E}^a + \mathbf{v} \times \mathbf{B}^a, \quad (6.2.26)$$

$$\mathbf{E}^{b'} = \mathbf{E}^b + \mathbf{v} \times \mathbf{B}^b, \quad (6.2.27)$$

we can rewrite (6.2.25) as

$$\mathbf{n} \times (\mathbf{E}^{a'} - \mathbf{E}^{b'}) = 0. \quad (6.2.28)$$

From the transformation of (6.1.38) we recognize that $\mathbf{E}^{a'}$ and $\mathbf{E}^{b'}$ are the electric fields that an observer will measure when he is in a coordinate system moving with the normal velocity of the boundary. In this coordinate system the boundary is at rest; consequently, as (6.2.24) indicates, the tangential component of electric field must be continuous, as it must be in any stationary system.‡ This idea can be used as the basis for an alternative derivation of the boundary condition on the tangential component of electric field, once the condition on a fixed boundary has been obtained. Note, however, that our

* $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$.

† $(\mathbf{a} \cdot \mathbf{b})\mathbf{c} = -\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b}(\mathbf{a} \cdot \mathbf{c})$.

‡ At least any stationary system in which $\mathbf{E}$ is finite everywhere.

transformations were derived for an inertial coordinate system. The boundary condition just derived is not restricted to boundaries that move with a constant velocity.

Example 6.2.2. Consider the system illustrated in Fig. 6.2.5 in which a surface Σ that is perpendicular to the y -axis moves with a speed v in the y -direction

$$\mathbf{v} = \mathbf{i}_y v.$$

We assume that the surface Σ has an infinite extent in the x - and z -directions so we can write its instantaneous position y_s as

$$y_s = vt.$$

We have chosen $t = 0$ as the instant when the surface contains the origin.

The surface Σ is immersed in vacuum and carries a uniform surface current density

$$\mathbf{K}_f = \mathbf{i}_z K_f.$$

The system is constrained so that to the right of the surface ($y > y_s$) the fields are zero. A simple experiment in which this physical situation arises is shown in Fig. 6.2.5b, in which a moving conductor shorts parallel electrodes driven by a current source. Here the moving short is modeled as being very thin and carrying a surface current K_f . In practice, the moving short could be a sheet of highly ionized gas moving down a shock tube.

The electric and magnetic fields in the region ($y < y_s$) to the left of the surface Σ are to be found.

We select as the normal vector $\mathbf{n}$ the vector $\mathbf{i}_y$,

$$\mathbf{n} = \mathbf{i}_y,$$

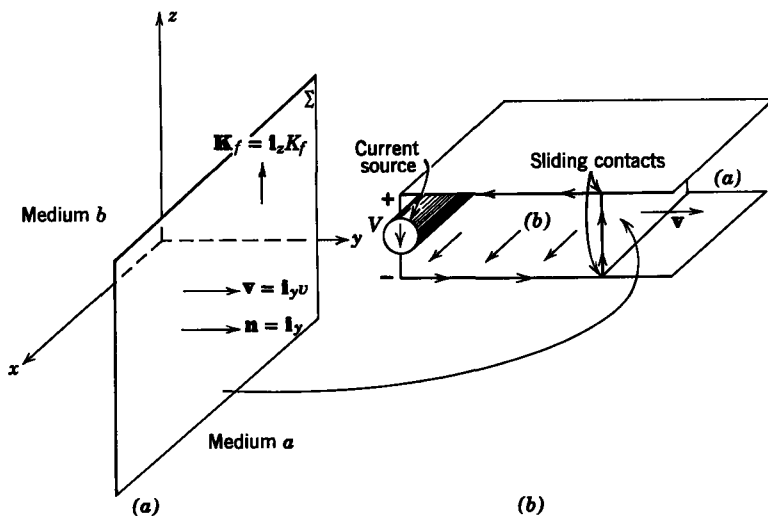


Fig. 6.2.5 (a) Plane surface in motion and carrying a surface current density; (b) the moving surface could constitute the moving conductor that short-circuits parallel plates excited by a current source distributed along the x -axis.

and thus medium b is $y < y_s$ and medium a is $y > y_s$. Our problem is specified such that

$$\mathbf{E}^a = 0 \quad \text{and} \quad \mathbf{H}^a = 0.$$

We first use (6.2.14) to find the magnetic field intensity $\mathbf{H}^b$:

$$\mathbf{i}_y \times (-\mathbf{H}^b) = \mathbf{i}_z K_f.$$

If we write $\mathbf{H}^b$ in component form,

$$\mathbf{H}^b = \mathbf{i}_x H_x^b + \mathbf{i}_y H_y^b + \mathbf{i}_z H_z^b,$$

we can write the boundary condition as

$$\mathbf{i}_z H_x^b - \mathbf{i}_x H_z^b = \mathbf{i}_z K_f.$$

Equating components in this expression yields

$$H_x^b = K_f,$$

$$H_z^b = 0.$$

Thus in vacuum $\mathbf{B} = \mu_0 \mathbf{H}$ and we can use the boundary condition on the normal component of $\mathbf{B}$ (6.2.7) with the given information that $\mathbf{H}^a = 0$ to arrive at the result

$$H_y^b = 0.$$

Thus the magnetic field intensity adjacent to the surface in region b is completely determined.

To find the electric field intensity to the left of the moving current sheet we use (6.2.22) to obtain

$$\mathbf{i}_y \times [-\mathbf{E}^b] = -v \mathbf{B}^b,$$

from which

$$\mathbf{E}^b = \mathbf{i}_z v \mu_0 K_f.$$

This is the electric field intensity to the left of the current sheet generated by the moving discontinuity in magnetic field intensity.

The discontinuity in electric field at the moving interface is necessary if concepts introduced earlier in this and preceding chapters are to remain consistent. We have already pointed out that the same boundary condition follows from the field transformation. If the moving surface is placed in the context of the problem shown in Fig. 6.2.5b, it is also possible to find the electric field behind the surface by using lumped parameter ideas to compute the voltage V , hence the electric field $\mathbf{E}$ between the plates.

6.2.2 Boundary Conditions for Electric Field Systems

For studying the boundary conditions in a quasi-static electric field system we assume a boundary surface Σ that carries a surface charge density σ , and a surface current density $\mathbf{K}$, (see Figs. 6.2.3 and 6.2.4). The surface charge density σ , is part of the primary excitation, whereas the surface current density $\mathbf{K}$, simply accounts for the conduction or convection of charge. The magnetic field generated by $\mathbf{K}$, can be computed once all of the other fields are known.

Two of the integral equations that determine the fields in a quasi-static electric field system are (see Table 1.2)

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0, \quad (6.2.29)$$

$$\oint_S \mathbf{D} \cdot \mathbf{i}_n da = \int_V \rho_f dV. \quad (6.2.30)$$

The results derived in the preceding section can be used to obtain the boundary conditions implied by these equations.

In the preceding section the boundary condition on the tangential component of $\mathbf{E}$ was derived from (6.2.15) by using the contour defined in Fig. 6.2.4 to obtain the result of (6.2.22). We note that (6.2.29) is simply (6.2.15) with the right side set equal to zero; consequently, by setting the right side of (6.2.22) equal to zero, we obtain

$$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = 0. \quad (6.2.31)$$

To derive the boundary condition on the normal component of $\mathbf{D}$ we use the pillbox-shaped surface of Fig. 6.2.3 with the same restrictions on relative geometry that were used in deriving (6.2.7) in the preceding section. By performing the integration (6.2.30) and taking the limit as $\delta \rightarrow 0$, we obtain the result

$$(\mathbf{D}^a \cdot \mathbf{n})S_1 - (\mathbf{D}^b \cdot \mathbf{n})S_1 = \sigma_f S_1, \quad (6.2.32)$$

where S_1 is the area of the top and bottom of the closed surface S . Division of both sides by S_1 yields the desired boundary condition

$$\mathbf{n} \cdot (\mathbf{D}^a - \mathbf{D}^b) = \sigma_f, \quad (6.2.33)$$

which is the same as the corresponding boundary condition for stationary systems.

When the conductivity of a material is uniform, the free charges have only a transient existence in the bulk of the material. Surface charges play an important role in such cases. As we shall see in Chapter 7, the conduction process in the region of an interface is an important factor in many electric field systems. For this reason the boundary condition associated with the conservation of charge equation (Table 1.2)

$$\oint_S \mathbf{J}_f \cdot \mathbf{i}_n da = -\frac{d}{dt} \int_V \rho_f dV \quad (6.2.34)$$

assumes primary significance.

The evaluation of this boundary condition is much like the evaluation of the boundary condition on current density derived for magnetic field systems in (6.2.8) to (6.2.10), except that now we must include the effects of volume

and surface charge densities. Using the pillbox-shaped volume of Fig. 6.2.3 and neglecting the contribution of volume current density $\mathbf{J}_f$ across the lateral surface S_2 , we find that (6.2.34) becomes

$$S_1 \mathbf{n} \cdot (\mathbf{J}_f^a - \mathbf{J}_f^b) + \oint_{C_2} \mathbf{K}_f \cdot \mathbf{i}_n dl = - \frac{d}{dt} [\rho_f^a S_1 (\delta - v_n t) + \rho_f^b S_1 v_n t + S_1 \sigma_f]. \quad (6.2.35)$$

In the limit, as $\delta \rightarrow 0$, then $S_1 \rightarrow 0$, this expression becomes

$$\mathbf{n} \cdot (\mathbf{J}_f^a - \mathbf{J}_f^b) + \nabla_\Sigma \cdot \mathbf{K}_f = v_n (\rho_f^a - \rho_f^b) - \frac{\partial \sigma_f}{\partial t}, \quad (6.2.36)$$

where we have used the definition of the surface divergence of surface current density ($\nabla_\Sigma \cdot \mathbf{K}_f$) given in (6.2.10). It should be pointed out here that surface current density occurs in electric field systems most often as the convection of free surface charge density, as indicated by the application of the transformation in (6.1.58) to surface current density and surface charge density.

Equation 6.2.36 is the boundary condition implied by the conservation of charge equation. An example will help to clarify the significance of the terms.

Example 6.2.3. An application of the conservation of charge boundary condition that is considered in Section 7.2 is shown in Fig. 6.2.6. Here two slightly conducting materials form a common boundary that moves to the right with the velocity $U\mathbf{i}_x$. There are no free charges in the bulk of the materials in which $\mathbf{J}_f = \sigma\mathbf{E}$ (see Section 7.2.2). Hence in this particular case the boundary condition (6.2.36) becomes

$$\sigma_a E_y^a - \sigma_b E_y^b + \frac{\partial}{\partial x} K_{fx} + \frac{\partial}{\partial z} K_{fz} = - \frac{\partial \sigma_f}{\partial t}. \quad (a)$$

On the interface the only surface current is due to the convection of free charge σ_f ; that is,

$$K_{fz} = 0; \quad K_{fx} = U\sigma_f. \quad (b)$$

Moreover, σ_f is related to the electric field through the boundary condition (6.2.33) (we assume that $\mathbf{D} = \epsilon\mathbf{E}$ in both materials):

$$\sigma_f = \epsilon_a E_y^a - \epsilon_b E_y^b. \quad (c)$$

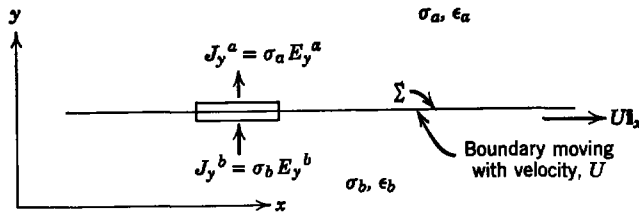


Fig. 6.2.6 A boundary between materials with conductivities σ_a and σ_b and permittivities ϵ_a and ϵ_b moves to the right with velocity U . Boundary condition (6.2.36) accounts for conservation of charge in a small section of the boundary.

It follows that (a) can be written as

$$\sigma_a E_y^a - \sigma_b E_y^b + \left(\frac{\partial \sigma_f}{\partial t} + U \frac{\partial \sigma_f}{\partial x} \right) = 0. \quad (d)$$

Note that (c) and (d) together constitute a single boundary condition on the electric field intensity at the moving surface. Remember that $\partial \sigma_f / \partial t + U \partial \sigma_f / \partial x$ is the rate of change with respect to time for an observer traveling with the velocity U [from (6.1.56) $\sigma_f = \sigma'_f$]. Hence (d) simply states that for such an observer the net current into a small section of the interface goes into an increase in the surface charge σ_f .

Equations 6.2.31, 6.2.33, and 6.2.36 are the only boundary conditions needed to solve problems for the electric fields in most quasi-static electric field systems. In these systems magnetic fields are generated by time-changing electric fields. Boundary conditions for these magnetic fields can be obtained from the integral form of Ampère's law (see Table 1.2)

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot \mathbf{i}_n da + \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{i}_n da, \quad (6.2.37)$$

where C and S are fixed as shown in Fig. 6.2.4. The process is analogous to that used in deriving (6.2.22) in the preceding section and leads to the boundary condition*

$$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K}_f + v_n \mathbf{n} \times [\mathbf{n} \times (\mathbf{D}^a - \mathbf{D}^b)]. \quad (6.2.38)$$

Note that this boundary condition is essentially that of (6.2.14) for the magnetic field system, with an added term to account for displacement current.

A summary of field transformations and boundary conditions is given in Table 6.1, which is arranged so that the correspondence of transformations and boundary conditions with differential equations is emphasized. One of the most important concepts related in this chapter is the consistency that must exist among differential equations, transformations, and boundary conditions.

The most obvious effects on boundary conditions from material motion are brought in through the normal velocity v_n . It must be remembered, however, that the boundary is itself part of a mechanical system that can often deform in the presence of magnetic or electric forces. This geometric effect of the boundary conditions is represented by the normal vector $\mathbf{n}$ and illustrated by Example 6.2.4.

Example 6.2.4. Boundary conditions provide a mechanism by which mechanical motions can alter electrostatic field solutions. In Fig. 6.2.7 one of a pair of perfectly conducting electrodes is plane, whereas the other has the sinusoidal dependence on (x, t)

$$y = A \sin \omega t \cos kx + B, \quad (a)$$

* To show this note that $-\mathbf{n} \times (\mathbf{n} \times \mathbf{A})$ is the component of $\mathbf{A}$ in the surface to which $\mathbf{n}$ is normal.

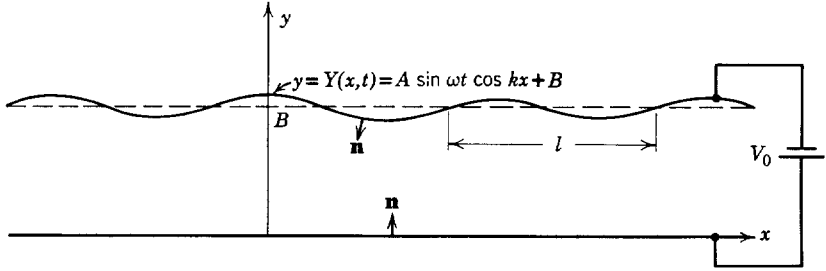


Fig. 6.2.7 Perfectly conducting electrodes at $y = 0$ and $y = A \sin \omega t \cos kx + B$ (where $k = 2\pi/l$) constrained to a constant potential difference.

where the dimension $A \ll B$, ω is the angular frequency, and $k = 2\pi/l$ (see Example 6.2.1). We wish to compute the electric field in the region between the electrodes when they are at the potential difference V_0 and to find the surface charge density on the lower electrode. It is assumed that the plates have infinite extent in the x - and z -directions.

The important boundary condition is (6.2.31). Because the electrodes are perfectly conducting, they can support no internal electric field. Hence at the surface of the electrodes

$$\mathbf{n} \times \mathbf{E} = 0, \quad (\text{b})$$

where we have set $\mathbf{E}^a = \mathbf{E}$ in (6.2.31). On the lower electrode $\mathbf{n} = \mathbf{i}_y$ and (b) reduces to

$$E_x(x, 0, t) = 0. \quad (\text{c})$$

Here we have used the fact that the upper surface position does not vary with z to set

$$E_z(x, y, t) = 0.$$

The normal vector $\mathbf{n}$ on the upper electrode is given by the negative of (c) of Example 6.2.1. We assume that the amplitude A is small enough ($4\pi^2 A^2/l^2 \ll 1$) to justify setting the denominator of this expression equal to 1. Then

$$\mathbf{n} = -\mathbf{i}_x k A \sin \omega t \sin kx - \mathbf{i}_y. \quad (\text{d})$$

The boundary condition (b), applied to the upper electrode, reduces to

$$n_y E_x(x, Y, t) = n_x E_y(x, Y, t), \quad (\text{e})$$

where $Y = A \sin \omega t \cos kx + B$ is the position of the upper electrode. If A were zero (two parallel flat plates) the electric field would be $\mathbf{E} = (V_0/B)\mathbf{i}_y$. Hence we define a perturbation electric field $\mathbf{e}(x, y, t)$ and let

$$\mathbf{E} = e_x \mathbf{i}_x + \left(\frac{V_0}{B} + e_y \right) \mathbf{i}_y. \quad (\text{f})$$

The perturbations e_x and e_y are proportional to the amplitude A . Introducing (f) into (e) [with n_x and n_y defined by (d)], we have

$$e_x(x, Y, t) = kA \sin \omega t \sin kx \left[\frac{V_0}{B} + e_y(x, Y, t) \right] \quad (\text{g})$$

and now, if we ignore terms that are proportional to A^2 (compared with terms proportional to A),

$$e_x(x, B, t) = \frac{kAV_0}{B} \sin \omega t \sin kx. \quad (\text{h})$$

Here e_x has been evaluated at $y = B$ rather than $y = Y$ because the difference in e_x evaluated at these points is proportional to A^2 . The approximate effect of the corrugated surface on the electric field is found by using (h) and (c) as boundary conditions.

Between the plates the divergence and curl of the electric field are zero [(1.1.11) and (1.1.12) with no free charge]. For the two-dimensional case under consideration this gives two expressions for the perturbation components.

$$\frac{\partial e_x}{\partial x} + \frac{\partial e_y}{\partial y} = 0, \quad (\text{i})$$

$$\frac{\partial e_x}{\partial y} - \frac{\partial e_y}{\partial x} = 0. \quad (\text{j})$$

If boundary condition (h) is to be satisfied, e_x must have the (x, t) dependence $\sin \omega t \sin kx$. Hence we assume (and later justify) that

$$e_x(x, y, t) = f(y) \sin \omega t \sin kx. \quad (\text{k})$$

Then (i) and (j) will be satisfied for all values of t and x only if e_y has the (x, t) dependence $\sin \omega t \cos kx$:

$$e_y(x, y, t) = g(y) \sin \omega t \cos kx. \quad (\text{l})$$

The dependence on (x, t) assumed for e_x and e_y is justified when we substitute (k) and (l) into (i) and (j) and find that the functions of x and t cancel out. After carrying out this process there remain the equations

$$fk + \frac{dg}{dy} = 0, \quad (\text{m})$$

$$\frac{df}{dy} + kg = 0. \quad (\text{n})$$

This pair of ordinary differential equations has the solution

$$f = C \sinh ky + D \cosh ky. \quad (\text{o})$$

where C and D are arbitrary constants. Remember that e_x is proportional to f [see (k)]. Hence, if (c) is to hold for *all* values of y , $D = 0$. From (k) and (o)

$$e_x(x, y, t) = C \sinh ky \sin \omega t \sin kx, \quad (\text{p})$$

where the constant C follows from boundary condition (h) as

$$C = \frac{kAV_0}{B \sinh kB}. \quad (\text{q})$$

Note that it was our foresight in guessing the (x, t) dependence of (k) that allowed us to satisfy condition (h) for all values of x and t . We now know f and therefore can find g from (n). Hence (l) becomes

$$e_y = -C \cosh ky \sin \omega t \cos kx. \quad (\text{r})$$

In the limiting case in which $B \ll l$ we can calculate these fields much more easily. This limiting case occurs when the wavelength l of the corrugation on the upper plate is long compared with the average spacing B between the plates. We expect a small section to have the same field as a parallel plate capacitor; that is,

$$E_y = \frac{V_0}{B + A \sin \omega t \cos kx} \cong \frac{V_0}{B} - \frac{V_0 A}{B^2} \sin \omega t \cos kx \quad (s)$$

and

$$E_x = 0. \quad (t)$$

If $B \ll l$, then $kB \ll 1$; and in this limit our solution [(f), (p), and (r)] reduces to the "long-wave" approximation of (s) and (t). In this limit all of the perturbation charges (those additional charges due to the corrugation) on one plate have image charges on the opposite plate. In the opposite extreme in which $kB \gg 1$ ($l \ll B$) there are few perturbation charges on the flat plate. To see this we can compute the surface charge density on the lower plate as

$$\sigma_f = \epsilon_0 \left[\frac{V_0}{B} + e_y(x, 0, t) \right], \quad (u)$$

which [from (q) and (r)] is

$$\sigma_f = \epsilon_0 \left[\frac{V_0}{B} - \frac{kA V_0 \sin \omega t \cos kx}{B \sinh kB} \right]. \quad (v)$$

As kB becomes large, the perturbation part of σ_f becomes small (as $kB \rightarrow \infty$, $\sinh kB \rightarrow \infty$).

A scheme for finding the deflection of a conducting surface would measure the charge on an electrode imbedded in the flat plate. Equation v shows that the perturbation surface charge density provides the (x, t) dependence of the deflection. The amplitude of σ_f however, would be inversely proportional to the wavelengths l to be detected.

6.3 CONSTITUENT RELATIONS FOR MATERIALS IN MOTION

Constituent relations, which are mathematical models of the electromagnetic properties of matter, were discussed briefly for stationary media in Section 1.1. At that point specific models that describe materials in a way that is useful in this study of electromechanical interactions were presented. As indicated in Section 6.0, however, the constituent relations expressed for stationary material may not be correct when the material is moving. In the next two sections we recast the constituent relations in forms that are correct when describing material that is in motion with respect to the reference frame in which electromagnetic quantities are measured. In general, a medium may be in motion relative to a particular inertial coordinate system in which we wish to define field and source quantities. We postulate that constituent relations, as conventionally defined for stationary media, still hold for moving media, provided they use source and field quantities defined in an inertial coordinate system with the same velocity as the material at the instant of time in question. It is therefore assumed that acceleration and rate of deformation do not affect local material properties. Constituent relations

obtained with this postulate yield predictions that agree to a high degree of accuracy with experimental results.* To apply this postulate we shall use the transformations of Sections 6.1.1 and 6.1.2 with the constituent relations for stationary media given in Section 1.1. Because the transformations are different for quasi-static electric and magnetic field systems we consider constituent relations for the two systems separately.

6.3.1 Constituent Relations for Magnetic Field Systems

Reference to the differential equations of Section 1.1 (see Table 1.2) and the boundary conditions of Section 6.2.1 shows that the fields in a quasi-static magnetic field system are excited by free current density $\mathbf{J}_f$, free surface current density $\mathbf{K}_f$, and magnetization density $\mathbf{M}$. Thus we need consider here only how $\mathbf{J}_f$, $\mathbf{K}_f$, and $\mathbf{M}$ are affected by field quantities in the presence of material motion.

Consider first a linear isotropic conducting medium, which, when stationary, has the constituent relation introduced as (1.1.9),

$$\mathbf{J}_f = \sigma \mathbf{E}, \quad (6.3.1)$$

where σ is the electrical conductivity.

We now define an inertial coordinate system $\mathbf{r}$ in which we measure the quantities $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{J}_f$ as functions of space ($\mathbf{r}$) and time (t). The material medium moves with respect to this coordinate system with a velocity $\mathbf{v}(\mathbf{r}, t)$. In general, the velocity $\mathbf{v}$ is different for each point within the material because it can be translating, rotating, and deforming. We wish to express the constituent relation describing electrical conduction in terms of quantities measured in the coordinate system $\mathbf{r}$. To do this we use the postulate given in the preceding section which states that the constituent relation for the material at rest is applicable in an inertial coordinate system with respect to which the material is instantaneously at rest. Thus to express the constituent relation for the material occupying position $\mathbf{r}$ at time t we define an inertial coordinate system having the velocity

$$\mathbf{v}^r = \mathbf{v}(\mathbf{r}, t), \quad (6.3.2)$$

that is, $\mathbf{v}^r$ is a constant with a value equal to the material velocity at position $\mathbf{r}$ at time t . We denote electromagnetic quantities as measured in this moving coordinate system with primes and apply our postulate along with (6.3.1) to write

$$\mathbf{J}'_f(\mathbf{r}, t) = \sigma(\mathbf{r}, t) \mathbf{E}'(\mathbf{r}, t). \quad (6.3.3)$$

* A case in which acceleration effects on conduction are computed is discussed in L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, Addison-Wesley, Reading, Mass., 1960, pp. 210–212. These effects are usually ignorable.

We now use the transformations of (6.1.36) and (6.1.38) to rewrite (6.3.3) as

$$\mathbf{J}_f(\mathbf{r}, t) = \sigma(\mathbf{r}, t)[\mathbf{E}(\mathbf{r}, t) + \mathbf{v}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t)]. \quad (6.3.4)$$

This equation is the desired result in that it is the relation imposed among electromagnetic variables expressed in the fixed frame by linear isotropic electrical conduction in a moving medium.

The functional notation $(\mathbf{r}, t)$ has been included in the preceding equations to make the meanings of the terms more explicit. Equation 6.3.4 is more compactly expressed without the functional notation as

$$\mathbf{J}_f = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (6.3.5)$$

Next consider surface conduction on a material with surface conductivity σ_s as described for stationary materials by (1.1.10).

$$\mathbf{K}_f = \sigma_s[-\mathbf{n} \times (\mathbf{n} \times \mathbf{E})], \quad (6.3.6)$$

where $\mathbf{n}$ is the normal to the surface and $[-\mathbf{n} \times (\mathbf{n} \times \mathbf{E})]$ is the component of $\mathbf{E}$ tangent to the surface. When the surface is moving, we use a process analogous to that used for volume conduction to obtain the result

$$\mathbf{K}_f = \sigma_s\{-\mathbf{n} \times [\mathbf{n} \times (\mathbf{E} + \mathbf{v} \times \mathbf{B})]\}. \quad (6.3.7)$$

The velocity $\mathbf{v}$ is the velocity of the surface with respect to the coordinate system in which the electromagnetic quantities are measured.

The final constituent relation that must be defined for magnetic field systems is the relation between magnetization density $\mathbf{M}$ and magnetic field intensity $\mathbf{H}$.

The constituent relation for an isotropic, linear, magnetic material was written as (1.1.6)

$$\mathbf{M} = \chi_m \mathbf{H}, \quad (6.3.8)$$

where χ_m is the magnetic susceptibility. Using (1.1.4) we wrote

$$\mathbf{B} = \mu \mathbf{H}, \quad (6.3.9)$$

where the permeability μ is defined as

$$\mu = \mu_0(1 + \chi_m). \quad (6.3.10)$$

Although the constituent relation as expressed by (6.3.8) or (6.3.9) was written for stationary material, the transformations of (6.1.35), (6.1.37), and (6.1.39) show that in a quasi-static magnetic field system $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{M}$ are unaffected by relative motion. Consequently, (6.3.8) and (6.3.9) hold also when the material is moving with respect to the coordinate system in which the electromagnetic quantities are to be measured.

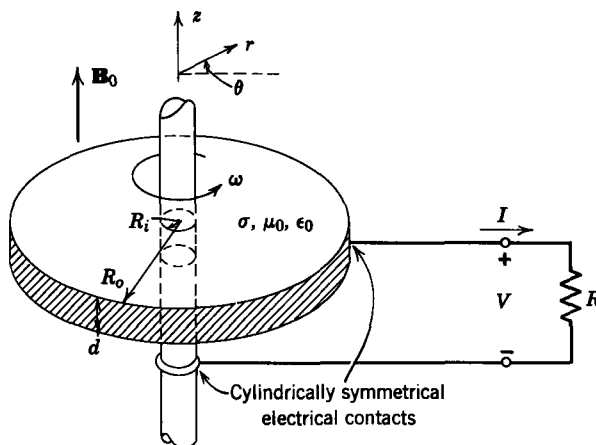


Fig. 6.3.1 A homopolar generator.

Example 6.3.1. As an example of transformations and constituent relations for moving materials, consider the device illustrated schematically in Fig. 6.3.1. This is a Faraday disk (also called a homopolar machine or an acyclic machine). Machines with the basic configuration of Fig. 6.3.1 or alternative configurations that operate physically in the same way are manufactured for supplying dc power at low voltage and high current.* A cutaway view of one such configuration is shown in Fig. 6.3.2.

With reference to Fig. 6.3.1, the device consists essentially of a right circular cylinder of conducting material that is rotated about its axis. Electrical contacts, usually made of liquid metal (see Fig. 6.3.2), are made symmetrically at inner and outer radii. Not shown in the figure is the electromagnet which produces a uniform axial flux density $\mathbf{B}_0$.

We specify that the applied flux density $\mathbf{B}_0$ is constant and that the shaft is driven by a constant angular velocity source ω . The electrical terminals are loaded by a resistance R . The material of the rotating disk is homogeneous, isotropic, and electrically linear with the material constants σ , μ_0 , ϵ_0 . The dimensions are defined in the figure.

We wish to find the terminal voltage and current for all values of load resistance R and steady-state operation.

It should be clear from an inspection of Fig. 6.3.1 that the current in the disk is radial and the current density is uniform around the periphery at any radius. Thus the magnetic field generated by this current density is tangential and has no effect on the terminal voltage. Hence we neglect the field due to current in the disk. The validity of this assumption becomes clearer in the analysis to follow.

We select the cylindrical coordinate system r , θ , z shown in Fig. 6.3.1. The cylindrical symmetry and the uniformity of variables in the z -direction indicate that we can assume

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial z} = 0,$$

* D. A. Watt, "Development and Operation of a 10KW Homopolar Generator with Mercury Brushes," *Proc. I.E.E. (London)*, **105A**, 33–40, (June 1958). A. K. Das Gupta, "Design of Self-Compensated High-Current Comparatively Higher Voltage Homopolar Generators," *Trans. AIEE*, **80**, Part III, 567–573, 1961–1962.

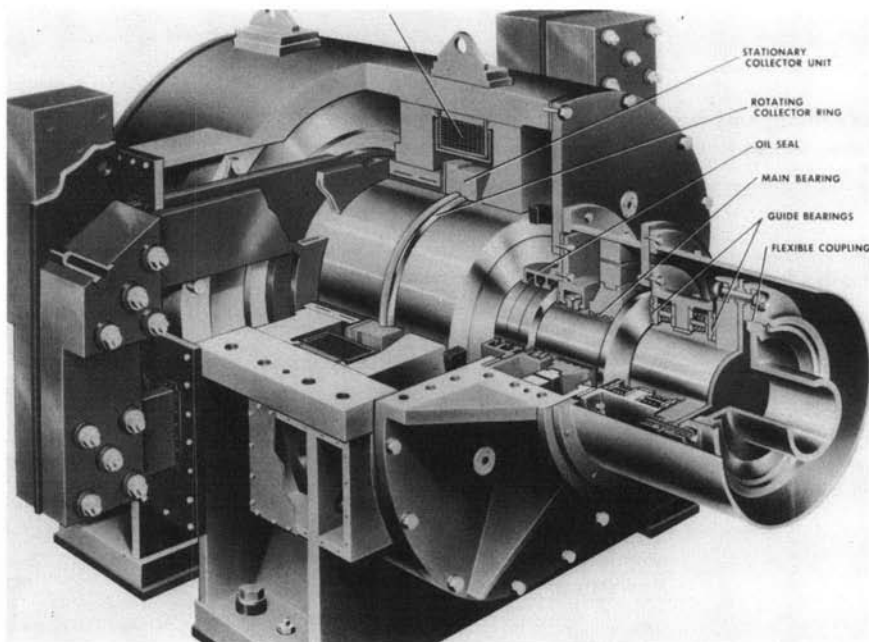


Fig. 6.3.2 Cutaway view of an acyclic generator. The solid rotor is made of magnetically soft steel, the flux density is radial, and current is axial between two liquid metal collector rings, one of which is shown. (Courtesy of General Electric Company.)

Courtesy of General Electric Company. Used with permission.

so that electromagnetic quantities of interest will vary with radius only. The electromagnetic equations for this quasi-static magnetic field system are those of Section 1.1.1a (see Table 1.2).

We first use the conservation of charge in integral form (1.1.22)

$$\oint \mathbf{J}_r \cdot \mathbf{n} \, da = 0, \quad (\text{a})$$

to establish that the radial component of current density is related to the terminal current by

$$J_r = \frac{I}{2\pi r d}. \quad (\text{b})$$

We next write Ohm's law for a grain of matter at the radius r by writing the r component of (6.3.5).

$$J_r = \sigma(E_r + \omega r B_0), \quad (\text{c})$$

where B_0 is the magnitude (z -component) of $\mathbf{B}_0$ and E_r is the radial component of electric field intensity. A tangential (θ) component of flux density is parallel to the material velocity and does not contribute a $\mathbf{v} \times \mathbf{B}$ term. Thus the neglect of the field generated by current in the disk is justified.

We now use (b) in (c) to find E_r :

$$E_r = \frac{I}{2\pi\sigma dr} - \omega B_0 r. \quad (d)$$

Recognizing that there is no time rate of change of magnetic field in the fixed reference frame, we can write the terminal voltage as

$$V = - \int_{R_i}^{R_o} E_r dr = - \frac{I}{2\pi\sigma d} \ln \frac{R_o}{R_i} + \frac{\omega B_0}{2} (R_o^2 - R_i^2). \quad (e)$$

Use of the terminal relation required by the resistance R yields

$$I = \frac{V_{oc}}{R + R_{int}} \quad (f)$$

where $V_{oc} = (\omega B_0/2)(R_o^2 - R_i^2)$ is the open circuit ($R \rightarrow \infty$) voltage of the generator, and $R_{int} = [\ln(R_o/R_i)]/2\pi\sigma d$ is the internal resistance of the generator.

To obtain some idea of the kinds of numbers obtainable with real materials consider a copper disk with the following parameters and dimensions:

$$\begin{array}{ll} \sigma = 5.9 \times 10^7 \text{ mhos/m} & \omega = 400 \text{ rad/sec} \\ d = 0.005 \text{ m} & B_0 = 1 \text{ Wb/m}^2 \\ R_i = 0.01 \text{ m} & R_o = 0.1 \text{ m} \end{array}$$

The open-circuit voltage is then

$$V_{oc} = 2 \text{ V}$$

and the internal resistance is

$$R_{int} = 1.25 \times 10^{-6} \Omega.$$

The short-circuit current is

$$I_{sc} = \frac{V_{oc}}{R_{int}} = 1.6 \times 10^6 \text{ A.}$$

The maximum power that can theoretically be delivered by this generator is

$$P_{max} = \frac{V_{oc} I_{sc}}{4} = 8 \times 10^5 \text{ W.}$$

For steady-state operation, however, the output power would be limited to a much lower value by allowable $I^2 R_{int}$ heating of the rotating disk. These figures indicate, though, that this device is suitable for supplying large pulses of power.*

We now use (d) and (f) to write the radial component of electric field intensity as

$$E_r = \left[\frac{R_{int} V_{oc}}{(R + R_{int}) \ln(R_o/R_i)} \right] \frac{1}{r} - \omega B_0 r. \quad (g)$$

In the spirit of the discussion of quasi-static systems in Section B.2.2 we can calculate the volume charge density necessary to satisfy Gauss's law:

$$\rho_f = \nabla \cdot \epsilon_0 E. \quad (h)$$

* T. J. Crawford, "Kinetic Energy Storage for Resistance Welding," *Welding Engineer*, 33, 36 (1948).

Using the equation for the divergence in cylindrical coordinates,* we have

$$\rho_f = -2\epsilon_0\omega B_0. \quad (i)$$

This charge density arises from the second term in (g); the first term is divergenceless. The finite volume charge density results because the electric field is generated by nonuniform motion in a uniform magnetic field. There is, however, no net charge on the disk because an equal amount of charge of opposite sign occurs as a surface charge density at $r = R_0$ and $r = R_i$.

As discussed in Section B.2.2 this charge density was derived *after* the field problem was solved and its presence has negligible effect on the field solutions. To illustrate, consider the current density that results from the convection of this charge density by the rotating disk. The result is a θ -component of current density that has the value

$$J_\theta = -2\epsilon_0\omega^2 B_0 r.$$

We use the θ -component of Ampere's law in cylindrical coordinates to find the change in B_z caused by this current:

$$\frac{\partial B_z}{\partial r} = -\mu_0 J_\theta = 2\mu_0\epsilon_0\omega^2 B_0 r.$$

Integrating this expression, we find the maximum possible fractional change in B_z as

$$\frac{\Delta B_z}{B_0} = \omega^2 R_0^2 \mu_0 \epsilon_0 = \frac{\omega^2 R_0^2}{c^2},$$

where c is the speed of light (Section B.2.1). For any disk made of real material the peripheral speed (ωR_0) must be much smaller than the speed of light; thus the change in B_z due to convection current is negligible.

We reconsider the homopolar machine and complete its terminal description as an electromechanical coupling device in Section 6.4.

6.3.2 Constituent Relations for Electric Field Systems

The differential equations in Section 1.1.1*b* (see Table 1.2) and the boundary conditions of Section 6.2.2 indicate that fields in quasi-static, electric field systems are excited by free charge density ρ_f , free surface charge density σ_f , and polarization density $\mathbf{P}$. The constituent relations for these source quantities are given for stationary media in Section 1.1.1*b*. We generalize those constituent relations appropriate for electric field systems to include the effects of material motion.

In Section 1.1.1*b* the conduction process in a stationary medium was modeled by (1.1.16):

$$\mathbf{J}_f = (\rho_{f+}\mu_+ + \rho_{f-}\mu_-)\mathbf{E}, \quad (6.3.11)$$

where ρ_{f+} and ρ_{f-} are the densities of free charge and μ_+ and μ_- are the mobilities of the free charges defined in Section B.3.3. When the material is moving, (6.3.11) must be modified according to the transformations of

* $[d(\epsilon_0 r E_r)/dr]/r$

(6.1.56) and (6.1.58). Thus, if the material is moving with velocity $\mathbf{v}$, we must write (6.3.11) in the form

$$\mathbf{J}_f = (\rho_{f+}\mu_+ + \rho_{f-}\mu_-)\mathbf{E} + \rho_f\mathbf{v}, \quad (6.3.12)$$

where the net charge density ρ_f is given by

$$\rho_f = \rho_{f+} + \rho_{f-}. \quad (6.3.13)$$

The first term $[(\rho_{f+}\mu_+ + \rho_{f-}\mu_-)\mathbf{E}]$ of (6.3.12) describes the motion of charge carriers with respect to the material and the second term $(\rho_f\mathbf{v})$ describes the convection of net charge by the motion of the material.

The transformations of (6.1.54), (6.1.55), and (6.1.59) show that in a quasi-static electric field system $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{P}$ are unaffected by relative motion. Consequently, (1.1.17) and (1.1.19), which were written for stationary material, are still valid when the material is moving with respect to the coordinate system in which electromagnetic quantities are measured. For convenience we repeat these two equations here:

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E}, \quad (6.3.14)$$

$$\mathbf{D} = \epsilon\mathbf{E}, \quad (6.3.15)$$

where the dielectric susceptibility χ_e and permittivity ϵ are related by

$$\epsilon = \epsilon_0(1 + \chi_e). \quad (6.3.16)$$

We conclude this section with an example that involves a particularly simple form of (6.3.12). Other examples of the use of these constituent relations are given in Chapter 7.

Example 6.3.2. A simple example in which the constituent relation (6.3.12) is used, is shown in Fig. 6.3.3. Here a cylindrical beam of charge carriers moves with the velocity V in the z -direction. We assume that the charge density is uniform throughout the beam and that the carriers (for example electrons) have zero mobility.

In a primed frame moving to the right at the velocity V the current is zero (6.3.12) and we have the simple fields associated with a uniform cylinder of charge density ρ_0 . Note that we have assumed $\rho_f = \rho'_f$, as given by (6.1.56). In this moving frame there is no magnetic field because there is no current $\mathbf{J}'_f$. Because the beam is uniform in the z -direction,

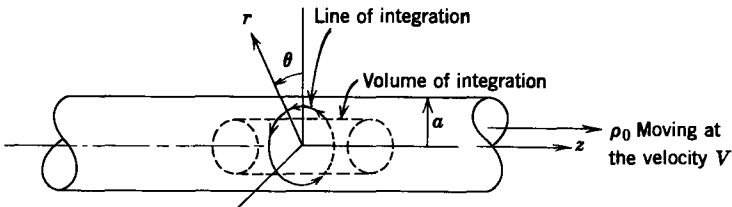


Fig. 6.3.3 Charged beam moving at velocity V to the right.

the electric field follows from Gauss's law (1.1.25) integrated over a cylinder of radius r and unit length in the z -direction:

$$\epsilon_0 E'_r(2\pi r) = \int_0^r \rho_f 2\pi r \, dr. \quad (a)$$

Hence inside the beam

$$E'_r = \frac{\rho_0 r}{2\epsilon_0}, \quad r < a, \quad (b)$$

and outside the beam

$$E'_r = \frac{\rho_0 a^2}{2\epsilon_0 r} \quad r > a. \quad (c)$$

To find the fields in the fixed frame is a simple matter, for (6.1.54) requires that $\mathbf{E} = \mathbf{E}'$ and (6.1.58) gives the current as

$$\mathbf{J}_f = \rho_0 V \mathbf{i}_z, \quad r < a. \quad (d)$$

The magnetic field follows from (6.1.57) and (6.1.54) as $\mathbf{H} = V \mathbf{i}_z \times \epsilon_0 \mathbf{E}$ or, by use of (b) and (c),

$$\mathbf{H} = \mathbf{i}_\theta \frac{V \rho_0 r}{2}, \quad r < a, \quad (e)$$

$$\mathbf{H} = \mathbf{i}_\theta \frac{V \rho_0 a^2}{2r}, \quad r > a.$$

The last result could be found alternatively by using the current density $\mathbf{J}_f$ from (d) in the integral form of Ampère's law (1.1.20). A line integral of $\mathbf{H}$ around the beam at a radius r gives

$$2\pi r H_\theta = \int_0^r J_z 2\pi r \, dr \quad (f)$$

or

$$H_\theta = \frac{V \rho_0 r}{2}, \quad r < a, \quad (g)$$

$$H_\theta = \frac{V \rho_0 a^2}{2r}, \quad r > a.$$

in agreement with (e).

6.4 DC ROTATING MACHINES

As stated in Section 4.1.6c, the dc machine is the most widely used rotating machine for control applications, especially when precise and versatile control of mechanical power or torque is required. Control can be achieved with high efficiency so that dc motors are used widely in high-power systems such as traction applications for driving locomotives and subway trains, rolling mills in steel plants, and ship propulsion. In electrically propelled ships and in diesel-electric locomotives the prime mover produces mechanical power and an electrical system is used for control rather than a mechanical

system of gears. The primary reason for this substitution is the versatility of the electrical power transmission system from the viewpoint of control.

The use of dc motors for controllable drives requires dc generators to supply the necessary power. Moreover, there are many applications of electric power in which direct current is necessary; for example, the production of aluminum is accomplished by the use of large quantities of direct current. Direct-current generators driven by alternating-current motors are used to supply power in many of these cases.

Because of the wide and extensive use of dc machines, we consider some of the important features of the more common types in use. Our treatment is introductory and intended to provide an understanding of the essential physics of dc machine operation and to indicate how the terminal behavior can be analyzed. In spite of our special attention to the topic at this point the reader should remember that although the dc machines we treat illustrate the basic material introduced in this chapter they are also specific examples of lumped-parameter, magnetic field-type, electromechanical devices introduced in Chapters 2 and 3. As we complete the analyses, we shall indicate the relation to the earlier chapters.

In the next two sections we treat two configurations of dc machines: commutator machines discussed briefly in Section 4.1.6c and homopolar machines, an example of which was introduced in Example 6.3.1.

6.4.1 Commutator Machines

6.4.1a Physical Characteristics

As discussed in Section 4.1.6c, a commutator can be viewed as a mechanically controlled frequency changer that causes rotor-current frequencies to satisfy automatically the condition for average power conversion (4.1.18) when rotor and stator electrical sources are at the same frequency (usually zero). To analyze the terminal behavior of a commutator machine, other viewpoints are used. (In some cases the techniques of Chapter 3 are employed, but this kind of treatment makes physical insight difficult.) We use a field approach to obtain equations of motion and indicate how the connection is made to the techniques of Chapter 3.*

To develop the equations of motion for a commutator machine from a field viewpoint we need to specify the geometry of the windings, the commutator, the brushes, and the magnetic material. To do this we use simplified schematic drawings. To put these representations in perspective a cutaway view

* For some alternative viewpoints on analytical techniques to be used with commutators see, for example, D. C. White and H. H. Woodson, *Electromechanical Energy Conversion*, Wiley, New York, 1959, Chapter 4; A. J. Thaler and M. I. Wilcox, *Electric Machines*, Wiley, New York, 1966, Chapters 3 and 4; A. E. Fitzgerald and C. Kingsley, Jr., *Electric Machinery*, 2nd ed., McGraw-Hill, New York, 1961, Chapter 3.

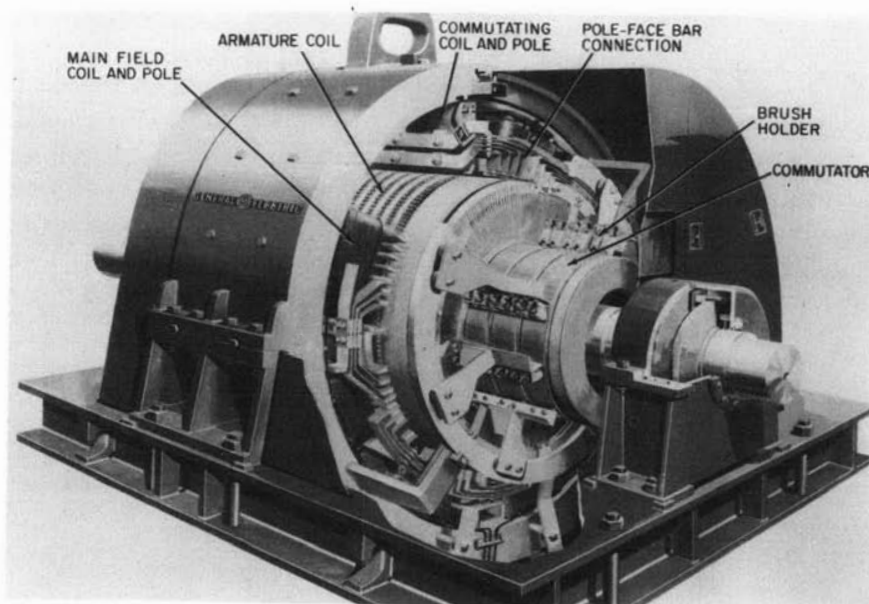


Fig. 6.4.1 Cutaway view of a 2250-hp, 300/600-rpm, 600-V, dc motor. (Courtesy of General Electric Company.)

Courtesy of General Electric Company. Used with permission.

of a commutator machine is shown in Fig. 6.4.1 with the principal parts labeled. When using the simplified schematic drawings, frequent reference should be made to this practical configuration.

First, consider the schematic end view of a two-pole commutator machine shown in Fig. 6.4.2. This is a salient-pole structure, as defined and discussed in Section 4.2, with the salient poles on the stator. A commutator machine can have any even number of poles (see Sections 4.1.8 and 4.2.4), but we treat a two-pole machine in the interest of simplicity. The rotor is essentially cylindrical with conductors placed in axial slots as indicated. The stator winding is excited directly at its terminals and the rotor conductors are excited through brushes (usually carbon) that make sliding contact with the commutator.

To follow the usual convention, we call the stator winding the *field* winding and denote quantities associated with it by the subscript f ; we call the rotor winding the *armature* winding and denote quantities associated with it by the subscript a .

Currents in the field windings, with directions indicated by dots and crosses in Fig. 6.4.2, produce a flux density distribution that is symmetrical about the field magnetic axis, as indicated in the developed view of the machine in Fig. 6.4.3. Currents in the armature conductors, maintained by the commutator

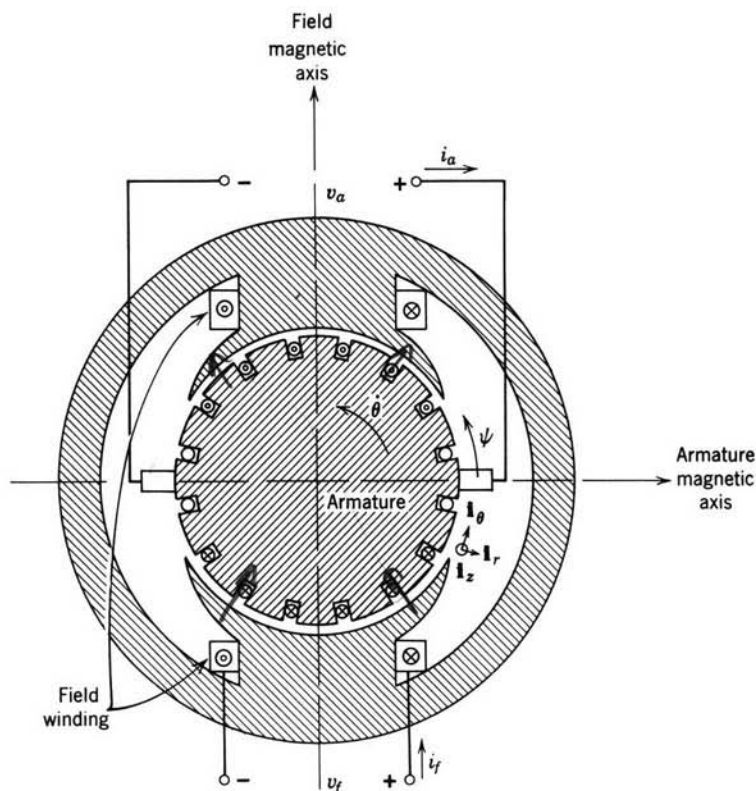


Fig. 6.4.2 Two-pole, commutator machine.

and brushes, have the directions indicated by the dots and crosses in Fig. 6.4.2, independent of armature (rotor) position or speed. Thus armature current produces a flux density distribution that is symmetrical about the armature magnetic axis, as indicated in Fig. 6.4.3.

To indicate qualitatively the shapes of the flux density distributions and to illustrate how armature conductors are connected to the commutator bars, a developed view of the machine is shown in Fig. 6.4.3. The shape of the field flux density distribution is understandable in terms of Ampère's law (1.1.20). For analytical purposes it is often assumed that fringing at the pole edges can be neglected because the air gap is small, that slot and teeth effects are negligible, and that there is no magnetic saturation. Then the flux density distribution has the square shape shown by dashed lines. As developed subsequently, the important quantity is the total magnetic flux per pole; consequently, the idealized curve is a good representation of the actual curve with respect to total flux (area under the curve).

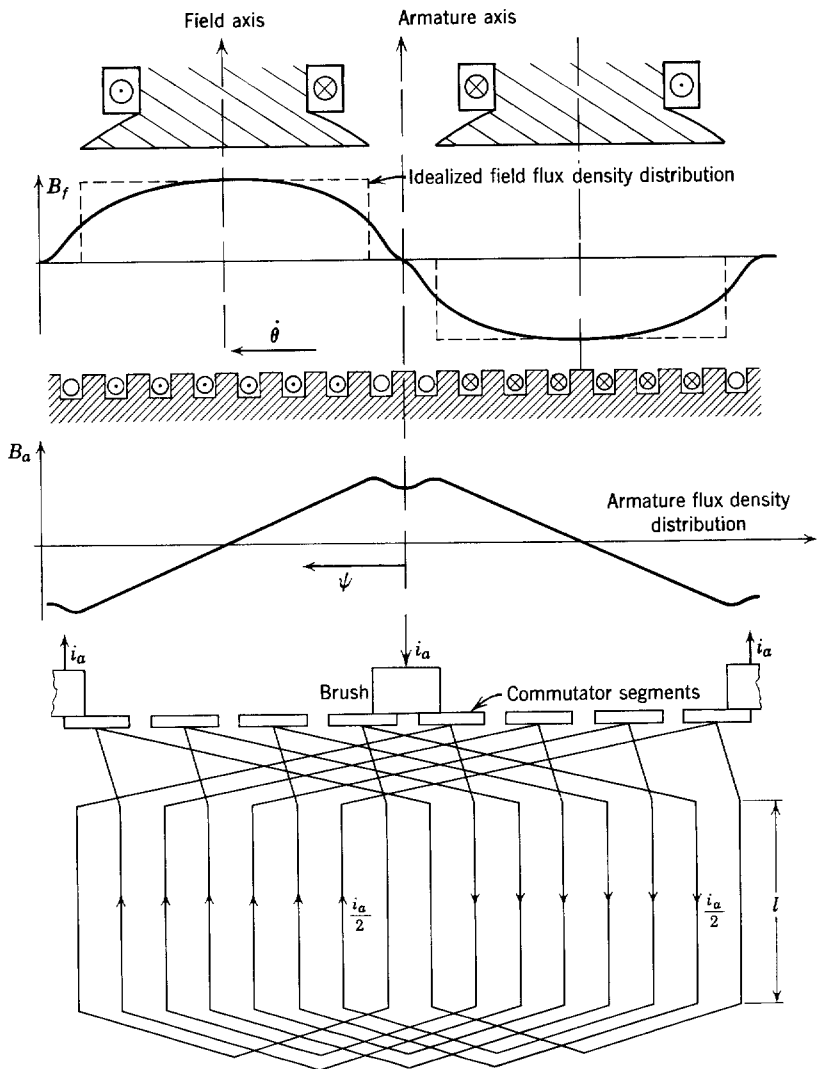


Fig. 6.4.3 Developed views of two-pole commutator machine showing flux distributions and armature connections.

The armature flux density distribution in Fig. 6.4.3 is idealized in two ways: the effects of slots and teeth are neglected and the armature current is assumed to be uniformly distributed along the armature surface. This approximation may seem rather gross with respect to the number and size of slots in Fig. 6.4.3; in practical machines, however, the number of slots is much greater and the slots are smaller; thus the approximate flux density distribution in Fig. 6.4.3 is quite accurate.

The developed view at the bottom of Fig. 6.4.3 shows how armature conductors are connected to commutator bars. A tracing of conducting paths will show that any one-turn coil has its ends connected to adjacent commutator bars and that there are two parallel conducting paths between brushes, each containing three coils in series. Two coils are short-circuited by the brushes. In practical machines with lap windings,* the coils may have one or more turns, with the coil terminals connected as shown in Fig. 6.4.3. Also, in practical machines each slot normally contains two coil sides, one each from two different coils, rather than the one we show for simplicity.

By visualizing what happens to the conductor currents in Fig. 6.4.3 as the armature conductors and commutator bars slide past the brushes, it will become clear that the armature current pattern will shift back and forth by about the distance between two slots. In a practical machine with a large number of armature conductors and commutator bars, this variation will be much less, and it is quite reasonable to assume that the current pattern is fixed at its average position. This switching of current pattern by the commutator can be interpreted as the electrical equivalent of a mechanical ratchet.

As indicated in Fig. 6.4.3 and noted earlier, two coils have no current because the brushes short-circuit them. As the armature turns, armature coils are successively shorted by the brushes. Before a coil is shorted it carries current in one direction, and after the short is removed the current direction is reversed. The process of current reversal is called commutation, and it is complicated by speed voltage and inductive voltages in the shorted coil and by arcing of the contact between brush and commutator. The process of commutation is complex and its practical realization imposes a limitation on the characteristics that can be achieved with commutator machines.† For the purpose of analyzing the terminal behavior of commutator machines we need to know only the geometry of the windings and the fact

* An alternative scheme is called a wave winding. Both schemes, lap and wave windings, essentially yield a continuous armature winding. For a discussion of the two schemes see, for example, A. E. Knowlton, ed., *Standard Handbook for Electrical Engineers*, 9th ed., McGraw-Hill, New York, 1957, Section 8.25.

† For a discussion of commutation and a list of good references see Knowlton, *op. cit.*, Sections 8.33 to 8.55.

that the commutator keeps the armature current pattern fixed in space with respect to the brushes, as illustrated in Figs. 6.4.2 and 6.4.3.

6.4.1b Equations of Motion

To develop the equations of motion for commutator machines we use the field transformations of Section 6.1.1 and the constituent relations of Section 6.3.1. We consider the effects of field flux density and armature flux density separately and superimpose the results. Such a process provides adequate accuracy; when the two flux density distributions in Fig. 6.4.3 are superimposed, however, they add in some regions and subtract in others. In the region in which they add, there may be saturation and resulting distortion of the flux patterns. When this occurs, the armature flux density distribution is skewed and there is a net linkage of armature flux with the field winding. This phenomenon is called armature reaction.* We neglect saturation, and thus the effects of armature reaction, in our analysis.

Consider first the field winding in Fig. 6.4.2. With reference to the flux density distributions of Fig. 6.4.3, it is clear that the armature produces no net flux linkage with the field winding because the axes of symmetry of the flux density distributions are orthogonal. Thus the field winding links only its own flux and we can write the equation (see Section 2.1.1)

$$v_f = R_f i_f + L_f \frac{di_f}{dt}, \quad (6.4.1)$$

where R_f = the field winding resistance,

L_f = the field winding self-inductance.

Effects of armature slots and teeth are neglected in defining the constant field inductance L_f .

Because the armature conductors are in motion with respect to the reference frame in which we are defining fields, we must be careful when writing the voltage equation for the armature circuit. We use Faraday's law in integral form with a *fixed* contour (see Table 1.2):

$$-\oint_C \mathbf{E} \cdot d\mathbf{l} = \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da. \quad (6.4.2)$$

The contour C is fixed so the $\mathbf{E}$ is measured in the fixed reference frame. The contour to be used (shown schematically in Fig. 6.4.4) follows one of the conducting paths through the armature conductors between the brushes (see Fig. 6.4.3). The contour is fixed and the conductors are moving; thus they coincide only instantaneously.

* See, for example, Knowlton, *op. cit.*, Sections 8.27 to 8.32.

To evaluate (6.4.2) we break the contour integration into two parts:

$$-\int_a^b \mathbf{E} \cdot d\mathbf{l} - \int_b^a \mathbf{E} \cdot d\mathbf{l} = \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} da. \quad (6.4.3)$$

The first term on the left is the integral between the terminals external to the machine and is

$$-\int_a^b \mathbf{E} \cdot d\mathbf{l} = v_a. \quad (6.4.4)$$

The second integral on the left is taken through the armature circuit of the machine. If the armature conductor material has a conductivity σ , then (6.3.5) indicates that we must write

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (6.4.5)$$

for the armature conductors in which $\mathbf{J}$ is the armature conductor current density and $\mathbf{v}$ is the armature conductor velocity. This expression can be used for all armature circuit conductors, including those from the terminals to the brushes, because $\mathbf{v} = 0$ and $\mathbf{J} = \sigma\mathbf{E}$, as it should be for a conductor at rest.

Solution of (6.4.5) for $\mathbf{E}$ and the use of that result in the second term of (6.4.3) yields

$$-\int_b^a \mathbf{E} \cdot d\mathbf{l} = -\int_b^a \frac{\mathbf{J}}{\sigma} \cdot d\mathbf{l} + \int_b^a (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}. \quad (6.4.6)$$

The first term on the right is just the drop in voltage across the armature resistance and can be written as

$$-\int_b^a \frac{\mathbf{J}}{\sigma} \cdot d\mathbf{l} = -i_a R_a, \quad (6.4.7)$$

where R_a is the armature circuit resistance. To show this for the armature conductors between brushes, assume that the current ($i_a/2$) (see Fig. 6.4.3) is distributed uniformly over the cross section A_w of the wire. Then the magnitude of $\mathbf{J}$ is

$$J = \frac{i_a}{2A_w}. \quad (6.4.8)$$

Specify the total length of wire between brushes as l_w and

$$-\int_b^a \frac{\mathbf{J}}{\sigma} \cdot d\mathbf{l} = -\int_0^{l_w} \frac{i_a}{2A_w\sigma} dl = \frac{-i_a l_w}{2A_w\sigma}. \quad (6.4.9)$$

The quantity $(l_w/2A_w\sigma)$ is just the resistance

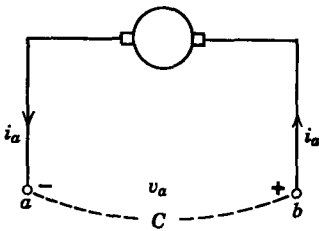


Fig. 6.4.4 Illustration of the contour for finding an armature voltage equation. The contour is completed on the armature, where it follows one of the two conducting paths joining the brushes in Fig. 6.4.3.

of two wires in parallel, each having length l_w , and A_w , and conductivity σ .

The second term on the right of (6.4.6) is a *speed voltage*, which is evaluated as follows. The velocity of a conductor is

$$\mathbf{v} = \mathbf{i}_\theta \dot{\theta} R, \quad (6.4.10)$$

where R = the radius measured from the axis of rotation to the conductor location,

$\dot{\theta}$ = the angular speed of the armature,

$\mathbf{i}_\theta$ = the unit vector in the tangential direction taken as positive in the counterclockwise direction in Fig. 6.4.2.

If we assume that appreciable flux density occurs only over the axial length l shown in Fig. 6.4.3 and that this flux density is radial and independent of axial position,* then

$$\mathbf{B} = \mathbf{i}_r B_r(\psi), \quad (6.4.11)$$

where $\mathbf{i}_r$ = the unit vector in the radial direction,

ψ = an angle measured with respect to the fixed reference frame, as indicated in Fig. 6.4.2.

Equations 6.4.10 and 6.4.11 are used to write

$$\mathbf{v} \times \mathbf{B} = -\mathbf{i}_z \dot{\theta} R B_r(\psi), \quad (6.4.12)$$

where $\mathbf{i}_z$ is the unit vector in the axial direction and is positive out of the paper in Fig. 6.4.2. Use of this term in the integral

$$\int_b^a (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l},$$

with the flux density distributions of Fig. 6.4.3 and the contour defined in Fig. 6.4.4, shows that there is no net contribution from armature flux density but there is from field flux density. To evaluate this contribution we can evaluate B_r at the value of ψ for each conductor, multiply by the length, and add up the contributions of each conductor to get the total. In real machines there are many armature conductors such that it is a good approximation to use the average flux density due to the field winding

$$(B_{rf})_{av} = \frac{\int_0^\pi B_{rf}(\psi) d\psi}{\pi} \quad (6.4.13)$$

and write

$$\mathbf{v} \times \mathbf{B} = -\mathbf{i}_z \dot{\theta} R (B_{rf})_{av}. \quad (6.4.14)$$

* This restriction is necessary only for simplicity. The method is general and can include axial variation of radial flux density and effects of the axial component of $\mathbf{B}$ on the radially directed end turns.

We then take twice the axial length ($2l$) for the voltage generated in one turn and the total number of turns N in one path between brushes to write

$$\int_b^a (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = -2lN \dot{\theta} R(B_{rf})_{av}. \quad (6.4.15)$$

Because $(B_{rf})_{av}$ is proportional to field current i_f , it is conventional to write this term as

$$\int_b^a (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = -G\dot{\theta}i_f, \quad (6.4.16)$$

where G is the *speed coefficient* that depends only on geometry and magnetic material properties.

To evaluate the final term, the right side of (6.4.3), we recognize that

$$\int_S \mathbf{B} \cdot \mathbf{n} \, da = \lambda_a \quad (6.4.17)$$

is the flux linking the armature circuit. As indicated earlier, all of this flux linkage is due to armature current, the system is assumed to be electrically linear, and the effects of slots and teeth are neglected; thus

$$\lambda_a = L_a i_a, \quad (6.4.18)$$

where L_a is the constant armature self-inductance and

$$\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da = \frac{d\lambda_a}{dt} = L_a \frac{di_a}{dt}. \quad (6.4.19)$$

The armature voltage equation is now written by combining (6.4.4), (6.4.6), (6.4.7), (6.4.16), and (6.4.19) in (6.4.3); thus

$$v_a = i_a R_a + L_a \frac{di_a}{dt} + G\dot{\theta}i_f. \quad (6.4.20)$$

This is the desired armature circuit equation.

It is clear from this result that the electrical consequence of mechanical motion is represented by the last term on the right of (6.4.20). This term was derived in (6.4.10) to (6.4.20) with a fixed contour and Ohm's law for a moving conductor. It could have been derived with a contour that moves with the armature conductor. In this case we assume a contour from b to a in Fig. 6.4.4 that is moving with the armature conductors and write (6.4.3) in the alternate form

$$-\int_a^b \mathbf{E} \cdot d\mathbf{l} - \int_b^a \mathbf{E}' \cdot d\mathbf{l} = \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da. \quad (6.4.21)$$

We must remember that the contour C is now moving and thus the surface S varies with time. In the frame moving with the conductor

$$\mathbf{J}' = \sigma \mathbf{E}'$$

and the transformation of (6.1.36) gives

$$\mathbf{J}' = \mathbf{J}.$$

Thus the second term on the left of (6.4.21) produces only the resistive voltage drop (6.4.7). The speed voltage (6.4.16) is now generated by the time-varying surface in the term on the right of (6.4.21). This can be verified in a straightforward manner and is not done here. The method corresponds to that of Chapter 3 in which speed voltages were obtained from time rates of change of fluxes. These alternate ways of computing the terminal voltage were also illustrated in Example 6.1.2.

To complete the terminal description of the commutator machine we must evaluate the torque of electric origin. There are several equally valid ways of doing this. The torque is evaluated here by using the Lorentz force density for magnetic field systems.

We shall use the force density (1.1.30)

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}, \quad (6.4.22)$$

with current density in the armature conductors and the radial component of flux density sketched in Fig. 6.4.3. It should be recognized that the teeth shield the conductors in the slots so that the conductors experience only a small fraction of the flux, most of which passes through the teeth. The result is that most of the torque is produced by magnetic forces on the magnetic material of the teeth. Nonetheless, we still get the correct answer by assuming that all the magnetic forces act on the conductors. That it is immaterial whether the force acts on the conductors or teeth but depends only on the magnetic fields in the air gap adjacent to the armature can be verified by using the Maxwell stress tensor to be introduced in Section 8.2.1.*

Making assumptions consistent with those in the derivation of the armature voltage equation, we assume (for mathematical convenience, not necessity) that appreciable flux density exists only along the length l of axial conductors (see Fig. 6.4.3 for definition of l) and that the flux density is radial and does not vary with axial position

$$\mathbf{B} = \mathbf{i}_r B_r(\psi). \quad (6.4.23)$$

* The fact that the force acts on the armature teeth and not on the conductors has the practical advantage that the mechanical forces applied to the conductor insulation are small and the insulation problem is more easily solved.

The current density in the region of interest is uniform across the conductor and axial

$$\mathbf{J} = \mathbf{i}_z J_z. \quad (6.4.24)$$

We use (6.4.23) and (6.4.24) in (6.4.22) to get the force density

$$\mathbf{F} = \mathbf{i}_\theta J_z B_r(\psi). \quad (6.4.25)$$

For uniform current density the magnitude of the current density in one armature conductor (see Fig. 6.4.3) is

$$|J_z| = \frac{i_a}{2A_w}, \quad (6.4.26)$$

where A_w is the cross-sectional area of the conductor. By integrating throughout the volume of one conductor over the active length l , the total force on one conductor is

$$\mathbf{F}_1 = \pm \mathbf{i}_\theta \frac{i_a l B_r(\psi)}{2}. \quad (6.4.27)$$

The $\pm$ is determined by the relative directions of current and flux density.

For the system of Fig. 6.4.2 we could simply superimpose the field and armature flux density distributions in Fig. 6.4.3, find the force on each conductor, add the forces, and multiply by the lever arm to get the torque. It is more instructive, however, to consider the two flux densities separately. First, it should be evident from the relative shapes of the armature current distribution and the armature flux density distribution (see Fig. 6.4.3) that no net torque results from their interaction. Furthermore, the relative directions of field flux density and armature current are such that each conductor produces a torque in the $+\theta$ -direction. Thus, with a lever arm (conductor radial position) of R , the torque from one conductor in the $+\theta$ -direction is

$$T_1 = \frac{i_a l R B_{rf}(\psi)}{2}. \quad (6.4.28)$$

We could add the contributions from the 12 active conductors (six coils) in Fig. 6.4.2; there are, however, many conductors, and the practice is to use the average field flux density defined in (6.4.13) and multiply by the number of active conductors, which is four times the number N of coils introduced in (6.4.15) (remember that there are two parallel paths between the brushes); thus the total torque is

$$T^e = 2Nl R(B_{rf})_{av} i_a. \quad (6.4.29)$$

Note that the coefficient of i_a in this expression is the same as the coefficient of $\hat{\theta}$ in (6.4.15), and we use the speed coefficient G defined by (6.4.15) and (6.4.16) to write

$$T^e = G i_f i_a. \quad (6.4.30)$$

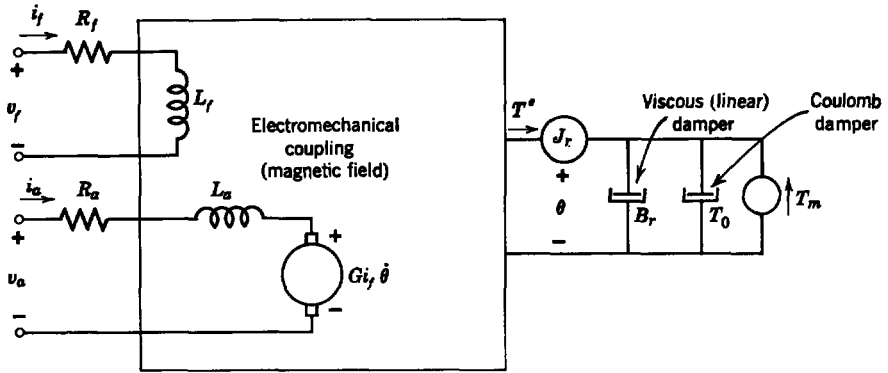


Fig. 6.4.5 Equivalent circuit of the commutator machine in Fig. 6.4.2.

In addition to the Maxwell stress tensor method mentioned earlier, this same torque can be derived by the energy method of Chapter 3, although with the commutator, care must be taken when applying this technique.

We have now completed the description of the electromechanical coupling properties of the commutator machine of Fig. 6.4.2. We complete the terminal description in the nomenclature introduced in Chapters 2 and 3 by drawing the equivalent circuit in Fig. 6.4.5. All losses are taken outside the coupling network which contains only magnetic field energy storage. On the mechanical side the rotor (armature) moment of inertia is J_r and both viscous (B_r) and coulomb (T_0) damping are included. A commutator machine normally has significant coulomb damping from the brushes sliding on the commutator. The sources T_m , v_a , and v_f are general and can be independent or dependent on some variable.

6.4.1c Machine Properties

We shall now study some properties of the commutator machine by using the equivalent circuit of Fig. 6.4.5. The instantaneous power converted to mechanical form by the coupling network is

$$p_m = T^e \dot{\theta} = G i_f i_a \dot{\theta}, \quad (6.4.31)$$

where (6.4.30) has been used for torque T^e . The instantaneous power absorbed from the armature circuit by the speed voltage $G \dot{\theta} i_f$ is

$$p_e = G \dot{\theta} i_f i_a, \quad (6.4.32)$$

which is equal to the mechanical power output. This leads to the following immediate conclusions:

1. Power conversion between electrical and mechanical form in a commutator machine occurs instantaneously between the mechanical system and the speed voltage source in the armature circuit.

2. The power conversion is proportional to field current, but there is no power conversion between the field circuit and the mechanical system.

Next, assume that the rotor is to be driven by a constant-speed source

$$\dot{\theta} = \omega_m, \quad (6.4.33)$$

open circuit the armature ($i_a = 0$), and apply a battery to the field circuit terminals

$$v_f = V_f. \quad (6.4.34)$$

The open-circuit armature voltage varies with field voltage, as can be seen by considering steady-state conditions and (6.4.1) to find

$$i_f = \frac{V_f}{R_f}. \quad (6.4.35)$$

From (6.4.20) with $i_a = 0$ we find

$$v_a = \frac{G\omega_m}{R_f} V_f. \quad (6.4.36)$$

The armature voltage is proportional to the field voltage. This has the makings of a linear amplifier.

Now with the constraints of (6.4.33) and (6.4.34) applied, put a load resistance R_L across the armature terminals; the armature terminal voltage in the steady-state is

$$v_a = \left(\frac{R_L}{R_L + R_a} \right) \frac{G\omega_m}{R_f} V_f. \quad (6.4.37)$$

The load voltage is linear with field voltage and all the load power comes from the mechanical source. This illustrates the basic mode of operation of a dc generator, but it also indicates that the machine can be operated as an electromechanical amplifier. Direct-current machines are used as power amplifiers in many control applications. The power gain in a single machine is usually in the range of 20 to 30 and the bandwidth over which the amplification factor is constant is limited by field inductance to a few Hertz. Nonetheless, for control applications in which devices and power requirements are large, the required bandwidth is often small. It is difficult to think of a more economical way to make a 100,000-W, linear dc amplifier.

Commutator machines are used widely as electromechanical amplifiers, especially when considerable power is to be handled in control applications. Some amplifiers involve special constraints on a machine like that in Fig. 6.4.2 (Rototrol and Regulex); others have special configurations that involve

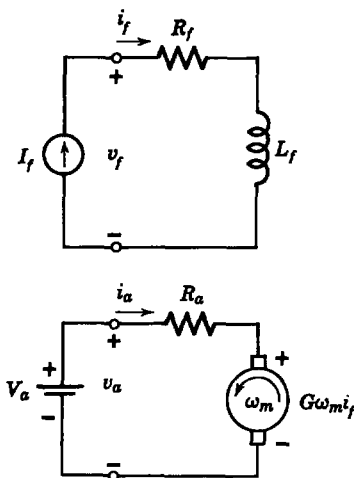


Fig. 6.4.6 Constraints for study of energy conversion properties in a dc machine.

a second set of brushes placed 90° from the original set to form a second armature circuit and a second field circuit placed 90° from the original field circuit (metadyne and amplidyne). All of these amplifiers are analyzed with the same basic techniques we used earlier.*

To learn some of the characteristics of the commutator machine as a dc energy converter we consider a steady-state problem with the constraints indicated in Fig. 6.4.6. The rotor position is constrained by a constant speed source

$$\dot{\theta} = \omega_m;$$

the field is constrained by a direct-current source

$$i_f = I_f;$$

and the armature terminals are constrained by a constant-voltage source

$$v_a = V_a.$$

We have left out armature inductance because we are treating the steady state.

The use of these constraints with the armature circuit equation (6.4.20) yields for the armature current

$$i_a = \frac{V_a - G\omega_m I_f}{R_a}. \quad (6.4.38)$$

* For a good discussion of rotating amplifiers in general and analyses of the specific configurations named see G. J. Thaler and M. L. Wilcox, *Electric Machines*, Wiley, New York, 1966, pp. 135–149.

The electric power fed into the armature terminals is

$$p_a = v_a i_a = V_a \left(\frac{V_a - G\omega_m I_f}{R_a} \right). \quad (6.4.39)$$

The torque of electric origin (6.4.30) is

$$T^e = G I_f \left(\frac{V_a - G\omega_m I_f}{R_a} \right) \quad (6.4.40)$$

and the mechanical power out of the coupling network (6.4.31) is

$$p_m = T^e \omega_m = G I_f \omega_m \left(\frac{V_a - G\omega_m I_f}{R_a} \right). \quad (6.4.41)$$

The quantities given by (6.4.38) to (6.4.41) are sketched as functions of ω_m for constant V_a and I_f in Fig. 6.4.7. As indicated in this figure, there are three regions of operation as defined by energy flow. Generator operation has mechanical power input ($p_m < 0$) and armature power output ($p_a < 0$) and occurs at values of speed at which the speed voltage ($G\omega_m I_f$) is greater than the armature voltage (V_a). The speed voltage then makes current flow to charge the battery on the armature terminals. Motor operation has armature power input ($p_a > 0$) and mechanical power output ($p_m > 0$) and occurs in a range of speed in which the armature terminal voltage V_a is greater than the speed voltage $G\omega_m I_f$ and can feed power into the speed voltage. Brake operation occurs when power is put into the machine both from the armature terminals ($p_a > 0$) and from the mechanical terminals ($p_m < 0$) and all of this power is dissipated in the armature resistance R_a . To see this refer to Fig. 6.4.6 and recognize that with $\omega_m < 0$ the speed voltage $G\omega_m I_f$ and armature battery V_a have aiding polarities and both feed power to R_a .

Figures 6.4.6 and 6.4.7 give the essential features of the operation of a dc machine with separate excitation; that is, the field winding is excited independently from a source separate from the armature excitation. The characteristics of a dc generator with separate excitation can be derived quite easily by using an equivalent circuit like that of Fig. 6.4.6. The transient performance of a separately excited dc generator is limited by field inductance, armature inductance, load inductance, or capacitance, and prime mover characteristics. The study of the characteristics is straightforward and is left to the problems at the end of the chapter.

The principal characteristics of a separately excited dc motor are shown in Fig. 6.4.8 in the sketch of torque-speed curves with armature voltage as parameter and field current constant. For practical machines the relatively small armature resistance leads to the steep slope of the curves. Thus at constant armature voltage there is little variation of speed with torque and the speed can be controlled quite closely by controlling the armature voltage.

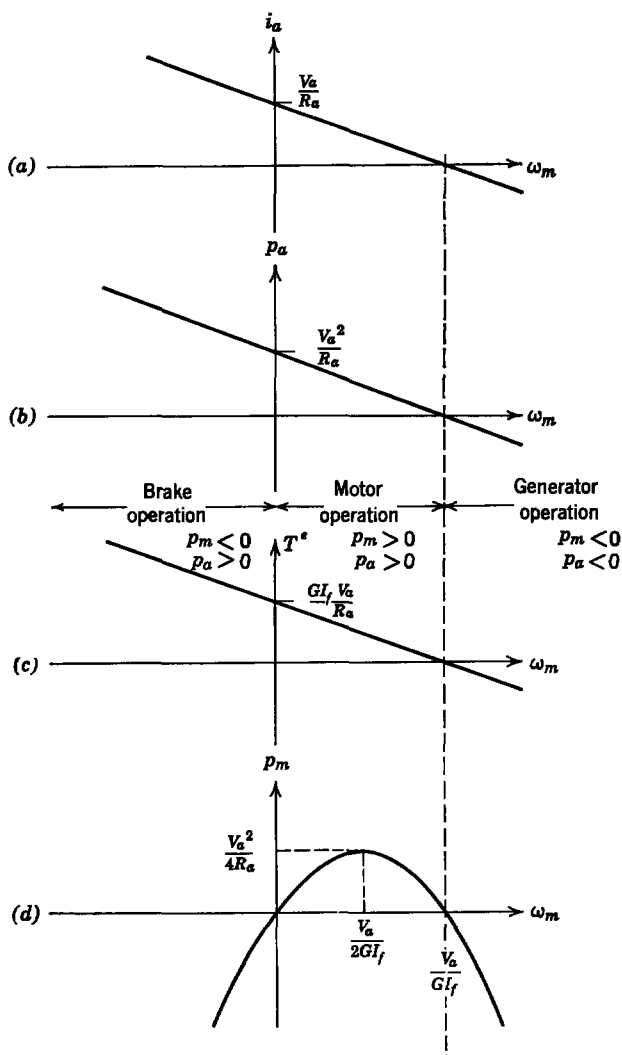


Fig. 6.4.7 Terminal characteristics of commutator machine with constant armature voltage and constant field current: (a) armature current; (b) armature power input; (c) torque of electric origin; (d) mechanical power out of coupling network.

This is the most common mode of operation of a dc motor when speed control is required. More precise speed regulation is obtained by using a feedback control system to sense the speed error and correct the armature voltage accordingly. When a dc generator is used to supply motor armature power in such a system it is called a Ward-Leonard system.*

* See, for example, Thaler and Wilcox, *op. cit.*, p. 291.

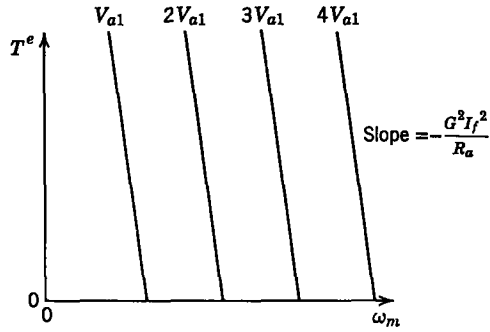


Fig. 6.4.8 Torque speed curves of separately excited dc motor.

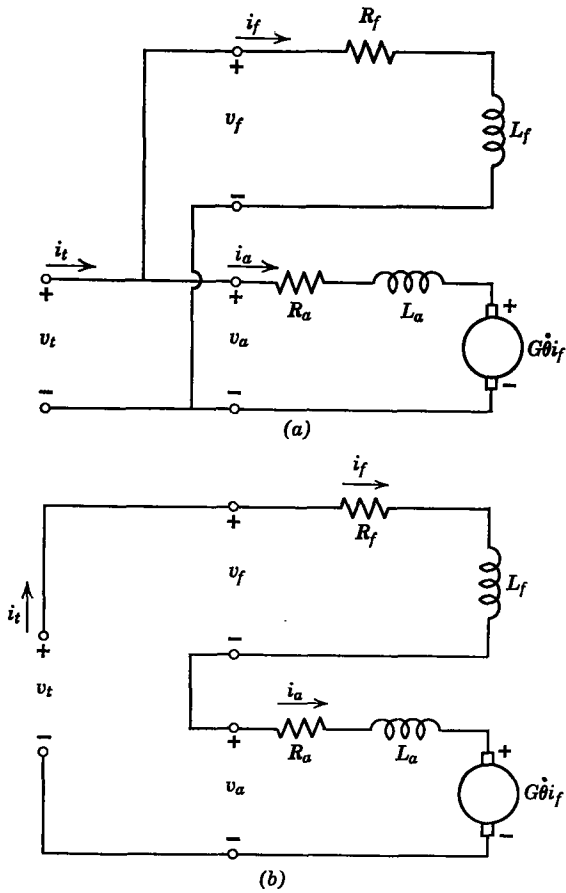


Fig. 6.4.9 Methods of self-exciting a dc motor: (a) shunt excitation; (b) series excitation.

There are two other fundamental ways of providing field excitation in dc motors; shunt excitation, illustrated schematically in Fig. 6.4.9a, and series excitation, illustrated in Fig. 6.4.9b. Considering shunt excitation with a steady state dc problem, we neglect the inductance and write

$$v_f = i_f R_f, \quad (6.4.42)$$

$$v_a = i_a R_a + G\theta i_f. \quad (6.4.43)$$

The shunt terminal constraints of Fig. 6.4.9a are

$$v_t = v_a = v_f,$$

$$i_t = i_a + i_f.$$

Using these constraints with (6.4.42) and (6.4.43) yields for the armature current and terminal current

$$i_a = \left(1 - \frac{G\theta}{R_f}\right) \frac{v_t}{R_a}, \quad (6.4.44)$$

$$i_t = \left(\frac{1}{R_f} + \frac{1}{R_a} - \frac{G\theta}{R_f R_a}\right) v_t. \quad (6.4.45)$$

The torque of electric origin is (6.4.30)

$$T^e = G \left(1 - \frac{G\theta}{R_f}\right) \frac{v_t^2}{R_f R_a}. \quad (6.4.46)$$

The torque-speed and terminal-current-speed curves for shunt excitation are sketched in Fig. 6.4.10. Because of the steep slope of the torque-speed curve, speed control is most effectively and efficiently achieved by the control of field resistance R_f because it determines the intercept with the speed axis.

Now consider a steady-state dc problem with series excitation as in Fig. 6.4.9b; (6.4.42) and (6.4.43) still hold but now the series connection imposes the constraints

$$v_t = v_a + v_f,$$

$$i_t = i_a = i_f.$$

Using these constraints with (6.4.42) and (6.4.43), we get for the terminal current

$$i_t = \frac{v_t}{R_f + R_a + G\theta}, \quad (6.4.47)$$

and the torque of electric origin is (6.4.30)

$$T^e = G i_t^2 = \frac{G v_t^2}{(R_f + R_a + G\theta)^2}. \quad (6.4.48)$$

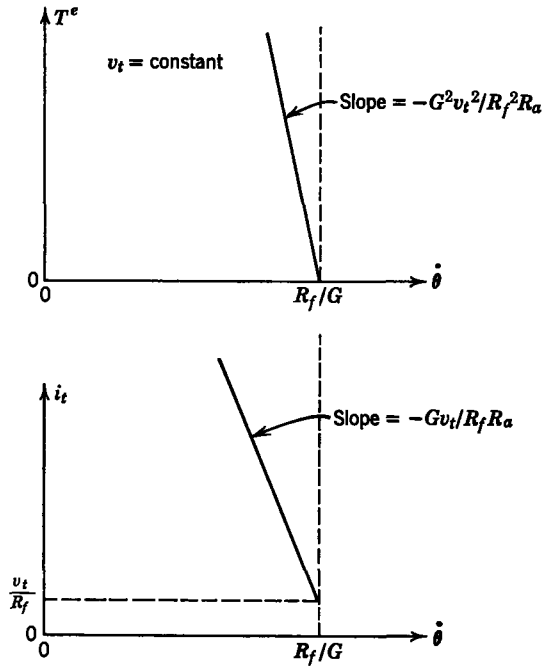


Fig. 6.4.10 Torque and terminal current as functions of speed in a shunt-excited dc motor with constant terminal voltage.

The torque and terminal current are sketched as functions of speed for constant terminal voltage in Fig. 6.4.11. Series-excited motors are used for traction drives and other applications in which high starting torque is required. The starting current is usually limited by variable series resistance, as is the running current.

A commutator machine like that of Fig. 6.4.2 may have more than one field winding on the same field structure. Oftentimes two field windings are used, one for shunt excitation and one for series excitation, in which case the machine is called a *compound* motor. It should be clear that combinations of the characteristics of Fig. 6.4.10 and 6.4.11 in varying amounts can lead to a wide variety of motor characteristics.*

We consider one final example of the possible steady-state characteristics of the commutator machine in Fig. 6.4.2, and that is as a shunt, self-excited, dc generator. By self-excited, we mean that only mechanical input is required to produce an electrical output. The field current is generated by the armature. The successful operation of this machine depends on saturation in the magnetic material, which shows up in the speed coefficient G , thus reducing

* See, for example, Fitzgerald and Kingsley, *op. cit.*, pp. 141–142.

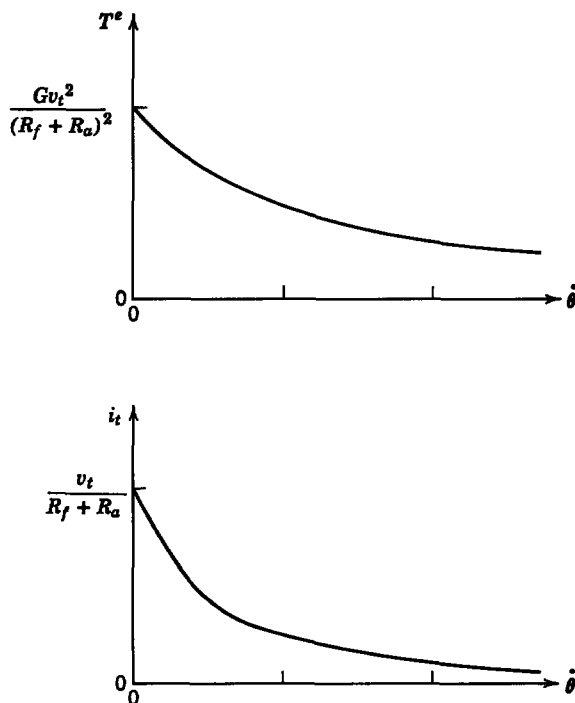


Fig. 6.4.11 Torque and terminal current as functions of speed for series-excited dc motor with constant terminal voltage.

the value of G as the excitation current i_f , hence the speed voltage increase. This saturation is usually represented by a plot of armature open-circuit voltage v_{oc} in Fig. 6.4.12 (the speed voltage $G\omega_m i_f$) as a function of field current with speed held fixed. Such a curve is sketched in Fig. 6.4.12a. Now connect the field winding in parallel with the armature terminals with no load connected, as shown in Fig. 6.4.12b; the field current is determined in the steady state by the field resistance line in Fig. 6.4.12a. The steady-state operating point at which the circuit equations are satisfied in the steady state is shown in Fig. 6.4.12a. If the field current is below the steady-state value, the excess of generated voltage over the iR drop goes into increasing the current as illustrated. Thus, if a small amount of voltage is produced (usually by residual field flux), the terminal voltage will build up automatically to the operating point, the rate of buildup being determined by the relative shapes of the two curves in Fig. 6.4.12a and by the machine inductances. The addition of a load causes a little additional voltage drop in R_a but does not change the essential features of the argument. A moderate range of voltage control is achieved by varying the field resistance R_f , the slope of the field resistance line, hence the intersection of the two curves in Fig. 6.4.12a.

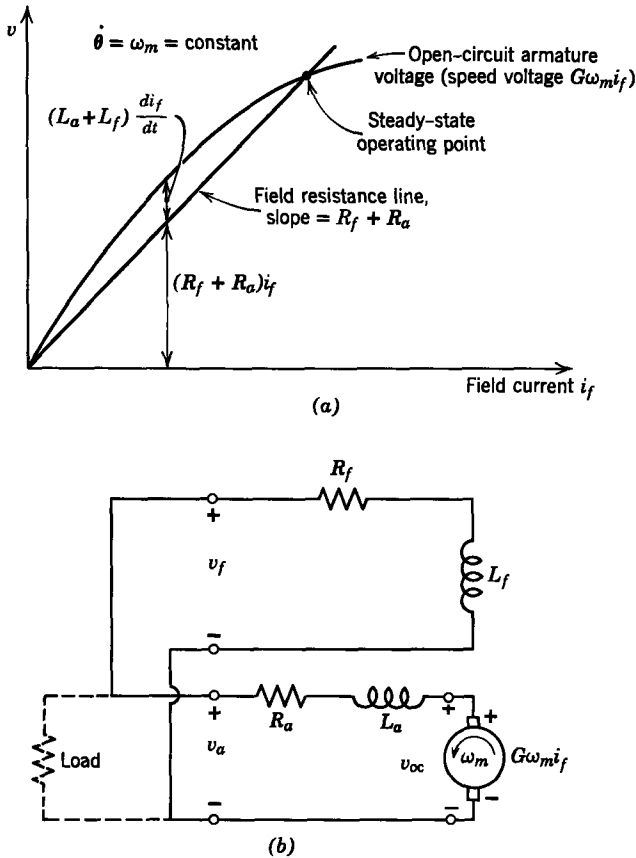


Fig. 6.4.12 Illustrating shunt self-excitation in a dc generator: (a) voltage versus field current curves; (b) shunt-connected generator with no load.

Generators can be series-excited and can use both series and shunt excitation. The techniques of analysis are similar to those used for shunt excitation and a great variety of terminal characteristics can be obtained.*

All of our examples have been based on dc excitation. Commutator machines are also used with alternating current. The techniques of analysis are essentially the same, but the inductances play a vital role in determining steady-state characteristics. Some examples of ac commutator machines are studied in the problems at the end of this chapter.

6.4.2 Homopolar Machines

A physically different type of dc rotating device is the homopolar machine, but, as we shall see, it has terminal behavior much like that of a commutator

* Fitzgerald and Kingsley, *op. cit.*, pp. 139–141.

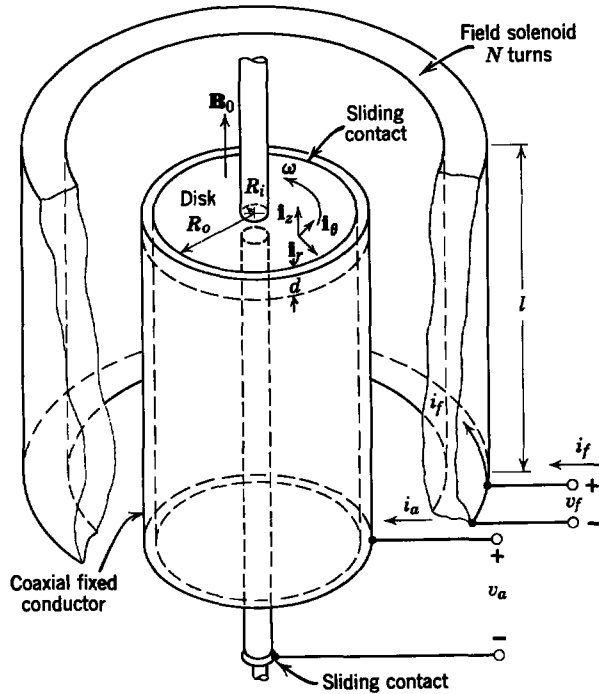


Fig. 6.4.13 Configuration for analysis of homopolar machine.

machine. One version of a homopolar machine is introduced in Example 6.3.1 to illustrate the application of Ohm's law for a moving conductor. A cutaway view of a practical homopolar generator appears in Fig. 6.3.2.

In Example 6.3.1 a limited analysis is made of the armature characteristics under steady-state conditions. In this section that analysis is generalized to include transients in the armature circuit and extended to find the terminal equations for the field circuit and the mechanical system.

For the analysis we use the configuration of Fig. 6.4.13 which is representative of a homopolar machine that would use a superconducting field coil. The system consists of a disk of the dimensions shown and connected to a shaft that is rotating with angular speed ω . A coaxial fixed conductor makes sliding electrical contact (conventionally through a liquid metal) with the outside surface of the disk. Another sliding contact is made with the shaft as shown and these two connections to the disk form the armature terminals. The whole assembly is surrounded by a coaxial solenoid that produces an axial flux density at the disk. The terminals of the solenoid are the field circuit terminals. The reason for the carefully specified cylindrical symmetry is mathematical simplicity and is not necessary for the analytical techniques to

be correct. In many practical cases, however, symmetry is desirable to cut down on harmful electrical and electromechanical effects like circulating currents, thrust on bearings, and forces on the solenoid.

Current to and from the disk in the shaft and in the coaxial conductor is axial and has cylindrical symmetry. (We assume that the armature terminals are far enough removed from the disk and solenoid that the end effects can be ignored.) Current in the disk is radial and distributed uniformly at any radius. Consequently, the flux density produced by armature current is tangential and none of it will link the field solenoid. Therefore, we can write the field circuit equation immediately as

$$v_f = R_f i_f + L_f \frac{di_f}{dt}, \quad (6.4.49)$$

where R_f = field circuit resistance,

L_f = field circuit inductance.

This is exactly the same form as (6.4.1) for the commutator machine.

The electromagnetic behavior of the disk was analyzed for steady-state conditions in Example 6.3.1. When we assume that any transients occur slowly enough not to disturb appreciably the uniformity of current density in the disk, (we pursue this point in Chapter 7), the results of Example 6.3.1 are still valid, provided we add a self-inductance term. (There is no net field-produced flux linkage with the armature.) Thus the armature voltage equation is

$$v_a = R_a i_a + L_a \frac{di_a}{dt} + G \omega i_f, \quad (6.4.50)$$

where R_a is the armature resistance found in Example 6.3.1 (where it is called R_{int})

$$R_a = \frac{\ln(R_o/R_i)}{2\pi\sigma d}; \quad (6.4.51)$$

L_a is the armature self-inductance that can be calculated from the geometry and G is a speed coefficient found from equating the open-circuit voltage in Example 6.3.1 to the speed voltage.

$$G \omega i_f = \frac{\omega B_0}{2} (R_o^2 - R_i^2). \quad (6.4.52)$$

As for the commutator machine, this speed coefficient G depends only on geometry and material properties.

We obtain an approximate value for G by assuming that the solenoid has N total turns and is axially long with small radial buildup. With an axial

length l , the value of flux density near the center is (from Ampère's law)

$$\mathbf{B}_0 = i_z \mathbf{B}_0 = i_z \frac{\mu_0 N i_f}{l}. \quad (6.4.53)$$

The use of this result in (6.4.52) and the solution for G yields

$$G = \frac{\mu_0 N}{2l} (R_o^2 - R_i^2). \quad (6.4.54)$$

Equation 6.4.50, which is the armature voltage equation for the homopolar machine, is the same form as the armature equation for the commutator machine (6.4.20). We could have derived this equation in a manner analogous to that used for the commutator machine; for example, if we had chosen a contour C , illustrated schematically in Fig. 6.4.14, then, with the contour fixed in the laboratory frame, we could write Faraday's law as

$$-\int_a^b \mathbf{E} \cdot d\mathbf{l} - \int_b^a \mathbf{E}' \cdot d\mathbf{l} = \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da. \quad (6.4.55)$$

The first term on the left is the terminal voltage, the second term on the left contains the resistive voltage drop and the speed voltage, and that on the right is the self-inductance term. To complete such an analysis we must simply follow the steps used in going from (6.4.3) to (6.4.20) for the commutator machine.

Alternatively, the contour C can be fixed to the disk in Fig. 6.4.14, and Faraday's law is written as

$$-\int_a^b \mathbf{E} \cdot d\mathbf{l} - \int_b^a \mathbf{E}' \cdot d\mathbf{l} = \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da. \quad (6.4.56)$$

In this case the first term is still the terminal voltage, but now the second term on the left contains only the resistance voltage drop, and because of the time-varying surface S the term on the right includes both the speed voltage and voltage of self-inductance.

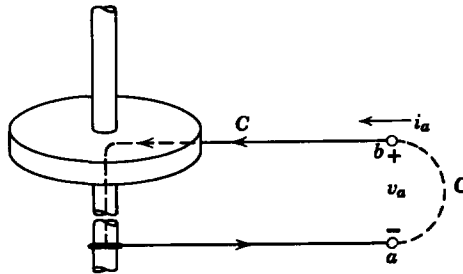


Fig. 6.4.14 Contour of integration for Faraday's law.

To complete the description of the homopolar machine, the torque of electric origin must be obtained. For this we use the force density of electric origin for magnetic field systems (1.1.30).

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}. \quad (6.4.57)$$

Using the unit vectors for a cylindrical coordinate system shown in Fig. 6.4.13, the current density in the disk at a radius r is

$$\mathbf{J} = \mathbf{i}_r J_r = -\mathbf{i}_r \frac{i_a}{2\pi r d}. \quad (6.4.58)$$

[See (b) of Example 6.3.1 and note that the assumed positive direction of armature current has been reversed.] The only flux density that interacts with this current density to produce a torque about the axis of rotation is the field flux density given by (6.4.53). The use of these two quantities in (6.4.57) yields

$$\mathbf{F} = \mathbf{i}_\theta \frac{\mu_0 N i_f i_a}{2\pi l dr}. \quad (6.4.59)$$

The force is tangential; thus we multiply by the lever arm r and integrate throughout the volume of the disk to find the torque T^e in the direction of positive rotation ω

$$T^e = \int_{R_i}^{R_o} \int_0^{2\pi} \left(\frac{\mu_0 N i_f i_a}{2\pi l} \right) r dr d\theta. \quad (6.4.60)$$

Evaluation of the integral yields

$$T^e = \frac{\mu_0 N}{2l} (R_o^2 - R_i^2) i_a i_f = G i_a i_f, \quad (6.4.61)$$

where the speed coefficient was defined in (6.4.54).

The results of (6.4.61) can be derived also by the energy methods of Chapter 3, provided great care is exercised in defining the moving circuit. It is much more reasonable and straightforward to evaluate the torque in the manner we did.

We now use the results of (6.4.49), (6.4.50), and (6.4.61) to redraw the equivalent circuit of Fig. 6.4.5. We have included the usual mechanical elements, and it would be well to remark that with liquid-metal brushes there is little coulomb friction (T_0) in a homopolar machine.

The equivalent circuit of Fig. 6.4.5 was originally drawn for the commutator machine discussed in Section 6.4.1. This emphasizes the similarity of commutator and homopolar machines. All of the discussion of the properties of commutator machines holds equally well for homopolar machines, with the qualification that relative parameter values in the two types of machine

are different; for example, the homopolar machine has essentially a one-turn armature and is always a low-voltage, high-current device. Thus matching a wire-wound coil to this low impedance is difficult and few homopolar machines can be self-excited, either with series field windings or with shunt field windings. Consequently, most homopolar machines have separately excited field windings.

Because of the similarity between homopolar and commutator machine characteristics we terminate the discussion here and treat homopolar machines further in the problems at the end of this chapter.

6.5 DISCUSSION

In this chapter we have made the necessary generalizations of electromagnetic theory that are needed for analyzing quasistatic systems with materials in relative motion. This has involved transformations for source and field quantities between inertial reference frames, boundary conditions for moving boundaries, and constituent relations for moving materials. In addition to some simple examples, we have made an extensive analysis of dc rotating machines because they are devices that are particularly amenable to analysis by the generalized field theory.

Having completed the generalization of field theory with illustrative examples of lumped-parameter systems, we are now prepared to proceed to continuum electromechanical problems. In Chapter 7 we consider systems with specified mechanical motion and in which electromagnetic phenomena must be described with a continuum viewpoint.

PROBLEMS

6.1. Two frames of reference have a relative angular velocity Ω , as shown in Fig. 6P.1. In the fixed frame a point in space is designated by the cylindrical coordinates (r, θ, z) . In the rotating frame the same point is designated by (r', θ', z') . Assume that $t = t'$.

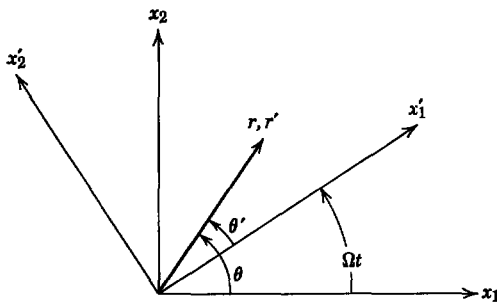


Fig. 6P.1

- (a) Write the transformation laws [like (6.1.6)] that relate primed coordinates to the unprimed coordinates.
- (b) Given that ψ is a function of (r, θ, z, t) , find $\partial\psi/\partial t'$ (the rate of change with respect to time of ψ for an observer in the rotating frame) in terms of derivatives with respect to (r, θ, z, t) .

6.2. A magnetic field distribution $\mathbf{B} = B_0 \sin kx_1 \mathbf{i}_2$ exists in the laboratory frame. What is the time rate of change for the magnetic field as viewed from the following:

- (a) An inertial frame traveling parallel to the x_1 -axis with speed V ?
- (b) An inertial frame traveling parallel to the x_2 -axis with speed V ?

6.3. A magnetic field traveling wave of the form $\mathbf{B} = \mathbf{i}_y B_0 \cos(\omega t - kx)$ is produced in the laboratory by two windings distributed in space such that the number of turns per unit length varies sinusoidally in space. The windings are identical except for a 90° separation. They are excited by currents of equal amplitude but 90° out of time phase. This is a linear version of Problem 4.10, which was cylindrical.

$$\omega = \text{radial frequency} = 2\pi f,$$

$$k = \text{wavenumber} = \frac{2\pi}{\lambda},$$

$$\lambda = \text{wavelength},$$

$$v_p = \frac{\omega}{k} = f\lambda = \text{phase velocity of wave}.$$

- (a) If an observer is in an inertial frame traveling with speed V in the x -direction, what is the apparent frequency of the magnetic wave?
- (b) For what velocity will the wave appear stationary?

6.4. The following equations describe the motions of an inviscid fluid in the absence of external forces:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = 0, \quad (1)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0, \quad (2)$$

$$p = p(\rho), \quad (3)$$

where p is the pressure, ρ , the mass density, and $\mathbf{v}$ the velocity of the fluid. Equation 1 is Newton's law for a fluid, (2) is the law of conservation of mass, and (3) is a constitutive relation relating the pressure and density. Are these equations invariant to a Galilean transformation to a coordinate system given by $\mathbf{r}' = \mathbf{r} - \mathbf{v}_0 t$? If so, find $\mathbf{v}'$, ρ' , p' as a function of the unprimed quantities $\mathbf{v}$, ρ , p .

6.5. A cylindrical beam of electrons has radius a , a charge density $\rho_0(1 - r/a)$ ($\rho_0 < 0$) in the stationary frame, and velocity $\mathbf{v} = v_0 \mathbf{i}_z$. (See Fig. 6P.5.)

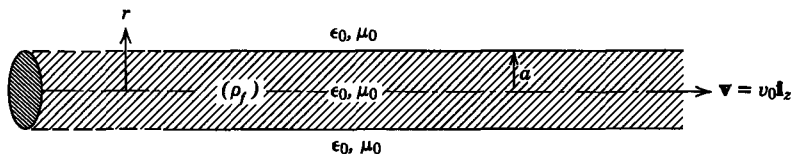


Fig. 6P.5

- Using only the transformation law for charge density, find the electric and magnetic fields in a reference frame that is at rest with the electrons.
- Without using any transformation laws, find the electric and magnetic fields in the stationary frame.
- Show that (a) and (b) are consistent with the electric field system transformation laws for $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{J}_r$.

6.6. A pair of cylinders coaxial with the z -axis, as shown in Fig. 6P.6, forms a capacitor. The inner and outer surfaces have the potential difference V and radii a and b , respectively. The cylinders are only very slightly conducting, so that as they rotate with the

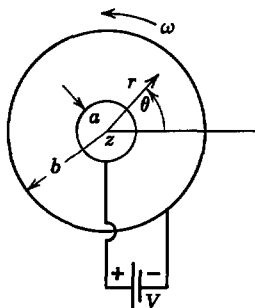


Fig. 6P.6

angular velocity ω they carry along the charges induced on their surfaces. As viewed from a frame rotating with the cylinders, the charges are stationary. We wish to compute the resulting fields.

- Compute the electric field between the cylinders and the surface charge densities σ_a and σ_b on the inner and outer cylinders, respectively.
- Use the transformation for the current density to compute the current density from the results of part (a).
- In turn, use the current density to compute the magnetic field intensity $\mathbf{H}$ between the cylinders.
- Now use the field transformation for the magnetic field intensity to check the result of part (c).

6.7. A pair of perfectly conducting electrodes traps a magnetic field, as shown in Fig. 6P.7. One electrode is planar and at $y = 0$, the other has a small sinusoidal variation given as a

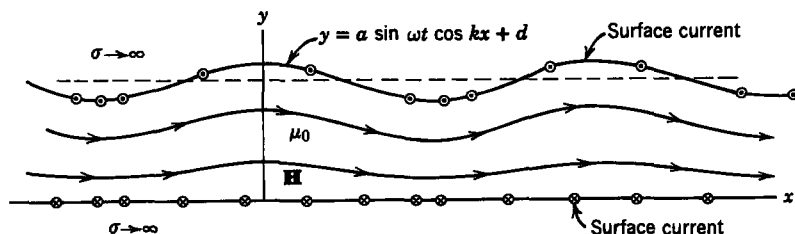


Fig. 6P.7

function of space and time. Both boundaries can be considered perfect conductors, so that $\mathbf{n} \cdot \mathbf{B} = 0$ on their surfaces. In what follows, assume that $a \ll d$ and that the magnetic field intensity between the plates takes the form

$$\mathbf{H} = \left[\frac{\Lambda}{\mu_0 d} + h_x(x, y, t) \right] \mathbf{i}_x + h_y(x, y, t) \mathbf{i}_y,$$

where Λ is the flux trapped between the plates (per unit length in the z -direction) and h_x and h_y are small compared with $\Lambda/\mu_0 d$.

- Find the perturbation components h_x and h_y .
- The solutions in part (a) must satisfy the boundary conditions: $\mathbf{n} \times \mathbf{E} = (\mathbf{n} \cdot \mathbf{v})\mathbf{B}$ [boundary condition (6.2.22) of Table 6.1]. Compute the electric field intensity by using the magnetic field found in part (a). Now check to see that this boundary condition is satisfied to linear terms.

6.8. The system shown in Fig. 6P.8 consists of a coaxial line of inner radius a and outer radius b , with an annular conductor that makes perfect electrical contact with the inner and outer conductors. The annular conductor is constrained to move to the right with constant velocity v . All conductors can be assumed to have infinite conductivity. A battery V_0 supplies a current I to the system through a resistance R . This excitation is physically arranged to

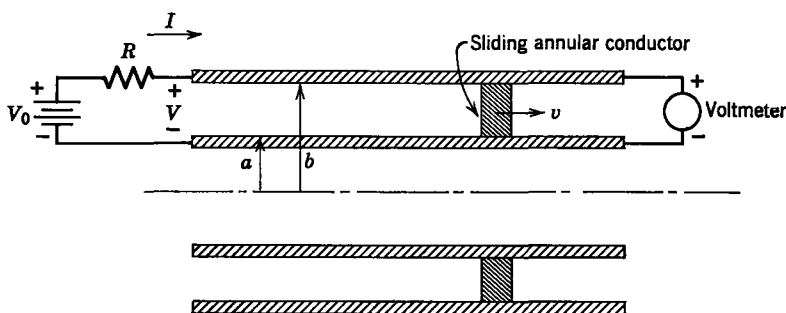


Fig. 6P.8

maintain cylindrical symmetry about the center line. Fringing fields at the left end can be neglected. A voltmeter is connected between the inner and outer conductors at the extreme right end as shown.

- In terms of the current I , find the fields $\mathbf{E}$ and $\mathbf{H}$ throughout the length of the line between the conductors.
- Using the constraints imposed by the battery and resistance, find $\mathbf{E}$, $\mathbf{H}$, and the terminal variables I and V in terms of V_0 and the other constants of the system.
- What does the voltmeter read? Why can this be different from the terminal voltage V ?
- Compute the rate of change of stored magnetic energy and the power $\mathbf{VI}$ into the line. If there is a discrepancy between these values, explain where the power goes (or comes from).
- Treat the problem as a lumped inductor with a mechanical variable. Find $L(x)$ and the force f^e acting on the sliding short. Use this to explain your answer to part (d).

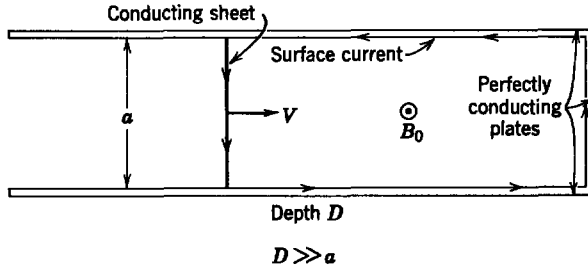


Fig. 6P.9

6.9. A sheet of metal (with surface conductivity σ_s) slides between plane-parallel, perfectly conducting plates, as shown in Fig. 6P.9. The perfectly conducting plates are shorted at the right end. At time $t = 0$ a current flows on the inside surfaces of the plates and through the metal sheet to generate the magnetic field B_0 . If the sheet were stationary, we would expect that the initial magnetic flux would decay with time. The sheet, however, moves to the right with a constant velocity V , hence tends to compress the magnetic field and prevent its decay. At what constant velocity V must the sheet move to keep B_0 constant?

6.10. A slab of conducting material with mass M and conductivity σ has the position $x(t)$, as shown in Fig. 6P.10. Fixed side and top plates form a closed circuit for currents

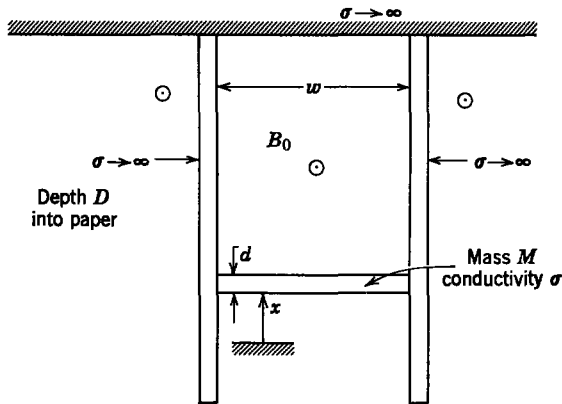


Fig. 6P.10

induced in the movable plate. A uniform magnetic field B_0 (directed out of the paper) extends throughout the system. The system is so constructed that $D \gg w$.

- Assume that contributions to the magnetic field induced by currents resulting from motions of the movable slab can be ignored. Find the magnetic force on the slab in the x -direction as a function of the velocity v .
- Given that the initial velocity of the slab is $(dx/dt)(t = 0) = v_0$, find $v(t)$ (ignore gravity).
- Under what conditions will the assumptions used in (a) be valid? (Remember, v_0 is given.)

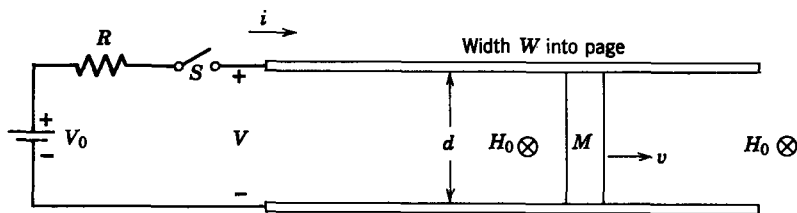


Fig. 6P.11

6.11. The system shown in Fig. 6P.11 is composed of a pair of parallel plates of width W and separation d , connected by a sliding conductor of mass M . The sliding conductor makes frictionless and perfect electrical contact with the plates. The entire system is immersed in a static magnetic field H_0 into the page and the plates are excited by the battery V_0 through the resistance R and the switch S . All conductors may be assumed to be perfect, fringing fields may be neglected, and you may assume that $i/W \ll H_0$.

- Find the force on the sliding conductor in terms of the current i .
- With the system at rest, the switch S is closed at $t = 0$. Find the velocity of the sliding conductor $v(t)$ for $t > 0$.

6.12. The system shown in Fig. 6P.12 consists of two parallel, perfectly conducting plates with depth D and separation W . Between these plates is placed a perfectly conducting short-circuit which has mass M and slides with viscous coefficient of friction B . You may assume that $W \ll D$ and that fringing fields may be neglected.

- Find $\lambda = \lambda(i, x)$.
- Find $W'_m(i, x)$ or $W_m(\lambda, x)$.
- Find the force of electric origin f^e (from $W'_m(i, x)$ or $W_m(\lambda, x)$) exerted by the fields on the sliding short.

Assume now that a battery is placed across the electric terminals so that $v = V_0 = \text{constant}$.

- Write a complete set of differential equations that would allow you to find $x(t)$.
- If the system has reached a state in which the velocity of the plate (dx/dt) is a constant, find (dx/dt) .
- Under the conditions of (e), find the current supplied by the battery $i(t)$.

You will now repeat this problem and solve it by using field theory. Do not assume that $v = V_0$ and $x = \text{constant}$ until part (l).

- Find the magnetic field H between the plates as a function of the current $i = i(t)$.

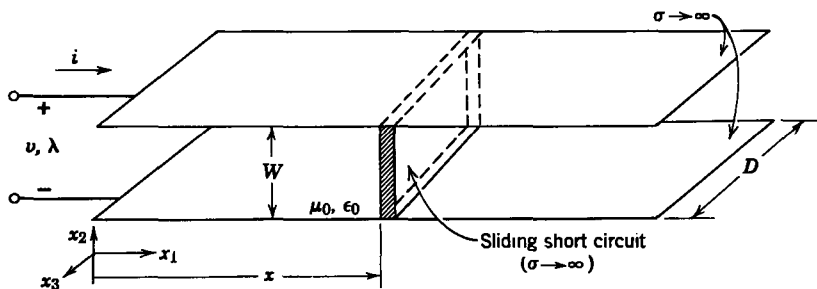


Fig. 6P.12

- (h) Find the force exerted by this $\mathbf{H}$ field on the sliding short in terms of i and x by using the Lorentz force law to show that it agrees with (c).
- (i) Compute the electric field everywhere between the plates. Evaluate the constant of integration by requiring that the voltage at the terminals be $v(t)$.
- (j) Relate the terminal voltage $v(t)$ to the current $i(t)$ and plate position $x(t)$ by explicitly using Faraday's law in integral form.
- (k) Show by using (i) and (j) that the boundary conditions on the electric field at the moving plate are satisfied.
- (l) Convince yourself that the results of (g) through (k) are formally equivalent to the lumped-parameter approach of parts (a) through (f); that is, again find (dx/dt) and $i(t)$ by assuming that $v = V_0$ and $(dx/dt) = \text{constant}$.
- (m) Under the conditions of part (l) evaluate the electric field of part (i) explicitly.

6.13. The device shown in Fig. 6P.13 is a model of an MHD system that is studied in greater detail in Chapters 12 to 14. It consists of a *rigid* annular ring of high but finite conductivity σ , depth D , and inner and outer radii R_1 and R_2 , respectively. The entire system is immersed in an externally applied, uniform, magnetic field $\mathbf{H} = H_0 \mathbf{i}_z$. A set of fixed, perfectly conducting electrodes, each subtending an angle 2α , closely fits the conducting shell. The conducting shell rotates without friction and has a moment of inertia K about the origin (z -axis). The position of the shell is described by the angle ψ . You may assume that the average radius R of the shell is large compared with its thickness $[R \gg (R_2 - R_1)]$.

1. Ignore the reaction $\mathbf{H}$ field produced by the flow of current in the shell between the electrodes.

2. Since $R \gg (R_2 - R_1)$, assume that both the current density *and* the electric field exist only between the electrodes (no fringing) and that they are uniform, radially directed, and independent of r and θ .

- (a) Write the differential equation of motion for the shell in terms of T_1^e and T_2^e , the torques of electric origin produced by the two electrode regions.

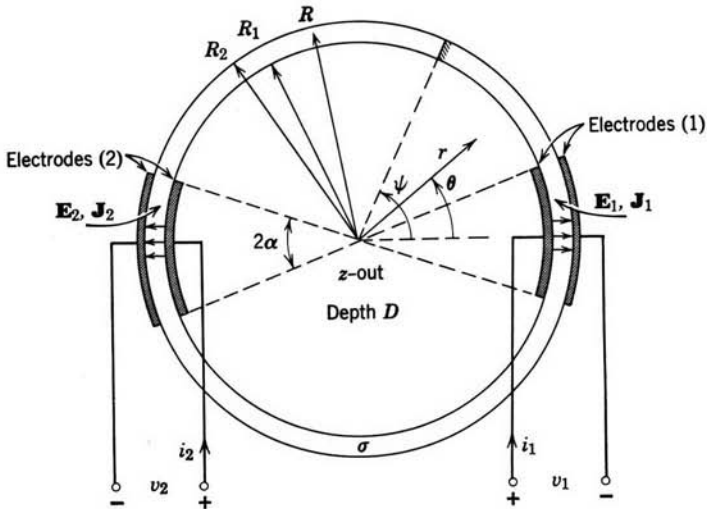


Fig. 6P.13

- (b) Find the Lorentz force *density* in the region between each pair of electrodes in terms of the currents i_1 and i_2 and H_0 .
- (c) Find the torques T_1^e and T_2^e you used in part (a) in terms of the currents i_1 , i_2 , and H_0 by appropriately integrating the force density you found in part (b).
- (d) Relate the electrode voltages v_1 and v_2 to the electric fields E_1 and E_2 as measured in a fixed frame.
- (e) Use the magnetic field system transformation laws with your answer to part (d) to find the electrode voltages v_1 and v_2 in terms of the currents i_1 and i_2 and the velocity of the cylindrical shell $R(d\psi/dt)$.

Your answers to parts (a), (c), and (e) should give you a set of three differential equations in ψ , v_1 , v_2 , i_1 , and i_2 . (If the electrical terminal constraints, namely, relations between v_1 and i_1 and v_2 and i_2 were specified, you should have as many equations as unknowns.)

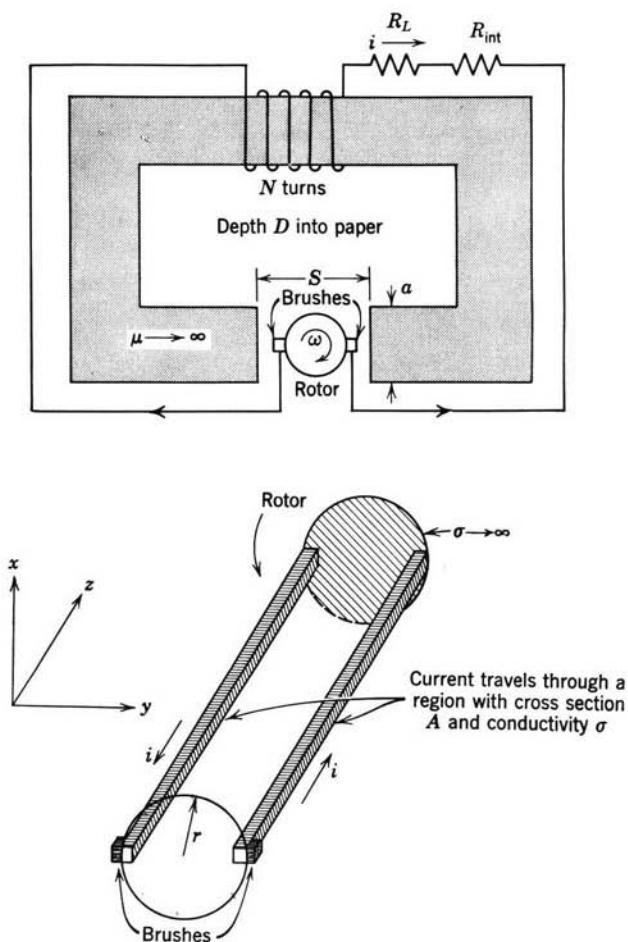


Fig. 6P.14

- (f) Suppose $i_2 = 0$ and $i_1(t) = i_0 u_{-1}(t)$; find and sketch $\psi(t)$, $v_2(t)$, and $v_1(t)$ for $t \geq 0$, assuming that for $t < 0$ the system is at rest ($\psi = \text{constant} = 0$).
- (g) Suppose $i_2 = 0$ and $v_1(t) = (v_0 \sin \omega t) u_{-1}(t)$; find and sketch $\psi(t)$, $v_2(t)$, and $i_1(t)$ for $t \geq 0$, again assuming that the system is at rest for $t < 0$.
- (h) A resistance R_0 is placed across terminal pair 2. Assuming that $i_1(t) = i_0 \cos \omega t$, find the steady-state current $i_2(t) = \text{Re} (i_2 e^{j\omega t})$. Plot $|i_2|$ versus ω to show that the device could be used as a transformer at low frequencies.

6.14. Figure 6P.14 shows a model for a self-excited dc machine. The rotor is laminated in such a way that i flows only in the z -direction. The brushes have an effective area A , hence the current density on either side of the rotor is i/A . The rotor has a conductivity σ over the area of the brushes. Neglect the thickness of the brushes compared with r . The far end of the rotor is assumed to be infinitely conducting, as shown. The rotor is driven at a constant angular velocity ω ; R_{int} does not include the effect of the conductivity of the rotor.

- (a) Find the differential equation for $i(t)$. Neglect the inductance of the rotor.
- (b) If $i(t = 0) = I_0$, calculate the power dissipated in the load resistor R_L as a function of time. For what values of the parameters is this power unbounded as $t \rightarrow \infty$?
- (c) In a real system what would prevent the current from becoming infinite?

6.15. A 150-kw, 240-V, separately-excited, dc generator has the following constants (see Fig. 6P.15):

$$\begin{array}{ll} R_f = 30 \text{ ohms} & R_a = 0.02 \text{ ohm,} \\ L_f = 12 \text{ H} & L_a = 0.001 \text{ H,} \\ & G = 0.30 \text{ H.} \end{array}$$

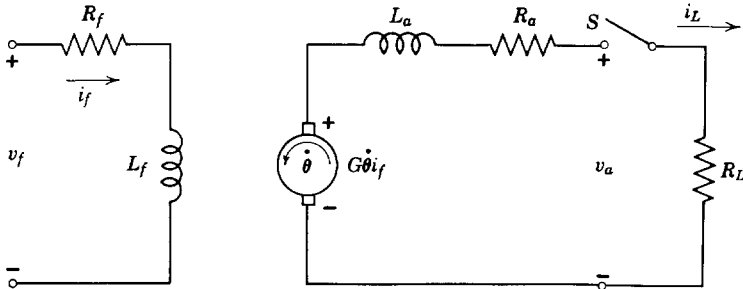


Fig. 6P.15

The generator is driven by a constant speed source at its rated speed of 900 rpm. The field current is constant at 9.0 A. The purely resistive load of $R_L = 0.385$ ohms is switched onto the armature at $t = 0$. Calculate and plot the ensuing transients in load current and shaft torque.

6.16. The generator of Problem 6.15 is connected as shown in Fig. 6P.16. The constants are the same as those given in Problem 6.15, plus $V_f = 270$ V. The machine is driven by a constant-speed source at 900 rpm.

- (a) With switch S_2 open, S_1 is closed at $t = 0$, calculate and plot the transient in open-circuit armature voltage v_a .
- (b) With switch S_2 closed, S_1 is closed at $t = 0$, calculate and plot the transient in armature voltage v_a .

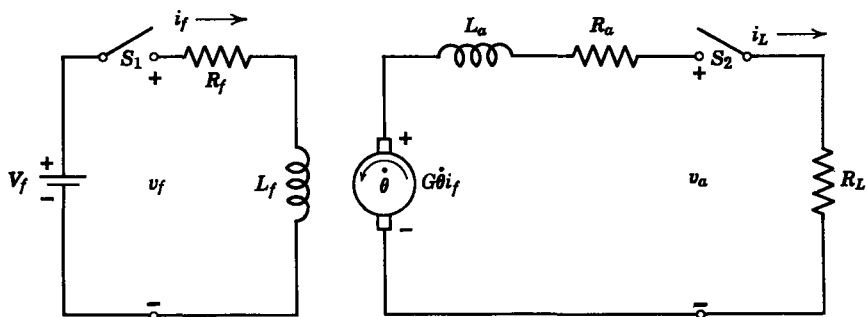


Fig. 6P.16

6.17. The generator of Problem 6.15 is connected in the circuit of Fig. 6P.17 and driven with a speed source having a torque-speed characteristic expressible as

$$T_m = T_0 \left(1 - \frac{\omega}{\omega_0} \right),$$

where $\omega_0 = 1000$ rpm, $\omega = \dot{\theta}$ = shaft speed, $T_0 = 1.6 \times 10^4$ N-m, T_m = torque applied to generator shaft. In addition the constant field voltage is $V_f = 270$ V and the rotor moment

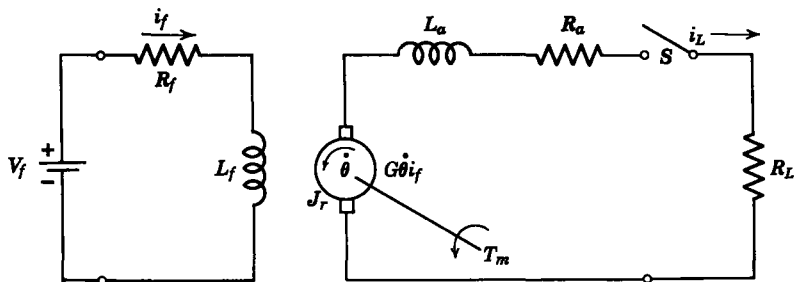


Fig. 6P.17

of inertia is $J_r = 12$ kg-m². Neglect mechanical damping. The switch in the armature circuit is closed at $t = 0$. Calculate and plot the ensuing transients in load current, shaft speed, and torque.

6.18. Consider a dc machine with moment of inertia J_0 and operating in the steady state at its rated speed ω_0 . Consider the stopping transient with electrical excitation removed. This is a purely mechanical transient. Consider three cases and calculate and plot speed as a function of time for each case.

- There is only coulomb friction.
- There is only viscous friction.
- Both types of friction are present and produce equal torques at rated speed.

6.19. A dc motor has the following constants and ratings:

Field resistance,	$R_f = 230$ ohms
Field inductance,	$L_f = 90$ H
Armature resistance,	$R_a = 0.15$ ohm
Armature inductance,	$L_a = 0.01$ H
Speed coefficient,	$G = 1.5$ H
Armature inertia,	$J_r = 0.5$ kg-m ²
Rated line voltage	230 V

The rated supply voltage of 230 V is applied to the field winding and the field current is 1 A. Assume that the motor shaft is unloaded and neglect mechanical losses.

- The 230-V supply is connected to the armature terminals at $t = 0$. Calculate and plot the ensuing transients in armature current and shaft speed.
- Repeat part (a), but with 1.35 ohms of resistance added in series with the armature.

6.20. A dc machine is often used as an energy storage element. With constant field current, negligible mechanical losses, and negligible armature inductance the machine, as viewed from the armature terminals, appears as the RC circuit of Fig. 6P.20.

- Find the equivalent capacitance C in terms of G , I_f , and J_r .
- Evaluate C for the machine in Problem 6.19 with $I_f = 1$ A.

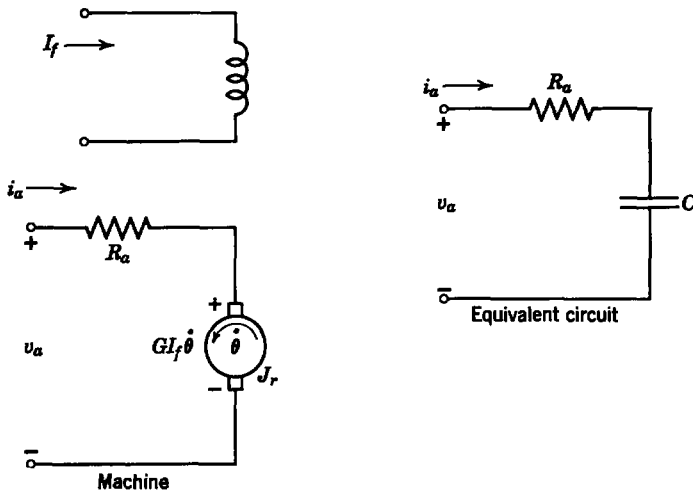


Fig. 6P.20

6.21. Discuss the problem of operating a commutator machine on alternating voltage from the viewpoint of armature and field currents drawn and torque produced. More specifically, contrast series and parallel operation of field and armature and explain why the series arrangement is used almost exclusively.

6.22. A homopolar generator is constructed as shown in Fig. 6P.22. The disk rotates with the angular velocity ω and has a conductivity σ . It is connected by slip rings to an N -turn

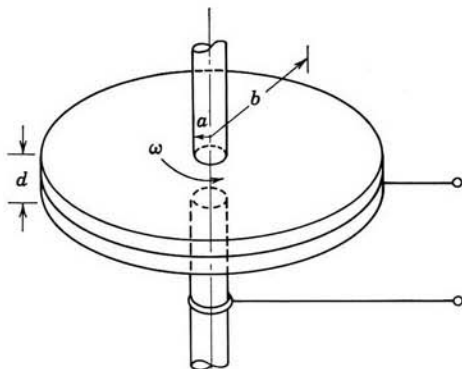
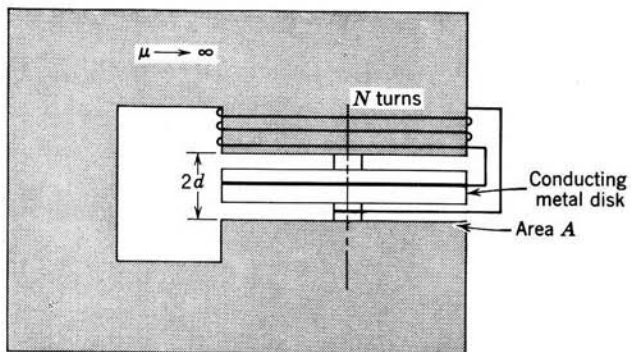
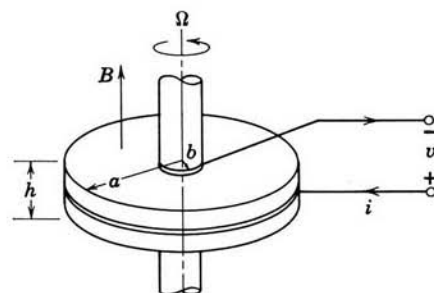
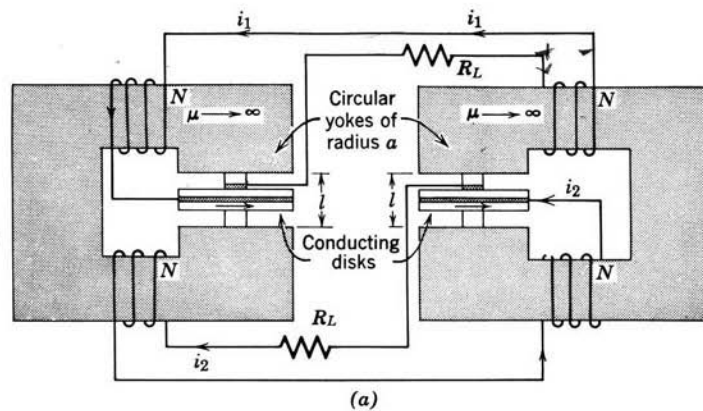


Fig. 6P.22



Detail of conducting disks

Fig. 6P.23

coil. Under what condition can this system be self-excited (generate its own magnetic field)? The coil resistance can be taken as negligible.

6.23. Disks of conductivity σ rotate with the constant angular velocity Ω , as shown in Fig. 6P.23. Each disk is symmetrically connected to slip rings at the inner radii b and outer radii a . Magnetic yokes ($\mu \rightarrow \infty$) impose uniform magnetic fields over the volume of the disks. These fields are generated by the disks interconnected with the windings.

- (a) Find an expression for the terminal voltage v for one of the disks as a function of Ω , i , and the magnetic flux density B .
- (b) Determine the condition under which the interconnected disks will deliver steady-state ac current to the load resistances R_L . The disks are constrained to have a *constant* angular velocity.
- (c) Determine the frequency of this current under the condition of part (b).

Appendix A

GLOSSARY OF COMMONLY USED SYMBOLS

Section references indicate where symbols of a given significance are introduced; grouped symbols are accompanied by their respective references. The absence of a section reference indicates that a symbol has been applied for a variety of purposes. Nomenclature used in examples is not included.

Symbol	Meaning	Section
A	cross-sectional area	
A_i	coefficient in differential equation	5.1.1
(A_n^+, A_n^-)	complex amplitudes of components of n th mode	9.2.1
A_w	cross-sectional area of armature conductor	6.4.1
a	spacing of pole faces in magnetic circuit	8.5.1
$a, (a_c, a_s)$	phase velocity of acoustic related waves	13.2.1, 11.4.1
a_b	Alfvén velocity	12.2.3
(a, b, c)	Lagrangian coordinates	11.1
a_i	constant coefficient in differential equation	5.1.1
$\mathbf{a}_p$	instantaneous acceleration of point p fixed in material	2.2.1c
B, B_r, B_s	damping constant for linear, angular and square law dampers	2.2.1b, 4.1.1, 5.2.2
$\mathbf{B}, \mathbf{B}_i, B_0$	magnetic flux density	1.1.1a, 8.1, 6.4.2
B_i	induced flux density	7.0
$(B_r, B_{ra}, B_{rb}, B_{rm})$	radial components of air-gap flux densities	4.1.4
$[B_{rf}, (B_{rf})_{av}]$	radial flux density due to field current	6.4.1
b	width of pole faces in magnetic circuit	8.5
b	half thickness of thin beam	11.4.2b
C	contour of integration	1.1.2a
$C, (C_a, C_b), C_o$	capacitance	2.1.2, 7.2.1a, 5.2.1
C	coefficient in boundary condition	9.1.1
$\mathbf{C}$	the curl of the displacement	11.4
(C^+, C^-)	designation of characteristic lines	9.1.1

Symbol	Meaning	Section
c_p	specific heat capacity at constant pressure	13.1.2
c_v	specific heat capacity at constant volume	13.1.2
D	electric displacement	1.1.1a
d	length	
da	elemental area	1.1.2a
$d\mathbf{f}_n$	total elemental force on material in rigid body	2.2.1c
$d\mathbf{l}$	elemental line segment	1.1.2a
$d\mathbf{T}_n$	torque on elemental volume of material	2.2.1c
dV	elemental volume	1.1.2b
E	constant of motion	5.2.1
E	Young's modulus or the modulus of elasticity	9.1
$\mathbf{E}, E_0$	electric field intensity	1.1.1a, 5.1.2d
E_f	magnitude of armature voltage generated by field current in a synchronous machine	4.1.6a
E_i	induced electric field intensity	7.0
e_{11}, e_{ij}	strain tensor	9.1, 11.2
e_{ij}	strain-rate tensor	14.1.1a
F	magnetomotive force (mmf)	13.2.2
F	force density	1.1.1a
$\hat{F}$	complex amplitude of $f(t)$	5.1.1
F_0	amplitude of sinusoidal driving force	9.1.3
f	equilibrium tension of string	9.2
f	driving function	5.1.1
$f, \mathbf{f}, f^e, f^s, f_j, f_i, f_1$	force	2.2.1, 2.2.1c, 3.1, 5.1.2a, 3.1.2b, 8.1, 9.1
f	arbitrary scalar function	6.1
f'	scalar function in moving coordinate system	6.1
f	three-dimensional surface	6.2
f	integration constant	11.4.2a
G	a constant	5.1.2c
G	shear modulus of elasticity	11.2.2
G	speed coefficient	6.4.1
G	conductance	3.1
g	air-gap length	5.2.1
$g, \mathbf{g}$	acceleration of gravity	5.1.2c, 12.1.3
$(\mathbf{H}, H_x, H_y, H_z)$	magnetic field intensity	1.1.1a
h	specific enthalpy	13.1.2
I, I, (I_r, I_s), I_f	electrical current	10.4.3, 12.2.1a, 4.1.2, 6.4.1
$(i, i_1, i_2, \dots, i_k),$ $(i_{ar}, i_{as}, i_{br}, i_{bs}),$ $i_a, (i_a, i_b, i_c),$ $(i_f, i_t), (i_r, i_s)$	electrical current	2.1, 4.1.3, 6.4.1, 4.1.7, 6.4.1, 4.1

Symbol	Meaning	Section
$\mathbf{i}_n$	unit vector perpendicular to area of integration	6.2.1
$\mathbf{i}_s$	unit vector normal to surface of integration	6.2.1
$(\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z), (\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$	unit vectors in coordinate directions	2.2.1c
$J, \mathbf{J}_f$	current density	7.0, 1.1.1a
$J, J_f, (J_x, J_y, J_z)$	moment of inertia	5.1.2b, 4.1.1, 2.2.1c
J_{xx}, J_{yz}	products of inertia	2.2.1c
j	$\sqrt{-1}$	4.1.6a
K	loading factor	13.2.2
$K, \mathbf{K}_f$	surface current density	7.0, 1.1.1a
K	linear or torsional spring constant	2.2.1a
K_t	induced surface current density	7.0
$k, k_e, (k_r, k_t)$	wavenumber	7.1.3, 10.1.3, 10.0
k	summation index	2.1.1
k	maximum coefficient of coupling	4.1.6b
k_n	n th eigenvalue	9.2
$(L, L_1, L_2), (L_a, L_f),$ $L_m, (L_0, L_2),$ $(L_r, L_s, L_{sr}), L_{ss}$	inductance	2.1.1, 6.4.1, 2.1.1, 4.2.1, 4.1.1, 4.2.4
L	length of incremental line segment	6.2.1
l	value of relative displacement for which spring force is zero	2.2.1a
l, l_w, l_y	length	
M	Hartmann number	14.2.2
M	mass of one mole of gas in kilograms	13.1.2
M	Mach number	13.2.1
M	mass	2.2.1c
M	number of mechanical terminal pairs	2.1.1
M, M_s	mutual inductance	4.1.1, 4.2.4
M	magnetization density	1.1.1a
m	mass/unit length of string	9.2
N	number of electrical terminal pairs	2.1.1
N	number of turns	5.2.2
n	number density of ions	12.3.1
n	integer	7.1.1
$\mathbf{n}$	unit normal vector	1.1.2
$\mathbf{P}$	polarization density	1.1.1a
P	power	12.2.1a
p	number of pole pairs in a machine	4.1.8
p	power per unit area	14.2.1
p	pressure	5.1.2d and 12.1.4
P_e, P_g, P_m, P_r	power	4.1.6a, 4.1.6b, 4.1.2, 4.1.6b
Q	electric charge	7.2.1a
q, q_i, q_k	electric charge	1.1.3 and 2.1.2, 8.1, 2.1.2
R, R_t, R_o	radius	

Symbol	Meaning	Section
$R, R_a, R_b, R_f, R_r, R_s$	resistance	
(R, R_g)	gas constant	13.1.2
R_e	electric Reynolds number	7.0
R_m	magnetic Reynolds number	7.0
r	radial coordinate	
$\mathbf{r}$	position vector of material	2.2.1c
$\mathbf{r}'$	position vector in moving reference frame	6.1
$\mathbf{r}_m$	center of mass of rigid body	2.2.1c
S	reciprocal modulus of elasticity	11.5.2c
S	surface of integration	1.1.2a
S	normalized frequency	7.2.4
S	membrane tension	9.2
S_s	transverse force/unit length acting on string	9.2
s	complex frequency	5.1.1
(s, s_{mT})	slip	4.1.6b
s_i	i th root of characteristic equation, a natural frequency	5.1.1
T	period of oscillation	5.2.1
T	temperature	13.1.2
$\mathbf{T}, T, T^e, T_{em}, T_m,$ T_0, T_1	torque	2.2.1c, 5.1.2b, 3.1.1, 4.1.6b, 4.1.1, 6.4.1, 6.4.1
$\mathbf{T}$	surface force	8.4
T_{ij}^m	mechanical stress tensor	13.1.2
T_{mn}	the component of the stress-tensor in the m th-direction on a cartesian surface with a normal vector in the n th-direction	8.1
T_{or}	constant of coulomb damping	4.1.1
T_o	initial stress distribution on thin rod	9.1.1
T	longitudinal stress on a thin rod	9.1.1
T_z	transverse force per unit area on membrane	9.2
T_2	transverse force per unit area acting on thin beam	11.4.2b
t	time	1.1.1
t'	time measured in moving reference frame	6.1
U	gravitational potential	12.1.3
U	longitudinal steady velocity of string or membrane	10.2
u	internal energy per unit mass	13.1.1
u	surface coordinate	11.3
$u_0(x - x_0)$	unit impulse at $x = x_0$	9.2.1
u	transverse deflection of wire in x -direction	10.4.3
$u_{-1}(t)$	unit step occurring at $t = 0$	5.1.2b
V, V_m	velocity	7.0, 13.2.3
V	volume	1.1.2
V, V_a, V_f, V_o, V_s	voltage	
V	potential energy	5.2.1

Symbol	Meaning	Section
$v, \mathbf{v}$	velocity	
$(v, v_1, \dots, v_k)$	voltage	2.1.1
$v', (v_a, v_b, v_c),$ v_f, v_{oc}, v_t	voltage	
v_n	velocity of surface in normal direction	6.2.1
v_o	initial velocity distribution on thin rod	9.1.1
v_p	phase velocity	9.1.1 and 10.2
$\mathbf{v}^r$	relative velocity of inertial reference frames	6.1
v_s	$\sqrt{f/m}$ for a string under tension f and having mass/unit length m	10.1.1
v	longitudinal material velocity on thin rod	9.1.1
v	transverse deflection of wire in y -direction	10.4.3
(W_e, W_m)	energy stored in electromechanical coupling	3.1.1
(W'_e, W'_m, W')	coenergy stored in electromechanical coupling	3.1.2b
W''	hybrid energy function	5.2.1
w	width	5.2.2
w	energy density	11.5.2c
w'	coenergy density	8.5
X	equilibrium position	5.1.2a
$(x, x_1, x_2, \dots, x_k)$	displacement of mechanical node	2.1.1
x	dependent variable	5.1.1
x_p	particular solution of differential equation	5.1.1
$(x_1, x_2, x_3), (x, y, z)$	cartesian coordinates	8.1, 6.1
(x', y', z')	cartesian coordinates of moving frame	6.1
(α, β)	constants along C^+ and C^- characteristics, respectively	9.1.1
(α, β)	see (10.2.20) or (10.2.27)	
α	transverse wavenumber	11.4.3
(α, β)	angles used to define shear strain	11.2
(α, β)	constant angles	4.1.6b
α	space decay parameter	7.1.4
α	damping constant	5.1.2b
α	equilibrium angle of torsional spring	2.2.1a
γ	ratio of specific heats	13.1.2
γ	piezoelectric constant	11.5.2c
$\gamma, \gamma_0, \gamma'$	angular position	
$\Delta_d(t)$	slope excitation of string	10.2.1b
Δ_0	amplitude of sinusoidal slope excitation	10.2.1b
$\Delta \mathbf{r}$	distance between unstressed material points	11.2.1a
$\Delta \mathbf{s}$	distance between stressed positions of material points	11.2.1a
$\delta()$	incremental change in $()$	8.5
$\delta, \delta_1, \delta_0$	displacement of elastic material	11.1, 9.1, 11.4.2a
δ	thickness of incremental volume element	6.2.1
δ	torque angle	4.1.6a

Symbol	Meaning	Section
δ_{ij}	Kronecker delta	8.1
(δ_+, δ_-)	wave components traveling in the $\pm x$ -directions	9.1.1
ϵ	linear permittivity	1.1.1b
ϵ_0	permittivity of free space	1.1.1a
η	efficiency of an induction motor	4.1.6b
η	second coefficient of viscosity	14.1.1c
$\theta, \theta_i, \theta_m$	angular displacement	2.1.1, 3.1.1, 5.2.1
θ	power factor angle; phase angle between current and voltage	4.1.6a
θ	equilibrium angle	5.2.1
$\dot{\theta}$	angular velocity of armature	6.4.1
θ_m	maximum angular deflection	5.2.1
$(\lambda, \lambda_1, \lambda_2, \dots, \lambda_k)$	magnetic flux linkage	2.1.1, 6.4.1, 4.1.7,
λ_a		4.1.3, 4.1
$(\lambda_a, \lambda_b, \lambda_c)$		
$(\lambda_{ar}, \lambda_{as}, \lambda_{br}, \lambda_{bs})$		
(λ_r, λ_s)		
λ	Lamé constant for elastic material	11.2.3
λ	wavelength	7.1.4
μ	linear permeability	1.1.1a
$\mu, (\mu_+, \mu_-)$	mobility	12.3.1, 1.1.1b
μ	coefficient of viscosity	14.1.1
μ_d	coefficient of dynamic friction	2.2.1b
μ_0	permeability of free space	1.1.1a
μ_s	coefficient of static friction	2.2.1b
ν	Poisson's ratio for elastic material	11.2.2
ν	damping frequency	10.1.4
(ξ, ξ)	continuum displacement	8.5
ξ_0	initial deflection of string	9.2
ξ_d	amplitude of sinusoidal driving deflection	9.2
$(\xi_n(x), \xi_n(x))$	n th eigenfunctions	9.2.1b
(ξ_+, ξ_-)	amplitudes of forward and backward traveling waves	9.2
$\dot{\xi}_0(x)$	initial velocity of string	9.2
ρ	mass density	2.2.1c
ρ_f	free charge density	1.1.1a
ρ_s	surface mass density	11.3
Σ	surface of discontinuity	6.2
σ	conductivity	1.1.1a
σ_f	free surface charge density	1.1.1a
σ_m	surface mass density of membrane	9.2
σ_o	surface charge density	7.2.3
σ_s	surface conductivity	1.1.1a
σ_u	surface charge density	7.2.3
τ	surface traction	8.2.1
τ, τ_d	diffusion time constant	7.1.1, 7.1.2a
τ	relaxation time	7.2.1a

Symbol	Meaning	Section
τ_e	electrical time constant	5.2.2
τ_m	time for air gap to close	5.2.2
τ_o	time constant	5.1.3
τ_t	traversal time	7.1.2a
ϕ	electric potential	7.2
ϕ	magnetic flux	2.1.1
ϕ	cylindrical coordinate	2.1.1
ϕ	potential for $\mathbf{H}$ when $\mathbf{J}_f = 0$	8.5.2
ϕ	flow potential	12.2
χ_e	electric susceptibility	1.1.1b
χ_m	magnetic susceptibility	1.1.1a
ψ	the divergence of the material displacement	11.4
ψ	angle defined in Fig. 6.4.2	6.4.1
ψ	angular position in the air gap measured from stator winding (a) magnetic axis	4.1.4
ψ	electromagnetic force potential	12.2
ψ	angular deflection of wire	10.4.3
Ω	equilibrium rotational speed	5.1.2b
Ω	rotation vector in elastic material	11.2.1a
Ω_n	real part of eigenfrequency (10.1.47)	10.1.4
$\omega, (\omega_r, \omega_s)$	radian frequency of electrical excitation	4.1.6a, 4.1.2
ω	natural angular frequency (Im s)	5.1.2b
ω, ω_m	angular velocity	2.2.1c, 4.1.2
ω_c	cutoff frequency for evanescent waves	10.1.2
ω_d	driving frequency	9.2
ω_n	n th eigenfrequency	9.2
ω_o	natural angular frequency	5.1.3
(ω_r, ω_i)	real and imaginary parts of ω	10.0
∇	nabla	6.1
∇_Σ	surface divergence	6.2.1

Appendix B

REVIEW OF ELECTROMAGNETIC THEORY

B.1 BASIC LAWS AND DEFINITIONS

The laws of electricity and magnetism are empirical. Fortunately they can be traced to a few fundamental experiments and definitions, which are reviewed in the following sections. The rationalized MKS system of units is used.

B.1.1 Coulomb's Law, Electric Fields and Forces

Coulomb found that when a charge q (coulombs) is brought into the vicinity of a distribution of *charge density* $\rho_e(\mathbf{r}')$ (coulombs per cubic meter), as shown in Fig. B.1.1, a force of repulsion $\mathbf{f}$ (newtons) is given by

$$\mathbf{f} = q\mathbf{E}, \quad (\text{B.1.1})$$

where the *electric field intensity* $\mathbf{E}$ (volts per meter) is evaluated at the position

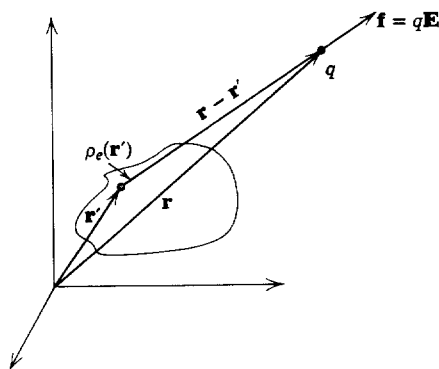


Fig. B.1.1 The force $\mathbf{f}$ on the point charge q in the vicinity of charges with density $\rho_e(\mathbf{r}')$ is represented by the electric field intensity $\mathbf{E}$ times q , where $\mathbf{E}$ is found from (B.1.2).

$\mathbf{r}$ of the charge q and determined from the distribution of charge density by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{V'} \rho_e(\mathbf{r}') \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV'. \quad (\text{B.1.2})$$

In the rationalized MKS system of units the permittivity ϵ_0 of free space is

$$\epsilon_0 = 8.854 \times 10^{-12} \approx \frac{1}{36\pi} \times 10^{-9} \text{ F/m}. \quad (\text{B.1.3})$$

Note that the integration of (B.1.2) is carried out over all the charge distribution (excluding q), hence represents a superposition (at the location $\mathbf{r}$ of q) of the electric field intensities due to elements of charge density at the positions $\mathbf{r}'$.

As an example, suppose that the charge distribution $\rho_e(\mathbf{r}')$ is simply a point charge Q (coulombs) at the origin (Fig. B.1.2); that is,

$$\rho_e = Q \delta(\mathbf{r}'), \quad (\text{B.1.4})$$

where $\delta(\mathbf{r}')$ is the *delta function* defined by

$$\begin{aligned} \delta(\mathbf{r}') &= 0, & \mathbf{r}' &\neq 0, \\ \int_{V'} \delta(\mathbf{r}') dV' &= 1. \end{aligned} \quad (\text{B.1.5})$$

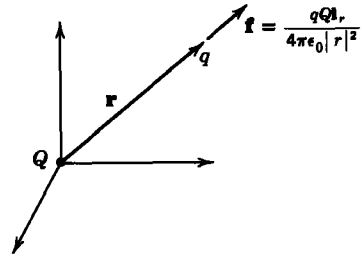


Fig. B.1.2 Coulomb's law for point charges Q (at the origin) and q (at the position $\mathbf{r}$).

For the charge distribution of (B.1.4) integration of (B.1.2) gives

$$\mathbf{E}(\mathbf{r}) = \frac{Q\mathbf{r}}{4\pi\epsilon_0 |\mathbf{r}|^3}. \quad (\text{B.1.6})$$

Hence the force on the point charge q , due to the point charge Q , is from (B.1.1)

$$\mathbf{f} = \frac{qQ\mathbf{r}}{4\pi\epsilon_0 |\mathbf{r}|^3}. \quad (\text{B.1.7})$$

This expression takes the familiar form of *Coulomb's law* for the force of repulsion between point charges of like sign.

We know that electric charge occurs in integral multiples of the electronic charge (1.60×10^{-19} C). The charge density ρ_e , introduced with (B.1.2), is defined as

$$\rho_e(\mathbf{r}) = \lim_{\delta V \rightarrow 0} \frac{1}{\delta V} \sum_i q_i, \quad (\text{B.1.8})$$

where δV is a small volume enclosing the point $\mathbf{r}$ and $\sum_i q_i$ is the algebraic sum of charges within δV . The charge density is an example of a continuum model. To be valid the limit $\delta V \rightarrow 0$ must represent a volume large enough to contain a large number of charges q_i , yet small enough to appear infinitesimal when compared with the significant dimensions of the system being analyzed. This condition is met in most electromechanical systems.

For example, in copper at a temperature of 20°C the number density of free electrons available for carrying current is approximately 10^{23} electrons/cm³. If we consider a typical device dimension to be on the order of 1 cm, a reasonable size for δV would be a cube with 1-mm sides. The number of electrons in δV would be 10^{20} , which certainly justifies the continuum model.

The force, as expressed by (B.1.1), gives the total force on a single test charge in vacuum and, as such, is not appropriate for use in a continuum model of electromechanical systems. It is necessary to use an *electric force density* $\mathbf{F}$ (newtons per cubic meter) that can be found by averaging (B.1.1) over a small volume.

$$\mathbf{F} = \lim_{\delta V \rightarrow 0} \frac{\sum_i \mathbf{f}_i}{\delta V} = \lim_{\delta V \rightarrow 0} \frac{\sum q_i \mathbf{E}_i}{\delta V}. \quad (\text{B.1.9})$$

Here q_i represents all of the charges in δV , $\mathbf{E}_i$ is the electric field intensity acting on the i th charge, and $\mathbf{f}_i$ is the force on the i th charge. As in the charge density defined by (B.1.8), the limit of (B.1.9) leads to a continuum model if the volume δV can be defined so that it is small compared with macroscopic dimensions of significance, yet large enough to contain many electronic charges. Further, there must be a sufficient amount of charge external to the volume δV that the electric field experienced by each of the test charges is essentially determined by the sources of field outside the volume. Fortunately these requirements are met in almost all physical situations that lead to useful electromechanical interactions. Because all charges in the volume δV experience essentially the same electric field $\mathbf{E}$, we use the definition of free charge density given by (B.1.8) to write (B.1.9) as

$$\mathbf{F} = \rho_e \mathbf{E}. \quad (\text{B.1.10})$$

Although the static electric field intensity $\mathbf{E}$ can be computed from (B.1.2), it is often more convenient to state the relation between charge density and field intensity in the form of *Gauss's law*:

$$\oint_S \epsilon_0 \mathbf{E} \cdot \mathbf{n} \, da = \int_V \rho_e \, dV. \quad (\text{B.1.11})$$

In this integral law $\mathbf{n}$ is the outward-directed unit vector normal to the surface S , which encloses the volume V . It is not our purpose in this brief review to show that (B.1.11) is implied by (B.1.2). It is helpful, however, to note that

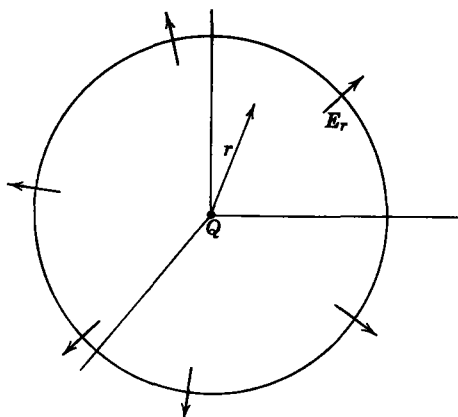


Fig. B.1.3 A hypothetical sphere of radius r encloses a charge Q at the origin. The integral of $\epsilon_0 \mathbf{E}_r$ over the surface of the sphere is equal to the charge Q enclosed.

in the case of a point charge Q at the origin it predicts the same electric field intensity (B.1.6) as found by using (B.1.2). For this purpose the surface S is taken as the sphere of radius r centered at the origin, as shown in Fig. B.1.3. By symmetry the only component of $\mathbf{E}$ is radial (E_r), and this is constant at a given radius r . Hence (B.1.11) becomes

$$4\pi r^2 E_r \epsilon_0 = Q. \quad (\text{B.1.12})$$

Here the integration of the charge density over the volume V enclosed by S is the total charge enclosed Q but can be formally taken by using (B.1.4) with the definition provided by (B.1.5). It follows from (B.1.12) that

$$E_r = \frac{Q}{4\pi\epsilon_0 r^2}, \quad (\text{B.1.13})$$

a result that is in agreement with (B.1.6).

Because the volume and surface of integration in (B.1.11) are arbitrary, the integral equation implies a differential law. This is found by making use of the *divergence theorem**

$$\oint_S \mathbf{A} \cdot \mathbf{n} \, da = \int_V \nabla \cdot \mathbf{A} \, dV \quad (\text{B.1.14})$$

to write (B.1.11) as

$$\int_V (\nabla \cdot \epsilon_0 \mathbf{E} - \rho_e) \, dV = 0. \quad (\text{B.1.15})$$

* For a discussion of the divergence theorem see F. B. Hildebrand, *Advanced Calculus for Engineers*, Prentice-Hall, New York, 1949, p. 312.

Since the volume of integration is arbitrary, it follows that

$$\nabla \cdot \epsilon_0 \mathbf{E} = \rho_e. \quad (\text{B.1.16})$$

From this discussion it should be apparent that this *differential* form of *Gauss's law* is implied by Coulomb's law, with the electric field intensity defined as a force per unit charge.

B.1.2 Conservation of Charge

Experimental evidence supports the postulate that electric charge is conserved. When a negative charge appears (e.g., when an electron is removed from a previously neutral atom), an equal positive charge also appears (e.g., the positive ion remaining when the electron is removed from the atom).

We can make a mathematical statement of this postulate in the following way. Consider a volume V enclosed by a surface S . If charge is conserved, the net rate of flow of electric charge out through the surface S must equal the rate at which the total charge in the volume V decreases. The current density $\mathbf{J}$ (coulombs per square meter-second) is defined as having the direction of flow of positive charge and a magnitude proportional to the net rate of flow of charge per unit area. Then the statement of conservation of charge is

$$\oint_S \mathbf{J} \cdot \mathbf{n} \, da = - \frac{d}{dt} \int_V \rho_e \, dV. \quad (\text{B.1.17})$$

Once again it follows from the arbitrary nature of S (which is fixed in space) and the divergence theorem (B.1.14) that

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho_e}{\partial t} = 0. \quad (\text{B.1.18})$$

It is this equation that is used as a *differential* statement of *conservation of charge*.

To express conservation of charge it has been necessary to introduce a new continuum variable, the current density $\mathbf{J}$. Further insight into the relation between this quantity and the charge density ρ_e is obtained by considering a situation in which two types of charge contribute to the current, charges q_+ with velocity $\mathbf{v}_+$ and charges q_- with velocity $\mathbf{v}_-$. The current density $\mathbf{J}_+$ that results from the flow of positive charge is

$$\mathbf{J}_+ = \lim_{\delta V \rightarrow 0} \frac{1}{\delta V} \sum_i q_{+i} \mathbf{v}_{+i}. \quad (\text{B.1.19})$$

If we define a *charge-average velocity* $\mathbf{v}_+$ for the positive charges as

$$\mathbf{v}_+ = \frac{\sum_i q_{+i} \mathbf{v}_{+i}}{\sum_i q_{+i}} \quad (\text{B.1.20})$$

and the density ρ_+ of positive charges from (B.1.8) as

$$\rho_+ = \lim_{\delta V \rightarrow 0} \frac{1}{\delta V} \sum_i q_{+i}, \quad (\text{B.1.21})$$

we can write the current density of (B.1.19) as

$$\mathbf{J}_+ = \rho_+ \mathbf{v}_+. \quad (\text{B.1.22})$$

Similar definitions for the charge-average velocity $\mathbf{v}_-$ and charge density ρ_- of negative charges yields the component of current density

$$\mathbf{J}_- = \rho_- \mathbf{v}_-. \quad (\text{B.1.23})$$

The total current density $\mathbf{J}$ is the vector sum of the two components

$$\mathbf{J} = \mathbf{J}_+ + \mathbf{J}_-. \quad (\text{B.1.24})$$

Now consider the situation of a material that contains charge densities ρ_+ and ρ_- which have charge-average velocities $\mathbf{v}_+$ and $\mathbf{v}_-$ with respect to the material. Assume further that the material is moving with a velocity $\mathbf{v}$ with respect to an observer who is to measure the current. The net average velocities of positive and negative charges as seen by the observer are $\mathbf{v}_+ + \mathbf{v}$ and $\mathbf{v}_- + \mathbf{v}$, respectively. The current density measured by the observer is then from (B.1.24)

$$\mathbf{J} = (\rho_+ \mathbf{v}_+ + \rho_- \mathbf{v}_-) + \rho_e \mathbf{v}, \quad (\text{B.1.25})$$

where the net charge density ρ_e is given by

$$\rho_e = \rho_+ + \rho_-. \quad (\text{B.1.26})$$

The first term of (B.1.25) is a net flow of charge with respect to the material and is normally called a *conduction current*. (It is often described by Ohm's law.) The last term represents the transport of net charge and is conventionally called a *convection current*. It is crucial that *net flow of charge* be distinguished from *flow of net charge*. The net charge may be zero but a current can still be accounted for by the conduction term. This is the case in metallic conductors.

B.1.3 Ampère's Law, Magnetic Fields and Forces

The *magnetic flux density* $\mathbf{B}$ is defined to express the force on a current element $i d\mathbf{l}$ placed in the vicinity of other currents. This element is shown in Fig. B.1.4 at the position $\mathbf{r}$. Then, according to Ampère's experiments, the force is given by

$$\mathbf{f} = i d\mathbf{l} \times \mathbf{B}, \quad (\text{B.1.27})$$

where

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV'. \quad (\text{B.1.28})$$

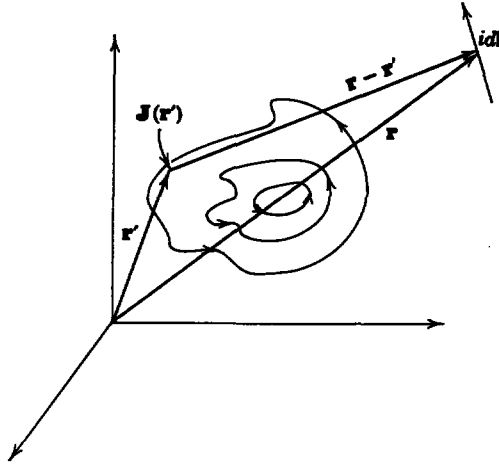


Fig. B.1.4 A distribution of current density $\mathbf{J}(\mathbf{r}')$ produces a force on the current element $i d\mathbf{l}$ which is represented in terms of the magnetic flux density $\mathbf{B}$ by (B.1.27) and (B.1.28).

Hence the flux density at the position $\mathbf{r}$ of the current element $i d\mathbf{l}$ is the superposition of fields produced by currents at the positions $\mathbf{r}'$. In this expression the permeability of free space μ_0 is

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m.} \quad (\text{B.1.29})$$

As an example, suppose that the distribution of current density $\mathbf{J}$ is composed of a current I (amperes) in the z direction and along the z -axis, as shown in Fig. B.1.5. The magnetic flux density at the position $\mathbf{r}$ can be computed

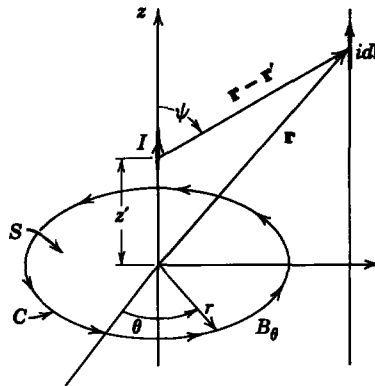


Fig. B.1.5 A current I (amperes) along the z -axis produces a magnetic field at the position $\mathbf{r}$ of the current element $i d\mathbf{l}$.

from (B.1.28), which for this case reduces to*

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{+\infty} \frac{\mathbf{i}_z \times (\mathbf{r} - z' \mathbf{i}_z)}{|\mathbf{r} - z' \mathbf{i}_z|^3} dz'. \quad (\text{B.1.30})$$

Here the coordinate of the source current I is z' , as shown in Fig. B.1.5, whereas the coordinate $\mathbf{r}$ that designates the position at which $\mathbf{B}$ is evaluated can be written in terms of the cylindrical coordinates (r, θ, z) . Hence (B.1.30) becomes

$$\mathbf{B} = \frac{\mu_0 I \mathbf{i}_\theta}{4\pi} \int_{-\infty}^{+\infty} \frac{\sin \psi \sqrt{(z - z')^2 + r^2}}{[(z - z')^2 + r^2]^{3/2}} dz', \quad (\text{B.1.31})$$

where, from Fig. B.1.5, $\sin \psi = r/\sqrt{(z - z')^2 + r^2}$. Integration on z' gives the magnetic flux density

$$\mathbf{B} = \frac{\mu_0 I \mathbf{i}_\theta}{2\pi r}. \quad (\text{B.1.32})$$

It is often more convenient to relate the magnetic flux density to the current density $\mathbf{J}$ by the integral of *Ampère's law* for static fields, which takes the form

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot \mathbf{n} \, da. \quad (\text{B.1.33})$$

Here C is a closed contour of line integration and S is a surface enclosed by C . We wish to present a review of electromagnetic theory and therefore we shall not embark on a proof that (B.1.33) is implied by (B.1.28). Our purpose is served by recognizing that (B.1.33) can also be used to predict the flux density in the situation in Fig. B.1.5. By symmetry we recognize that $\mathbf{B}$ is azimuthally directed and independent of θ and z . Then, if we select the contour C in a plane z equals constant and at a radius r , as shown in Fig. B.1.5, (B.1.33) becomes

$$2\pi r B_\theta = \mu_0 I. \quad (\text{B.1.34})$$

Solution of this expression for B_θ gives the same result as predicted by (B.1.28). [See (B.1.32).]

The contour C and surface S in (B.1.33) are arbitrary and therefore the equation can be cast in a differential form. This is done by using Stokes' theorem†,

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S \mathbf{n} \cdot (\nabla \times \mathbf{A}) \, da, \quad (\text{B.1.35})$$

* Unit vectors in the coordinate directions are designated by $\mathbf{i}$. Thus $\mathbf{i}_z$ is a unit vector in the z -direction.

† See F. B. Hildebrand, *Advanced Calculus for Engineers*, Prentice-Hall, New York, 1949, p. 318.

to write (B.1.33) as

$$\int_S (\nabla \times \mathbf{B} - \mu_0 \mathbf{J}) \cdot \mathbf{n} \, da = 0, \quad (\text{B.1.36})$$

from which the differential form of Ampère's law follows as

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \quad (\text{B.1.37})$$

So far the assumption has been made that the current $\mathbf{J}$ is constant in time. Maxwell's contribution consisted in recognizing that if the sources ρ_e and $\mathbf{J}$ (hence the fields $\mathbf{E}$ and $\mathbf{B}$) are time varying the displacement current $\epsilon_0 \partial \mathbf{E} / \partial t$ must be included on the right-hand side of (B.1.37). Thus for dynamic fields Ampère's law takes the form

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \frac{\partial \epsilon_0 \mathbf{E}}{\partial t}. \quad (\text{B.1.38})$$

This alteration of (B.1.37) is necessary if conservation of charge expressed by (B.1.18) is to be satisfied. Because the divergence of any vector having the form $\nabla \times \mathbf{A}$ is zero, the divergence of (B.1.38) becomes

$$\nabla \cdot \mathbf{J} + \frac{\partial (\nabla \cdot \epsilon_0 \mathbf{E})}{\partial t} = 0. \quad (\text{B.1.39})$$

Then, if we recall that ρ_e is related to $\mathbf{E}$ by Gauss's law (B.1.16), the conservation of charge equation (B.1.18) follows. The displacement current in (B.1.38) accounts for the rate of change of ρ_e in (B.1.18).

We shall make considerable use of Ampère's law, as expressed by (B.1.38), with Maxwell's displacement current included. From our discussion it is clear that the static form of this law results from the force law of interaction between currents. The magnetic flux density is defined in terms of the force produced on a current element. Here we are interested primarily in a continuum description of the force, hence require (B.1.27) expressed as a force density. With the same continuum restrictions implied in writing (B.1.10), we write the magnetic force density (newtons per cubic meter) as

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}. \quad (\text{B.1.40})$$

In view of our remarks it should be clear that this force density is not something that we have derived but rather arises from the definition of the flux density $\mathbf{B}$. Further remarks on this subject are found in Section 8.1.

B.1.4 Faraday's Law of Induction and the Potential Difference

Two extensions of static field theory are required to describe dynamic fields. One of these, the introduction of the displacement current in Ampère's law, was discussed in the preceding section. Much of the significance of this

generalization stems from the apparent fact that an electric field can lead to the generation of a magnetic field. As a second extension of static field theory, Faraday discovered that, conversely, time-varying magnetic fields can lead to the generation of electric fields.

Faraday's law of induction can be written in the integral form

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = - \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da, \quad (\text{B.1.41})$$

where again C is a contour that encloses the surface S . The contour and surface are arbitrary; hence it follows from Stokes' theorem (B.1.35) that Faraday's law has the differential form

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}. \quad (\text{B.1.42})$$

Note that in the static case this expression reduces to $\nabla \times \mathbf{E} = 0$, which is, in addition to Gauss's law, a condition on the static electric field. That this further equation is consistent with the electric field, as given by (B.1.2), is not shown in this review. Clearly the one differential equation represented by Gauss's law could not alone determine the three components of $\mathbf{E}$.

In regions in which the magnetic field is either static or negligible the electric field intensity can be derived as the gradient of a scalar potential ϕ :

$$\mathbf{E} = -\nabla\phi. \quad (\text{B.1.43})$$

This is true because the curl of the gradient is zero and (B.1.42) is satisfied. The difference in potential between two points, say a and b , is a measure of the line integral of $\mathbf{E}$, for

$$\int_a^b \mathbf{E} \cdot d\mathbf{l} = - \int_a^b \nabla\phi \cdot d\mathbf{l} = \phi_a - \phi_b. \quad (\text{B.1.44})$$

The potential difference $\phi_a - \phi_b$ is referred to as the voltage of point a with respect to b . If there is no magnetic field $\mathbf{B}$ in the region of interest, the integral of (B.1.44) is independent of path. In the presence of a time-varying magnetic field the integral of $\mathbf{E}$ around a closed path is not in general zero, and if a potential is defined in some region by (B.1.43) the path of integration will in part determine the measured potential difference.

The physical situation shown in Fig. B.1.6 serves as an illustration of the implications of Faraday's law. A magnetic circuit is excited by a current source $I(t)$ as shown. Because the magnetic material is highly permeable, the induced flux density $B(t)$ is confined to the cross section A which links a circuit formed by resistances R_a and R_b in series. A cross-sectional view of the

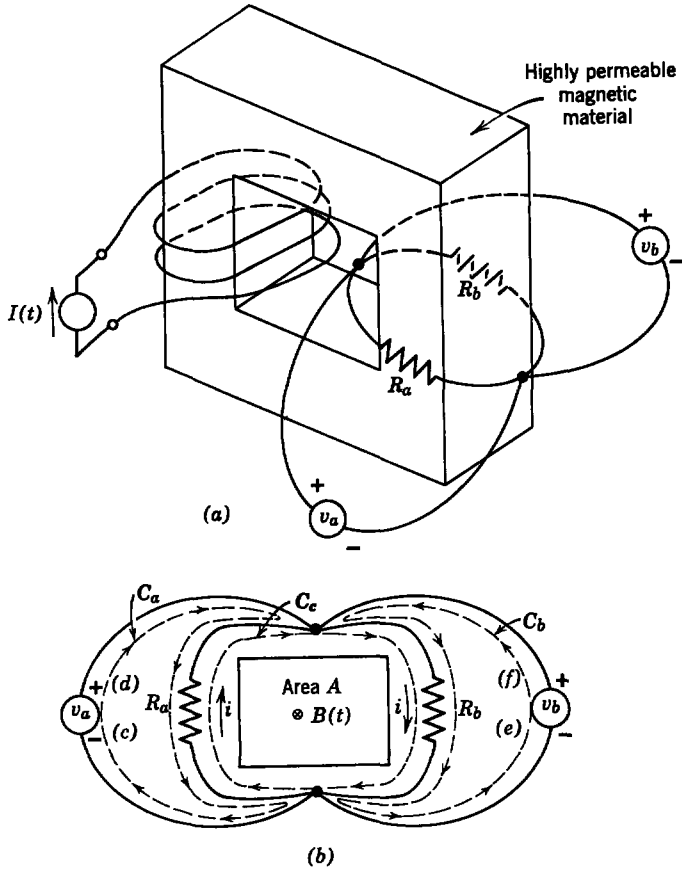


Fig. B.1.6 (a) A magnetic circuit excited by $I(t)$ so that flux $AB(t)$ links the resistive loop (b) a cross-sectional view of the loop showing connection of the voltmeters.

circuit is shown in Fig. B.1.6b, in which high impedance voltmeters v_a and v_b are shown connected to the same nodes. Under the assumption that no current is drawn by the voltmeters, and given the flux density $B(t)$, we wish to compute the voltages that would be indicated by v_a and v_b .

Three contours of integration C are defined in Fig. B.1.6b and are used with Faraday's integral law (B.1.41). The integral of $\mathbf{E}$ around the contour C_c is equal to the drop in potential across both of the resistances, which carry the same current i . Hence, since this path encloses a total flux $AB(t)$, we have

$$i(R_a + R_b) = - \frac{d}{dt} [AB(t)]. \quad (\text{B.1.45})$$

The paths of integration C_a and C_b do not enclose a magnetic flux; hence for

these paths (B.1.41) gives

$$v_a = -iR_a = \frac{R_a}{R_a + R_b} \frac{d}{dt} [AB(t)] \quad \text{for } C_a, \quad (\text{B.1.46})$$

$$v_b = iR_b = \frac{-R_b}{R_a + R_b} \frac{d}{dt} [AB(t)] \quad \text{for } C_b, \quad (\text{B.1.47})$$

where the current i is evaluated by using (B.1.45). The most obvious attribute of this result is that although the voltmeters are connected to the same nodes they do not indicate the same values. In the presence of the magnetic induction the contour of the voltmeter leads plays a role in determining the voltage indicated.

The situation shown in Fig. B.1.6 can be thought of as a transformer with a single turn secondary. With this in mind, it is clear that Faraday's law plays an essential role in electrical technology.

The divergence of an arbitrary vector $\nabla \times \mathbf{A}$ is zero. Hence the divergence of (B.1.42) shows that the divergence of $\mathbf{B}$ is constant. This fact also follows from (B.1.28), from which it can be shown that this constant is zero. Hence an additional differential equation for $\mathbf{B}$ is

$$\nabla \cdot \mathbf{B} = 0. \quad (\text{B.1.48})$$

Integration of this expression over an arbitrary volume V and use of the divergence theorem (B.1.14) gives

$$\oint_S \mathbf{B} \cdot \mathbf{n} \, da = 0. \quad (\text{B.1.49})$$

This integral law makes more apparent the fact that there can be no net magnetic flux emanating from a given region of space.

B.2 MAXWELL'S EQUATIONS

The generality and far-reaching applications of the laws of electricity and magnetism are not immediately obvious; for example, the law of induction given by (B.1.42) was recognized by Faraday as true when applied to a conducting circuit. The fact that (B.1.42) has significance even in regions of space unoccupied by matter is a generalization that is crucial to the theory of electricity and magnetism. We can summarize the differential laws introduced in Section B.1 as

$$\nabla \cdot \epsilon_0 \mathbf{E} = \rho_e, \quad (\text{B.2.1})$$

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho_e}{\partial t} = 0, \quad (\text{B.2.2})$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \frac{\partial \epsilon_0 \mathbf{E}}{\partial t}, \quad (\text{B.2.3})$$

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}, \quad (\text{B.2.4})$$

$$\nabla \cdot \mathbf{B} = 0. \quad (\text{B.2.5})$$

Taken together, these laws are called *Maxwell's equations* in honor of the man who was instrumental in recognizing that they have a more general significance than any one of the experiments from which they originate. For example, we can think of a time-varying magnetic flux that induces an electric field according to (B.2.4) even in the absence of a material circuit. Similarly, (B.2.3) is taken to mean that even in regions of space in which there is no circuit, hence $\mathbf{J} = 0$, a time-varying electric field leads to an induced magnetic flux density $\mathbf{B}$.

The coupling between time-varying electric and magnetic fields, as predicted by (B.2.1 to B.2.5), accounts for the existence of electromagnetic waves, whether they be radio or light waves or even gamma rays. As we might guess from the electromechanical origins of electromagnetic theory, the propagation of electromagnetic waves is of secondary importance in the study of most electromechanical phenomena. This does not mean that electromechanical interactions are confined to frequencies that are low compared with radio frequencies. Indeed, electromechanical interactions of practical significance extend into the gigahertz range of frequencies.

To take a mature approach to the study of electromechanics it is necessary that we discriminate at the outset between essential and nonessential aspects of interactions between fields and media. This makes it possible to embark immediately on a study of nontrivial interactions. An essential purpose of this section is the motivation of approximations used in this book.

Although electromagnetic waves usually represent an unimportant consideration in electromechanics and are not discussed here in depth, they are important to an understanding of the quasi-static approximations that are introduced in Section B.2.2. Hence we begin with a brief simplified discussion of electromagnetic waves.

B.2.1 Electromagnetic Waves

Consider fields predicted by (B.2.3) and (B.2.4) in a region of free space in which $\mathbf{J} = 0$. In particular, we confine our interest to situations in which the fields depend only on (x, t) (the fields are one-dimensional) and write the y -component of (B.2.3) and the z -component of (B.2.4)

$$-\frac{\partial B_z}{\partial x} = \mu_0 \epsilon_0 \frac{\partial E_y}{\partial t}, \quad (\text{B.2.6})$$

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}. \quad (\text{B.2.7})$$

This pair of equations, which make evident the coupling between the dynamic electric and magnetic fields, is sufficient to determine the field components B_z and E_y . In fact, if we take the time derivative of (B.2.6) and use the resulting

expression to eliminate B_z from the derivative with respect to x of (B.2.7), we obtain

$$\frac{\partial^2 E_y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2}, \quad (\text{B.2.8})$$

where

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ (m/sec)}.$$

This equation for E_y is called the *wave equation* because it has solutions in the form of

$$E_y(x, t) = E_+(x - ct) + E_-(x + ct). \quad (\text{B.2.9})$$

That this is true may be verified by substituting (B.2.9) into (B.2.8). Hence solutions for E_y can be analyzed into components E_+ and E_- that represent waves traveling, respectively, in the $+x$ - and $-x$ -directions with the *velocity of light* c , given by (B.2.8). The prediction of electromagnetic wave propagation is a salient feature of Maxwell's equations. It results, as is evident from the derivation, because time-varying magnetic fields can induce electric fields [Faraday's law, (B.2.7)] while at the same time dynamic electric fields induce magnetic fields [Ampère's law with the displacement current included (B.2.6)]. It is also evident from the derivation that if we break this two-way coupling by leaving out the displacement current *or* omitting the magnetic induction term electromagnetic waves are not predicted.

Electromechanical interactions are usually not appreciably affected by the propagational character of electromagnetic fields because the velocity of propagation c is very large. Suppose that we are concerned with a system whose largest dimension is l . The time l/c required for the propagation of a wave between extremes of the system is usually short compared with characteristic dynamical times of interest; for example, in a device in which $l = 0.3$ m the time l/c equals 10^{-9} sec. If we were concerned with electromechanical motions with a time constant of a microsecond (which is extremely short for a device characterized by 30 cm), it would be reasonable to ignore the wave propagation. In the absence of other dynamic effects this could be done by assuming that the fields were established everywhere within the device instantaneously.

Even though it is clear that the propagation of electromagnetic waves has nothing to do with the dynamics of interest, it is not obvious how to go about simplifying Maxwell's equations to remove this feature of the dynamics. A pair of particular examples will help to clarify approximations made in the next section. These examples, which are considered simultaneously so that they can be placed in contrast, are shown in Fig. B.2.1.

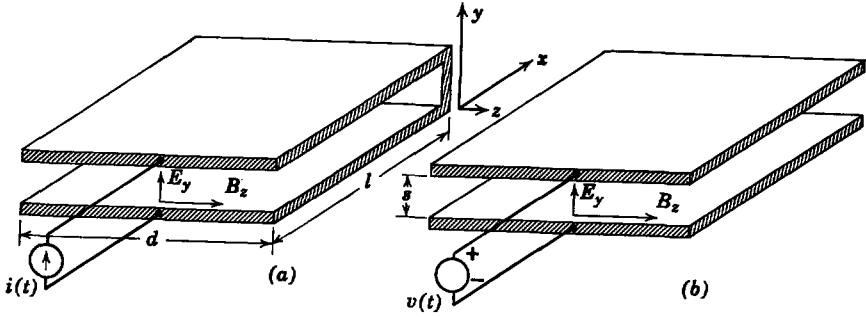


Fig. B.2.1 Perfectly conducting plane-parallel electrodes driven at $x = -l$: (a) $i(t) = i_0 \cos \omega t$; (b) $v(t) = v_0 \cos \omega t$.

A pair of perfectly conducting parallel plates has the spacing s which is much smaller than the x - z dimensions l and d . The plates are excited at $x = -l$ by

a current source

$$i(t) = i_0 \cos \omega t \text{ (amperes). (B.2.10a)}$$

a voltage source

$$v(t) = v_0 \cos \omega t \text{ (volts). (B.2.10b)}$$

At $x = 0$, the plates are terminated in

a perfectly conducting short circuit plate.

an open circuit.

If we assume that the spacing s is small enough to warrant ignoring the effects of fringing and that the driving sources at $x = -l$ are distributed along the z -axis, the one-dimensional fields B_z and E_y predicted by (B.2.6) and (B.2.7) represent the fields between the plates. Hence we can think of the current and voltage sources as exciting electromagnetic waves that propagate along the x -axis between the plates. The driving sources impose conditions on the fields at $x = -l$. They are obtained by

integrating (B.1.33) around the contour C (Fig. B.2.2a) which encloses the upper plate adjacent to the current source. (The surface S enclosed by C is very thin so that negligible displacement current links the loop).

$$B_z(-l, t) = -\mu_0 K = -\frac{\mu_0 i(t)}{d} \quad (\text{B.2.11a})$$

integrating the electric field between (a) and (b) in Fig. B.2.2b to relate the potential difference of the voltage source to the electric field intensity $E_y(-l, t)$.

$$\int_s^0 E_y dy = -s E_y(-l, t) = v(t). \quad (\text{B.2.11b})$$

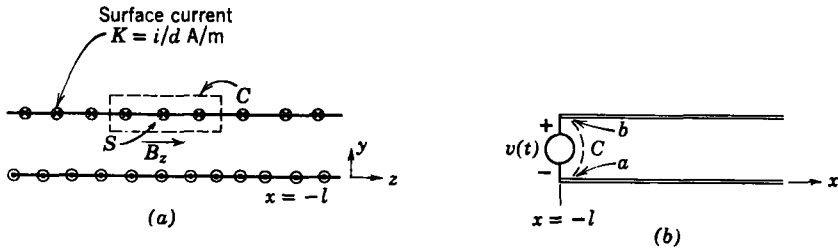


Fig. B.2.2 Boundary conditions for the systems in Fig. B.2.1

Similar conditions used at $x = 0$ give the boundary conditions

$$E_y(0, t) = 0 \quad (\text{B.2.12a}) \quad | \quad B_z(0, t) = 0 \quad (\text{B.2.12b})$$

It is not our purpose in this chapter to become involved with the formalism of solving the wave equation [or (B.2.6) and (B.2.7)] subject to the boundary conditions given by (B.2.11) and (B.2.12). There is ample opportunity to solve boundary value problems for electromechanical systems in the text, and the particular problem at hand forms a topic within the context of transmission lines and waveguides. For our present purposes, it suffices to guess solutions to these equations that will satisfy the appropriate boundary conditions. Then direct substitution into the differential equations will show that we have made the right choice.

$$E_y = -i_o \frac{\sin \omega t \sin (\omega x/c)}{d \epsilon_0 c \cos (\omega l/c)}, \quad (\text{B.2.13a}) \quad \left| \quad E_y = -\frac{v_o \cos \omega t \cos (\omega x/c)}{s \cos (\omega l/c)}, \quad (\text{B.2.13b})\right.$$

$$B_z = -\frac{\mu_0 i_o \cos \omega t \cos (\omega x/c)}{d \cos (\omega l/c)}, \quad (\text{B.2.14a}) \quad \left| \quad B_z = -\frac{v_o \sin \omega t \sin (\omega x/c)}{cs \cos (\omega l/c)}, \quad (\text{B.2.14b})\right.$$

Note that at $x = -l$ the boundary conditions B.2.11 are satisfied, whereas at $x = 0$ the conditions of (B.2.12) are met. One way to show that Maxwell's equations are satisfied also (aside from direct substitution) is to use trigonometric identities* to rewrite these standing wave solutions as the superposition of two traveling waves in the form of (B.2.9). Our solutions are sinusoidal, steady-state solutions, so that with the understanding that the amplitude of the field at any point along the x -axis is varying sinusoidally with time we can obtain an impression of the dynamics by plotting the instantaneous amplitudes, as shown in Fig. B.2.3. In general, the fields have the sinusoidal distribution along the x -axis of a standing wave. From (B.2.13 to B.2.14) it

* For example in (B.2.13a) $\sin \omega t \sin (\omega x/c) \equiv \frac{1}{2} \{ \cos [\omega(t - x/c)] - \cos [\omega(t + x/c)] \}$.

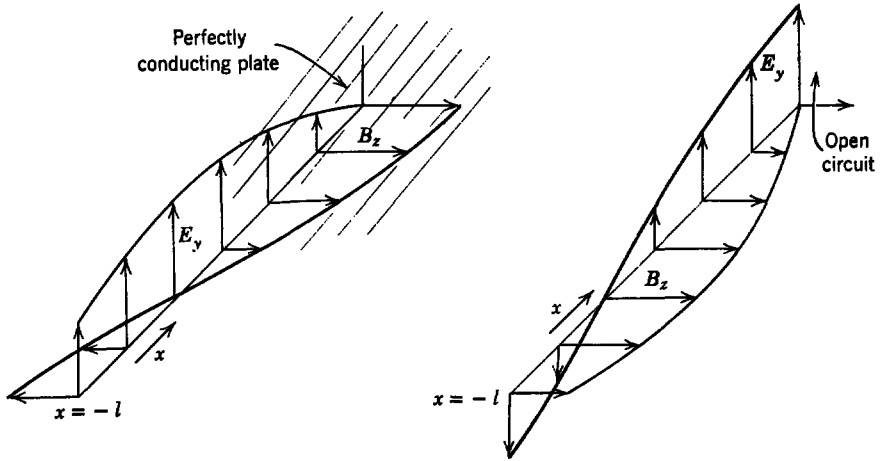


Fig. B.2.3 Amplitude of the electric field intensity and magnetic flux density along the x -axis of the parallel-plate structures shown in Fig. B.2.1 For these plots $\omega l/c = 3\pi/4$.

is clear that as a function of time the electric field reaches its maximum amplitude when $B_x = 0$ and vice versa. Hence the amplitudes of E_y and B_z shown in Fig. B.2.3 are for different instants of time. The fields near $x = 0$ do not in general have the same phase as those excited at $x = -l$. If, however, we can make the approximation that times of interest (which in this case are $1/\omega$) are much longer than the propagation time l/c ,

$$\frac{l/c}{1/\omega} = \frac{\omega l}{c} \ll 1. \quad (\text{B.2.15})$$

The sine functions can then be approximated by their arguments (which are small compared with unity) and the cosine functions are essentially equal to unity. Hence, when (B.2.15) is satisfied, the field distributions (B.2.13) and (B.2.14) become

$$E_y \simeq -\frac{i_0 \sin \omega t}{d \epsilon_0 c} \left(\frac{\omega x}{c} \right), \quad (\text{B.2.16a})$$

$$B_z \simeq -\frac{\mu_0 i_0 \cos \omega t}{d}, \quad (\text{B.2.17a})$$

$$E_y \simeq -\frac{v_0}{s} \cos \omega t, \quad (\text{B.2.16b})$$

$$B_z \simeq -\frac{v_0}{cs} \sin \omega t \left(\frac{\omega x}{c} \right). \quad (\text{B.2.17b})$$

The distribution of field amplitudes in this limit is shown in Fig. B.2.4. The most significant feature of the limiting solutions is that

the magnetic field between the short-circuited plates has the same distribution as if the excitation current were static.

the electric field between the open-circuited plates has the same distribution as if the excitation voltage were constant.

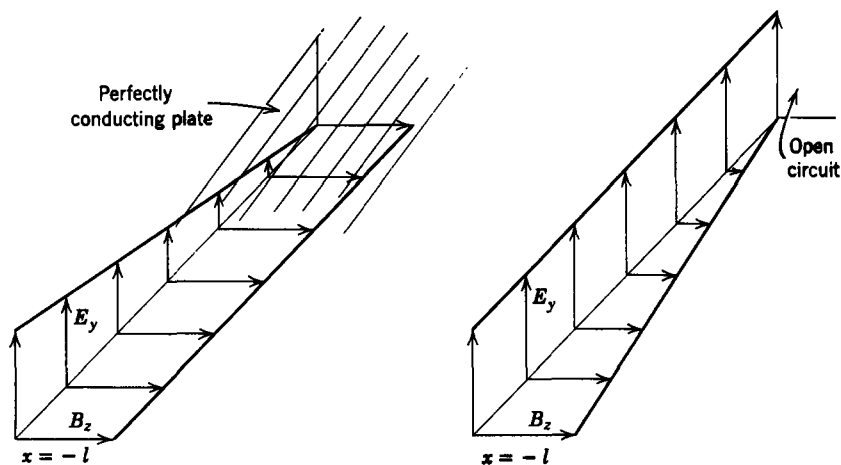


Fig. B.2.4 The distribution of field amplitudes between the parallel plates of Fig. B.2.1 in the limit in which $(\omega l/c) \ll 1$.

Note that the fields as they are excited at $x = -l$ retain the same phase everywhere between the plates. This simply reflects the fact that according to the approximate equations there is no time lag between an excitation at $x = -l$ and the field response elsewhere along the x -axis. It is in this limit that the ideas of circuit theory are applicable, for if we now compute

the voltage $v(t)$ at $x = -l$

$$v(t) = -sE_y(-l, t) \quad (\text{B.2.18a})$$

we obtain the terminal equation for an inductance

$$v = L \frac{d}{dt} (i_o \cos \omega t), \quad (\text{B.2.19a})$$

where the inductance L is

$$L = \frac{s\mu_0}{d}.$$

the current $i(t)$ at $x = -l$

$$i(t) = -B_z(-l, t) \frac{d}{\mu_0} \quad (\text{B.2.18b})$$

we obtain the terminal equation for a capacitance

$$i(t) = C \frac{d}{dt} (v_o \cos \omega t), \quad (\text{B.2.19b})$$

where the capacitance C is

$$C = \frac{\epsilon_0 dl}{s}.$$

A comparison of the examples will be useful for motivating many of the somewhat subtle ideas introduced in the main body of the book. One of the most important points that we can make here is that even though we have solved the same pair of Maxwell's equations (B.2.6) and (B.2.7) for both examples, subject to the same approximation that $\omega l/c \ll 1$ (B.2.15), we have been led to very different physical results. The difference between these

two examples arises from the boundary condition at $x = 0$. In the case of

a short circuit a static excitation leads to a uniform magnetic field but no electric field. The electric field is generated by Faraday's law because the magnetic field is in fact *only quasi-static* and varies slowly with time.

an open circuit a static excitation results in a uniform electric field but no magnetic field. The magnetic field is induced by the displacement current in Ampère's law because the electric field is, in fact, *only quasi-static* and varies slowly with time.

B.2.2 Quasi-Static Electromagnetic Field Equations

As long as we are not interested in phenomena related to the propagation of electromagnetic waves, it is helpful to recognize that most electromechanical situations are in one of two classes, exemplified by the two cases shown in Fig. B.2.1. In the situation in which the plates are short-circuited together (Fig. B.2.1a) the limit $\omega l/c \ll 1$ means that the *displacement current* is of *negligible* importance. A characteristic of this system is that with a static excitation a large current results; hence there is a large static magnetic field. For this reason it exemplifies a *magnetic field system*. By contrast, in the case in which the plates are open-circuited, as shown in Fig. B.2.1b, a static excitation gives rise to a static electric field but no magnetic field. This example exemplifies an *electric field system*, in which the *magnetic induction* of Faraday's law is of *negligible* importance. To emphasize these points consider how we can use these approximations at the outset to obtain the approximate solutions of (B.2.19). Suppose that the excitations in Fig. B.2.1 were static. The fields between the plates are then independent of x and given by

$$E_y = 0, \quad (\text{B.2.20a}) \quad \left| \quad E_y = -\frac{v}{s}, \quad (\text{B.2.20b}) \right.$$

$$B_z = -\frac{\mu_0 i}{d}, \quad (\text{B.2.21a}) \quad \left| \quad B_z = 0. \quad (\text{B.2.21b}) \right.$$

Now suppose that the fields vary slowly with time [the systems are quasi-static in the sense of a condition like (B.2.15)]. Then i and v in these equations are time-varying, hence

B_z is a function of time.

From Faraday's law of induction as expressed by (B.2.7)

$$\frac{\partial E_y}{\partial x} = \frac{\mu_0}{d} \frac{di}{dt}. \quad (\text{B.2.22a})$$

E_y is a function of time.

From Ampère's law, as expressed by (B.2.6)

$$\frac{\partial B_z}{\partial x} = \frac{\mu_0 \epsilon_0}{s} \frac{dv}{dt}. \quad (\text{B.2.22b})$$

Now the right-hand side of each of these equations is independent of x ; hence they can be integrated on x . At the same time, we recognize that

$$E_y(0, t) = 0, \quad (\text{B.2.23a}) \quad \left| \quad B_z(0, t) = 0, \quad (\text{B.2.23b}) \right.$$

so that integration gives

$$E_y = \frac{\mu_0 x}{d} \frac{di}{dt}, \quad (\text{B.2.24a}) \quad \left| \quad B_z = \frac{\mu_0 \epsilon_0 x}{s} \frac{dv}{dt}. \quad (\text{B.2.24b}) \right.$$

Recall how the terminal voltage and current are related to these field quantities (B.2.18) and these equations become

$$v(t) = L \frac{di}{dt}, \quad (\text{B.2.25a}) \quad \left| \quad i(t) = C \frac{dv}{dt}, \quad (\text{B.2.25b}) \right.$$

where again the inductance L and capacitance C are defined as following (B.2.19). Hence making these approximations at the outset has led to the same approximate results as those found in the preceding section by computing the exact solution and taking the limits appropriate to $\omega l/c \ll 1$.

The simple example in Fig. B.2.1 makes it plausible that Maxwell's equations can be written in two quasi-static limits appropriate to the analysis of two major classes of electromechanical interaction:

Magnetic Field Systems	Electric Field Systems
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad (\text{B.2.26a})$	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad (\text{B.2.26b})$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (\text{B.2.27a})$	$\nabla \times \mathbf{E} = 0, \quad (\text{B.2.27b})$
$\nabla \cdot \mathbf{B} = 0, \quad (\text{B.2.28a})$	$\nabla \cdot \epsilon_0 \mathbf{E} = \rho_e, \quad (\text{B.2.28b})$
$\nabla \cdot \mathbf{J} = 0, \quad (\text{B.2.29a})$	$\nabla \cdot \mathbf{J} + \frac{\partial \rho_e}{\partial t} = 0. \quad (\text{B.2.29b})$

Here the displacement current has been omitted from Ampère's law in the magnetic field system, whereas the magnetic induction has been dropped from Faraday's law in the electric field system. Note that if the displacement current is dropped from (B.2.26a) the charge density must be omitted from the conservation of charge equation (B.2.29a) because the latter expression is the divergence of (B.2.26a).

We have not included Gauss's law for the charge density in the magnetic field system or the divergence equation for $\mathbf{B}$ in the electric field system because in the respective situations these expressions are of no interest. In fact, only the divergence of (B.2.26b) is of interest in determining the dynamics of most electric field systems and that is (B.2.29b).

It must be emphasized that the examples of Fig. B.2.1 serve only to motivate the approximations introduced by (B.2.26 to B.2.29). The two systems of equations have a wide range of application. The recognition that a given physical situation can be described as a magnetic field system, as opposed to an electric field system, requires judgment based on experience. A major intent of this book is to establish that kind of experience.

In the cases of Fig. B.2.1 we could establish the accuracy of the approximate equations by calculating effects induced by the omitted terms; for example, in the magnetic field system of Fig. B.2.1a we ignored the displacement current to obtain the quasi-static solution of (B.2.21a) and (B.2.24a). We could now compute the correction B_z^c to the quasi-static magnetic field induced by the displacement current by using (B.2.6), with $\mathbf{E}$ given by (B.2.24a). This produces

$$\frac{\partial B_z^c}{\partial x} = -\frac{\mu_0^2 \epsilon_0 x}{d} \frac{d^2 i}{dt^2}. \quad (\text{B.2.30})$$

Because the right-hand side of this expression is a known function of x , it can be integrated. The constant of integration is evaluated by recognizing that the quasi-static solution satisfies the driving condition at $x = -l$; hence the correction field B_z^c must be zero there and

$$B_z^c = -\frac{\mu_0^2 \epsilon_0 (x^2 - l^2)}{2d} \frac{d^2 i}{dt^2}. \quad (\text{B.2.31})$$

Now, to determine the error incurred in ignoring this field we take the ratio of its largest value (at $x = 0$) to the quasi-static field of (B.2.21a):

$$\frac{|B_z^c|}{|B_z|} = \frac{l^2}{2c^2} \frac{|d^2 i/dt^2|}{|i|}. \quad (\text{B.2.32})$$

If this ratio is small compared with 1, the quasi-static solution is adequate. It is evident that in this case the ratio depends on the time rate of change of the excitation. In Section B.2.1, in which $i = i_0 \cos \omega t$, (B.2.32) becomes

$$\frac{|B_z^c|}{|B_z|} = \frac{1}{2} \left(\frac{\omega l}{c} \right)^2 \ll 1, \quad (\text{B.2.33})$$

which is essentially the same condition given by (B.2.15).

Once the fields have been determined by using either the magnetic field or the electric field representation it is possible to calculate the effects of the omitted terms. This procedure results in a condition characterized by (B.2.33). For this example, if the device were 30 cm long and driven at 1 MHz (this

is an extremely high frequency for anything 30 cm long to respond to electro-mechanically) (B.2.33) becomes

$$\frac{1}{2} \left(\frac{\omega l}{c} \right)^2 = \frac{1}{2} \left(\frac{2 \cdot \pi \cdot 10^6 \cdot 0.3}{3 \times 10^8} \right)^2 = 2\pi^2 \times 10^{-6} \ll 1, \quad (\text{B.2.34})$$

and the quasi-static approximation is extremely good.

It is significant that the magnetic and electric field systems can be thought of in terms of their respective modes of electromagnetic energy storage. In the quasi-static systems the energy that can be attributed to the electromagnetic fields is stored either in the magnetic or electric field. This can be seen by using (B.2.26 to B.2.27) to derive Poynting's theorem for the conservation of electromagnetic energy. If the equations in (B.2.27) are multiplied by $\mathbf{B}/\mu_0$ and subtracted from the equations in (B.2.26) multiplied by $\mathbf{E}/\mu_0$, it follows that

$$\left. \begin{aligned} \frac{\mathbf{E}}{\mu_0} \cdot \nabla \times \mathbf{B} - \frac{\mathbf{B}}{\mu_0} \cdot \nabla \times \mathbf{E} &= \mathbf{E} \cdot \mathbf{J} \\ + \frac{\mathbf{B}}{\mu_0} \cdot \frac{\partial \mathbf{B}}{\partial t} & \end{aligned} \right| \left. \begin{aligned} \frac{\mathbf{E}}{\mu_0} \cdot \nabla \times \mathbf{B} - \frac{\mathbf{B}}{\mu_0} \cdot \nabla \times \mathbf{E} &= \mathbf{E} \cdot \mathbf{J} \\ + \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} & \end{aligned} \right. \quad (\text{B.2.35a}) \quad (\text{B.2.35b})$$

Then, because of a vector identity,* these equations take the form

$$\left. \begin{aligned} -\nabla \cdot \left(\mathbf{E} \times \frac{\mathbf{B}}{\mu_0} \right) &= \mathbf{E} \cdot \mathbf{J} \\ + \frac{\partial}{\partial t} \left(\frac{1}{2} \frac{\mathbf{B} \cdot \mathbf{B}}{\mu_0} \right) & \end{aligned} \right| \left. \begin{aligned} -\nabla \cdot \left(\mathbf{E} \times \frac{\mathbf{B}}{\mu_0} \right) &= \mathbf{E} \cdot \mathbf{J} \\ + \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon_0 \mathbf{E} \cdot \mathbf{E} \right) & \end{aligned} \right. \quad (\text{B.2.36a}) \quad (\text{B.2.36b})$$

Now, if we integrate these equations over a volume V enclosed by a surface S , the divergence theorem (B.1.14) gives

$$-\oint_S \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \cdot \mathbf{n} \, da = \int_V \mathbf{E} \cdot \mathbf{J} \, dV + \frac{\partial}{\partial t} \int_V w \, dV, \quad (\text{B.2.37})$$

where

$$\left. w = \frac{1}{2} \frac{\mathbf{B} \cdot \mathbf{B}}{\mu_0} \right| \left. w = \frac{1}{2} \epsilon_0 \mathbf{E} \cdot \mathbf{E} \right. \quad (\text{B.2.38a}) \quad (\text{B.2.38b})$$

The term on the left in (B.2.37) (including the minus sign) can be interpreted as the flux of energy into the volume V through the surface S . This energy is either dissipated within the volume V , as expressed by the first term on the right, or stored in the volume V , as expressed by the second term. Hence

$$* \nabla \cdot (\mathbf{A} \times \mathbf{C}) = \mathbf{C} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{C}.$$

(w) can be interpreted as an electromagnetic energy density. The electromagnetic energy of the magnetic field system is stored in the magnetic field alone. Similarly, an electric field system is one in which the electromagnetic energy is stored in the electric field.

The familiar elements of electrical circuit theory illustrate the division of interactions into those defined as magnetic field systems and those defined as electric field systems. From the discussion in this and the preceding section it is evident that the short-circuited plates in Fig. B.2.1 constitute an inductor, whereas the open-circuited plates can be represented as a capacitor. This fact is the basis for the development of electromechanical interactions undertaken in Chapter 2. From this specific example it is evident that the magnetic field system includes interactions in which we can define lumped-parameter variables like the inductance, but it is not so evident that this model also describes the magnetohydrodynamic interactions of a fluid and some plasmas with a magnetic field and the magnetoelastic interactions of solids in a magnetic field, even including electromechanical aspects of microwave magnetics.

Similarly, the electric field system includes not only the electromechanics of systems that can be modeled in terms of circuit concepts like the capacitance but ferroelectric interactions between solids and electric fields, the electrohydrodynamics of a variety of liquids and slightly ionized gases in an electric field, and even the most important oscillations of an electron beam. Of course, if we are interested in the propagation of an electromagnetic wave through an ionospheric plasma or through the slightly ionized wake of a space vehicle, the full set of Maxwell's equations must be used.

There are situations in which the propagational aspects of the electromagnetic fields are not of interest, yet neither of the quasi-static systems is appropriate. This is illustrated by short-circuiting the parallel plates of Fig. B.2.1 at $x = 0$ by a resistive sheet. A static current or voltage applied to the plates at $x = -l$ then leads to both electric and magnetic fields between the plates. If the resistance of the sheet is small, the electric field between the plates is also small, and use of the exact field equations would show that we are still justified in ignoring the displacement current. In this case the inductance of Fig. B.2.1a is in series with a resistance. In the opposite extreme, if the resistance of the resistive sheet were very high, we would still be justified in ignoring the magnetic induction of Faraday's law. The situation shown in Fig. B.2.1b would then be modeled by a capacitance shunted by a resistance. The obvious questions are, when do we make a transition from the first case to the second and why is not this intermediate case of more interest in electromechanics?

The purpose of practical electromechanical systems is either the conversion of an electromagnetic excitation into a force that can perform work on a

mechanical system or the reciprocal generation of electromagnetic energy from a force of mechanical origin. From (B.1.10) and (B.1.40) there are two fundamental types of electromagnetic force. Suppose that we are interested in producing a force of electrical origin on the upper of the two plates in Fig. B.2.1. We have the option of imposing a large current to interact with its induced magnetic field or of using a large potential to create an electric field that would interact with induced charges on the upper plate. Clearly, we are not going to impose a large potential on the plates if they are terminated in a small resistance or attempt to drive a large current through the plates with an essentially open circuit at $x = 0$. The electrical dissipation in both cases would be prohibitively large. More likely, if we intended to use the force $\mathbf{J} \times \mathbf{B}$, we would make the resistance as small as possible to minimize the dissipation of electric power and approach the case of Fig. B.2.1a. The essentially open circuit shown in Fig. B.2.1b would make it possible to use a large potential to create a significant force of the type $\rho_e \mathbf{E}$ without undue power dissipation. In the intermediate case the terminating resistance could be adjusted to make the electric and magnetic forces about equal. As a practical matter, however, the resulting device would probably melt before it served any useful electromechanical function. The power dissipated in the termination resistance would be a significant fraction of any electric power converted to mechanical form.*

The energy densities of (B.2.38) provide one means of determining when the problem shown in Fig. B.2.1 (but with a resistive sheet terminating the plates at $x = 0$) is intermediate between a magnetic and an electric field system. In the intermediate case the energy densities are equal

$$\frac{1}{2} \epsilon_0 \mathbf{E} \cdot \mathbf{E} = \frac{1}{2} \frac{\mathbf{B} \cdot \mathbf{B}}{\mu_0}. \quad (\text{B.2.39})$$

Now, if the resistive sheet has a total resistance of R , then from (B.2.18a) applied at $x = 0$

$$E_y s = -iR. \quad (\text{B.2.40})$$

The current can be evaluated in terms of the magnetic field at $x = 0$ by using (B.2.18b):

$$E_y s = B_z \frac{dR}{\mu_0}. \quad (\text{B.2.41})$$

Substitution of the electric field, as found from this expression into (B.2.39), gives

$$\frac{\epsilon_0}{2} B_z^2 \left(\frac{Rd}{s\mu_0} \right)^2 = \frac{1}{2} \frac{B_z^2}{\mu_0}. \quad (\text{B.2.42})$$

* It is interesting that for this particular intermediate case the electric force tends to pull the plates together, whereas the magnetic force tends to push them apart. Hence, because the two forces are equal in magnitude, they just cancel.

Hence, if the energy densities are equal, we obtain the following relation among the physical parameters of the system:

$$\frac{dR}{s} = \left(\frac{\mu_0}{\epsilon_0} \right)^{1/2}. \quad (\text{B.2.43})$$

It would be a digression to pursue this point here, but (B.2.43) is the condition that must be satisfied if an electromagnetic wave launched between the plates at $x = -l$ is to be absorbed, without reflection, by the resistive sheet*; that is, the intermediate case is one in which all the power fed into the system, regardless of the frequency or time constant, is dissipated by the resistive sheet.

B.3 MACROSCOPIC MODELS AND CONSTITUENT RELATIONS

When solids, liquids, and gases are placed in electromagnetic fields, they influence the field distribution. This is another way of saying that the force of interaction between charges or between currents is influenced by the presence of media. The effect is not surprising because the materials are comprised of charged particles.

Problems of physical significance can usually be decomposed into parts with widely differing scales. At the molecular or submolecular level we may be concerned with the dynamics of individual charges or of the atoms or molecules to which they are attached. These systems tend to have extremely small dimensions when compared with the size of a physical device. On the macroscopic scale we are not interested in the detailed behavior of the microscopic constituents of a material but rather only a knowledge of the average behavior of variables, since only these averages are observable on a macroscopic scale. The charge and current densities introduced in Section B.1 are examples of such variables, hence it is a macroscopic picture of fields and media that we require here.

There are three major ways in which media influence macroscopic electromagnetic fields. Hence the following sections undertake a review of magnetization, polarization, and conduction in common materials.

B.3.1 Magnetization

The macroscopic motions of electrons, even though associated with individual atoms or molecules, account for aggregates of charge and current

* The propagation of an electromagnetic wave on structures of this type is discussed in texts concerned with transmission lines or TEM wave guide modes. For a discussion of this matching problem see R. B. Adler, L. J. Chu, and R. M. Fano, *Electromagnetic Energy Transmission and Radiation*, Wiley, New York, 1960, p. 111, or S. Ramo, J. R. Whinnery, and T. Van Duzer, *Fields and Waves in Communication Electronics*, Wiley, New York, p. 27.

(when viewed at the macroscopic level) that induce electric and magnetic fields. These field sources are not directly accessible; for example, the equivalent currents within the material cannot be circulated through an external circuit. The most obvious sources of magnetic field that are inaccessible in this sense are those responsible for the field of a permanent magnet. The earliest observations on magnetic fields involved the lodestone, a primitive form of the permanent magnet. Early investigators such as Oersted found that magnetic fields produced by a permanent magnet are equivalent to those induced by a circulating current. In the formulation of electromagnetic theory we must distinguish between fields due to sources within the material and those from applied currents simply because it is only the latter sources that can be controlled directly. Hence we divide the source currents into *free currents* (with the density $\mathbf{J}_f$) and *magnetization currents* (with the density $\mathbf{J}_m$). Ampère's law then takes the form

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} \right) = \mathbf{J}_m + \mathbf{J}_f. \quad (\text{B.3.1})$$

By convention it is also helpful to attribute a fraction of the field induced by these currents to the magnetization currents in the material. Hence (B.3.1) is written as

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right) = \mathbf{J}_f, \quad (\text{B.3.2})$$

where the *magnetization density* $\mathbf{M}$ is defined by

$$\nabla \times \mathbf{M} = \mathbf{J}_m. \quad (\text{B.3.3})$$

Up to this point in this chapter it has been necessary to introduce only two field quantities to account for interactions between charges and between currents. To account for the macroscopic properties of media we have now introduced a new field quantity, the magnetization density $\mathbf{M}$, and in the next section similar considerations concerning electric polarization of media lead to the introduction of the polarization density $\mathbf{P}$. It is therefore apparent that macroscopic field theory is formulated in terms of four field variables. In our discussion these variables have been $\mathbf{E}$, $\mathbf{B}$, $\mathbf{M}$, and $\mathbf{P}$. An alternative representation of the fields introduces the *magnetic field intensity* $\mathbf{H}$, in our development *defined* as

$$\mathbf{H} = \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right). \quad (\text{B.3.4})$$

From our definition it is clear that we could just as well deal with $\mathbf{B}$ and $\mathbf{H}$ as the macroscopic magnetic field vectors rather than with $\mathbf{B}$ and $\mathbf{M}$. This is

particularly appealing, for then (B.3.2) takes the simple form

$$\nabla \times \mathbf{H} = \mathbf{J}_f. \quad (\text{B.3.5})$$

When the source quantities $\mathbf{J}_f$ and $\mathbf{M}$ are specified independently, the magnetic field intensity $\mathbf{H}$ (or magnetic flux density $\mathbf{B}$) can be found from the quasi-static magnetic field equations. A given constant magnetization density corresponds to the case of the permanent magnet. In most cases, however, the source quantities are functions of the field vectors, and these functional relations, called *constituent relations*, must be known before the problems can be solved. The constituent relations represent the constraints placed on the fields by the internal physics of the media being considered. Hence it is these relations that make it possible to separate the microscopic problem from the macroscopic one of interest here.

The simplest form of constituent relation for a magnetic material arises when it can be considered *electrically linear* and *isotropic*. Then the *permeability* μ is constant in the relation

$$\mathbf{B} = \mu \mathbf{H}. \quad (\text{B.3.6})$$

The material is isotropic because $\mathbf{B}$ is collinear with $\mathbf{H}$ and a particular constant (μ) times $\mathbf{H}$, regardless of the direction of $\mathbf{H}$. A material that is *homogeneous* and isotropic will in addition have a permeability μ that does not vary with position in the material. Another way of expressing (B.3.6) is to define a magnetic susceptibility χ_m (dimensionless) such that

$$\mathbf{M} = \chi_m \mathbf{H}, \quad (\text{B.3.7})$$

where

$$\mu = \mu_0(1 + \chi_m). \quad (\text{B.3.8})$$

Magnetic materials are commonly found with $\mathbf{B}$ not a linear function of $\mathbf{H}$ and the constitutive law takes the general form

$$\mathbf{B} = \mathbf{B}(\mathbf{H}). \quad (\text{B.3.9})$$

We deal with some problems involving materials of this type, but with few exceptions confine our examples to situations in which $\mathbf{B}$ is a single-valued function of $\mathbf{H}$. In certain magnetic materials in some applications the $\mathbf{B}$ - $\mathbf{H}$ curve must include hysteresis and (B.3.9) is not single-valued.*

The differential equations for a magnetic field system in the presence of moving magnetized media are summarized in Table 1.2.

B.3.2 Polarization

The force between a charge distribution and a test charge is observed to change if a dielectric material is brought near the region occupied by the test

* G. R. Slemon, *Magnetoelectric Devices*, Wiley, New York, 1966, p. 115.

charge. Like the test charge, the charged particles which compose the dielectric material experience forces due to the applied field. Although these charges remain identified with the molecules of the material, their positions can be distorted incrementally by the electric force and thus lead to a polarization of the molecules.

The basic sources of the electric field are charges. Hence it is natural to define a *polarization charge density* ρ_p as a source of a fraction of the electric field which can be attributed to the inaccessible sources within the media. Thus Gauss's law (B.1.16) is written

$$\nabla \cdot \epsilon_0 \mathbf{E} = \rho_f + \rho_p, \quad (\text{B.3.10})$$

where the *free charge density* ρ_f resides on conducting electrodes and other parts of the system capable of supporting conduction currents. The free charges do not remain attached to individual molecules but rather can be conducted from one point to another in the system.

In view of the form taken by Gauss's law, it is convenient to identify a field induced by the polarization charges by writing (B.3.10) as

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f, \quad (\text{B.3.11})$$

where the *polarization density* $\mathbf{P}$ is related to the polarization charge density by

$$\rho_p = -\nabla \cdot \mathbf{P}. \quad (\text{B.3.12})$$

As in Section B.3.1, it is convenient to define a new vector field that serves as an alternative to $\mathbf{P}$ in formulating the electrodynamics of polarized media. This is the *electric displacement* $\mathbf{D}$, defined as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad (\text{B.3.13})$$

In terms of this field, Gauss's law for electric fields (B.3.11) becomes

$$\nabla \cdot \mathbf{D} = \rho_f. \quad (\text{B.3.14})$$

The simple form of this expression makes it desirable to use $\mathbf{D}$ rather than $\mathbf{P}$ in the formulation of problems.

If a polarization charge model is to be used to account for the effects of polarizable media on electric fields, we must recognize that the motion of these charges can lead to a current. In fact, now that two classes of charge density have been identified we must distinguish between two classes of current density. The free current density $\mathbf{J}_f$ accounts for the conservation of free charge so that (B.1.18) can be written as

$$\nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t} = 0. \quad (\text{B.3.15})$$

In view of (B.3.11), this expression becomes

$$\nabla \cdot \mathbf{J}_f + \frac{\partial}{\partial t} \nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = 0. \quad (\text{B.3.16})$$

Now, if we write Ampère's law (B.2.26*b*) as

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} \right) = \mathbf{J}_f + \mathbf{J}_p + \frac{\partial}{\partial t} \epsilon_0 \mathbf{E}, \quad (\text{B.3.17})$$

where $\mathbf{J}_p$ is a current density due to the motion of polarization charges, the divergence of (B.3.17) must give (B.3.16). Therefore

$$\nabla \cdot \mathbf{J}_p + \frac{\partial}{\partial t} (-\nabla \cdot \mathbf{P}) = 0. \quad (\text{B.3.18})$$

which from (B.3.12) is an expression for the conservation of polarization charge. This expression does not fully determine the polarization current density $\mathbf{J}_p$, because in general we could write

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{A}, \quad (\text{B.3.19})$$

where $\mathbf{A}$ is an arbitrary vector, and still satisfy (B.3.18). At this point we could derive the quantity $\mathbf{A}$ (which would turn out to be $\mathbf{P} \times \mathbf{v}$, where $\mathbf{v}$ is the velocity of the polarized medium). It is important, however, to recognize that this represents an unnecessary digression. In the electric field system the magnetic field appears in only one of the equations of motion—Ampère's law. It does not appear in (B.2.27*b*) to (B.2.29*b*), nor will it appear in any constitutive law used in this book. For this reason the magnetic field serves simply as a quantity to be calculated once the electromechanical problem has been solved. We might just as well lump the quantity $\mathbf{A}$ with the magnetic field in writing Ampère's law. In fact, if we are consistent, the magnetic field intensity $\mathbf{H}$ can be defined as given by

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}, \quad (\text{B.3.20})$$

with no loss of physical significance. In an electric field system the magnetic field is an alternative representation of the current density $\mathbf{J}_f$. A review of the quasi-static solutions for the system in Fig. B.2.1*b* illustrates this point.

In some materials (ferroelectrics) the polarization density $\mathbf{P}$ is constant. In most common dielectrics, however, the polarization density is a function of $\mathbf{E}$. The simplest constituent relation for a dielectric is that of linear and isotropic material,

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad (\text{B.3.21})$$

where χ_e is the *dielectric susceptibility* (dimensionless) that may be a function of space but not of $\mathbf{E}$. For such a material we define the *permittivity* ϵ as

$$\epsilon = \epsilon_0(1 + \chi_e). \quad (\text{B.3.22})$$

and then write the relation between $\mathbf{D}$ and $\mathbf{E}$ as [see (B.3.13)]

$$\mathbf{D} = \epsilon \mathbf{E}. \quad (\text{B.3.23})$$

This mathematical model of polarizable material is used extensively in this book.

The differential equations for the electric field system, in the presence of moving polarized media, are summarized in Table 1.2.

B.3.3 Electrical Conduction

In both magnetic and electric field systems the conduction process accounts for the free current density $\mathbf{J}_f$ in a fixed conductor. The most common model for this process is appropriate in the case of an isotropic, linear, conducting medium which, when stationary, has the constituent relation (often called *Ohm's law*)

$$\mathbf{J}_f = \sigma \mathbf{E}. \quad (\text{B.3.24})$$

Although (B.3.24) is the most widely used mathematical model of the conduction process, there are important electromechanical systems for which it is not adequate. This becomes apparent if we attempt to derive (B.3.24), an exercise that will contribute to our physical understanding of Ohm's law.

In many materials the conduction process involves two types of charge carrier (say, ions and electrons). As discussed in Section B.1.2, a macroscopic model for this case would recognize the existence of free charge densities ρ_+ and ρ_- with charge average velocities $\mathbf{v}_+$ and $\mathbf{v}_-$, respectively. Then

$$\mathbf{J}_f = \rho_+ \mathbf{v}_+ + \rho_- \mathbf{v}_-. \quad (\text{B.3.25})$$

The problem of relating the free current density to the electric field intensity is thus a problem in electromechanics in which the velocities of the particles carrying the free charge must be related to the electric fields that apply forces to the charges.

The charge carriers have finite mass and thus accelerate when subjected to a force. In this case there are forces on the positive and negative carriers, respectively, given by (B.1.10) (here we assume that effects from a magnetic field are ignorable):

$$\mathbf{F}_+ = \rho_+ \mathbf{E}, \quad (\text{B.3.26})$$

$$\mathbf{F}_- = \rho_- \mathbf{E}. \quad (\text{B.3.27})$$

As the charge carriers move, their motion is retarded by collisions with other particles. On a macroscopic basis the retarding force of collisions can be thought of as a viscous damping force that is proportional to velocity. Hence we can picture the conduction process in two extremes. With no collisions between particles the electric force densities of (B.3.26 and B.3.27) continually accelerate the charges, for the only retarding forces are due to acceleration expressed by Newton's law. In the opposite extreme a charge carrier suffers collisions with other particles so frequently that its average velocity quickly reaches a limiting value, which in view of (B.3.26 and B.3.27) is proportional to the applied electric field. It is in this second limiting case that Ohm's law assumes physical significance. By convention *mobilities* μ_+ and μ_- which relate these limiting velocities to the field $\mathbf{E}$ are defined

$$\mathbf{v}_+ = \mu_+ \mathbf{E}, \quad (\text{B.3.28})$$

$$\mathbf{v}_- = \mu_- \mathbf{E}. \quad (\text{B.3.29})$$

In terms of these quantities, (B.3.25) becomes

$$\mathbf{J}_f = (\rho_+ \mu_+ + \rho_- \mu_-) \mathbf{E}. \quad (\text{B.3.30})$$

It is important to recognize that it is only when the collisions between carriers and other particles dominate the accelerating effect of the electric field that the conduction current takes on a form in which it is dependent on the instantaneous value of $\mathbf{E}$. Fortunately, (B.3.30) is valid in a wide range of physical situations. In fact, in a metallic conductor the number of charge carriers is extremely high and very nearly independent of the applied electric field. The current carriers in most metals are the electrons, which are detached from atoms held in the lattice structure of the solid. Therefore the negatively charged electrons move in a background field of positive charge and, to a good approximation, $\rho_+ = -\rho_-$. Then (B.3.30) becomes

$$\mathbf{J} = \sigma \mathbf{E}, \quad (\text{B.3.31})$$

where the conductivity is defined as

$$\sigma = \rho_+ (\mu_+ - \mu_-). \quad (\text{B.3.32})$$

The usefulness of the conductivity as a parameter stems from the fact that both the number of charges available for conduction and the net mobility (essentially that of the electrons) are constant. This makes the conductivity essentially independent of the electric field, as assumed in (B.3.24).*

* We assume here that the temperature remains constant. A worthwhile qualitative description of conduction processes in solids is given in J. M. Ham and G. R. Slemon, *Scientific Basis of Electrical Engineering*, Wiley, New York, 1961, p. 453.

In some types of material (notably slightly ionized gases) which behave like insulators, the conduction process cannot be described simply by Ohm's law. In such materials the densities of charge carriers and even the mobilities may vary strongly with electric field intensity.

B.4 INTEGRAL LAWS

The extensive use of circuit theory bears testimony to the usefulness of the laws of electricity and magnetism in integral form. Physical situations that would be far too involved to describe profitably in terms of field theory have a lucid and convenient representation in terms of circuits. Conventional circuit elements are deduced from the integral forms of the field equations. The description of lumped-parameter electromechanical systems, as undertaken in Chapter 2, requires that we generalize the integral laws to include time-varying surfaces and contours of integration. Hence it is natural that we conclude this appendix with a discussion of the integral laws.

B.4.1 Magnetic Field Systems

Faraday's law of induction, as given by (B.1.42), has the differential form

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}. \quad (\text{B.4.1})$$

This expression can be integrated over a surface S enclosed by the contour C . Then, according to Stokes's theorem,

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \, da. \quad (\text{B.4.2})$$

Now, if S and C are fixed in space, the time derivative on the right can be taken either before or after the surface integral of $\mathbf{B} \cdot \mathbf{n}$ is evaluated. Note that $\int_S \mathbf{B} \cdot \mathbf{n} \, da$ is only a function of time. For this reason (B.1.41) could be written with the total derivative outside the surface integral. It is implied in the integral equation (B.1.41) that S is fixed in space.

Figure B.4.1 shows an example in which it is desirable to be able to use (B.4.2), with S and C varying in position as a function of time. The contour C is a rectangular loop that encloses a surface S which makes an angle $\theta(t)$ with the horizontal. Although the induction law is not limited to this case, the loop could comprise a one-turn coil, in which case it is desirable to be able to use (B.4.2) with C fixed to the coil. The integral law of induction would be much more useful if it could be written in the form

$$\oint_C \mathbf{E}' \cdot d\mathbf{l} = - \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da. \quad (\text{B.4.3})$$

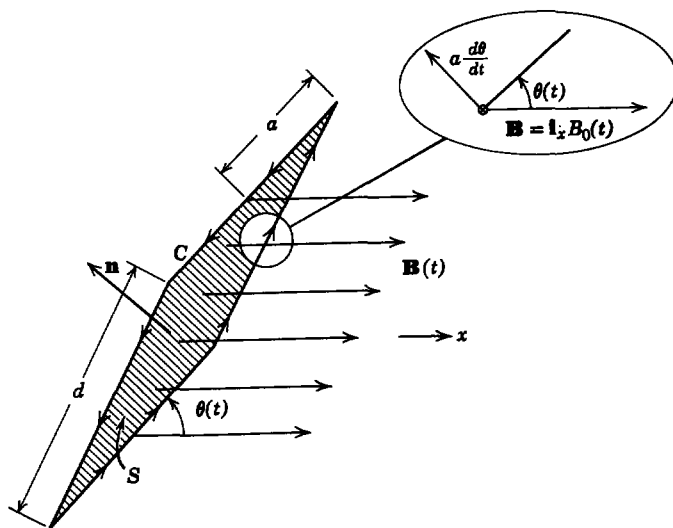


Fig. B.4.1 Contour C enclosing a surface S which varies as a function of time. The rectangular loop links no magnetic flux when $\theta = 0, \pi, \dots$

In this form the quantity on the right is the negative time rate of change of the flux linked by the contour C , whereas $\mathbf{E}'$ is interpreted as the electric field measured in the moving frame of the loop. An integral law of induction in the form of (B.4.3) is essential to the lumped-parameter description of magnetic field systems. At this point we have two choices. We can accept (B.4.3) as an empirical result of Faraday's original investigations or we can show mathematically that (B.4.2) takes the form of (B.4.3) if

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \quad (\text{B.4.4})$$

where $\mathbf{v}$ is the velocity of $d\mathbf{l}$ at the point of integration. In any case this topic is pursued in Chapter 6 to clarify the significance of electric and magnetic fields measured in different frames of reference.

The mathematical connection between (B.4.2) and (B.4.3) is made by using the integral theorem

$$\frac{d}{dt} \int_S \mathbf{A} \cdot \mathbf{n} \, da = \int_S \left[\frac{\partial \mathbf{A}}{\partial t} + (\nabla \cdot \mathbf{A}) \mathbf{v} \right] \cdot \mathbf{n} \, da + \oint_C (\mathbf{A} \times \mathbf{v}) \cdot d\mathbf{l}, \quad (\text{B.4.5})$$

where $\mathbf{v}$ is the velocity of S and C and in the case of (B.4.3), $\mathbf{A} \rightarrow \mathbf{B}$. Before we embark on a proof of this theorem, an example will clarify its significance.

Example B.4.1. The coil shown in Fig. B.4.1 rotates with the angular deflection $\theta(t)$ in a uniform magnetic flux density $\mathbf{B}(t)$, directed as shown. We wish to compute the rate of change of the flux linked by the coil in two ways: first by computing $\int_S \mathbf{B} \cdot \mathbf{n} \, da$ and taking

its derivative [the left-hand side of (B.4.5)], then by using the surface and contour integrations indicated on the right-hand side of (B.4.5). This illustrates how the identity allows us to carry out the surface integration before rather than after the time derivative is taken. From Fig. B.4.1 we observe that

$$\int_S \mathbf{B} \cdot \mathbf{n} \, da = -B_0(t)2ad \sin \theta, \quad (\text{a})$$

so that the first calculation gives

$$\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da = -2ad \sin \theta \frac{dB_0}{dt} - B_0 2ad \cos \theta \frac{d\theta}{dt}. \quad (\text{b})$$

To evaluate the right-hand side of (B.4.5) observe that $\nabla \cdot \mathbf{B} = 0$ and [from (a)]

$$\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \, da = -2ad \sin \theta \frac{dB_0}{dt}. \quad (\text{c})$$

The quantity $\mathbf{B} \times \mathbf{v}$ is collinear with the axis of rotation in Fig. B.4.1; hence there is no contribution to the line integral along the pivoted ends of the loop. Because both the velocity $\mathbf{v} = \mathbf{i}_\theta a (d\theta/dt)$ and line elements $d\mathbf{l}$ are reversed in going from the upper to the lower horizontal contours, the line integral reduces to twice the value from the upper contour.

$$\oint_C \mathbf{B} \times \mathbf{v} \cdot d\mathbf{l} = -2B_0 a d \cos \theta \frac{d\theta}{dt} \quad (\text{d})$$

From (c) and (d) it follows that the right-hand side of (B.4.5) also gives (b). Thus, at least for this example, (B.4.5) provides alternative ways of evaluating the time rate of change of the flux linked by the contour C .

The integral theorem of (B.4.5) can be derived by considering the deforming surface S shown at two instants of time in Fig. B.4.2. In the incremental time interval Δt the surface S moves from S_1 to S_2 , and therefore by

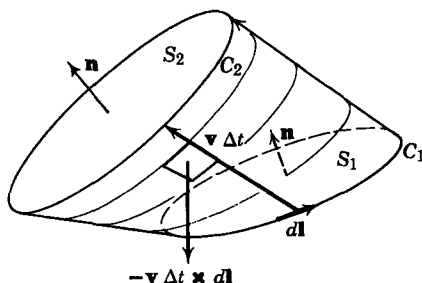


Fig. B.4.2 When $t = t$, the surface S enclosed by the contour C is as indicated by S_1 and C_1 . By the time $t = t + \Delta t$ this surface has moved to S_2 , where it is enclosed by the contour C_2 .

definition

$$\frac{d}{dt} \int_S \mathbf{A} \cdot \mathbf{n} \, da = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left(\int_{S_2} \mathbf{A} \Big|_{t+\Delta t} \cdot \mathbf{n} \, da - \int_{S_1} \mathbf{A} \Big|_t \cdot \mathbf{n} \, da \right). \quad (\text{B.4.6})$$

Here we have been careful to show that when the integral on S_2 is evaluated $t = t + \Delta t$, in contrast to the integration on S_1 , which is carried out when $t = t$.

The expression on the right in (B.4.6) can be evaluated at a given instant in time by using the divergence theorem (B.1.14) to write

$$\int_V \nabla \cdot \mathbf{A} \, dV \cong \int_{S_2} \mathbf{A} \Big|_{t+\Delta t} \cdot \mathbf{n} \, da - \int_{S_1} \mathbf{A} \Big|_t \cdot \mathbf{n} \, da - \Delta t \oint_{C_1} \mathbf{A} \cdot \mathbf{v} \times d\mathbf{l} \quad (\text{B.4.7})$$

for the volume V traced out by the surface S in the time Δt . Here we have used the fact that $-\mathbf{v} \, \Delta t \times d\mathbf{l}$ is equivalent to a surface element $\mathbf{n} \, da$ on the surface traced out by the contour C in going from C_1 to C_2 in Fig. B.4.2. To use (B.4.7) we make three observations. First, as $\Delta t \rightarrow 0$,

$$\int_{S_2} \mathbf{A} \Big|_{t+\Delta t} \cdot \mathbf{n} \, da \cong \int_{S_2} \mathbf{A} \Big|_t \cdot \mathbf{n} \, da + \int_{S_1} \frac{\partial \mathbf{A}}{\partial t} \Big|_t \cdot \Delta t \cdot \mathbf{n} \, da + \cdots \quad (\text{B.4.8})$$

Second, it is a vector identity that

$$\mathbf{A} \cdot \mathbf{v} \times d\mathbf{l} = \mathbf{A} \times \mathbf{v} \cdot d\mathbf{l}. \quad (\text{B.4.9})$$

Third, an incremental volume dV swept out by the surface da is essentially the base times the perpendicular height or

$$dV = \Delta t \mathbf{v} \cdot \mathbf{n} \, da. \quad (\text{B.4.10})$$

From these observations (B.4.7) becomes

$$\begin{aligned} \Delta t \int_{S_1} (\nabla \cdot \mathbf{A}) \mathbf{v} \cdot \mathbf{n} \, da &\cong \int_{S_2} \mathbf{A} \Big|_{t+\Delta t} \cdot \mathbf{n} \, da - \int_{S_1} \Delta t \frac{\partial \mathbf{A}}{\partial t} \Big|_t \cdot \mathbf{n} \, da \\ &\quad - \int_{S_1} \mathbf{A} \Big|_t \cdot \mathbf{n} \, da - \Delta t \oint_{C_1} \mathbf{A} \times \mathbf{v} \cdot d\mathbf{l}. \end{aligned} \quad (\text{B.4.11})$$

This expression can be solved for the quantity on the right in (B.4.6) to give

$$\frac{d}{dt} \int_S \mathbf{A} \cdot \mathbf{n} \, da = \lim_{\Delta t \rightarrow 0} \left\{ \int_{S_1} \left[(\nabla \cdot \mathbf{A}) \mathbf{v} + \frac{\partial \mathbf{A}}{\partial t} \right] \cdot \mathbf{n} \, da + \oint_{C_1} \mathbf{A} \times \mathbf{v} \cdot d\mathbf{l} \right\}. \quad (\text{B.4.12})$$

The limit of this expression produces the required relation (B.4.5).

Use of (B.4.5) to express the right-hand side of (B.4.2) results in

$$\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \, da = \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da - \int_S (\nabla \cdot \mathbf{B}) \mathbf{v} \cdot \mathbf{n} \, da - \oint_C (\mathbf{B} \times \mathbf{v}) \cdot d\mathbf{l}. \quad (\text{B.4.13})$$

Because $\nabla \cdot \mathbf{B} = 0$, (B.4.2) then reduces to (B.4.3), with $\mathbf{E}'$ given by (B.4.4).

The integral laws for the magnetic field system are summarized in Table 1.2 at the end of Chapter 1. In these equations surfaces and contours of integration can, in general, be time-varying.

B.4.2 Electric Field System

Although the integral form of Faraday's law can be taken as an empirical fact, we require (B.4.5) to write Ampère's law in integral form for an electric field system. If we integrate (B.3.20) over a surface S enclosed by a contour C , by Stokes's theorem it becomes

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot \mathbf{n} \, da + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{n} \, da. \quad (\text{B.4.14})$$

As with the induction law for the magnetic field system, this expression can be generalized to include the possibility of a deforming surface S by using (B.4.13) with $\mathbf{B} \rightarrow \mathbf{D}$ to rewrite the last term. If, in addition, we use (B.3.14) to replace $\nabla \cdot \mathbf{D}$ with ρ_f , (B.4.14) becomes

$$\oint_C \mathbf{H}' \cdot d\mathbf{l} = \int_S \mathbf{J}'_f \cdot \mathbf{n} \, da + \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{n} \, da, \quad (\text{B.4.15})$$

where

$$\mathbf{H}' = \mathbf{H} - \mathbf{v} \times \mathbf{D}, \quad (\text{B.4.16})$$

$$\mathbf{J}'_f = \mathbf{J}_f - \rho_f \mathbf{v}. \quad (\text{B.4.17})$$

The fields $\mathbf{H}'$ and $\mathbf{J}'_f$ can be interpreted as the magnetic field intensity and free current density measured in the moving frame of the deforming contour. The significance of these field transformations is discussed in Chapter 6. Certainly the relationship between $\mathbf{J}'_f$ (the current density in a frame moving with a velocity $\mathbf{v}$) and the current density $\mathbf{J}_f$ (measured in a fixed frame), as given by (B.4.17), is physically reasonable. The free charge density appears as a current in the negative $\mathbf{v}$ -direction when viewed from a frame moving at the velocity $\mathbf{v}$. It was reasoning of this kind that led to (B.1.25).

As we have emphasized, it is the divergence of Ampère's differential law that assumes the greatest importance in electric field systems, for it accounts for conservation of charge. The integral form of the conservation of charge

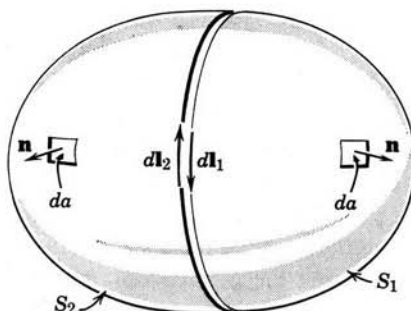


Fig. B.4.3 The sum of two surfaces S_1 and S_2 "spliced" together at the contour to enclose the volume V .

equation, including the possibility of a deforming surface of integration, is obtained by using (B.4.15). For this purpose integrations are considered over two deforming surfaces, S_1 and S_2 , as shown in Fig. B.4.3. These surfaces are chosen so that they are enclosed by the same contour C . Hence, taken together, S_1 and S_2 enclose a volume V .

Integration of (B.4.15) over each surface gives

$$\oint_C \mathbf{H}' \cdot d\mathbf{l}_1 = \int_{S_1} \mathbf{J}' \cdot \mathbf{n} da + \frac{d}{dt} \int_{S_1} \mathbf{D} \cdot \mathbf{n} da. \quad (\text{B.4.18})$$

$$\oint_C \mathbf{H}' \cdot d\mathbf{l}_2 = \int_{S_2} \mathbf{J}' \cdot \mathbf{n} da + \frac{d}{dt} \int_{S_2} \mathbf{D} \cdot \mathbf{n} da. \quad (\text{B.4.19})$$

Now, if $\mathbf{n}$ is defined so that it is directed out of the volume V on each surface, the line integral enclosing S_1 will be the negative of that enclosing S_2 . Then the sum of (B.4.18 and B.4.19) gives the desired integral form of the conservation of charge equation:

$$\oint_S \mathbf{J}' \cdot \mathbf{n} da + \frac{d}{dt} \int_V \rho_f dV = 0. \quad (\text{B.4.20})$$

In writing this expression we have used Gauss's theorem and (B.3.14) to show the explicit dependence of the current density through the deforming surface on the enclosed charge density.

The integral laws for electric field systems are summarized in Table 1.2 at the end of Chapter 1.

B.5 RECOMMENDED READING

The following texts treat the subject of electrodynamics and provide a comprehensive development of the fundamental laws of electricity and magnetism.

R. M. Fano, L. J. Chu, and R. B. Adler, *Electromagnetic Fields, Energy, and Forces*, Wiley, New York, 1960; J. D. Jackson, *Classical Electrodynamics*, Wiley, New York, 1962; S. Ramo, J. R. Whinnery, and T. Van Duzer, *Fields and Waves in Communication Electronics*, Wiley, New York, 1965; W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, Addison-Wesley, Reading, Mass., 1956; J. A. Stratton, *Electromagnetic Theory*, McGraw-Hill, New York, 1941.

Many questions arise in the study of the effects of moving media on electric and magnetic fields concerning the macroscopic representation of polarized and magnetized media; for example, in this appendix we introduced the fields $\mathbf{E}$ and $\mathbf{B}$ as the quantities defined by the force law. Then $\mathbf{P}$ and $\mathbf{M}$ (or $\mathbf{D}$ and $\mathbf{H}$) were introduced to account for the effects of polarization and magnetization. Hence the effect of the medium was accounted for by equivalent polarization charges ρ_p and magnetization currents $\mathbf{J}_m$. Other representations can be used in which a different pair of fundamental vectors is taken, as defined by the force law (say, $\mathbf{E}$ and $\mathbf{H}$), and in which the effects of media are accounted for by an equivalent magnetic charge instead of an equivalent current. If we are consistent in using the alternative formulations of the field equations, they predict the same physical results, including the force on magnetized and polarized media. For a complete discussion of these matters see P. Penfield, and H. Haus, *Electrodynamics of Moving Media*, M.I.T. Press, Cambridge, Mass., 1967.

Appendix C

SUMMARY OF PARTS I AND II AND USEFUL THEOREMS

IDENTITIES

$$\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{B} \times \mathbf{C},$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi,$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B},$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B},$$

$$\nabla(\phi\psi) = \phi \nabla\psi + \psi \nabla\phi,$$

$$\nabla \cdot (\psi\mathbf{A}) = \mathbf{A} \cdot \nabla\psi + \psi \nabla \cdot \mathbf{A},$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B},$$

$$\nabla \cdot \nabla\phi = \nabla^2\phi,$$

$$\nabla \cdot \nabla \times \mathbf{A} = 0,$$

$$\nabla \times \nabla\phi = 0,$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A},$$

$$(\nabla \times \mathbf{A}) \times \mathbf{A} = (\mathbf{A} \cdot \nabla)\mathbf{A} - \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A}),$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

$$\nabla \times (\phi\mathbf{A}) = \nabla\phi \times \mathbf{A} + \phi \nabla \times \mathbf{A},$$

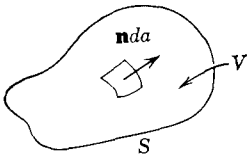
$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}.$$

THEOREMS

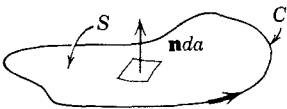
$$\int_a^b \nabla \phi \cdot d\mathbf{l} = \phi_b - \phi_a.$$



Divergence theorem $\oint_S \mathbf{A} \cdot \mathbf{n} \, da = \int_V \nabla \cdot \mathbf{A} \, dV$



Stokes's theorem $\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} \, da$



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ELECTROMECHANICAL DYNAMICS

Part II: Fields, Forces, and Motion

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To our parents

PREFACE

Part II: Fields, Forces, and Motion

In the early 1950's the option structure was abandoned and a common core curriculum was instituted for all electrical engineering students at M.I.T. The objective of the core curriculum was then, and is now, to provide a foundation in mathematics and science on which a student can build in his professional growth, regardless of the many opportunities in electrical engineering from which he may choose. In meeting this objective, core curriculum subjects cannot serve the needs of any professional area with respect to nomenclature, techniques, and problems unique to that area. Specialization comes in elective subjects, graduate study, and professional activities.

To be effective a core curriculum subject must be broad enough to be germane to the many directions an electrical engineer may go professionally, yet it must have adequate depth to be of lasting value. At the same time, the subject must be related to the real world by examples of application. This is true because students learn by seeing material in a familiar context, and engineering students are motivated largely by the relevance of the material to the realities of the world around them.

In the organization of the core curriculum in electrical engineering at M.I.T. electromechanics is one major component. As our core curriculum has evolved, there have been changes in emphasis and a broadening of the topic. The basic text in electromechanics until 1954, when a new departure was made, was *Electric Machinery* by Fitzgerald and Kingsley. This change produced *Electromechanical Energy Conversion* by White and Woodson, which was used until 1961. At that time we started the revision that resulted in the present book. During this period we went through many versions of notes while teaching the material three semesters a year.

Our objective has always been to teach a subject that combines classical mechanics with the fundamentals of electricity and magnetism. Thus the subject offers the opportunity to teach both mechanics and electromagnetic theory in a context vital to much of the electrical engineering community.

Our choice of material was to some extent determined by a desire to give the student a breadth of background sufficient for further study of almost any type of electromechanical interaction, whether in rotating machinery,

plasma dynamics, the electromechanics of biological systems, or magnetoelasticity. It was also chosen to achieve adequate depth while maintaining suitable unity, but, most important, examples were chosen that could be enlivened for the engineering student interested in the interplay of physical reality and the analytical model. There were many examples from which to choose, but only a few satisfied the requirement of being both mathematically lucid *and* physically demonstrable, so that the student could “push it or see it” and directly associate his observations with symbolic models. Among the areas of electrical engineering, electromechanics excels in offering the opportunity to establish that all-important “feel” for a physical phenomenon. Properly selected electromechanical examples can be the basis for discerning phenomena that are remote from human abilities to observe.

Before discussing how the material can be used to achieve these ends, a review of the contents is in order. The student who uses this book is assumed to have a background in electrostatics and magnetostatics. Consequently, Chapter 1 and Appendix B are essentially a review to define our starting point.

Chapter 2 is a generalization of the concepts of inductance and capacitance that are necessary to the treatment of electromechanical systems; it also provides a brief introduction to rigid-body mechanics. This treatment is included because many curricula no longer cover mechanics, other than particle mechanics in freshman physics. The basic ideas of Chapter 2 are repeated in Chapter 3 to establish some properties of electromechanical coupling in lumped-parameter systems and to obtain differential equations that describe the dynamics of lumped-parameter systems.

Next, the techniques of Chapters 2 and 3 are used to study rotating machines in Chapter 4. Physical models are defined, differential equations are written, machine types are classified, and steady-state characteristics are obtained and discussed. A separate chapter on rotating machines has been included not only because of the technological importance of machines but also because rotating machines are rich in examples of the kinds of phenomena that can be found in lumped-parameter electromechanical systems.

Chapter 5 is devoted to the study, with examples, of the dynamic behavior of lumped-parameter systems. Virtually all electromechanical systems are mathematically nonlinear; nonetheless, linear incremental models are useful for studying the stability of equilibria and the nature of the dynamical behavior in the vicinity of an equilibrium. The second half of this chapter develops the classic potential-well motions and loss-dominated dynamics in the context of electromechanics. These studies of nonlinear dynamics afford an opportunity to place linear models in perspective while forming further insights on the physical significance of, for example, flux conservation and state functions.

Chapter 6 represents our first departure from lumped-parameter systems into continuum systems with a discussion of how observers in relative motion will define and measure field quantities and the related effects of material motion on electromagnetic fields. It is our belief that dc rotating machines are most easily understood in this context. Certainly they are a good demonstration of field transformations at work.

As part of any continuum electromechanics problem, one must know how the electric and magnetic fields are influenced by excitations and motion. In quasi-static systems the distribution of charge and current are controlled by magnetic diffusion and charge relaxation, the subjects of Chapter 7. In Chapter 7 simple examples isolate significant cases of magnetic diffusion or charge relaxation, so that the physical processes involved can be better understood.

Chapters 6 and 7 describe the electrical side of a continuum electromechanical system with the material motion predetermined. The mechanical side of the subject is undertaken in Chapter 8 in a study of force densities of electric and magnetic origin. Because it is a useful concept in the analysis of many systems, we introduce the Maxwell stress tensor. The study of useful properties of tensors sets the stage for later use of mechanical stress tensors in elastic and fluid media.

At this point the additional ingredient necessary to the study of continuum electromechanics is the mechanical medium. In Chapter 9 we introduce simple elastic continua—longitudinal motion of a thin rod and transverse motion of wires and membranes. These models are used to study simple continuum mechanical motions (nondispersive waves) as excited electromechanically at boundaries.

Next, in Chapter 10 a string or membrane is coupled on a continuum basis to electric and magnetic fields and the variety of resulting dynamic behavior is studied. The unifying thread of this treatment is the dispersion equation that relates complex frequency ω with complex wavenumber k . Without material convection there can be simple nondispersive waves, cut off or evanescent waves, absolute instabilities, and diffusion waves. The effect of material convection on evanescent waves and oscillations and on wave amplification are topics that make a strong connection with electron beam and plasma dynamics. The method of characteristics is introduced as a convenient tool in the study of wave propagation.

In Chapter 11 the concepts and techniques of Chapters 9 and 10 are extended to three-dimensional systems. Strain displacement and stress-strain relations are introduced, with tensor concepts, and simple electromechanical examples of three-dimensional elasticity are given.

In Chapter 12 we turn to a different mechanical medium, a fluid. We first study electromechanical interactions with inviscid, incompressible

fluids to establish essential phenomena in the simplest context. It is here that we introduce the basic notions of MHD energy conversion that can result when a conducting fluid flows through a transverse magnetic field. We also bring in electric-field interactions with fluids, in which ion drag phenomena are used as an example. In addition to these basically conducting processes, we treat the electromechanical consequences of polarization and magnetization in fluids. We demonstrate how highly conducting fluids immersed in magnetic fields can propagate Alfvén waves.

In Chapter 13 we introduce compressibility to the fluid model. This can have a marked effect on electromechanical behavior, as demonstrated with the MHD conduction machine. With compressibility, a fluid will propagate longitudinal disturbances (acoustic waves). A transverse magnetic field and high electrical conductivity modify these disturbances to magnetoacoustic waves.

Finally, in Chapter 14 we add viscosity to the fluid model and study the consequences in electromechanical interactions with steady flow. Hartmann flow demonstrates the effect of viscosity on the dc magnetohydrodynamic machine.

To be successful a text must have a theme; the material must be inter-related. Our philosophy has been to get into the subject where the student is most comfortable, with lumped-parameter (circuit) concepts. Thus many of the subtle approximations associated with quasi-statics are made naturally, and the student is faced with the implications of what he has assumed only after having become thoroughly familiar with the physical significance and usefulness of his approximations. By the time he reaches Chapter 4 he will have drawn a circle around at least a class of problems in which electromagnetic fields interact usefully with media in motion.

In dealing with physical and mathematical subjects, as we are here, in which the job is incomplete unless the student sees the physical laws put to work in some kind of physical embodiment, it is necessary for the thread of continuity to be woven into the material in diverse and subtle ways. A number of attempts have been made, to which we can add our early versions of notes, to write texts with one obvious, pedagogically logical basis for evolving the material; for example, it can be recognized that classes of physical phenomena could be grouped according to the differential equation that describes the pertinent dynamics. Thus we could treat magnetic diffusion, diffusion waves on elastic continua, and viscous diffusion waves in one chapter, even though the physical embodiments are entirely different. Alternatively, we could devise a subject limited to certain technological applications or cover superficially a wide range of basically unrelated topics such as "energy conversion" under one heading. This was the prevalent approach in engineering education a decade or so ago, even at the

undergraduate level. It seems clear to us that organizing material in a teachable and meaningful fashion is far more demanding than this. To confess our own mistakes, our material went originally from the general to the specific; it began with the relativistic form of Maxwell's equations, including the effects of motion, and ended with lumped-parameter devices as special cases. Even if this were a pedagogically tenable approach, which we found it was not, what a bad example to set for students who should be learning to distinguish between the essential and the superfluous! Ideas connected with the propagation of electromagnetic waves (relativistic ideas) must be included in the curriculum, but their connection with media in motion should be made after the student is aware of the first-order issues.

A meaningful presentation to *engineers* must interweave and interrelate mathematical concepts, physical characteristics, the modeling process, and the establishment of a physical "feel" for the world of reality. Our approach is to come to grips with each of these goals as quickly as possible (let the student "get wet" within the first two weeks) and then, while reinforcing what he has learned, continually add something new. Thus, if one looks, he will see the same ideas coming into the flow of material over and over again.

For the organization of this book one should look for many threads of different types. We can list here only a few, in the hope that the subtle reinforcing interplay of mathematical and physical threads will be made evident. Probably the essential theme is Maxwell's equations and the ideas of quasi-statics. The material introduced in Chapter 1 is completely abstract, but it is reinforced in the first few chapters with material that is close to home for the student. By the time he reaches Chapter 10 he will have learned that waves exist which intimately involve electric and magnetic fields that are altogether quasistatic. (This is something that comes as a surprise to many late in life.) Lumped-parameter ideas are based on the integral forms of Maxwell's equations, so that the dynamical effects found with lumped-parameter time constants L/R and RC in Chapter 5 are easily associated with the subjects of magnetic diffusion and charge relaxation. A close tie is made between the "speed voltage" of Chapter 5 and the effects of motion on magnetic fields, as described by field transformations in Chapters 6 to 14. Constant flux dynamics of a lumped coil in Chapter 5 are strongly associated with the dynamics of perfectly conducting continuous media; for example, Alfvén waves in Chapter 12.

Consider another thread of continuity. The book begins with the mathematics of circuit theory. The machines of Chapter 4 are essentially circuits in the sinusoidal steady state. In Chapter 5 we linearize to pursue lumped-parameter ideas of stability and other transient responses and then proceed to nonlinear dynamics, potential-well theory, and other approaches that should form a part of any engineer's mathematical background. By the time

the end of Chapter 10 is reached these ideas will have been carried into the continuum with the addition of tensor concepts, simple cases of the method of characteristics, and eigenvalue theory. The ω - k plot and its implication for all sorts of subjects in modern electrical engineering can be considered as a mathematical or a physical objective. The ideas of stability introduced with ordinary differential equations ($\exp st$) in Chapter 5 evolve into the continuum stability studies of Chapter 10 [$\exp j(\omega t - kx)$] and can be regarded as a mathematical or a physical thread in our treatment. We could list many other threads: witness the evolution of energy and thermodynamic notions from Chapters 3 to 5, 5 to 8, and 8 to 13.

We hope that this book is not just one more in the mathematics of electrical engineering or the technical aspects of rotating machines, transducers, delay lines, MHD converters, and so on, but rather that it is the mathematics, the physics, and, most of all, the engineering combined into one.

The material brought together here can be used in a variety of ways. It has been used by Professors C. N. Weygandt and F. D. Ketterer at the University of Pennsylvania for two subjects. The first restricts attention to Chapters 1 to 6 and Appendix B for a course in lumped-parameter electromechanics that both supplants the traditional one on rotating machines in the electrical engineering curriculum and gives the background required for further study in a second term (elective) covering Chapter 7 and beyond. Professors C. D. Hendricks and J. M. Crowley at the University of Illinois have used the material to follow a format that covers up through Chapter 10 in one term but omits much of the material in Chapter 7. Professor W. D. Getty at the University of Michigan has used the material to follow a one-term subject in lumped-parameter electromechanics taught from a different set of notes. Thus he has been able to use the early chapters as a review and to get well into the later chapters in a one-term subject.

At M.I.T. our curriculum seems always to be in a state of change. It is clear that much of the material, Chapters 1 to 10, will be part of our required (core) curriculum for the foreseeable future, but the manner in which it is packaged is continually changing. During the fall term, 1967, we covered Chapters 1 to 10 in a one-semester subject taught to juniors and seniors. The material from Chapters 4 and 6 on rotating machines was used selectively, so that students had "a foot solidly in the door" on this important subject but also that the coverage could retain an orientation toward the needs of all the diverse areas found in electrical engineering today. We have found the material useful as the basis for early graduate work and as a starting point in several courses related to electromechanics.

Finally, to those who open this book and then close it with the benediction, "good material but unteachable," we apologize because to them we have not made our point. Perhaps not as presented here, but certainly as it is

represented here, this material is rich in teaching possibilities. The demands on the teacher to see the subject in its total context, especially the related problems that lie between the lines, are significant. We have taught this subject many times to undergraduates, yet each term has been more enjoyable than the last. There are so many ways in which drama can be added to the material, and we do not need to ask the students (bless them) when we have been successful in doing so.

In developing this material we have found lecture demonstrations and demonstration films to be most helpful, both for motivation and for developing understanding. We have learned that when we want a student to see a particular phenomenon it is far better for us to do the experiment and let the student focus his attention on what he should see rather than on the wrong connections and blown fuses that result when he tries to do the experiment himself. The most successful experiments are often the simplest—those that give the student an opportunity to handle the apparatus himself. Every student should “chop up some magnetic field lines” with a copper “axe” or he will never really appreciate the subject. We have also found that some of the more complex demonstrations that are difficult and expensive to store and resurrect each semester come through very well in films. In addition to our own short films, three films have been produced professionally in connection with this material for the National Committee on Electrical Engineering Films, under a grant from the National Science Foundation, by the Education Development Center, Newton, Mass.

Synchronous Machines: Electromechanical Dynamics by H. H. Woodson

Complex Waves I: Propagation, Evanescence and Instability by J. R. Melcher

Complex Waves II: Instability, Convection and Amplification by J. R. Melcher

An additional film is in the early stages of production. Other films that are useful have been produced by the Education Development Center for the National Committee on Fluid Mechanics Films and for the College Physics Film Program. Of particular interest, from the former series, is *Magnetohydrodynamics* by Arthur Shercliff.

A book like this can be produced only with plenty of assistance. We gratefully acknowledge the help we received from many directions and hope we have forgotten no one after seven years of work. First of all we want to acknowledge our students with whom we worked as the material developed. They are the one most essential ingredient in an effort of this sort. Next we want to thank Dr. S. I. Freedman, Professor H. H. Richardson, and Dr. C. V. Smith, Jr., for their assistance in framing worthwhile approaches to several of our key topics. In seven years we have had the help of many able

teachers in presenting this material to students. Their discussions and advice have been most useful. In this category we want particularly to mention Professors H. A. Haus, P. L. Penfield, D. C. White, G. L. Wilson, R. Gallager, and E. Pierson and Doctors J. Reynolds, W. H. Heiser, and A. Kusko. Professor Ketterer, who has taught this material at M.I.T. and the University of Pennsylvania, Professors C. D. Hendricks and J. M. Crowley, who have taught it at M.I.T. and the University of Illinois, and Professor W. D. Getty, who has taught it at M.I.T. and the University of Michigan, have been most generous with their comments. Messrs. Edmund Devitt, John Dressler, and Dr. Kent Edwards have checked the correctness of many of the mathematical treatments. Such a task as typing a manuscript repeatedly is enough to try the patience of anyone. Our young ladies of the keyboard, Miss M. A. Daly, Mrs. D. S. Figgins, Mrs. B. S. Morton, Mrs. E. M. Holmes, and Mrs. M. Mazroff, have been gentle and kind with us.

A lengthy undertaking of this sort can be successful only when it has the backing of a sympathetic administration. This work was started with the helpful support of Professor P. Elias, who was then head of the Department of Electrical Engineering at M.I.T. It was finished with the active encouragement of Professor L. D. Smullin, who is presently head of the Department.

Finally, and most sincerely, we want to acknowledge the perseverance of our families during this effort. Our wives, Blanche S. Woodson and Janet D. Melcher, have been particularly tolerant of the demands of this work.

This book appears in three separately bound, consecutively paged parts that can be used individually or in any combination. Flexibility is ensured by including with each part a complete Table of Contents and Index. In addition, for convenient reference, Parts II and III are supplemented by brief appendices which summarize the relevant material from the preceding chapters. Part II, Chapters 7 to 10, develops interactions between moving media and fields with simple mechanical models that illustrate the dynamics of continuum electromechanical systems.

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Cambridge, Massachusetts
January 1968

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ELECTROMECHANICAL DYNAMICS

Part II: Fields, Forces, and Motion

Chapter 7

MAGNETIC DIFFUSION AND CHARGE RELAXATION

7.0 INTRODUCTION

There are two important reasons for introducing a distributed or continuum description of electromechanical interactions. The most obvious is that the mechanical system may be an elastic or fluid medium, with the electric or magnetic force distributed throughout. In this case the electrical and mechanical equations of motion are likely to have both time and space coordinates as independent variables. As we shall see in later chapters, there is a need for this model when the dynamical times of interest (say the period of a sinusoidal excitation) are on the same order as the time required for a disturbance to propagate from one extreme to another in the mechanical system.*

The second reason for introducing a continuum model concerns characteristic times that arise because of competing energy storage and dissipation mechanisms in the electrical system; for example, a lumped inductance L , shunted by a resistance R , constitutes a simple magnetic field system. In the absence of motion (which could change the inductance L) this circuit has a time constant L/R . The response of the circuit to an excitation depends greatly on the time constant (or period) of the excitation relative to L/R . Similarly, the behavior of magnetic field systems involving conducting materials depends considerably on the relative values of times associated with the motion or with electrical excitation and times that characterize the competing energy storage and dissipation phenomena. This point can best be made in terms of a simple example.

* Note that we neglected similar electromagnetic propagational effects starting in Chapter 1, when the quasi-static electric and magnetic field systems were introduced. This is an extremely good approximation in most systems because the velocity of light is usually much greater than any velocity of propagation for a mechanical or an electromechanical disturbance.

Figure 7.0.1 shows what can be thought of as a one-turn inductor (the perfectly conducting plates short-circuited at the end by a perfectly conducting plate) shunted by a resistance (the block with a conductivity σ). Suppose that the plates are excited at the left end by a sinusoidally varying current source with the frequency ω . At very low frequencies (essentially direct current) we know that this current will flow through the perfectly conducting end plate. As the frequency is raised, however, the rate of change of the time-varying magnetic field will induce a voltage across the block and a current will flow through it between the top and bottom plates. In terms of lumped-parameter models, we could determine this current by attributing a resistance to the block. The block itself, however, has finite dimensions, and we could think of breaking it into two sections and modeling each section by

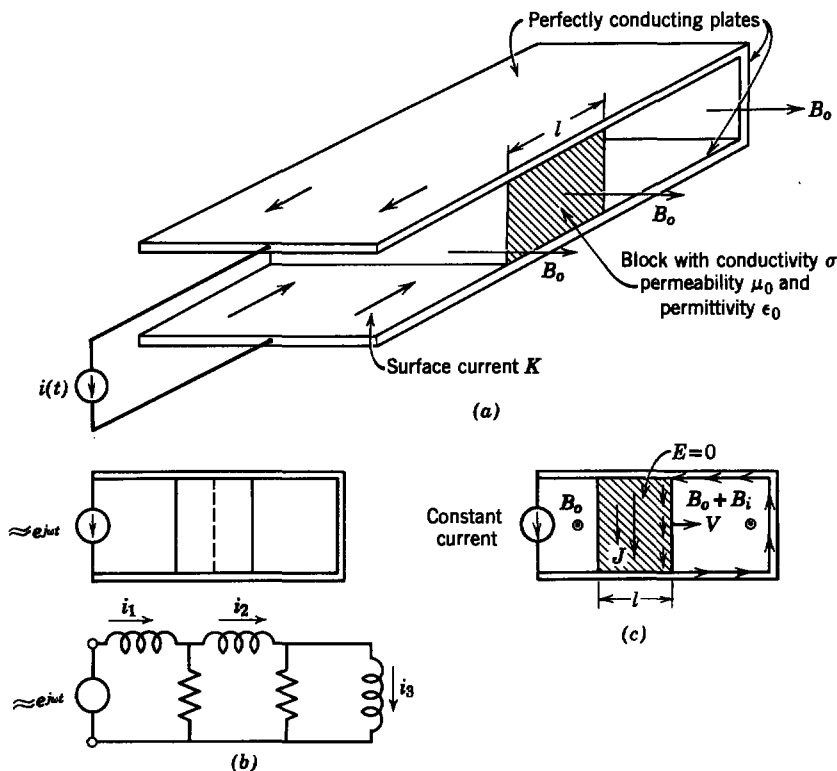


Fig. 7.0.1 (a) A pair of perfectly conducting plates short-circuited at one end by a perfectly conducting plate and driven at the other end by a current source: a block of conductivity σ makes electrical contact between the plates as shown; (b) equivalent circuit that shows the effect of magnetic diffusion on the currents induced in the block when the source is time-varying; (c) block moving with a velocity V induces currents that can alter the imposed magnetic field significantly.

a resistance. We now have the equivalent circuit shown in Fig. 7.0.1*b*, in which it is easy to see that at very low frequencies $i_1 = i_2 = i_3$. As the frequency is raised, the reactances become significant and $i_1 > i_2 > i_3$. In fact, at very high frequencies we expect that current i_3 will be very small. This simple problem points up why a continuum model is required. If the currents i_1 and i_2 are not equal (i.e., if the current through the block is not uniform) to a degree that depends on the dynamical nature of the excitation, what value of resistance do we use to characterize the block? A similar problem exists in attributing one or more equivalent inductances to the system. Our dilemma is brought about by not knowing where the currents flow or, to put it another way, of not knowing how the magnetic field is distributed in space. In the following sections a knowledge of the distribution of magnetic field is our objective, as we discuss the physical phenomenon of *magnetic diffusion*.

In the situation just discussed the time that determined the appropriate system model was $2\pi/\omega$, or the period of excitation. In electromechanical problems the motion is responsible for introducing characteristic times which also play an important role in determining the distribution of magnetic field. This can be illustrated by considering again the system of Fig. 7.0.1, this time with the block moving with the velocity V , as shown in Fig. 7.0.1*c*, and with constant excitation current.

We might model this system by assuming that the constant magnetic flux density B_o , induced by the current source, is the only magnetic field everywhere between the plates. Then, because of the perfectly conducting end plate, the electric field E between the plates would be zero. It follows [from (6.3.5)] (Ohm's law for moving media is $\mathbf{J}' = \sigma \mathbf{E}'$, where field transformations (Table 6.1, Appendix E) require that $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$ and $\mathbf{J}'_f = \mathbf{J}_f$.) that the motion induces a current density J of

$$J = VB_o\sigma. \quad (7.0.1)$$

(Here for purposes of illustration we assume that the plates have sufficient extent to justify plane parallel geometry.) Now, we could use this current, together with the imposed magnetic field B_o , to compute the magnetic force on the block. In doing so, however, we assume that the magnetic field induced by the current J is negligible compared with the imposed field B_o . We can establish when this assumption is valid by computing the addition B_i to the magnetic flux density to the right of the moving block. Constraints on the problem require that J flow through the end plate to the right, where there is an addition to the surface current K of

$$K_i = Jl. \quad (7.0.2)$$

Hence the induced magnetic field in the region to the right is

$$B_i = \mu_0 Jl. \quad (7.0.3)$$

It follows from (7.0.1) and (7.0.3) that the ratio of the induced magnetic field to the imposed magnetic field is

$$\frac{B_i}{B_o} = \mu_0 \sigma V l = R_m. \quad (7.0.4)$$

This dimensionless quantity is called the *magnetic Reynolds number*.^{*} From our example we know that if R_m is significant compared with unity the magnetic field will be altered appreciably by the motion. Of course, this means that the distribution of current within the block will also be dependent on the motion. From (7.0.4) a block with a large dimension l , a large velocity V , or a high conductivity σ will require a continuum model to determine the current distribution. One of our objectives in the sections that follow is to indicate electromechanical models that can be used under various conditions of the excitation and mechanical motion.

In electric field systems we are concerned about the distribution of charge and the associated electric field. Here we can think of a simple system in which a resistance R is connected in series with a capacitance C . The time constant is then RC , and when the period of an excitation is on the order of this time we expect that lumped parameter models may be of dubious significance.

Consider, as an example, the system shown in Fig. 7.0.2. Here, a resistive sheet is placed adjacent to a pair of perfectly conducting electrodes with the potential difference v . A circuit model for this system is shown in Fig. 7.0.2b, where the plates form capacitors with the resistive sheet and the sheet is modeled by a single resistance. A more refined model, also shown in Fig. 7.0.2b, divides each of the capacitors into two sections and recognizes that the outermost capacitances are connected to a larger series resistance than the inner capacitances.

It is apparent from this latter model that the distribution of charge among the capacitances (hence on the surface of the sheet) is dependent on the frequency of excitation. At very low frequencies the charge would be evenly distributed among the capacitances. As the frequency is raised, the outside capacitances, which have larger series resistances, will support less charge than the inside capacitances. Thus we are in a dilemma about how the capacitances and resistances should be computed and a continuum model is appropriate.

A case in which material motion can have an effect on the charge distribution is shown in Fig. 7.0.2c. Here the resistive sheet moves to the right with the velocity V . If the motion has a negligible effect on the surface charge,

^{*} See, for example, W. F. Hughes, and F. J. Young, *The Electromagnetodynamics of Fluids*, Wiley, New York (1966), p. 154.

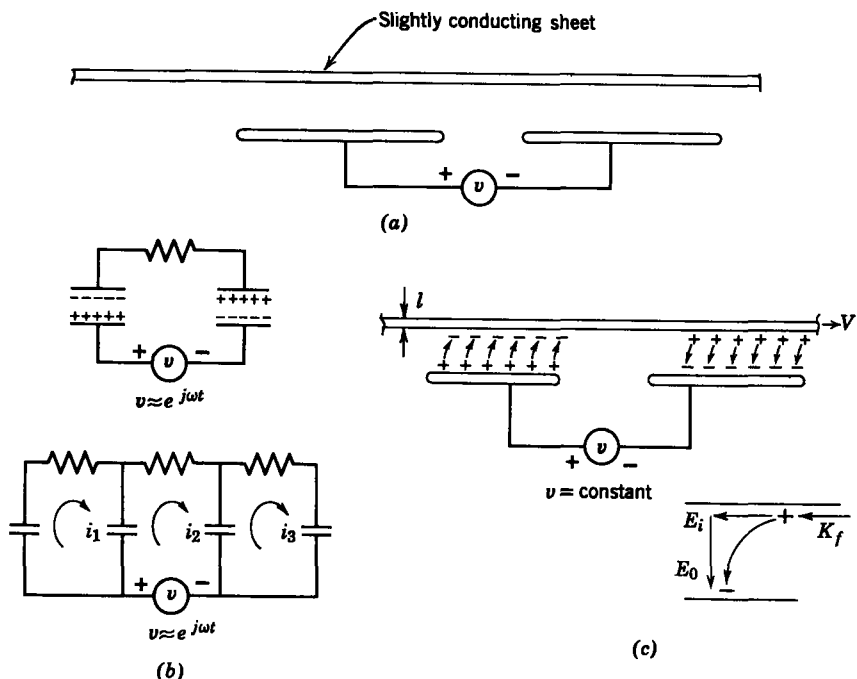


Fig. 7.0.2 (a) A slightly conducting sheet is placed adjacent to a pair of perfectly conducting plates with the potential difference v ; (b) equivalent circuit for case in which excitation is sinusoidal: each section of the sheet can be thought of as forming a capacitor with the opposite electrode, but each capacitance has a different series resistance, as shown by the bottom, more refined, circuit; (c) the motion of the sheet, moving with velocity V and a constant potential difference on the plates, induces a shift in the charge distribution and an electric field E_i .

the electric field intensity E_0 in the section to the right is related to the surface charge density σ_f by

$$\sigma_f = \epsilon_0 E_0. \quad (7.0.5)$$

We model the resistive material as a sheet having conductivity σ ; hence Ohm's law written in the moving frame of the material requires that $\mathbf{K}'_f = \sigma l \mathbf{E}'_i$, where l is the sheet thickness. Field transformations* then require $\mathbf{K}'_f = \mathbf{K}_f - \sigma_f \mathbf{V}$ and $\mathbf{E}'_i = \mathbf{E}_i$ so that

$$K_f = \sigma l E_i - \sigma_f V. \quad (7.0.6)$$

In the steady state there is no current to the right; hence $K_f = 0$. From (7.0.6) there is then an induced electric field intensity given by

$$E_i = \frac{\sigma_f V}{\sigma l}. \quad (7.0.7)$$

* Table 6.1, Appendix E.

It follows from this expression and (7.0.5) that the ratio of the induced electric field intensity E_i to the imposed electric field intensity E_0 is

$$\frac{E_i}{E_0} = \frac{\epsilon_0 V}{\sigma l} = R_e \quad (7.0.8)$$

This dimensionless ratio R_e is called the *electric Reynolds number*.^{*} When this number is significant compared with unity, motion has an appreciable effect on the charge distribution. To understand situations of this kind a continuum model is required.

In the sections that follow we first undertake a study of magnetic field systems (magnetic diffusion) and then electric field systems (charge relaxation). As we have illustrated, our objective is to use a series of examples as vehicles for establishing an understanding of situations in which the distributions of currents and charges are not known until after the problem has been solved.

7.1 MAGNETIC FIELD DIFFUSION

In quasi-static magnetic field systems, which consist primarily of magnetizable and conducting materials, magnetic field diffusion is the principal phenomenon that determines what further approximations can be made. Because a general problem is virtually impossible to solve, we will explore diffusion by using a set of simple examples.

To set the stage for studying these examples we first review the mathematical description of a magnetic field system. We assume all materials (magnetizable and/or conducting) to be electrically linear and isotropic. Thus we write the field equations as [see Table 1.2 and (6.3.5.)][†]

$$\nabla \times \mathbf{H} = \mathbf{J}_f, \quad (7.1.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (7.1.2)$$

$$\mathbf{B} = \mu \mathbf{H}, \quad (7.1.3)$$

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}, \quad (7.1.4)$$

$$\mathbf{J}_f = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (7.1.5)$$

In these equations the velocity $\mathbf{v}$ of the conducting material is measured with respect to the inertial coordinate system in which all the field quantities are measured. For our present treatment we assume that the velocity $\mathbf{v}$ is independently specified as a function of space and time. After we have introduced

^{*} O. M. Stuetzer, "Magnetohydrodynamics and Electrohydrodynamics," *Phys. Fluids*, 5, No. 3, 534-544 (May 1962).

[†] See Table 1.2, Appendix E, and footnote on p. 332.

continuum mechanics, in later chapters, we shall be prepared to treat situations in which the velocity $\mathbf{v}$ is a dependent variable.

To illustrate the different phenomena that can occur it is helpful to eliminate all but one variable from (7.1.1) to (7.1.5) and study the resulting expression. For this purpose we assume that the materials are homogeneous (this means that μ and σ are not functions of space but that they may be functions of time) and eliminate all variables except the flux density $\mathbf{B}$. We use (7.1.5) to eliminate $\mathbf{E}$ from (7.1.4).

$$\frac{1}{\sigma} \nabla \times \mathbf{J}_f - \nabla \times (\mathbf{v} \times \mathbf{B}) = - \frac{\partial \mathbf{B}}{\partial t}. \quad (7.1.6)$$

Next, we use (7.1.3) to eliminate $\mathbf{H}$ from (7.1.1):

$$\frac{1}{\mu} \nabla \times \mathbf{B} = \mathbf{J}_f. \quad (7.1.7)$$

The curl of this expression, substituted into (7.1.6), yields

$$\frac{1}{\mu\sigma} \nabla \times (\nabla \times \mathbf{B}) - \nabla \times (\mathbf{v} \times \mathbf{B}) = - \frac{\partial \mathbf{B}}{\partial t}. \quad (7.1.8)$$

We use a vector identity* and (7.1.2) to rewrite the first term on the left of (7.1.8) and obtain the desired expression involving a single dependent variable, the magnetic flux density $\mathbf{B}$:

$$- \frac{1}{\mu\sigma} \nabla^2 \mathbf{B} + \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}). \quad (7.1.9)$$

This equation describes the distribution of magnetic field in the conducting medium.† It includes both the effects of the time-varying magnetic field and the material motion, hence it can be used to analyze the situations shown in Fig. 7.0.1. In a continuum electromechanical problem (7.1.9) plays the same role as the electrical circuit equations played in the lumped-parameter magnetic field systems of the preceding chapters. In fact, there is considerable analogy between (7.1.9) and electrical circuit equations that we encountered in Chapters 3 to 6; for example, the dynamics of a coil in a magnetic field were described in Section 5.1.3 by (5.1.31):

$$-i_1 R - L_1 \frac{di_1}{dt} = AB_o \cos \theta \frac{d\theta}{dt}, \quad (7.1.10)$$

* $\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$.

† This equation assumes great importance in the determination of magnetic field origins in the liquid core of the earth. In this context it is sometimes referred to as Bullard's equation. (See T. G. Cowling, *Magnetohydrodynamics*, Interscience, New York, 1957, pp. 4, 77.)

where R and L_1 are the resistance and inductance of a coil that rotates in the flux density B_0 with the angular displacement θ . This expression states that the voltage developed by the motion of the coil in the magnetic field (the "speed voltage" on the right-hand side) is taken up by the voltage drop across the resistance and the inductance of the coil. Similarly, the right-hand side of (7.1.9) is a rate of change of the flux density caused by the motion, and this change is taken up by a flux change due to ohmic dissipation $-(1/\mu\sigma)\nabla^2\mathbf{B}$ and by a time rate of change of flux density $(\partial\mathbf{B}/\partial t)$ at the point.

In the absence of material motion (7.1.9) reduces to the *diffusion equation*.*

$$\frac{1}{\mu\sigma} \nabla^2 \mathbf{B} = \frac{\partial \mathbf{B}}{\partial t}. \quad (7.1.11)$$

This expression is appropriate to the analysis of a problem such as that shown in Fig. 7.0.1b, in which the conductor is fixed and the distribution of magnetic field is required. We can think of it as a continuum representation of the dynamics of a distributed system of inductances and resistances. In Section 7.1.1 an example is presented to establish the physical phenomena represented by (7.1.11).

A second limiting case of (7.1.9) of considerable importance results when there is motion but steady-state conditions have been established. Then (7.1.9) reduces to

$$-\frac{1}{\mu\sigma} \nabla^2 \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}). \quad (7.1.12)$$

Here the increase in flux density due to the motion is entirely absorbed by the ohmic dissipation in the material. Physical situations consistent with this limit are considered in Section 7.1.2.

Some of the most important engineering applications involving magnetic diffusion are made possible because there are time-varying excitations in the presence of material motion. The induction machine of Section 4.1.6b is an example of an interaction of this type. In general, (7.1.9) is required to analyze these problems, which are exemplified in Sections 7.1.3 and 7.1.4. In fact, the example considered in Section 7.1.4 illustrates the significance of the lumped parameter model for an induction machine.

* This form of equation is found to describe a variety of dissipative continuum phenomena, including heat conduction, neutron diffusion, and the diffusion of fluid vorticity. See, for example, R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, Wiley, New York, 1960, p. 352. A. M. Weinberg and E. P. Wigner, *The Physical Theory of Neutron Chain Reactors*, University of Chicago Press, Chicago, 1958, p. 194. H. Schlichting, *Boundary Layer Theory*, McGraw-Hill, New York, 1960, p. 58.

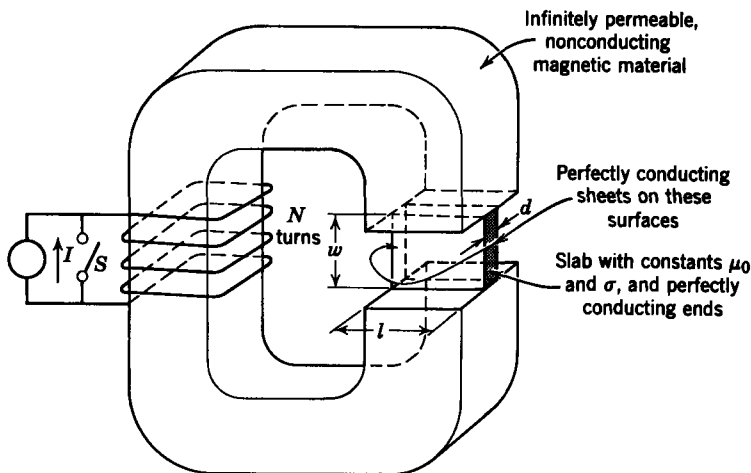


Fig. 7.1.1 System for studying magnetic diffusion as an electrical transient. A slab of conducting material is placed in the gap of a magnetic circuit excited by a step function of current. An end view of the slab is shown in Fig. 7.1.2.

7.1.1 Diffusion as an Electrical Transient

To obtain a firm understanding of the basic electromagnetic phenomena that occur in magnetic field systems we consider first a simple example in which all materials are at rest and we study the diffusion of a magnetic field into (or out of) a rectangular slab of material. The physical situation is shown in Fig. 7.1.1. The system consists of an electromagnet made of infinitely permeable, nonconducting magnetic material with an air gap of length w and excited through the N -turn winding by a constant-current source I which can be turned on or off by the switch S . In the air gap is a slab of non-magnetizable ($\mu = \mu_0$) material of the dimensions shown and with constant electrical conductivity σ .

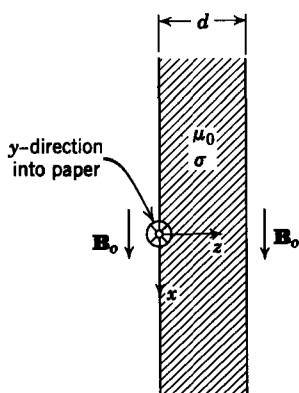


Fig. 7.1.2 Coordinate system for studying one-dimensional diffusion. At time $t = 0^+$ the magnetic field is distributed as shown in Fig. 7.1.3.

We assume that the switch S is initially closed so that there is no current through the winding and no flux in the magnet. At $t = 0$ switch S is opened, and we wish to determine the time history of the flux density near the center of the slab. The coordinate system is defined in Fig. 7.1.2. We assume that the slab width w and length l are large enough compared with the thickness d to consider the slab as

infinitely large in the x - and y -directions. Moreover, we neglect fringing at the edges of the air gap; consequently, the flux density will have only an x -component. With switch S opened at $t = 0$, the boundary conditions on the slab are for $t > 0$, $z < 0$, $z > d$,

$$\mathbf{B} = \mathbf{i}_x B_o, \quad (7.1.13)$$

where B_o (see Section 2.1.1) is given by

$$B_o = \frac{\mu_0 N I}{w}. \quad (7.1.14)$$

In this one-dimensional problem there is only an x -component of $\mathbf{B}$ and it varies only with z ; thus we write the x -component of (7.1.11) as

$$\frac{1}{\mu_0 \sigma} \frac{\partial^2 B_x}{\partial z^2} = \frac{\partial B_x}{\partial t}. \quad (7.1.15)$$

Within the slab ($0 < z < d$) the flux density is zero immediately after the switch S is opened

$$\text{at } t = 0^+; \quad B_x = 0. \quad (7.1.16)$$

This condition follows from (7.1.6) with $\mathbf{v} = 0$, which we integrate around an arbitrary fixed contour within the conductor. Then by Stokes's theorem we have

$$\frac{1}{\sigma} \oint_C \mathbf{J}_t \cdot d\mathbf{l} = - \frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{n} \, da.$$

If $\mathbf{B}$ has a finite value just after the switch is opened, the time derivative on the right (assuming that the integral encloses some $\mathbf{B}$ field) will be infinite. It follows that (because the contour of integration is arbitrary) the current density $\mathbf{J}_t$ will be infinite. Because this is not physically reasonable, we argue that the flux density $\mathbf{B}$ remains zero just after the switch S is opened or that the condition of (7.1.16) holds. If the contour encloses the boundary at $z = 0$ or $z = d$, the term on the right is infinite, thus indicating an infinite $\mathbf{J}_t$ which is actually a finite surface current density necessary to terminate the B_o field. This mathematical idealization results from the assumption that the external flux density B_o is established in zero time. In a real situation the flux density B_o will be established rapidly but at a finite rate. This idealization is better understood after the solution is completed. The process is analogous to the use of singularity functions in circuit theory to simplify mathematical analyses.

It should be clear that after a sufficiently long time all transient currents in the slab will die out and the flux density will be uniform throughout the slab,

$$\text{as } t \rightarrow \infty, \quad B_x = B_o, \quad (7.1.17)$$

for then $\partial B_x / \partial t = 0$ in (7.1.15) for steady-state conditions.

To effect a solution of this problem we assume a separable solution of the form

$$B_x = \hat{B}(z)e^{-\alpha t} + B_0. \quad (7.1.18)$$

Substitution of this expression into 7.1.15 and cancellation of the exponential factor yields

$$\frac{1}{\mu_0 \sigma} \frac{d^2 \hat{B}}{dz^2} = -\alpha \hat{B}. \quad (7.1.19)$$

This equation has a solution

$$\hat{B} = C_1 \sin \sqrt{\mu_0 \sigma \alpha} z + C_2 \cos \sqrt{\mu_0 \sigma \alpha} z. \quad (7.1.20)$$

Thus the general solution is

$$B_x = (C_1 \sin \sqrt{\mu_0 \sigma \alpha} z + C_2 \cos \sqrt{\mu_0 \sigma \alpha} z) e^{-\alpha t} + B_0. \quad (7.1.21)$$

This equation must satisfy the initial conditions illustrated in the sketch of B_x as a function of z in Fig. 7.1.3. We can achieve this by setting

$$C_2 = 0$$

and expanding the rectangular distribution in a Fourier sine series.

Using the initial condition expressed by (7.1.16) and illustrated in Fig. 7.1.3, we write $B_x(z, 0^+)$ as

$$0 = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi z}{d} + B_0. \quad (7.1.22)$$

Comparison of (7.1.21) with $C_2 = 0$ and (7.1.22) shows that each term in the series has

$$\alpha_n = \frac{n^2 \pi^2}{\mu_0 \sigma d^2} \quad (7.1.23)$$

We evaluate the coefficients a_n in the standard way by multiplying both sides

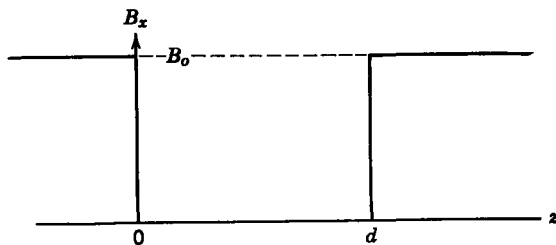


Fig. 7.1.3 Distribution of B_x at $t = 0^+$ in the conducting slab of Fig. 7.1.2. The magnetic field distribution for $t > 0$ is shown in Fig. 7.1.4.

(7.1.22) by $\sin m\pi z/d$ and integrating

$$\int_0^d -B_o \sin \frac{m\pi z}{d} dz = \int_0^d \sum_{n=1}^{\infty} a_n \sin \frac{n\pi z}{d} \sin \frac{m\pi z}{d} dz. \quad (7.1.24)$$

All of the terms on the right for which $m \neq n$ integrate to zero. Thus (7.1.24) reduces to

$$\int_0^d -B_o \sin \frac{n\pi z}{d} dz = \int_0^d a_n \sin^2 \frac{n\pi z}{d} dz. \quad (7.1.25)$$

Evaluation of the integral yields

$$\begin{aligned} a_n &= -\frac{4}{n\pi} B_o, & n \text{ odd}, \\ a_n &= 0, & n \text{ even}, \end{aligned} \quad (7.1.26)$$

and we write the complete solution for B_x as

$$B_x = B_o \left(1 - \sum_{n \text{ odd}} \frac{4}{n\pi} \sin \frac{n\pi z}{d} e^{-a_n t} \right). \quad (7.1.27)$$

In the solution (7.1.27) we see that each space harmonic damps at a different rate, the higher harmonics damping faster. We define the fundamental time constant τ as

$$\tau = \frac{1}{\alpha_1} = \frac{\mu_0 \sigma d^2}{\pi^2} \quad (7.1.28)$$

and write (7.1.27) as

$$B_x = B_o \left(1 - \sum_{n \text{ odd}} \frac{4}{n\pi} \sin \frac{n\pi z}{d} e^{-n^2 t/\tau} \right). \quad (7.1.29)$$

The fundamental time constant of (7.1.28), which is the longest time constant of the series, is usually called the *diffusion time constant* of the system.

The distribution of flux density, as expressed by (7.1.29), is plotted for several instants of time in Fig. 7.1.4. At $t = 0$ the magnetic flux is excluded completely from the interior of the slab. As time progresses the flux diffuses into the slab. Because the higher space harmonics damp out faster, only three terms in the series are needed to calculate the curve at $t = 0.1\tau$; only two terms are necessary for $t = 0.3\tau$; and only the first term has appreciable value for $t > \tau$. Thus it is clear why the fundamental diffusion time τ is the controlling time constant in the diffusion process. It is also clear from Fig. 7.1.4 that the field has almost completely diffused into the slab in a time $t > 3\tau$.

It is also instructive to investigate the behavior of the current density inside the slab during this transient diffusion process. The currents created by

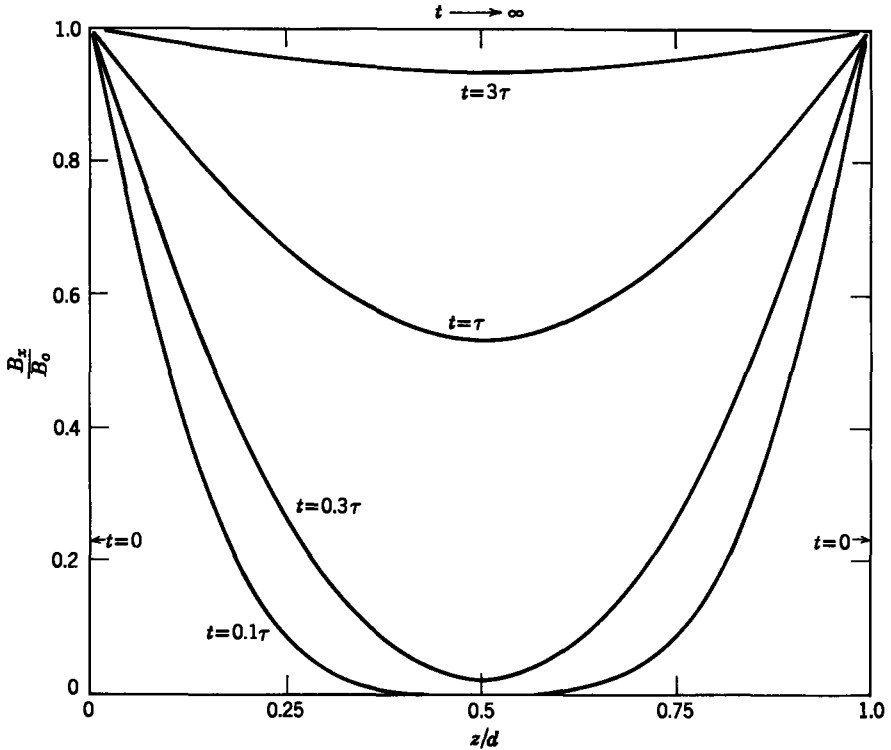


Fig. 7.1.4 Transient, one-dimensional diffusion of a magnetic field into a conducting material. The slab of conductor shown in Figs. 7.1.1 and 7.1.2 is viewed here from the end, as shown in Fig. 7.1.2.

induction in this way are often called *eddy currents*. We use (7.1.7) with the definitions of our present one-dimensional example (see Fig. 7.1.2) to obtain

$$J_y = \frac{1}{\mu_0} \frac{\partial B_z}{\partial z} = -\frac{B_0}{\mu_0 d} \sum_{n \text{ odd}} 4 \cos \frac{n\pi z}{d} e^{-n^2 t/\tau}. \quad (7.1.30)$$

The space distribution of this current density is plotted for several instants of time in Fig. 7.1.5. Note that at $t = 0$ the series does not converge at $z = 0$ and $z = d$. This shows that at $t = 0$ the magnetic field is excluded completely from the slab and, to terminate the external tangential field, a surface current density is required [see (6.2.14)]*. A surface current density implies an infinite volume current density; thus at $t = 0$ there are impulses of J_y at $z = 0$ and $z = d$, as shown in Fig. 7.1.5. As time progresses, the currents diffuse into the slab and decay. At the surface of the slab ($z = 0$ or $z = d$) the current density decays continuously, but at an interior point the current density increases from zero to some maximum value and decays back to zero.

* Table 6.1, Appendix E.

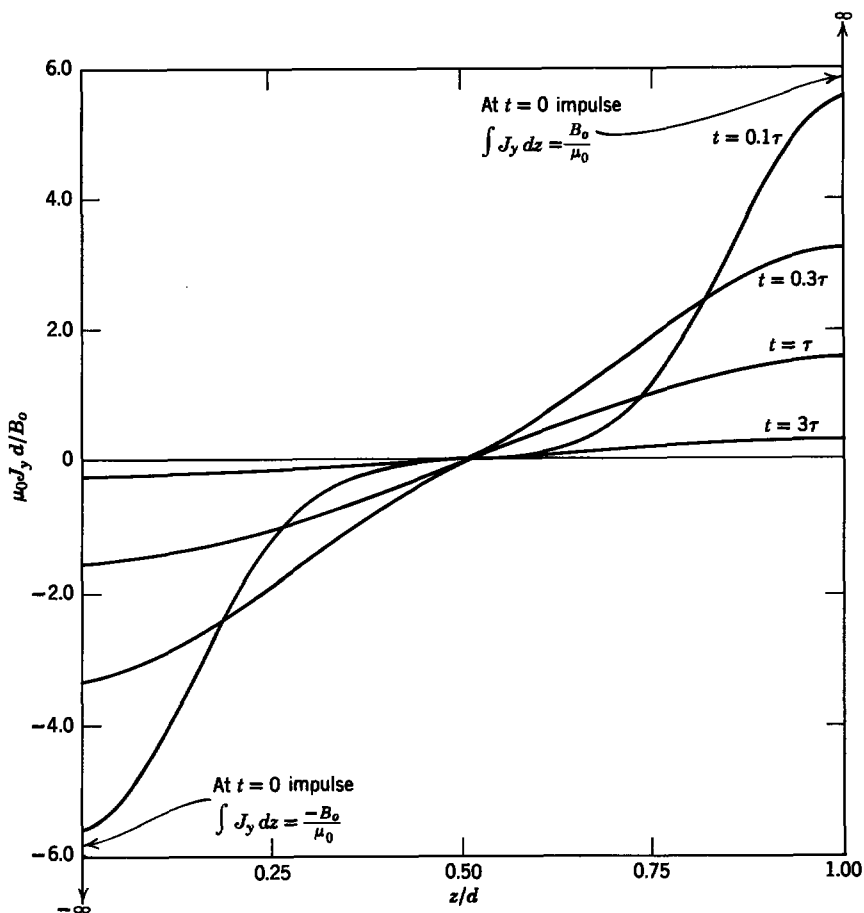


Fig. 7.1.5 Current distribution in transient one-dimensional diffusion.

Conservation of current requires that the current that flows down one side of the block must return on the opposite side. Hence the current distribution of Fig. 7.1.5 has odd symmetry about $z = d/2$. Note that the one-dimensional solution implies perfectly conducting end plates on the slab to provide a return path for the currents.

It should be clear from a study of Figs. 7.1.4 and 7.1.5 that if we are interested in electromagnetic phenomena that occur outside the slab and that have characteristic times much shorter than the diffusion time τ we can approximate the properties of the slab by saying that $\sigma \rightarrow \infty$. In this approximation the flux is excluded from the interior of the slab and currents flow on the surfaces.

On the other hand, when we are concerned with phenomena with characteristic times that are long compared with the diffusion time τ , we can

neglect perturbations due to diffusion effects for field calculations. These two limiting cases are illuminated more thoroughly in additional examples to follow. They are analogous to the constant current and constant flux conditions of the lumped-parameter system described in Section 5.1.3.

We can now state the physical conditions necessary for our mathematical idealization to be accurate. If the flux density B_0 produced by current in the winding in Fig. 7.1.1 builds up in a time much smaller than the diffusion time constant, then our approximation of the boundary conditions at $t = 0^+$ is valid. In any case, the mathematical model is least accurate for very small time, as should be evident from a careful comparison of the results of the analysis with physical reasoning about the sequence of events in a real system.

To illustrate the range of diffusion time constants that may be encountered with conducting media conventionally used in electromechanical systems (7.1.28) is used to obtain the plots of diffusion time constants as functions of slab thickness shown in Fig. 7.1.6. For a slab of copper 1 cm thick the diffusion time constant is approximately 1 msec, whereas for a seeded combustion gas, 1 m thick, the time constant is approximately 10 μ sec. Note that although the conductivity of silicon iron is more than an order of magnitude less than that of copper the diffusion time is longer because the permeability is so high. [In general $\mu_0 \rightarrow \mu$ in (7.1.28) for magnetizable materials.] It is this long diffusion time that makes lamination of iron cores necessary in ac equipment of all types.

Because magnetic diffusion is described basically by Maxwell's equations, it occurs in both lumped-parameter and continuum systems. The analyses of Section 5.1.3 described magnetic diffusion in lumped-parameter systems, although this terminology was not used there. It is often helpful when considering continuum systems to reason qualitatively in terms of lumped-parameter models. Consequently, we consider a lumped-parameter system that provides an approximation of the diffusion process just discussed.

The conducting block of Fig. 7.1.1 is replaced with two perfectly conducting sheets short-circuited at one end and connected at the other end by a conductance G , as shown in Fig. 7.1.7. If, as before, we ignore fringing, the flux densities B_1 and B_2 are related to the currents by

$$B_2 = \frac{\mu_0 N i_2}{w}, \quad (7.1.31)$$

$$B_1 = \frac{\mu_0 N i_2}{w} - \frac{\mu_0 i_1}{w}, \quad (7.1.32)$$

where i_2 is the coil current in Fig. 7.1.1. Then the flux linked by the terminals

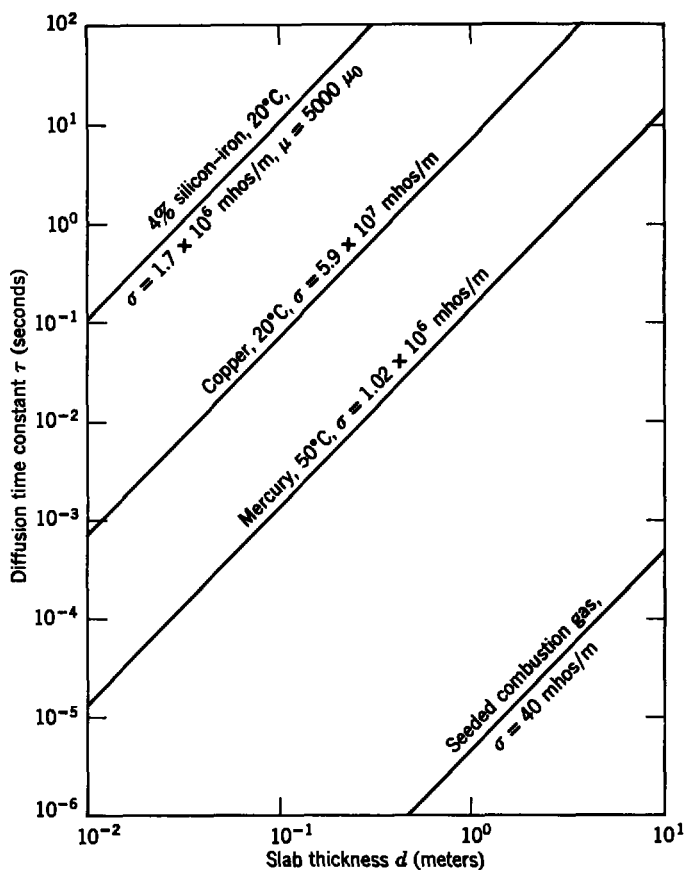


Fig. 7.1.6 Diffusion time constants $\mu\sigma d^2/\pi^2$.

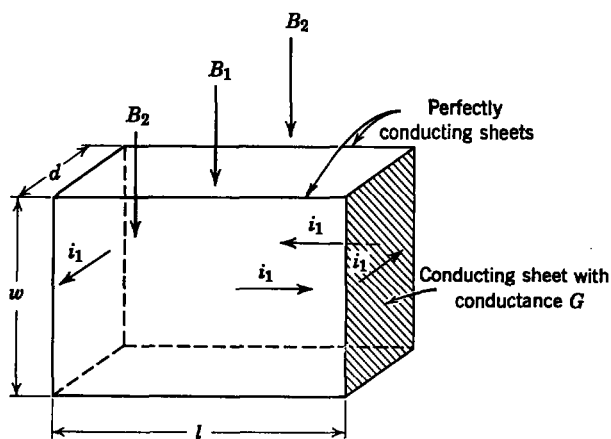


Fig. 7.1.7 A one-turn inductor used to replace the conducting block in Fig. 7.1.1.

of the inductor formed by the conducting sheets is

$$\lambda_1 = -dlB_1 = Li_1 - Mi_2, \quad (7.1.33)$$

where

$$L = \frac{dl\mu_0}{w},$$

$$M = \frac{dl\mu_0 N}{w}.$$

The circuit equation is then

$$GL \frac{di_1}{dt} + i_1 = GM \frac{di_2}{dt}. \quad (7.1.34)$$

In our example the current i_2 is a step function of magnitude I . Hence the solution to (7.1.34) for i_1 is

$$i_1 = INe^{-t/GL}, \quad (7.1.35)$$

where the definitions of L and M have been used.

The flux densities follow from (7.1.31) and (7.1.32). Outside the conductors for $t > 0$,

$$B = B_2 = \frac{\mu_0 NI}{w}, \quad (7.1.36)$$

whereas between the conductors

$$B = B_1 = \frac{\mu_0 NI}{w} (1 - e^{-t/GL}). \quad (7.1.37)$$

This distribution of flux is shown in Fig. 7.1.8. When the switch is closed, there is no magnetic field between the conductors and a current exists to exclude the field. Then the interior region fills up with magnetic field at a

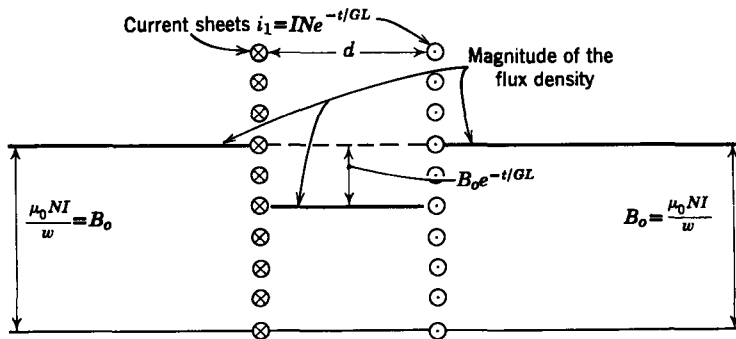


Fig. 7.1.8 Distribution of flux density as predicted by a lumped-parameter model.

rate characterized by the time constant GL . These distributions of current and flux density should be compared with those shown in Figs. 7.1.4 and 7.1.5.

If the equivalent G were computed as the dc conductance of the two sheets with length l and width w by using as the thickness $2d/\pi^2 = 0.203d$, we have

$$G = \left[\frac{\sigma(2d/\pi^2)w}{2l} \right],$$

and using the value of L (7.1.33) the time constant is

$$GL = \frac{\mu_0 \sigma d^2}{\pi^2}.$$

or that predicted by the diffusion equation as the fundamental time constant of the system (7.1.28). Obviously, we would not know what thickness to use unless we had solved the exact problem, hence our lumped-parameter model can at best serve to interpret the diffusion process in terms of familiar lumped-parameter concepts. For engineering purposes, however, judiciously chosen lumped parameters provide a convenient and powerful model for many situations.

7.1.2 Diffusion and Steady Motion

As pointed out in the introduction, if the magnetic Reynolds number (7.0.4) is significant compared with unity, material motion can be responsible for altering current paths, hence redistributing the magnetic field. This type of phenomenon is best emphasized by considering situations in which the fields and motion are in a steady-state condition. Then the appropriate expression for the flux density is (7.1.12) with the velocity $\mathbf{v}$ independent of time.

7.1.2a Steady-State in the Fixed Frame

The system depicted schematically in Fig. 7.1.9 is an example of the kind of steady-state diffusion problem we must consider when studying magneto-hydrodynamic (MHD) generators (see Chapter 12). It involves a continuous strip of material with constant σ , ϵ_0 , μ_0 which slides with constant velocity between a parallel pair of highly conducting electrodes. The sliding strip makes perfect electrical contact with the electrodes. The dimensions and coordinate system used are shown in Fig. 7.1.9. The system is excited at the end $z = l$ by two current sources that supply a total current I to the electrodes. We wish to find the distribution of magnetic flux density in the sliding strip and study how it varies with system parameters.

To simplify the problem we assume that l/d and w/d are large enough

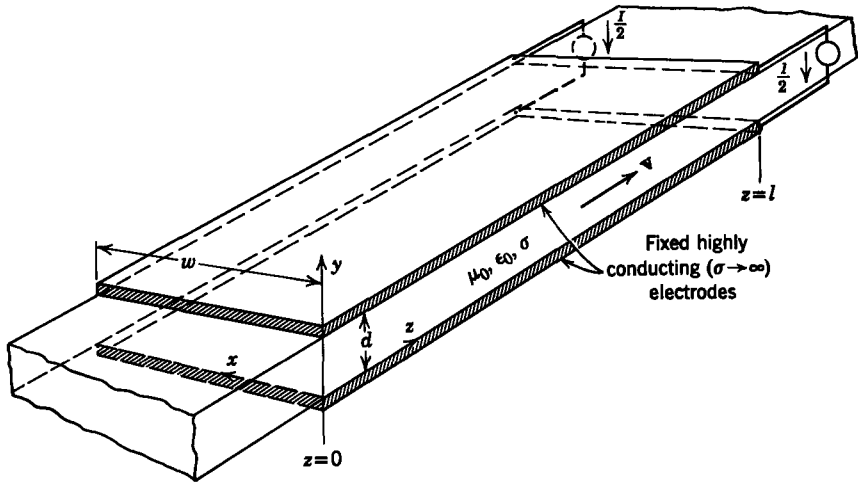


Fig. 7.1.9 Geometry for studying diffusion in the presence of steady motion.

that end and edge effects can be neglected.* Thus the flux density will have only an x -component

$$\mathbf{B} = \mathbf{i}_x B_x, \quad (7.1.38)$$

the electric field intensity and current density will have only y -components

$$\mathbf{J} = \mathbf{i}_y J_y, \quad (7.1.39)$$

$$\mathbf{E} = \mathbf{i}_y E_y, \quad (7.1.40)$$

and the variables will not be functions of x or y ,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0.$$

We have already specified that the strip is moving with constant velocity in the z -direction

$$\mathbf{v} = \mathbf{i}_z v_z. \quad (7.1.41)$$

Finally, interest is confined to a steady-state problem ($\partial/\partial t = 0$).

With the given specifications, the x -component of (7.1.12) is

$$\frac{d^2 B_x}{dz^2} - \mu_0 \sigma v_z \frac{dB_x}{dz} = 0. \quad (7.1.42)$$

We have written total derivatives because B_x is a function only of z .

* Standard practice in the analysis of MHD machines is to assume no fringing, solve the problem, and then add end effects as a perturbation. The end effect analysis is done by using conformal mapping techniques. See, for example, G. W. Sutton and A. Sherman, *Engineering Magnetohydrodynamics*, McGraw-Hill, New York, 1965, pp. 504–508.

Using the excitation of Fig. 7.1.9 and the boundary conditions of Table 6.1*, we arrive at the conditions on the flux density

$$\text{at } z = 0, \quad B_x = 0, \quad (7.1.43a)$$

$$\text{at } z = l, \quad B_x = \frac{\mu_0 I}{w}. \quad (7.1.43b)$$

First consider the system with the velocity equal to zero. The solution for the flux distribution is

$$B_x = \frac{\mu_0 I}{w} \frac{z}{l}. \quad (7.1.44)$$

The distribution of current density is found from the y -component of (7.1.7),

$$J_y = \frac{1}{\mu_0} \frac{dB_x}{dz}, \quad (7.1.45)$$

and for zero velocity it is

$$J_y = \frac{I}{wl}. \quad (7.1.46)$$

This uniform current density and linearly varying flux density are what we expect in the absence of motion

When the velocity is finite, solutions for (7.1.42) are of the form

$$B_x = C e^{rz}$$

Substitution into (7.1.42) and cancellation of common factors gives

$$r^2 - \mu_0 \sigma v_z r = 0.$$

The solutions for this equation are

$$r = \mu_0 \sigma v_z, 0,$$

which yield the general solution for B_x

$$B_x = C_1 + C_2 e^{R_m(z/l)}, \quad (7.1.47)$$

where we have again defined the *magnetic Reynolds number* R_m as

$$R_m = \mu_0 \sigma v_z l. \quad (7.1.48)$$

The significance of this number in the present context is discussed later.

Using the boundary conditions of (7.1.43), the constants C_1 and C_2 in (7.1.47) are evaluated to obtain the solution

$$B_x = \frac{\mu_0 I}{w} \left[\frac{e^{R_m(z/l)} - 1}{e^{R_m} - 1} \right]. \quad (7.1.49)$$

* See Appendix E.

The distribution of current density is evaluated by using this result in (7.1.45):

$$J_y = \frac{I}{wl} \left[\frac{R_m e^{R_m(z/l)}}{e^{R_m} - 1} \right]. \tag{7.1.50}$$

Note that as $R_m \rightarrow 0$ these expressions reduce to those of (7.1.44) and (7.1.46), which were derived by assuming zero velocity.

The flux density and current density are plotted as functions of position for several values of magnetic Reynolds number R_m in Fig. 7.1.10. It is evident from these curves that the higher R_m , the more flux density and current density are distorted, both being decreased at the entrance to the electrodes. It was shown in Section 7.1.1 that when we try to change the flux density in a conductor eddy currents flow to oppose the flux change. The eddy currents decay and the flux diffuses into the conductor in the process characterized by a diffusion time constant.

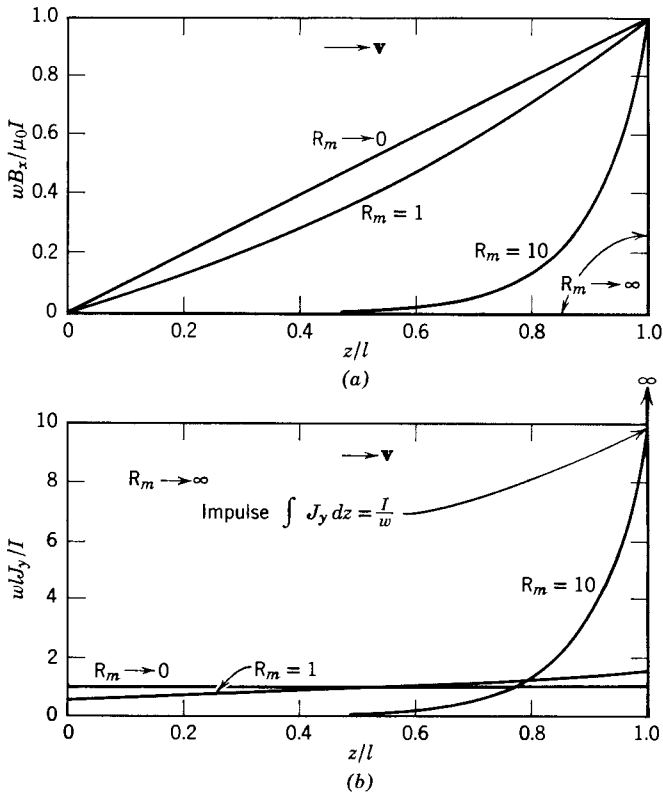


Fig. 7.1.10 Distribution of flux density and current density in the system of Fig. 7.1.9: (a) flux density; (b) current density.

If, in the system of Fig. 7.1.9, we view what happens to a particular sample of the moving strip, we see the competing processes of the sample trying to sweep the flux and current in the direction of motion and the diffusion of flux and current in the opposite direction. The magnitude of the magnetic Reynolds number is an indicator of the relative effectiveness of the two processes; a large R_m indicates a relatively slow diffusion process. We can interpret the magnetic Reynolds number as being proportional to the ratio of the diffusion time constant to the time it takes a sample of material to traverse the length of the electrodes. Equation 7.1.28 is used to write the diffusion time constant for the length l as

$$\tau_d = \frac{\mu_0 \sigma l^2}{\pi^2}.$$

The time taken for a sample of material to traverse the length l with speed v_z is

$$\tau_t = \frac{l}{v_z}.$$

The ratio of diffusion time constant to traversal time is then

$$\frac{\tau_d}{\tau_t} = \frac{\mu_0 \sigma v_z l}{\pi^2} = \frac{R_m}{\pi^2}.$$

The magnetic Reynolds number was defined in a particular way for this example. We find in our further studies of quasi-static magnetic field systems that it is useful to define magnetic Reynolds numbers in other ways; the fundamental interpretation, however, is always the same, namely that a magnetic Reynolds number indicates the relative importance of the diffusion ($\nabla^2 \mathbf{B}$) and convection $\nabla \times (\mathbf{v} \times \mathbf{B})$ terms in (7.1.12).

7.1.2b Steady State in the Moving Frame

The steady-state diffusion situation considered in the preceding subsection could be viewed from a frame moving with the conductor, in which case the magnetic field would be in a transient condition but the velocity v_z would be zero. Here, we turn this relationship around and consider a problem that is in a transient state in the fixed frame but in the steady state in the moving frame. The example provides further insight into the significance of the magnetic Reynolds number.

In the introduction to this chapter (see Fig. 7.0.1c) we discussed an example in which a block of conducting material moves with the velocity V between short-circuited, perfectly conducting parallel plates. If the current excitation in that problem is used to establish the field and is then instantaneously removed, currents will continue to flow through the block and the shorting

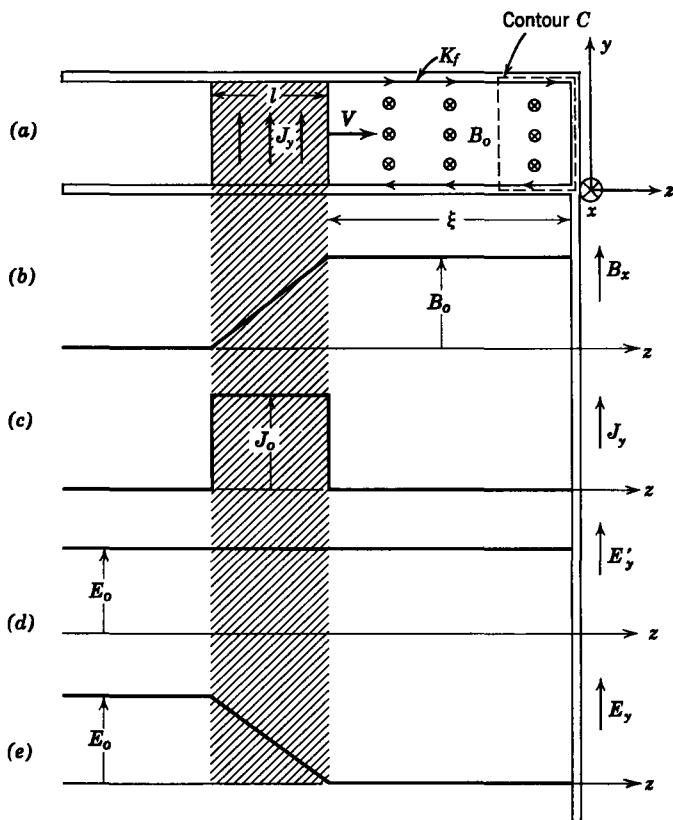


Fig. 7.1.11 The conducting block moves with just the velocity V required to maintain the flux density B_0 constant. This velocity is such that $R_m = 1$.

end plate, as shown in Fig. 7.1.11a. Then there is a magnetic field ahead of the moving block of conductor but none behind.

Now, if the block were stationary, the magnetic field within the “inductor” formed by the block and the perfectly conducting plates would decay, much as in the example in Section 7.1.1. It is possible, however, for the block to move to the right with a sufficient velocity V to maintain the flux density B_0 ahead of the block constant. Under this circumstance the fields can be found by observing that in the frame of the block the situation is no different than for a static case in which the field to the right is constant and the field to the left is zero; that is, in the frame of the moving block,

$$\mathbf{J}' = J_y' \mathbf{i}_y = \text{constant} = \sigma E_y' \mathbf{i}_y \quad (7.1.51)$$

for $-(\xi + l) < z < -\xi$.

Remember that the flux density generated by this current is the same in either the fixed or moving frame (see Table 6.1)*. Hence

$$\mathbf{B}' = \mathbf{B} = \mathbf{i}_x B_o \frac{(z + \xi + l)}{l} \quad (7.1.52)$$

for $-(\xi + l) < z < -\xi$ and

$$\mathbf{B} = B_o \mathbf{i}_x \quad (7.1.53)$$

for $-\xi < z < 0$. The flux density and current density are sketched in Fig. 7.1.11*b, c*. Here the gap length $\xi = -Vt + \text{constant}$, where V is a constant.

There is no time rate of change of the flux enclosed by a fixed contour (such as C in Fig. 7.1.11*a*) that passes through the perfectly conducting plates and the region ahead of the moving block. Hence

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \quad (7.1.54)$$

around this contour and E_y ahead of the block is zero. It follows from the transformation for the electric field (see Table 6.1)* that

$$E'_y(z = -\xi) = (\mathbf{v} \times \mathbf{B})_y = VB_o. \quad (7.1.55)$$

A sketch of the distribution of E'_y is shown in Fig. 7.1.11*d*.

Of course, the surface current density K_f (which is equal to K'_f in a magnetic field system) is related to the flux density B_o (Ampère's law)

$$B_o = \mu_o K_f \quad (7.1.56)$$

and the surface current density in the perfectly conducting plates is also related to the current density in the block

$$K_f = J_y l. \quad (7.1.57)$$

If we now combine (7.1.51) with these last three equations, we obtain

$$\sigma V B_o = \frac{B_o}{\mu_o l}, \quad (7.1.58)$$

from which it follows that to obtain a constant magnetic flux density ahead of the moving block we must have

$$\mu_o \sigma V l = R_m = 1. \quad (7.1.59)$$

The distribution of electric field in the moving and fixed frames is shown in Fig. 7.1.11*d, e*. Note that in the region to the left $E'_y = E_y$, because there is no magnetic field in that region.

Now that the fields have been found, it is possible to write them in the fixed frame [in which they are functions of (z, t)] and see that they satisfy (7.1.9).

Of course, what we have found is the solution for a particular value of V . The postulated steady-state condition in the moving frame prevails only if

* See Appendix E.

(7.1.59) is satisfied. We know that the field decays to zero if the block is stationary, hence we conclude that if the velocity V is less than required to make $R_m = 1$ the field ahead of the block will decay. More interesting is the fact that if V exceeds the value required to make $R_m = 1$ the magnetic field ahead of the block will increase. In this case the block literally acts as a magnetic piston that compresses the magnetic field in the region bounded by the perfectly conducting plates and increases the flux density. This provides a mechanism for mechanical amplification of flux density which has been used with chemical explosives to achieve amplification by more than a factor of 10 (from 2 Wb/m² to 30 Wb/m²).*

To establish a numerical idea when the conditions of (7.1.59) are met, it is helpful to recognize that this condition can also be written as

$$\frac{l}{V} = \pi^2 \tau,$$

where τ is the diffusion time, defined by (7.1.28), and plotted in Fig. 7.1.6 for several conductors as a function of the material thickness. The quantity l/V is the time required for the material to traverse a distance equal to its thickness. Hence, if we have a copper block with the thickness of 10 cm, it follows from Fig. 7.1.6 that a velocity of

$$V = \frac{l}{\pi^2 \tau} = 0.13 \text{ m/sec}$$

is required just to maintain the flux ahead of the block constant.

The use of a solid block in motion to concentrate magnetic flux mechanically implies a transient situation. For steady-state flux concentration a system has been proposed and tested in which the steady flow of a liquid metal conductor with $R_m > 1$ concentrates the flux.†

Other instances of flux concentration by mechanical motion occur in plasma devices used in research on controlled thermonuclear reactions. In these devices energy from a capacitor bank drives and heats a plasma and the mechanical motion of the plasma compresses the magnetic field and increases the flux density.‡

In a very different physical situation superconducting solids have essentially infinite electrical conductivity; hence at even low velocities the magnetic

* C. M. Fowler et al., "Flux Concentration by Implosion," *High Magnetic Fields*, H. Kolm et al., eds. M.I.T. Press, Cambridge, Mass., and Wiley, New York, 1962, pp. 269–276.

† H. H. Kolm and O. K. Mawardi, "Hydromagnet: A Self-Generating Liquid Conductor Electromagnet," *J. Appl. Phys.*, **32**, No. 7, 1296–1304 (July 1961).

‡ D. J. Rose and Melville Clark, Jr., *Plasmas and Controlled Fusion*, M.I.T. Press and Wiley, New York, 1961, p. 360.

Reynolds number R_m is essentially infinite. As a result, superconducting solids are used for mechanical amplification of flux density.*

7.1.3 The Sinusoidal Steady State in the Presence of Motion

Some of the most significant practical applications of magnetic diffusion phenomena are made possible by introducing two characteristic dynamical times into a single system; for example, in Fig. 7.1.12 a thin sheet of conducting metal (conductivity σ) moves to the right with the constant velocity V .

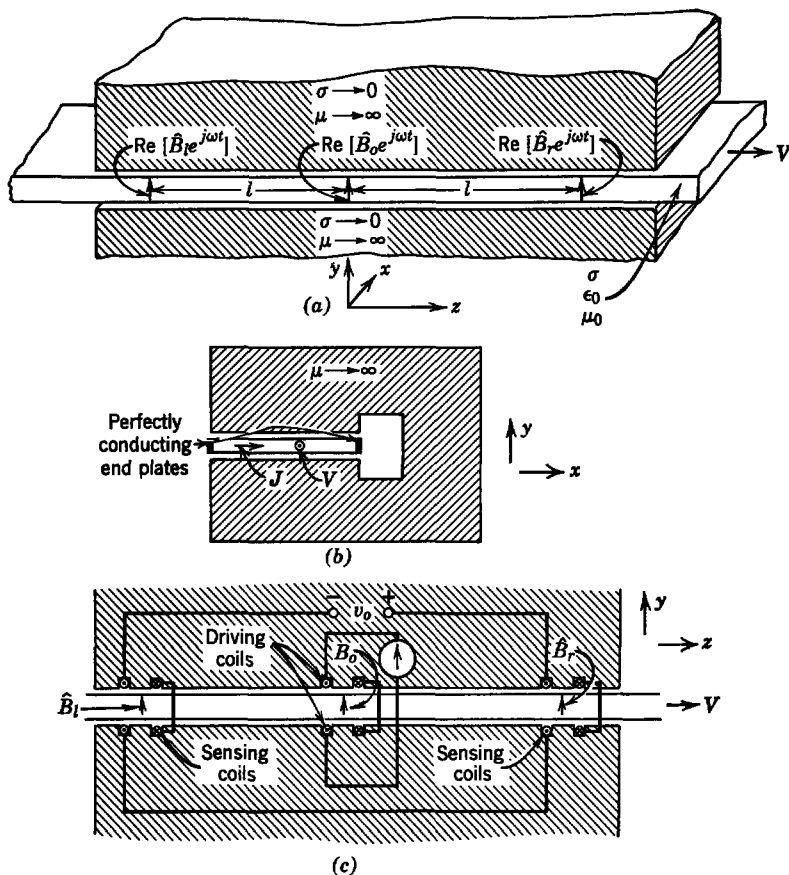


Fig. 7.1.12 (a) A thin slab of conductor moves to the right with velocity V in the air gap of a magnetic circuit; (b) end view of magnetic circuit showing a slab with perfectly conducting end plates moving out of paper; (c) excitation and detection circuits used to measure velocity V of the slab.

* P. S. Swartz and C. H. Rosner, "Characteristics and a New Application of High-Field Superconductors," *J. Appl. Phys.*, 33, No. 7, 2292-2300 (July 1962).

External coils are used to impose the magnetic flux density

$$\mathbf{B} = \mathbf{i}_y \operatorname{Re} \left(\hat{B}_0 e^{i\omega t} \right) \quad (7.1.60)$$

at $z = 0$, as shown in Fig. 7.1.12*a*, *b*, *c*. In addition, coils are arranged symmetrically upstream and downstream from the excitation of (7.1.60). These coils are connected to a high impedance voltmeter, which measures the rms value of v_o , as shown in Fig. 7.1.12*c*. The coils are connected in series and in such a way that the voltage v_o is proportional to the difference between the time rates of change of B_r and B_l . With no motion, the device is symmetrical, and we expect that the output voltage v_o is zero. With motion we expect from the results of the preceding section that the conductor will tend to increase the flux density B_r and decrease B_l . If this is the case, the voltage v_o is a measure of the material velocity. Hence the system shown in Fig. 7.1.12 has the possibility of being a device to measure the velocity of the moving conducting sheet without making mechanical or electrical contact with the sheet.

The problem is particularly appropriate to our present discussion because it can be modeled by a one-dimensional diffusion problem. The sheet is assumed to move in the air gap of an infinitely permeable magnetic circuit so that fields B_y are returned through the circuit. Then currents are in the x -direction through the sheet and for our present purposes we can think of the sheet as having perfectly conducting end plates (Fig. 7.1.12*b*) which allow the currents to return through the sheet at other positions along the z -axis.

There are two characteristic times for the dynamics of this situation. First the excitation has the period $2\pi/\omega$, and we know from Section 7.1.1 that the relative value of this time and the diffusion time will be important. In addition, the motion introduces the characteristic time l/V , and from the studies undertaken in Section 7.1.2 we expect that this time in relation to the diffusion time will also be important.

We are interested in sinusoidal steady-state conditions, hence we require an expression for $\mathbf{B}$ that includes both the effects of time varying fields and material motion. The appropriate equation is the y -component of (7.1.9), written with the assumption that $\partial/\partial x = \partial/\partial y = 0$. (Effects of the slots in which coils are placed are neglected.)

$$\frac{1}{\mu\sigma} \frac{\partial^2 B_y}{\partial z^2} = \frac{\partial B_y}{\partial t} + V \frac{\partial B_y}{\partial z}. \quad (7.1.61)$$

Note that these same assumptions require that a current be in the x -direction which from (7.1.7) is related to B_y by

$$J_x = -\frac{1}{\mu} \frac{\partial B_y}{\partial z}. \quad (7.1.62)$$

It is worthwhile to compare (7.1.61) to the one-dimensional diffusion equation for stationary media (7.1.15). The effect of the motion is to replace the time derivative with the convective derivative. Because B_y is the same, whether measured in the moving or the fixed frame, this result is reasonable. As we learned in Section 6.1, the terms on the right in 7.1.61 constitute the time rate of change of B_y for an observer moving with the material at the velocity V . Hence in the frame of the material (7.1.61) has the same physical significance as (7.1.15). In the present example, of course, the excitation is in the fixed frame and we cannot ignore the effect of the motion by solving the problem in the moving frame.

Because of the sinusoidal steady-state excitation (7.1.60), solutions to (7.1.61) are assumed to have the form

$$B_y = \text{Re} [\hat{B}(z)e^{j\omega t}]. \quad (7.1.63)$$

Substitution into (7.1.61) shows that this assumption is justified if

$$\frac{d^2 \hat{B}}{dz^2} - \mu\sigma V \frac{d\hat{B}}{dz} - j\omega\mu\sigma \hat{B} = 0. \quad (7.1.64)$$

This ordinary constant-coefficient equation can be solved by assuming solutions of the form $\hat{B}(z) = e^{-jkz}$. Then

$$k^2 - jk\mu\sigma V + j\omega\mu\sigma = 0. \quad (7.1.65)$$

Note that we could have obtained this expression, which is called the *dispersion equation*, by assuming both the temporal and spacial dependence at the outset and letting

$$B_y(z, t) = \text{Re} [\hat{B}_y e^{j(\omega t - kz)}]. \quad (7.1.66)$$

Solutions of this form satisfy (7.1.61), provided that ω and k are related by (7.1.65). In the chapters that follow we make extensive use of solutions with this form. In the present situation ω is a given real number. Then it is clear from (7.1.65) that the *wavenumber* k is complex. In solving (7.1.65), it is convenient to normalize k to the length l . There are then two possible wavenumbers ($k = k^-$ and $k = k^+$) for a given frequency ω .

$$l(k^\mp) = \frac{jR_m}{2} \pm j \left[\left(\frac{R_m}{2} \right)^2 + 2j \left(\frac{l}{\delta} \right)^2 \right]^{1/2} \quad (7.1.67)$$

Here the magnetic Reynolds number

$$R_m = \mu\sigma V l \quad (7.1.68)$$

and the *skin depth* δ , defined by

$$\delta = \left(\frac{2}{\omega\mu\sigma} \right)^{1/2}, \quad (7.1.69)$$

have been introduced. By these definitions R_m represents the effect of the material velocity V , whereas δ brings in the excitation frequency ω . As can be seen from (7.1.67), the diffusion with no motion ($R_m = 0$) is described by δ , and it is worthwhile to consider this limiting case so that the effect of the motion can be fully appreciated.

7.1.3a Sinusoidal Steady State with No Motion (Skin Effect)

The sinusoidal steady-state diffusion of a magnetic field into a conductor assumes considerable importance outside the area of electromechanics; for example, the ac resistance of a stationary wire is significantly dependent on frequency,* as are the losses in the walls of a waveguide.† In each of these cases the phenomenon is conventionally referred to as *skin effect* because the currents in the conductor tend to crowd into a region near the surface.

Skin-effect is a magnetic diffusion phenomenon. Section 7.1.1 was concerned with transient skin effect. It is exemplified here as the limiting case in which $V = 0$ ($R_m = 0$). Then from (7.1.67) the wavenumbers are (remember that $\sqrt{j} = (1 + j)/\sqrt{2}$)

$$k^{\mp} = \mp \frac{1 - j}{\delta}. \quad (7.1.70)$$

Now, if we substitute these wavenumbers into (7.1.66), we find that one of the solutions increases exponentially with z as the other decreases. For the present purposes we assume that the extremities of the system in the z -direction are sufficiently remote that the fields there have decayed to zero. Then, in the region to the right in Fig. 7.1.12, the appropriate solution is (here the wavenumber k^+ is used)

$$B_y = \text{Re} [B_0 e^{-z/\delta} e^{j(\omega t - z/\delta)}]. \quad (7.1.71)$$

From this expression it is clear that there are two parts to the solution, as shown in Fig. 7.1.13. To illustrate some of the properties of this *diffusion wave*, the two factors of (7.1.71) are plotted in Fig. 7.1.13. Here, the excitation constant B_0 is assumed to be real. Then the factor $\cos(\omega t - z/\delta)$ represents a wave of constant amplitude, traveling in the positive z -direction. We define a *phase velocity* v_p for this constant-amplitude wave by calculating the velocity of a point of constant phase. The phase of an arbitrary point on the wave (e.g., point A in Fig. 7.1.13) remains constant for an observer at the position z such that

$$\omega t - z/\delta = \text{constant}. \quad (7.1.72)$$

* S. Ramo, J. R. Whinnery, and T. Van Duzer, *Fields and Waves in Communication Electronics*. Wiley, New York, 1965, p. 194.

† *Ibid.* p. 379.

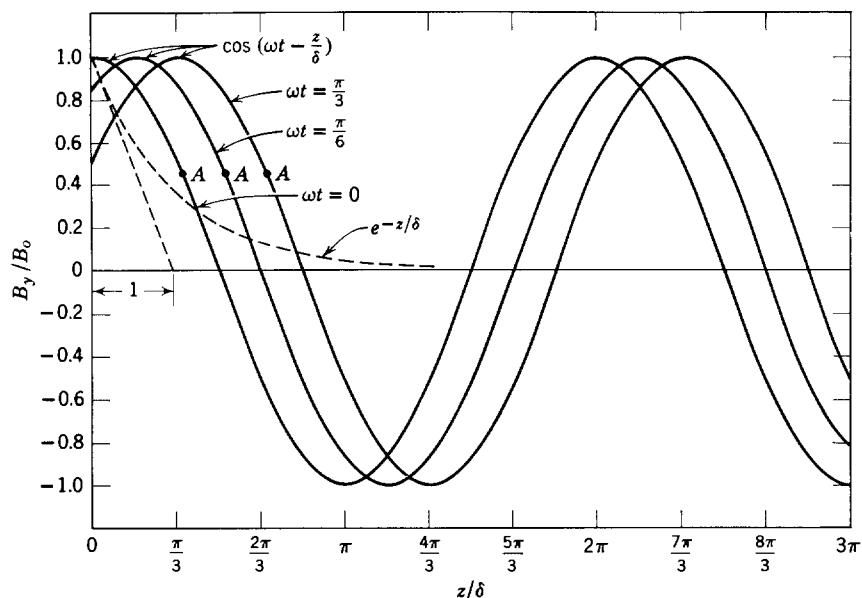


Fig. 7.1.13 Components of a diffusion wave.

To find the phase velocity we differentiate this expression with respect to time to obtain

$$v_p = \omega\delta. \quad (7.1.73)$$

Thus the point *A* or any other point of constant phase on the constant amplitude part of the wave will travel with a phase velocity $\omega\delta$.

The two factors plotted separately in Fig. 7.1.13 are multiplied and replotted to show the complete diffusion wave in Fig. 7.1.14. With the attenuation it is difficult to identify an arbitrary point of constant phase on the wave.

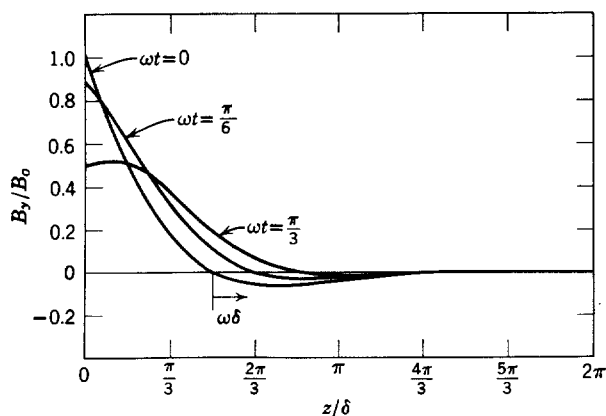


Fig. 7.1.14 A diffusion wave.

We can, however, always be unambiguous by defining the phase velocity in terms of the velocity of a zero-crossing.

A salient property of the diffusion wave is that the spacial rate of decay and the wavelength are both directly proportional to δ ; that is, as can be seen from Fig. 7.1.13, the wave envelope has an extrapolated decay distance equal to the skin depth δ and the wavelength $\lambda = 2\pi\delta$. Hence in Fig. 7.1.14 it is difficult to discern more than one point of zero amplitude.

It is important to recognize that a diffusion wave describes the dynamics of an essentially dissipative interaction. As we shall see in Chapter 10, sinusoidal steady-state waves can be excited in cases in which decay (or evanescence) occurs but there is no dissipation of energy.

In Section 10.1.4 we shall have occasion to return to diffusion waves in another context. There the “mushy” nature of these waves is emphasized.

Because the skin depth δ is such a critical parameter in the electromagnetic

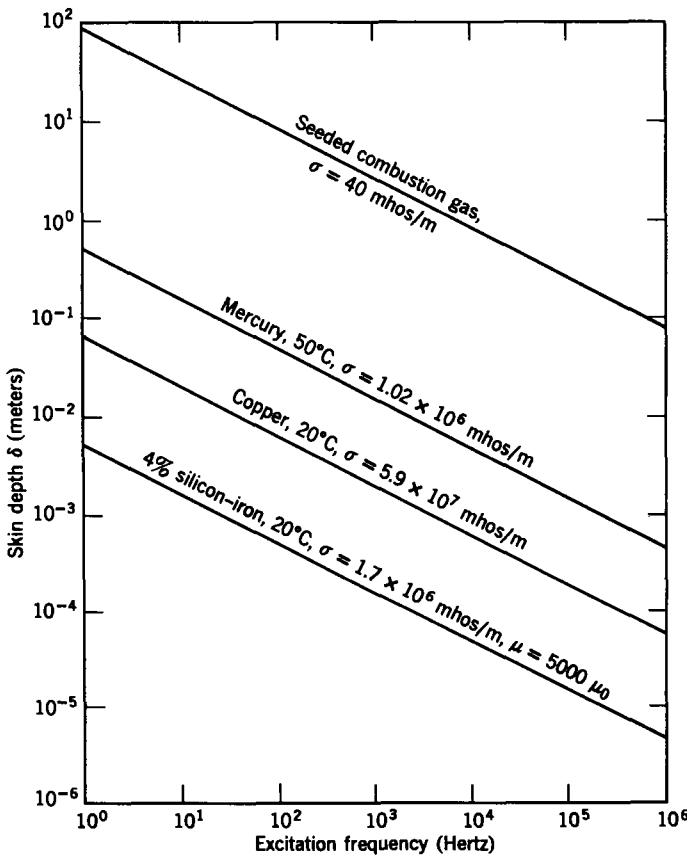


Fig. 7.1.15 Skin depth as a function of frequency for several typical materials.

behavior of conducting materials, it is worthwhile to calculate the skin depth for some typical materials. Equation 7.1.69 is used with the material constants in Fig. 7.1.6 to plot the curves of skin depths as functions of frequency shown in Fig. 7.1.15. Note the tremendous range of skin depths obtainable with useful materials; for example, in silicon iron, even at the low frequency of 1 Hz the skin depth is approximately $\frac{1}{2}$ cm. It is quite clear from these plots that many situations exist in which it is appropriate to assume the material infinitely conducting ($\delta \approx 0$). On the other hand, it is also clear that situations exist in which it is appropriate to neglect the skin effect and the induced currents. A comparison should be made between Figs. 7.1.15 and 7.1.6 because the skin depth and the diffusion time constant are closely related.

7.1.3b The Effect of Motion

To assess the effect of material motion on diffusion waves, the wavenumbers given by (7.1.67) must be considered for finite values of R_m . To determine the relative magnitudes of the real (k_r) and imaginary (k_i) parts of the wavenumber, it is helpful to construct the solution of (7.1.67) graphically (Fig. 7.1.16). From this construction it is clear that the real parts of

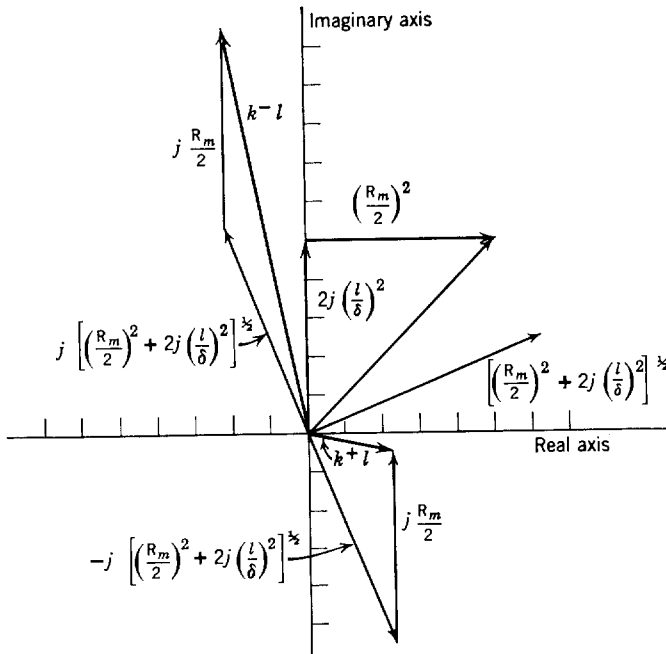


Fig. 7.1.16 Graphical construction of solutions to (7.1.67) for the normalized complex wavenumbers $k+l$ and $k-l$.

k^+ and k^- are negatives,

$$k_r^- = -k_r^+, \quad (7.1.74)$$

that k_i^- is positive and k_i^+ is negative,

$$k_i^- > 0 > k_i^+, \quad (7.1.75)$$

and that the magnitude of k_i^- is greater than that of k_i^+ ,

$$|k_i^-| > |k_i^+|. \quad (7.1.76)$$

The arguments in Section 7.1.3a concerning the extent of the system are then used to arrive at the solutions

$$B_y = \text{Re} [\hat{B}_o e^{-|k_i^+|z} e^{j(\omega t - |k_r^+|z)}] \quad (7.1.77)$$

for $z > 0$ and

$$B_y = \text{Re} [\hat{B}_o e^{|k_i^-|z} e^{j(\omega t + |k_r^-|z)}] \quad (7.1.78)$$

for $z < 0$. From these solutions it is clear that to the right a diffusion wave still propagates in the $+z$ -direction with a phase velocity ω/k_r^+ that is modified by the material motion. Moreover, in spite of the motion, a wave propagates to the left with this same phase velocity. The effect of the motion is most apparent in the rate of decay for these waves. The spacial rate of attenuation to the right is decreased by the motion, whereas that to the left is increased. An instantaneous view of the magnetic flux density amplitude to the right and left would appear as shown in Fig. 7.1.17.

As a practical application of the magnetic diffusion phenomenon we have considered in this section, the device shown in Fig. 7.1.12 is now used to measure the velocity of the material. The output voltage from the sensing coils has the form $v_o = \text{Re} (\hat{v}_o e^{j\omega t})$. The coils are connected so that in the absence of motion there is no output signal; that is,

$$\hat{v}_o = j\omega A [\hat{B}_y(l) - \hat{B}_y(-l)], \quad (7.1.79)$$

where A is a geometric constant that depends on the coil area and number of turns. The $j\omega$ in (7.1.79) accounts for the voltage being the time rate of change of the flux. (Here we assume that not only do the coils support a negligible current so that they do not disturb the magnetic field but they have dimensions in the z -direction that are small compared with a wavelength $2\pi/k_r$.) In substituting the solutions given by (7.1.77) and (7.1.78) into (7.1.79), it is convenient to use first the complex wavenumbers k^+ and k^- ,

$$\hat{v}_o = j\omega A \hat{B}_o (e^{-jk^+l} - e^{+jk^-l}) \quad (7.1.80)$$

because then we can use (7.1.67) to write this expression as

$$\hat{v}_o = j\omega 2A \hat{B}_o \sinh \left(\frac{R_m}{2} \right) \exp \left[- \left[\left(\frac{R_m}{2} \right)^2 + 2j \left(\frac{l}{\delta} \right)^2 \right]^{1/2} \right]. \quad (7.1.81)$$

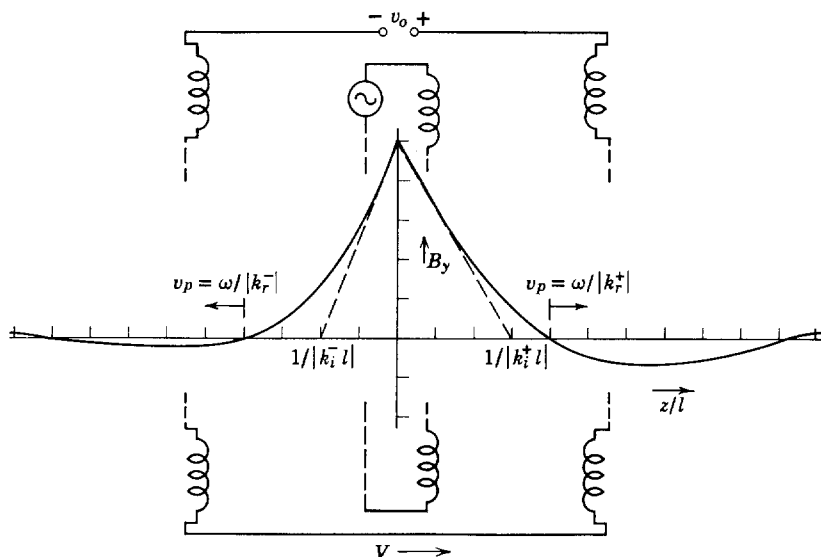


Fig. 7.1.17 Instantaneous distribution of magnetic flux density in the moving slab of conductor shown in Fig. 7.1.12. Points of constant phase move to the right and left with equal velocities but attenuation to the right is decreased by motion, whereas that to the left is increased. Note that detailed account of the excitation at $z = 0$ by the coil has not been made. Hence distribution of B_y at $z = 0$ has a discontinuous slope.

From this equation it is apparent that as R_m approaches zero, the output voltage from the sensing coils does also. For small values of $R_m/2$ compared with unity and l/δ and for fixed frequency ω the output voltage takes the form

$$\hat{v}_o = (\text{constant}) R_m; \quad (7.1.82)$$

that is, in this range of parameters, the voltage v_o is directly proportional to the velocity V of the moving conducting sheet.

The mechanism made available by this phenomenon for measuring the velocity of a material is attractive because it does not require mechanical or electrical contact with the moving medium. Also by varying the frequency the output voltage can be made to give information about the material conductivity.

Devices that exploit the interaction described here are used to measure rotational velocity with drag-cup tachometers* and in flowmeters to measure

* C. W. Blachford, "Induction Cup Parameters from Electromagnetic Field Theory and Experimental Analysis," *IEEE Trans. Power Apparatus Systems*, **PAS. 84**, No. 11, 189-193 (November 1965).

the velocity of liquid metals.* Note that the output voltage is proportional to the conductivity of the moving medium ($R_m = \mu_0 \sigma V l$). Hence there is a lower limit on the material conductivity that will lead to a useful output voltage.

7.1.4 Traveling Wave Diffusion in Moving Media

In Section 7.1.3 attention was given to the effect of material motion on the diffusion of magnetic fields excited in the sinusoidal steady state. These same ingredients are also present in the topic to be undertaken in this section, with attention now being given to currents and forces induced in the moving material. Figure 7.1.18 shows a slab of conducting material that can be moving to the right with velocity V . Adjacent to the lower surface, coils are distributed and interconnected in such a way that a surface current

$$\mathbf{K}_f = \text{Re} [K e^{j(\omega t - kz)} \mathbf{i}_x], \quad (K \text{ real}), \quad (7.1.83)$$

is excited. Hence at any point in the x - z plane this current varies sinusoidally with the frequency ω , whereas at any instant it varies sinusoidally in space. Points of constant phase on the current sheet move to the right with the velocity ω/k , where both ω and k are given.

This example has attributes that make it possible to demonstrate the basic mechanism responsible for several practical magnetic induction-type interactions. As we shall see, currents are induced in the conducting material.

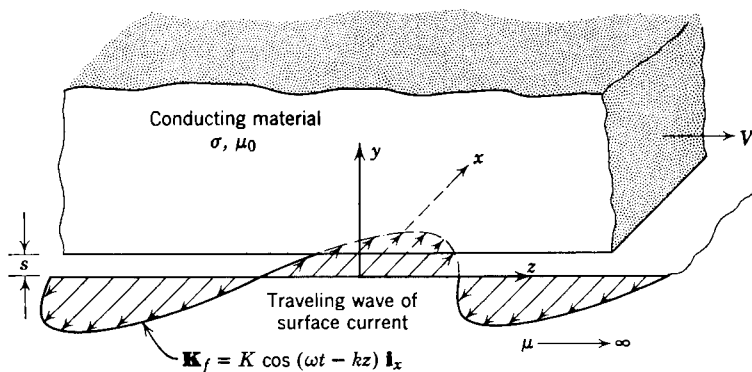


Fig. 7.1.18 A slab of conducting material moves in the z -direction with velocity V . Just below the slab sinusoidally distributed windings are driven by a traveling wave of surface current density which has the phase velocity ω/k to the right. It is assumed that an infinitely permeable material is just below the current sheet and that the extent of the conducting slab in the y -direction is many wavelengths $2\pi/k$.

* For a discussion of magnetohydrodynamic flow measuring devices see J. A. Shercliff, *The Theory of Electromagnetic Flow Measurement*, Cambridge University Press (1962). In particular, the interaction described is basic to material beginning on p. 103.

These currents, and the associated magnetic fields, can produce a time average force on the material in the y -direction. This type of force is often used to levitate solid and liquid metals. Other forces are induced in the z -direction, and under certain conditions this makes it possible to pull the conducting material in the z -direction. This induced force is the basis for the induction machine discussed in Section 4.1.6b and is often used to pump liquid metals.

One of our main objectives in this section is to establish further insight into the manner in which currents and the magnetic fields distribute themselves in a moving conducting medium. The magnetic flux density is given by (7.1.9), which has the y -component

$$\frac{1}{\mu\sigma} \left(\frac{\partial^2 B_y}{\partial y^2} + \frac{\partial^2 B_y}{\partial z^2} \right) = \frac{\partial B_y}{\partial t} + V \frac{\partial B_y}{\partial z} \quad (7.1.84)$$

Hence we assume that the field distribution does not depend on the x -coordinate, and B_y is predicted by a diffusion equation with the same form as (7.1.61), except that now the magnetic field diffusion occurs in two dimensions. Once the component B_y has been found from (7.1.84), the remaining component B_z can be found from the relation $\nabla \cdot \mathbf{B} = 0$, which requires

$$\frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0. \quad (7.1.85)$$

The magnetic field is driven by the surface current of (7.1.83). Hence we look for solutions with the same traveling wave dependence on (z, t) ; that is, we assume that the flux density and the current density take the forms

$$\begin{aligned} \mathbf{B} &= \text{Re} [(\hat{B}_y(y)\mathbf{i}_y + \hat{B}_z(y)\mathbf{i}_z)e^{j(\omega t - kz)}], \\ \mathbf{J} &= \text{Re} [\hat{J}_x(y)\mathbf{i}_xe^{j(\omega t - kz)}]. \end{aligned} \quad (7.1.86)$$

It follows from (7.1.84) that in order for the solutions to take these forms $\hat{B}_y$ must satisfy the equation

$$\frac{d^2 \hat{B}_y}{dy^2} - \alpha^2 \hat{B}_y = 0, \quad (7.1.87)$$

where

$$\begin{aligned} \alpha &= k\sqrt{1 + jS}, \\ S &= \frac{\mu\sigma}{k^2}(\omega - kV). \end{aligned}$$

The solutions to (7.1.87) are $\exp \pm \alpha y$. We have defined α as having a positive real part. The y -dimension of the slab is assumed to be very large compared with a wavelength $2\pi/k$, and it is therefore appropriate to assume that the fields approach zero as $y \rightarrow \infty$ with the dependence

$$\hat{B}_y = \hat{A}e^{-\alpha y}. \quad (7.1.88)$$

From (7.1.85) and the assumed form of solutions $\hat{B}_z$ is given by

$$\hat{B}_z = \frac{1}{jk} \frac{d\hat{B}_y}{dy} = -\frac{\alpha\hat{A}}{jk} e^{-\alpha y}. \quad (7.1.89)$$

The constant $\hat{A}$ is determined by the boundary condition imposed by the current sheet at $y = 0$. For purposes of simplicity it is assumed that the distance s between the current sheet and the lower surface of the conducting slab is small compared with the wavelength $2\pi/k$ and that the current sheet is bounded from below by a highly permeable material. This means that

$$\hat{B}_z(0) = \mu_0 K. \quad (7.1.90)$$

These last two equations serve to define the constant $\hat{A}$, and it follows that the magnetic flux density in the moving conductor is

$$\mathbf{B} = \text{Re} \left[\mu_0 K \left(\frac{-jk}{\alpha} \mathbf{i}_y + \mathbf{i}_z \right) e^{-\alpha y} e^{j(\omega t - kz)} \right]. \quad (7.1.91)$$

The current density implied by this flux density can be found from the relation $\mathbf{J} = \nabla \times \mathbf{H}$. In the present two-dimensional situation the only component of $\mathbf{J}$ is

$$J_x = \frac{1}{\mu_0} (jk\hat{B}_y - \alpha\hat{B}_z), \quad (7.1.92)$$

and it follows from (7.1.91) that this is

$$J_x = \text{Re} \left[\frac{-jKk^2 S}{\alpha} e^{-\alpha y} e^{j(\omega t - kz)} \right] \quad (7.1.93)$$

The component B_x of the flux density and the current density J_x are distributed in the conducting slab, as shown in Fig. 7.1.19. It is clear that the flux and current distributions are strongly dependent on the parameter S . To understand the significance of this parameter recall from Section 7.1.1 that the magnetic diffusion time based on the wavelength $\lambda = 2\pi/k$ is (within a factor of 4)

$$\frac{\mu\sigma}{k^2},$$

whereas the rate of change with respect to time of B_y (say) for an observer moving with the velocity V of the slab is

$$\left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial z} \right) B_y \rightarrow j(\omega - kV) \hat{B}_y;$$

that is, $\omega - kV$ is the frequency of the magnetic flux density for an observer moving with the conducting slab. Hence S is (proportional to) the ratio of the magnetic diffusion time to the period of excitation in the frame of the moving medium.

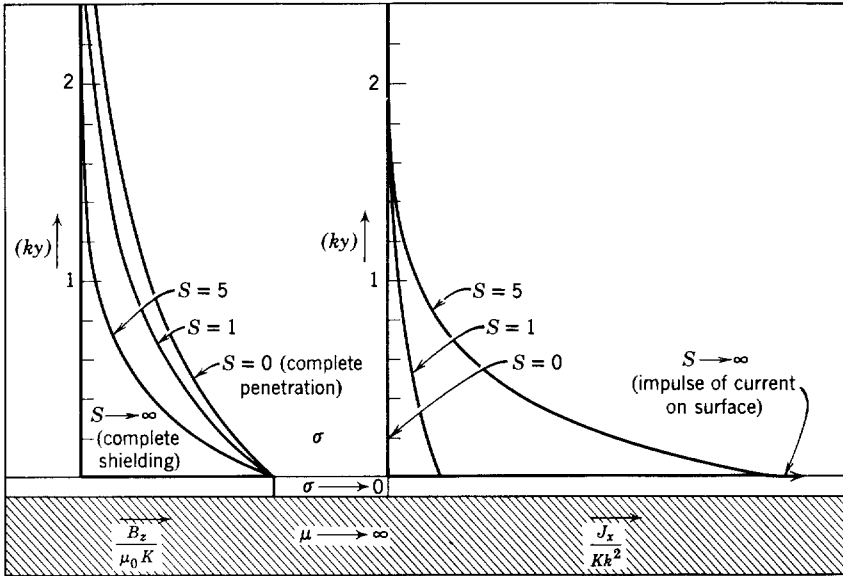


Fig. 7.1.19 Distribution of the instantaneous magnitudes of B_z and J_x in the moving conducting slab of Fig. 7.1.18; S is defined following (7.1.87).

Because the phase velocity of the traveling wave is ω/k , S is zero when the velocity V of the conductor is equal to the phase velocity of the wave. It is clear from (7.1.93) that under this synchronous condition there is no interaction between the conducting slab and the traveling current sheet. No currents are induced and the flux density completely penetrates the medium, with the characteristic exponential decay found in free space (Fig. 7.1.19). This figure shows how finite values of S give rise to currents that tend to exclude the magnetic flux density from the conductor. As S approaches infinity, the magnetic field is completely excluded from the conductor by a current density that is confined to a region very near the lower surface of the conductor. This is as would be expected, for, when $V = 0$, it is similar to the skin effect described in Section 7.1.3a. Here, however, the fields can also be excluded from the conductor because of the motion through a nonuniform field. To illustrate this point suppose that the current sheet is stationary so that $\omega = 0$. Then S is proportional to the velocity V , and it is apparent that a large material velocity leads to a shielding of the fields from the conductor. Note that in this limit S is a magnetic Reynolds number based on the wavelength of the current sheet. Once again a large magnetic Reynolds number indicates that induced field effects are significant.

The currents induced in the conducting slab give rise to time average forces

in the y - and z -directions. Both forces are significant because they are basic to a number of practical applications. The force in the y -direction makes it possible to support or levitate the slab on the magnetic field, whereas the force in the z -direction makes it possible to accelerate the slab in the z -direction.

It is a straightforward matter to compute these forces from the solutions we have obtained, since the force density in the material is $\mathbf{F} = \mathbf{J} \times \mathbf{B}$. The time average force $\langle T_z \rangle$ in the z -direction (per unit x - z area) is

$$\langle T_z \rangle = \int_0^\infty \langle F_z \rangle dy, \quad (7.1.94)$$

where*

$$\langle F_z \rangle = \frac{1}{2} \operatorname{Re} (\hat{J}_x \hat{B}_y^*)$$

with $\hat{B}_y^*$ the complex conjugate of $\hat{B}_y$. The solutions of (7.1.91) and (7.1.93) make it possible to carry out the integration of (7.1.94) to obtain

$$\langle T_z \rangle = \frac{1}{4} \frac{\mu_0 K^2 S}{\sqrt{1 + S^2} \operatorname{Re} \sqrt{1 + jS}}. \quad (7.1.95)$$

As expected, when the currents are not induced in the slab ($S = 0$) there is no force in the z -direction. It is significant that the sign of the z -directed force depends on the sign of S . If S is positive (the traveling wave has a phase velocity that exceeds V), the traveling wave interaction tends to pull the slab in the z -direction. Conversely, if the phase velocity is slower than that of the slab, the induced currents tend to retard the motion of the slab. These properties are similar to those discussed for the induction machine in Section 4.1.6b. In fact, this interaction is the basis for a distributed linear induction machine.

The dependence of the time average force $\langle T_z \rangle$ on the parameter S is shown in Fig. 7.1.20. Here it is apparent that there is an optimum value of S at which the maximum force per unit area $\langle T_z \rangle$ is produced.† This maximum exists between low values of S , in which little current is induced in the medium, and large values of S in which the induced currents have the wrong phase relationship with respect to the imposed current sheet to produce maximum force. Compare this discussion and Fig. 7.1.20 with Fig. 4.1.20 and the associated discussion for an induction machine. There is a direct correspondence between the parameter S used here and the slip s used in the induction-machine analysis. The physical phenomena that govern the shape of the force (or torque) as a function of S (or slip s) are the same.

* Here we use the identity $\langle \operatorname{Re} \hat{A} e^{j\omega t} \operatorname{Re} \hat{B} e^{j\omega t} \rangle \equiv \frac{1}{2} \operatorname{Re} \hat{A} \hat{B}^*$.

† This phenomenon is also used to achieve desired characteristics in deep-bar rotors of induction machines as discussed in Fitzgerald and Kingsley, *op. cit.*, p. 411.

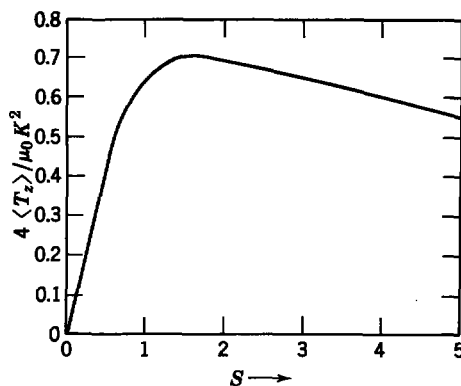


Fig. 7.1.20 Normalized time average force $\langle T_z \rangle$ per unit x - z area on the conducting slab of Fig. 7.1.18; S is defined following (7.1.87).

The time-average force density in the vertical (y)-direction is

$$\langle F_y \rangle = \frac{1}{2} \operatorname{Re} (\hat{J}_x \hat{B}_z^*). \quad (7.1.96)$$

The force per unit x - z area $\langle T_y \rangle$ can be computed in a manner similar to that used to find (7.1.95):

$$\langle T_y \rangle = \frac{\mu_0 K^2}{4} \left(\frac{\sqrt{1 + S^2} - 1}{\sqrt{1 + S^2}} \right) \quad (7.1.97)$$

The dependence of this levitating force on S is shown in Fig. 7.1.21. As S is increased, the upward force approaches an asymptotic value of $\mu_0 K^2/4$. As discussed in Chapter 8, this is the average magnetic pressure that acts on a perfectly conducting surface. A levitation experiment is shown in Fig. 7.1.22.

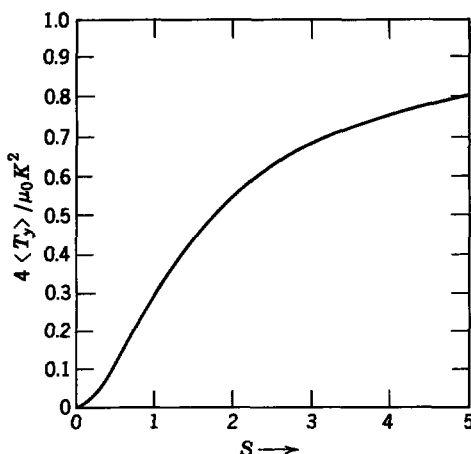


Fig. 7.1.21 Normalized time average force $\langle T_y \rangle$ per unit x - z area.

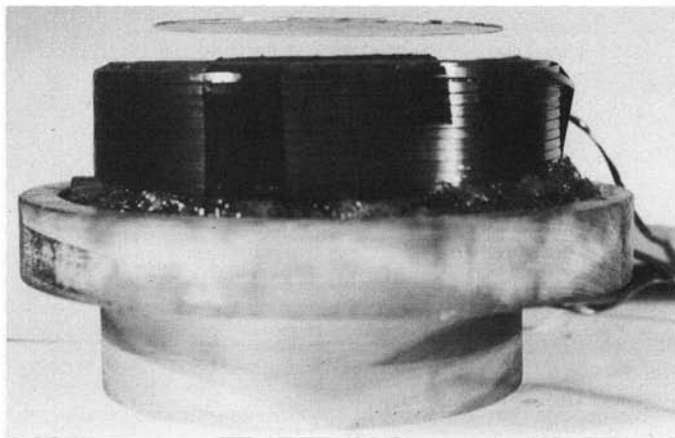


Fig. 7.1.22 Laboratory project in which an aluminum disk (about 6 in. in diameter) is levitated by an alternating magnetic field. The coil is driven by a 400-Hz current. The experiment demonstrates qualitatively the induced force given by (7.1.97) with $V = 0$. Of course, in the experiment the finite thickness of the disk is highly significant (see Fig. 7.1.15) with $\sigma = 3.7 \times 10^7$ mhos/m.

7.2 CHARGE RELAXATION

As discussed in the introduction to this chapter, the relaxation of charge in slightly conducting media constitutes the mechanism by which motion has an effect on electric field distributions in electric field systems. This is illustrated in the following sections by a series of examples. First, the relaxation of charge in systems involving media at rest is considered. Here the relaxation time is of fundamental importance in determining volume and surface charge densities that result from initial conditions and from excitations. In Section 7.2.2 emphasis is given to the effect of steady motion on the relaxation of free charge. In this case the electric Reynolds number based on the material velocity is important. Sections 7.2.3 and 7.2.4 then consider systems in which relaxation is affected by two characteristic times, a time that characterizes an electrical excitation and a time based on the material velocity and characteristic length.

The field equations for an electric field system (see Table 1.2)* are

$$\nabla \times \mathbf{E} = 0, \quad (7.2.1)$$

$$\nabla \cdot \epsilon \mathbf{E} = \rho_f, \quad (7.2.2)$$

$$\nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t} = 0 \quad (7.2.3)$$

where we have assumed $\mathbf{D} = \epsilon \mathbf{E}$.

* See Appendix E.

In this section we confine our attention to situations in which the conduction of free charge can be accounted for by a constitutive law in the form

$$\mathbf{J}_f = \sigma \mathbf{E} + \rho_f \mathbf{v}. \quad (7.2.4)$$

Here we have combined Ohm's law $\mathbf{J}'_f = \sigma \mathbf{E}'$ with the field transformations $\mathbf{E}' = \mathbf{E}$ and $\mathbf{J}'_f = \mathbf{J}_f - \rho_f \mathbf{v}$.^{*} This law describes the conduction process in a wide range of solids, liquids, and gases but is by no means of general applicability.

There is a considerable analogy between the way in which we develop the subject of charge relaxation in this section and the way in which magnetic diffusion is developed in Section 7.1; for example, we begin by developing an equation that, together with appropriate boundary conditions, defines the distribution of the electric field, given the velocity of the medium.[†] From (7.2.3) and (7.2.4)

$$\nabla \cdot \sigma \mathbf{E} + \nabla \cdot \rho_f \mathbf{v} + \frac{\partial \rho_f}{\partial t} = 0. \quad (7.2.5)$$

The free charge density ρ_f can be eliminated from this expression by using (7.2.2).

$$\nabla \cdot \sigma \mathbf{E} + \nabla \cdot (\mathbf{v} \nabla \cdot \epsilon \mathbf{E}) + \frac{\partial}{\partial t} \nabla \cdot \epsilon \mathbf{E} = 0. \quad (7.2.6)$$

Given the velocity $\mathbf{v}$, this expression involves only the electric field intensity $\mathbf{E}$. It is a scalar equation, hence does not in general define the three components of $\mathbf{E}$. From (7.2.1), however, we can define a potential ϕ such that

$$\mathbf{E} = -\nabla \phi. \quad (7.2.7)$$

Then (7.2.6) becomes one equation in one unknown.

$$\nabla \cdot \sigma \nabla \phi + \nabla \cdot (\mathbf{v} \nabla \cdot \epsilon \nabla \phi) = -\frac{\partial}{\partial t} \nabla \cdot \epsilon \nabla \phi. \quad (7.2.8)$$

Physically, this equation accounts for the conservation of free charge. The first term accounts for the flow of free charges into a small volume due to conduction. The second term appears because convection of the medium can give rise to the transport of free charge into a given region. Then the term on the right is the rate of increase of the local free charge density. Equation 7.2.8 serves the same purpose in a distributed electric field system as the electrical equation of motion served in the lumped-parameter descriptions of Chapters 2 to 5. In the sections that follow we confine our attention to the effect of the motion (if there is motion) on the fields; that is, the velocity $\mathbf{v}$ is prescribed. One objective is an indication of models that can

^{*} Table 6.1, Appendix E.

[†] In the development of this section σ and ϵ are assumed to be given functions of the space coordinates.

be used to simplify the analysis of electromechanical coupling, in which the velocity $\mathbf{v}$ is not known until the fields are known.

7.2.1 Charge Relaxation as an Electrical Transient

We first consider the relaxation of charge in the absence of motion. A common situation is one in which the conductivity σ and permittivity ϵ of each medium are uniform; thus changes in conductivity and permittivity occur only at surfaces. In these important situations it is essential to recognize that in the absence of a free charge source in the medium there will be no steady-state volume free-charge density. This is one of the important points to be made in the following subsection.

7.2.1a Media with Uniform Properties

Suppose that we are concerned with the free-charge density in a medium at rest. Then the second term in (7.2.8) is zero and (7.2.2) and (7.2.7) can be used to write this expression as

$$\frac{\partial \rho_f}{\partial t} + \frac{\sigma}{\epsilon} \rho_f = 0. \quad (7.2.9)$$

A more direct derivation of this equation simply combines (7.2.2) to (7.2.4) with $\mathbf{v} = 0$. A general solution to (7.2.9) is

$$\rho_f(x, y, z, t) = \rho_0(x, y, z)e^{-(\sigma/\epsilon)t}, \quad (7.2.10)$$

as can be seen by substitution into (7.2.9); that is, given an initial free-charge distribution ρ_0 at $t = 0$, the free charge density at each point in space decays to zero with the *relaxation time* τ .

$$\tau = \frac{\epsilon}{\sigma}. \quad (7.2.11)$$

Unless there are sources of free charge in the volume of the material [which would contribute a driving term to (7.2.9)], there are no steady-state charges in the bulk. Note that (7.2.9) can be derived without recourse to (7.2.1), so that these conclusions do not depend on the field equations being quasi-static. Of course, in any physical situation the uniformly conducting medium is of finite extent and conservation of charge requires that those charges initially distributed throughout the volume relax to the surfaces that bound the volume. This can best be shown by means of an example.

Example 7.2.1. A conducting shell contains a uniformly conducting sphere at its center (Fig. 7.2.1). The region between the sphere and the shell is perfectly insulating. At time $t = 0$ there is a free charge density ρ_0 distributed uniformly throughout a spherical region of radius R at the center of the conducting sphere; that is, if we call the total charge supported by the center sphere Q ,

$$\rho_f(r < R, t = 0) = \rho_0 = Q/(4/3)\pi R^3. \quad (a)$$

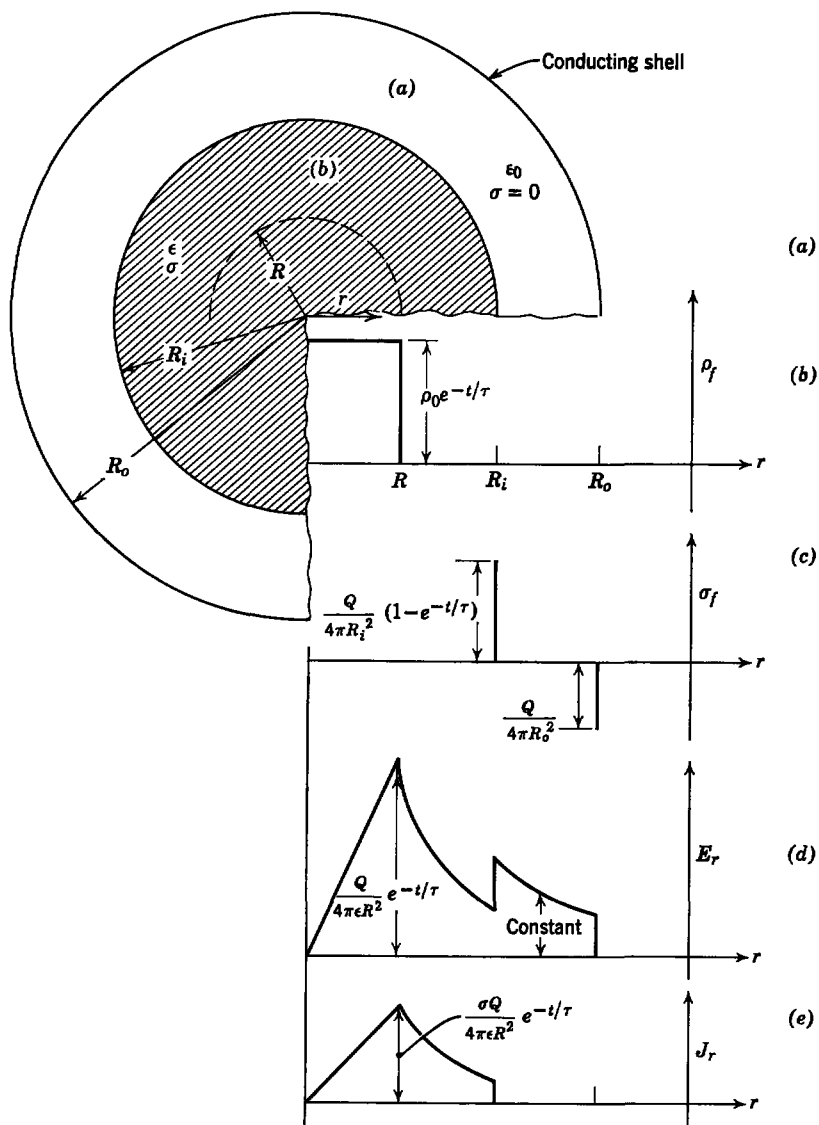


Fig. 7.2.1 (a) A uniformly conducting sphere of radius R_i is bounded by an insulating medium and a perfectly conducting shell of radius R_o ; (b) when $t = 0$, the sphere supports a volume charge distributed uniformly over the region $r < R$; (c) for $t > 0$ charges accumulate at $r = R_i$; (d) radial electric field E_r ; (e) current density J_r .

The transient charge density follows from (7.2.10)

$$\begin{aligned}\rho_f &= \rho_0 e^{-t/\tau} & \text{for } r < R, \\ \rho_f &= 0 & \text{for } r > R.\end{aligned}\quad (b)$$

This distribution of volume charge density is shown in Fig. 7.2.1b.

The problem has radial symmetry. Hence the charge distribution of (b) can be used with (7.2.2) to determine the radial component of the electric field intensity since

$$\frac{1}{r^2} \frac{d}{dr} (r^2 E_r) = \frac{\rho_0}{\epsilon} e^{-t/\tau}. \quad (c)$$

This expression can be integrated to obtain

$$E_r^b = \frac{Q}{4\pi\epsilon R^2} \left(\frac{r}{R}\right) e^{-t/\tau}; \quad r < R \quad (d)$$

The total charge within the radius $r = R$ is $Qe^{-t/\tau}$. Then the integral form of Gauss's theorem shows that

$$E_r^b = \frac{Q}{4\pi\epsilon r^2} e^{-t/\tau}; \quad R < r < R_i \quad (e)$$

Similarly, since the total charge on the sphere must be conserved, the electric field outside the conducting sphere is

$$E_r^a = \frac{Q}{4\pi\epsilon_0 r^2}; \quad R_i < r < R_o \quad (f)$$

These last two electric field intensities can be used to determine the amount of free charge on the surface of the inner sphere, since

$$\sigma_f = \epsilon_0 E_r^a (r = R_i) - \epsilon E_r^b (r = R_i) = \frac{Q}{4\pi R_i^2} (1 - e^{-t/\tau}). \quad (g)$$

Of course, the surface charge on the outer shell is constant, for the electric field in the insulating section adjacent to the shell is constant. The surface charge density and distribution of E_r are shown in Fig. 7.2.1c,d. It is clear from the solutions that the initial free-charge density relaxes to the outer surface of the conducting sphere. In the steady-state condition there is no electric field within the sphere, but rather the field is shielded out of the sphere by the surface charge at $r = R_i$.

Remember that the current density is

$$J_r = \sigma E_r \quad (h)$$

and is therefore distributed as shown in Fig. 7.2.1e. It is this current that accounts for the conduction of free charge to the surface of the sphere. Note that there is a conduction current in the region $R < r < R_i$, even though at no time during the transient is there a free-charge density in this region. Conduction currents modeled by a constitutive law in the form of (7.2.4) can be present in a medium in the absence of ρ_f .

Without injection of charges directly into the bulk of the material, a uniformly conducting medium with uniform permittivity does not support a volume charge density as illustrated by Example 7.2.1. In such systems free charges are confined to surfaces. These surface charges can relax through a

medium and this surface-charge relaxation is accounted for through boundary conditions. Lumped-parameter models often can be used to represent this process, as illustrated by the following example.

Example 7.2.2. The important role of surface-charge relaxation through dielectrics can be demonstrated by means of the experiment shown in Fig. 7.2.2. Three plane parallel metallic electrodes are immersed in a liquid dielectric as shown. A voltage source V_o is used to establish a positive charge on the middle electrode. At time $t = 0$ the switch S connects the battery to the terminal (2), leaving the middle plate isolated from the other two plates except for conduction through the dielectric. We wish to compute the electric fields defined in Fig. 7.2.2 as functions of time, to show the relaxation of charge from the middle plate.

We assume that the fields are uniformly distributed between the plates so that for $t < 0$

$$E_a = \frac{V_o}{a}, \quad (a)$$

$$E_b = 0. \quad (b)$$

We integrate the conservation of charge equation (7.2.3) over the volume that encloses the middle plate to write

$$(J_a - J_b) = -\frac{\partial}{\partial t} \epsilon(E_a - E_b). \quad (c)$$

This equation guarantees that the rate of change of charge on the middle plate is accounted for by current to either of the other plates. The current densities can be replaced by the

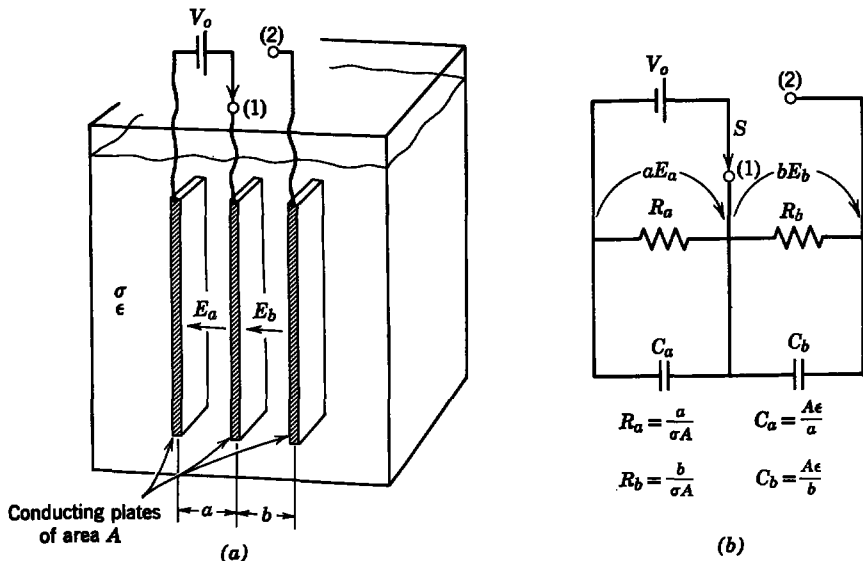


Fig. 7.2.2 System of three parallel-plane metallic electrodes immersed in a dielectric of conductivity σ and permittivity ϵ : at $t = 0$ the switch is connected to terminal (2); (b) equivalent circuit.

electric field intensities by using (7.2.4) with $v = 0$:

$$\sigma(E_a - E_b) = -\frac{d}{dt} \epsilon(E_a - E_b). \quad (d)$$

This expression is recognized as having the same form as (7.2.9) except that the charge density is replaced by the jump in electric field intensity, which is proportional to the charge density on the middle plate. If we use the initial conditions of (a) and (b) the solution to (d) is

$$E_a - E_b = \frac{V_o}{a} e^{-t/\tau}; \quad \tau = \frac{\epsilon}{\sigma}. \quad (e)$$

For $t > 0$ the voltage across the outside plates is constrained to be V_o , hence

$$E_a a + E_b b = V_o. \quad (f)$$

We solve for both E_a and E_b from these last two equations.

$$E_a = \frac{V_o}{a+b} \left(1 + \frac{b}{a} e^{-t/\tau} \right), \quad (g)$$

$$E_b = \frac{V_o}{a+b} (1 - e^{-t/\tau}). \quad (h)$$

These solutions are shown in Fig. 7.2.3. Initially, all of the charge is retained on the middle plate. [Recall that the charge on the middle plate is at first $(A\epsilon V_o/a)$ and is always $A\epsilon(E_a - E_b)$.] All the charge relaxes to the outer plates with the characteristic relaxation time τ . The relaxation of charge is a fundamental way of viewing the decay of the RC circuit shown in Fig. 7.2.2. The solution using this circuit with the equivalent parameter values is the same as the solution just given for the continuum model. This example provides the answer to the question posed in Fig. 7.2.4.

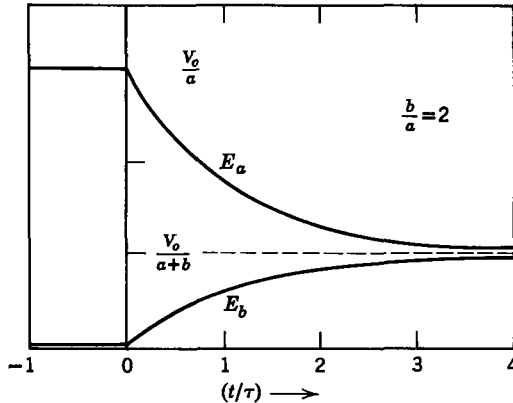


Fig. 7.2.3 Relaxation of the electric field intensities between the plates in Fig. 7.2.2 when S is switched from (1) to (2). The surface charge density on the middle plate is proportional to the difference between E_a and E_b .

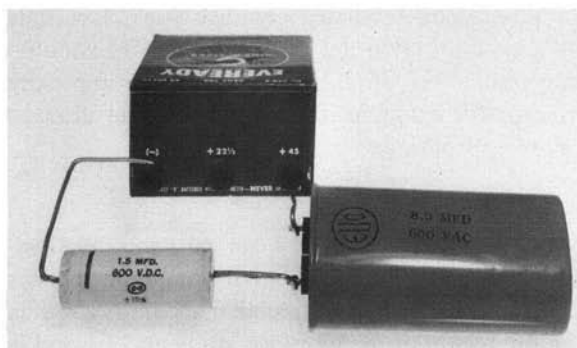


Fig. 7.2.4 The capacitors are connected in series with the battery. The circuit has been completed for a sufficient length of time to establish steady-state conditions. Why is it not possible, on the basis of the given information, to determine the steady-state charge stored in each of the capacitors? (See Example 7.2.2.)

Whether we are concerned with relaxation of charge from the bulk or from an interface, the relaxation time ϵ/σ is of fundamental importance. In Table 7.2.1 we see that the relaxation time depends greatly on the material used. In metals the volume distribution of charge would be difficult to measure, let alone ever be large enough to include in a physical model. On the other hand, there are many materials in which the relaxation time can be measured in minutes or even hours. In slightly conducting materials, however, the conductivity is often extremely sensitive to impurities and sources of

Table 7.2.1 Characteristic Values of Relaxation Time $\tau = \epsilon/\sigma$ for Conductors and Insulators

Material	σ (mhos/meter)	ϵ	τ (sec)
Silver*	6.17×10^7	ϵ_0	1.43×10^{-19}
Copper*	5.80×10^7	ϵ_0	1.52×10^{-19}
Aluminum*	3.72×10^7	ϵ_0	2.38×10^{-19}
Mercury†	1.06×10^6	ϵ_0	8.35×10^{-18}
Seawater*	4	$80 \epsilon_0$	1.77×10^{-10}
Water†	4×10^{-6}	$80 \epsilon_0$	1.77×10^{-4}
Nitrobenzene†	5×10^{-7}	$36.1 \epsilon_0$	6.40×10^{-4}
Corn oil	5×10^{-11}	$2.7 \epsilon_0$	4.8
Carbon tetrachloride†	4×10^{-16}	$2.24 \epsilon_0$	4.95×10^4

* S. Ramo, and J. R. Whinnery, *Fields and Waves in Modern Radio*, Wiley, New York, 1958, pp. 240 and 312.

† N. A. Lange, *Handbook of Chemistry*, Handbook Pub., Inc., Sandusky, Ohio, 1952, pp. 1240 and 1253.

ionization. For this reason tabulated conductivities, especially for liquids, are likely to indicate only the order of magnitude. We should bear in mind also that the constitutive law of (7.2.4) may be a poor approximation for the conduction process; for example, the mobility model discussed in Section 6.3.2 may be more appropriate.

7.2.1b Media with Nonuniform Properties

A steady-state volume charge density can exist in a medium at rest when there are gradients of either the conductivity or permittivity in the bulk of the material. The interface between dissimilar media is a special case of this situation, for in the region of the interface gradients of ϵ and σ are singular. It is for this reason that surface charges accumulate on the surfaces of uniformly conducting media.

With σ and ϵ functions of position the potential in a stationary medium must satisfy the expression

$$\nabla \cdot \sigma \nabla \phi + \frac{\partial}{\partial t} \nabla \cdot \epsilon \nabla \phi = 0, \quad (7.2.12)$$

as we can see from (7.2.8). Note that this expression does not take the simple form of (7.2.9) unless σ and ϵ are constant. A one-dimensional example serves to illustrate the consequences of nonuniform properties in electric field systems.

Example 7.2.3. Plane parallel electrodes bound a material of nonuniform properties shown in Fig. 7.2.5. For purposes of illustration we consider the case in which an external current source drives a current through the material in the x -direction. The material properties depend only on the x -dimension. We wish to determine the free charge density and distribution of electric field intensity between the plates. Physically, this situation would be realized by simply placing electrodes with a temperature difference in an organic liquid. The conductivity is a strong function of temperature in many liquids and there would therefore be a variation in σ as a function of x .

In one dimension the gradient of ϕ in (7.2.12) becomes $(-E_x)$; hence (7.2.12) can be written as

$$\frac{\partial}{\partial x} \left[\frac{\partial}{\partial t} (\epsilon E_x) + \sigma E_x \right] = 0. \quad (a)$$

This expression can be integrated to obtain

$$\frac{\partial}{\partial t} (\epsilon E_x) + \sigma E_x = f(t). \quad (b)$$

This result is not surprising, for it simply states that the sum of the displacement and conduction currents passing any given y - z plane is the same as that found at any other y - z plane; That is, $f(t)$ is simply the current density $i(t)/A$:

$$\frac{\partial}{\partial t} (\epsilon E_x) + \sigma E_x = \frac{i(t)}{A}. \quad (c)$$

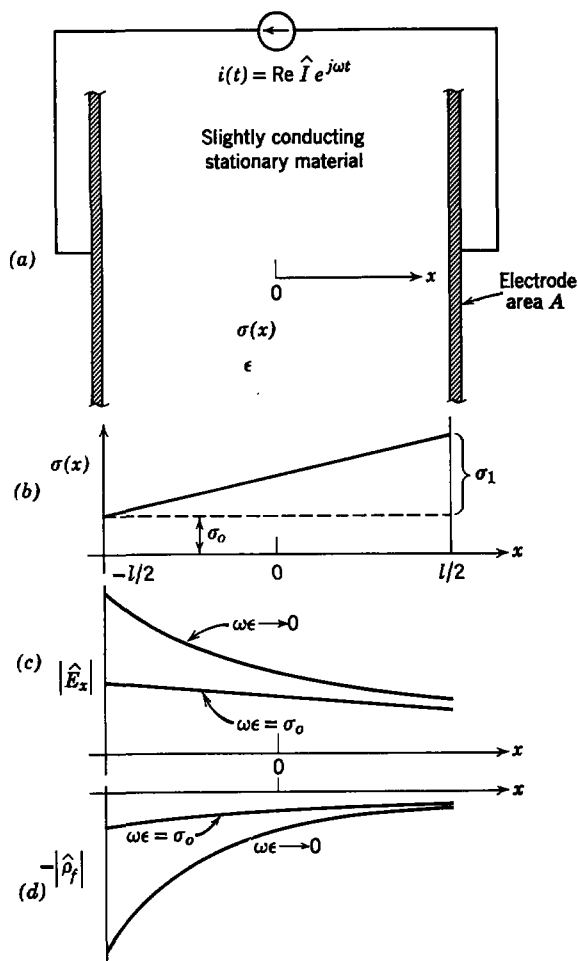


Fig. 7.2.5 (a) A slightly conducting material of uniform permittivity is bounded by plane-parallel electrodes of area A ; (b) conductivity distribution between plates; (c) distribution of the electric field magnitude; (d) free charge density magnitude.

To emphasize the fact that under steady-state conditions there are bulk charges in the nonuniform medium consider a sinusoidal driving current

$$i(t) = \text{Re} (\hat{I} e^{j\omega t}), \quad (d)$$

hence sinusoidal responses in the form

$$E_x = \text{Re} [\hat{E}_x(x) e^{j\omega t}]. \quad (e)$$

Then substitution into (c) shows that

$$\hat{E}_x = \frac{\hat{I}}{A(j\omega\epsilon + \sigma)}. \quad (f)$$

Here σ and ϵ are in general functions of x , hence so also is E_x . The charge density is found by using (7.2.2):

$$\hat{\rho}_f = \frac{d\epsilon \hat{E}_x}{dx} = -\frac{\epsilon \hat{I}}{A} \frac{[j\omega(d\epsilon/dx) + d\sigma/dx]}{(j\omega\epsilon + \sigma)^2} + \frac{\hat{I}(d\epsilon/dx)}{A(j\omega\epsilon + \sigma)}. \quad (g)$$

As a more particular example consider the case in which the conductivity has the linear distribution shown in Fig. 7.2.5b and the permittivity is constant.

$$\sigma(x) = \sigma_0 + \frac{\sigma_1}{l} x. \quad (h)$$

Then it follows from (f) and (g) that

$$|\hat{E}_x| = \frac{|\hat{I}|}{A\sqrt{\sigma^2 + (\omega\epsilon)^2}} \quad (i)$$

$$|\hat{\rho}_f| = \frac{\epsilon |\hat{I}| \sigma_1 / Al}{[\sigma^2 + (\omega\epsilon)^2]}. \quad (j)$$

The distribution of electric field intensity and charge density magnitudes between the plates is shown in Fig. 7.2.5c, d. Note that the greatest amount of free charge exists in the bulk of the medium when the frequency is zero (the current is constant). The charge density tends to zero as the frequency is made large compared with the local reciprocal of the relaxation time ϵ/σ . In the high frequency limit one period of excitation is insufficient time for the free charge to accumulate in the bulk of the material.

Bulk charge accumulation due to nonuniform conductivity occurs in many practical situations; for example, in oil-insulated, high-voltage dc cables for the transmission of power a temperature gradient in the oil can make the conductivity vary markedly with position. A bulk charge accumulates; and, when the applied voltage is reversed, the bulk charge may cause overly high electrical stresses and insulation failure before it can relax to a new steady-state distribution.*

7.2.2 Charge Relaxation in the Presence of Steady Motion

We are now in a position to undertake a study of the effects of material motion on charge relaxation. In the sections that follow interest is confined to systems in which changes in material properties occur at surfaces and can be accounted for by the boundary conditions. Then, because σ and ϵ are constant in the bulk of the material, (7.2.8) can be written as

$$\frac{\sigma}{\epsilon} \rho_f + \nabla \cdot \rho_f \mathbf{v} + \frac{\partial \rho_f}{\partial t} = 0, \quad (7.2.13)$$

where use has been made of (7.2.2) and (7.2.7) to state this equation in terms of the free charge density.

A great deal can be said about the implications of (7.2.13) in the important case in which the media are *incompressible*. In such situations the material density remains constant and the net flux of material into a given region is

* See, for example, K. Kojima, S. Tanaka, and K. Matsuura, "Potential Distribution in Dielectrics of Oil-Filled, D-C Cable," *Elec. Eng. Japan*, August 1964.

zero. By analogy with a similar condition on the magnetic flux density ($\nabla \cdot \mathbf{B} = 0$) the condition that the material be incompressible is stated as

$$\nabla \cdot \mathbf{v} = 0. \quad (7.2.14)$$

(More is said about this equation in Chapter 12, in which incompressible fluids are discussed.)

Expansion of the second term in (7.2.13) gives

$$\nabla \cdot \rho_f \mathbf{v} = \mathbf{v} \cdot \nabla \rho_f + \rho_f (\nabla \cdot \mathbf{v}), \quad (7.2.15)$$

where, in view of (7.2.14), the last term is zero. Hence (7.2.13) can be written as

$$\frac{\sigma}{\epsilon} \rho_f + \frac{\partial \rho_f}{\partial t} + \mathbf{v} \cdot \nabla \rho_f = 0. \quad (7.2.16)$$

Written in this form, it is clear that (7.2.16) expresses the same relaxation conditions as (7.2.9) did for stationary media, except that now the charge relaxation occurs with respect to the frame of the moving medium; that is, the last two terms in (7.2.16) (the convective derivative) represent the time rate of change of ρ_f for an observer moving with the velocity $\mathbf{v}$ of the material. Hence the material motion simply serves to transport the free charge as it relaxes with the time constant ϵ/σ .

Example 7.2.4. The relaxation of an initial distribution of charge placed in the fluid stream shown in Fig. 7.2.6 is to be considered. Note that the velocity $\mathbf{v} = v_x(y)\mathbf{i}_x$ satisfies (7.2.14).

Remember that (7.2.16), which must be satisfied by the distribution of ρ_f , expresses the same relaxation phenomenon as (7.2.9), except that the relaxation occurs with respect to the moving medium. Hence we look for a solution to (7.2.16) that has the form of (7.2.10), with x replaced by x' , where

$$x' = x - v_x(y)t. \quad (a)$$

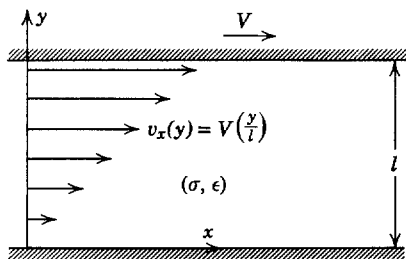


Fig. 7.2.6 A fluid with uniform conductivity and permittivity is confined by parallel plates, one of which is fixed as the other moves to the right with the velocity V . The fluid, which then moves to the right with the velocity profile shown, is given an initial charge distribution which relaxes as shown in Fig. 7.2.7.

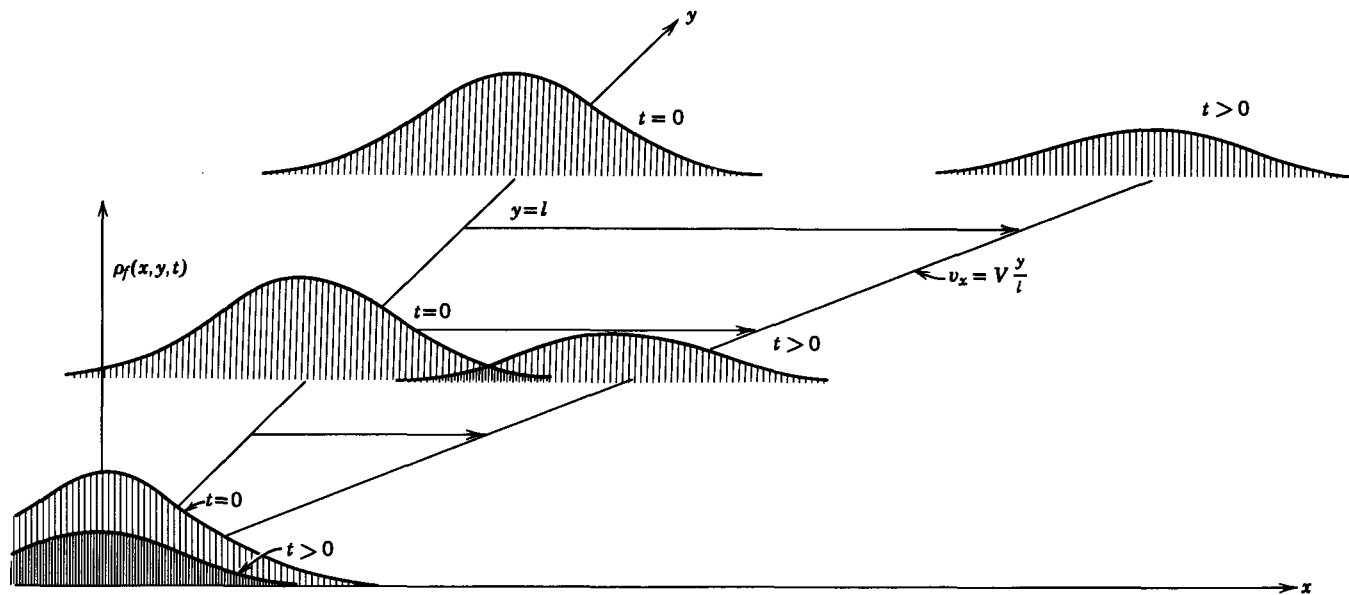


Fig. 7.2.7 An initial distribution of charge is transported downstream with the material velocity as it relaxes with the time constant ϵ/σ .

By this reasoning the desired solution to (7.2.16) is

$$\rho_f = \rho_0[x - v_x(y)t, y]e^{-(\sigma/\epsilon)t}, \quad (b)$$

which can be checked by direct substitution. From this result it is clear that an initial charge distribution is simply translated downstream, attenuating in amplitude with the relaxation time ϵ/σ but otherwise retaining its initial shape; for example, if at $t = 0$ the charge distribution is

$$\rho_f(x, y, t = 0) = \rho_0 e^{-(x/l)^2}, \quad (c)$$

as shown in Fig. 7.2.7, the subsequent charge distribution will be

$$\rho_f(x, y, t) = \rho_0 \exp \left\{ - \left[\frac{x - v_x(y)t}{l} \right]^2 \right\} e^{-(\sigma/\epsilon)t}. \quad (d)$$

This redistribution of the charge is also shown in Fig. 7.2.7. It is important to see that even in the presence of convection the steady-state charge density in source-free media with uniform electrical properties tends to zero.

It must be recognized that there is an electric potential associated with the charge density defined by (7.2.16). In the simple bulk relaxation problem considered here it is possible to determine the charge density without recourse to the boundary conditions, hence to the potential ϕ . Many practical devices exploit the energy conversion process that takes place when the charge is transported from a region of one potential to a region of another. To determine whether energy is taken from the moving medium (a generator) or put into the medium (motor or pump) it is necessary to know the potential distribution. A simple illustration of this kind of problem is made next. The conditions required within the medium for appreciable charge transport are apparent from the developments of this section. The time required for a given initial distribution of charge to relax (with respect to the medium) is ϵ/σ , whereas the time required for the transport of the charge over a distance l with the velocity V is l/V . For an appreciable convection of charge the transport time must be short compared with the relaxation time; that is

$$R_e = \frac{\epsilon/\sigma}{l/V} = \frac{\epsilon V}{\sigma l} \gg 1 \quad (7.2.17)$$

or the electric Reynolds number R_e must be large compared with one if the convection is to compete with the relaxation process in determining the location of the volume charge density.

One highly developed application of the ideas introduced in this section is to the generation of extremely high voltages. Here the basic electro-mechanical system is a Van de Graaff generator that uses the mechanical motion of an insulator to transport free charge against an electric field and generate high-voltage dc power.* A simple model for this generator is the subject of the following example.

* A description of this and other sources of power is given in J. G. Trump, "Electrostatic Sources of Electric Power," *Elec. Eng.*, June 1947.

Example 7.2.5. The basic system, illustrated in Fig. 7.2.8a, consists of a continuous belt made of slightly conducting material, which is driven by rollers at the constant velocity

$$\mathbf{v} = \mathbf{i}_z V.$$

The electrode at $z = 0$ feeds positive ions onto the surface of the belt and the electrode at $z = l$ removes the positive ions. The two electrodes are the electrical terminals that will remain open-circuited for this example. The belt material has a constant conductivity σ and constant permittivity ϵ but the positive ions put on the belt by the electrode are immobile.

The analysis of the system begins with the following assumptions:

1. We consider only that portion of the belt that carries charge from $z = 0$ to $z = l$.
2. The electric field intensity and current density have only z components:

$$\mathbf{E} = \mathbf{i}_z E,$$

$$\mathbf{J}_f = \mathbf{i}_z J.$$

3. All variables are functions of z alone.
4. In the spirit of the one-dimensional model we assume that the net effect of the positive ions applied to the belt and the induced charge in the belt can be represented by an effective free charge density ρ_f which is a function of z only.
5. We assume that the electrode at $z = 0$ maintains the boundary condition

$$\text{at } z = 0, \quad \rho_f = \rho_0. \quad (\text{a})$$

6. The system is operating in the steady state.

With the conductivity constant, the constituent relation for the transport of free charge is (7.2.4)

$$J = \sigma E + \rho_f V. \quad (\text{b})$$

The other equations necessary for the analysis are Gauss's law (7.2.2) written as

$$\epsilon \frac{dE}{dz} = \rho_f \quad (\text{c})$$

and the condition imposed by the open circuit

$$J = 0. \quad (\text{d})$$

We differentiate (b) with respect to z and use (c) to eliminate E and obtain

$$V \frac{d\rho_f}{dz} + \frac{\sigma}{\epsilon} \rho_f = 0. \quad (\text{e})$$

The general solution of this equation is

$$\rho_f = C_1 e^{-\sigma z / \epsilon V}. \quad (\text{f})$$

We use boundary condition (a) to evaluate the constant C_1 and obtain the solution

$$\rho_f = \rho_0 e^{-z/R_e l}; \quad R_e = \frac{\epsilon V}{\sigma l}. \quad (\text{g})$$

Finally, we calculate the open-circuit voltage by using (b) and (d):

$$V_{oc} = - \int_0^l E_z dz = R_e^2 \frac{\rho_0 l^2}{\epsilon} (1 - e^{-1/R_e}). \quad (\text{h})$$

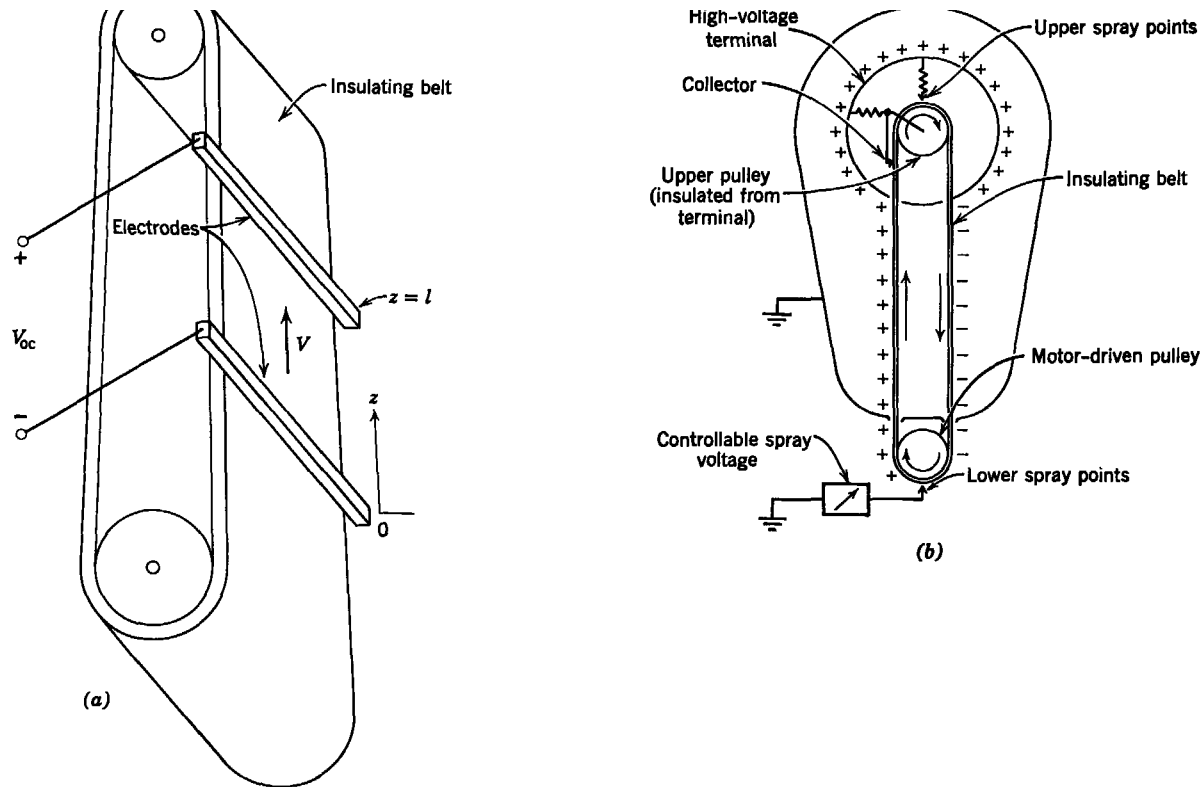


Fig. 7.2.8 (a) A simple model for a Van de Graaff generator. Charges are conveyed to the belt at $z = 0$ and transported in the presence of relaxation to the electrode at $z = l$ through a potential difference V_{0c} . (b) A more realistic schematic diagram of the generator shows how the charge is injected and removed with corona sources. In a practical device the conduction along the belt does not represent a significant limitation.

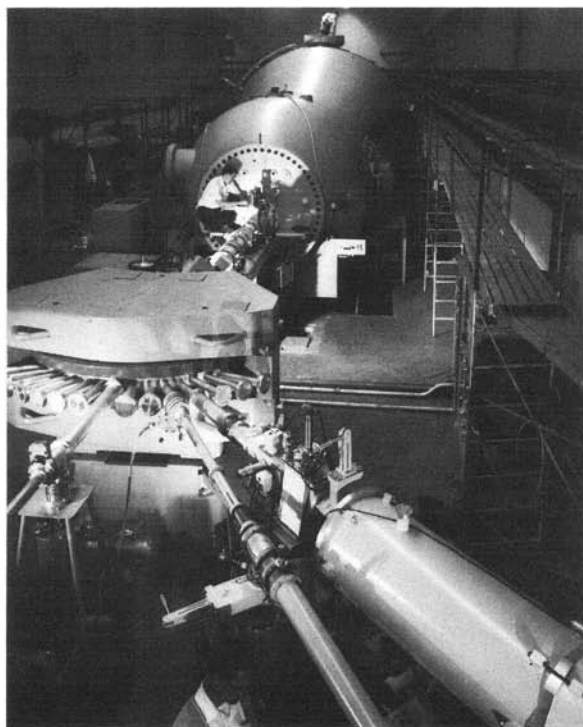


Fig. 7.2.8c Van de Graaff tandem accelerator at High Voltage Engineering Corporation, Burlington, Mass. Capable of a terminal potential of 10 MV, this machine weighs 188 tons, is 18 ft in diameter, and is 81 ft long.

The distributions of free charge density ρ_f and field intensity E along the belt are shown in Figs. 7.2.9 and 7.2.10. To interpret these curves, consider that ϵ and l are fixed and that R_e is changed by varying σ and/or V . The current density J , as given by (b), indicates that free charge is transported by conduction (σE) and by convection ($\rho_f V$). As indicated earlier in this section, the presence of conduction makes the free charge tend to relax to the surfaces of the conductor. Thus in this system we have two competing processes: (a) convection of the charge in the positive z -direction by the moving belt and (b) relaxation of free charge in the belt to neutralize the injected charge. It is apparent from the curves of Fig. 7.2.9 that the electric Reynolds number R_e is a measure of the relative effectiveness of these two processes. For a large R_e the relaxation process is relatively ineffective and the charge density varies very little along the length of the device. On the other hand, for small R_e , the relaxation process predominates and the net charge density decays rapidly along the length because induced charge can build up in the belt to neutralize the injected charge.

It is clear that because the amount of net free charge between the electrodes varies significantly with R_e so will the electric field intensity. As indicated by (b) and (d), E has the same space variation as the charge density but its magnitude is a function of R_e also.

For the three values of R_e for which curves are plotted in Figs. 7.2.9 and 7.2.10 we calculate the open-circuit voltage from (h) and give the values in Table 7.2.2.

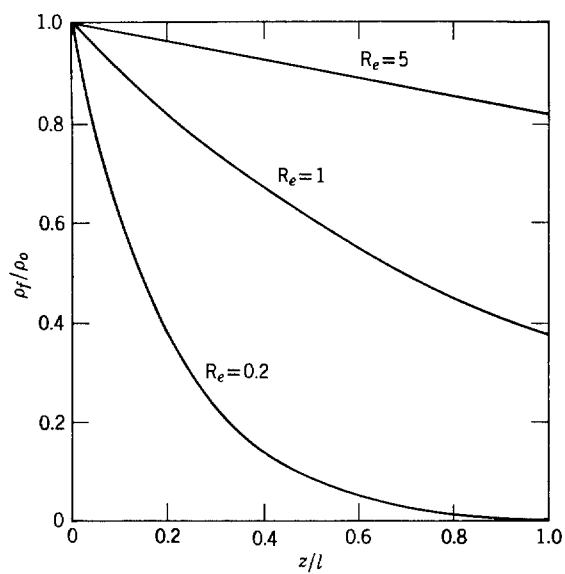


Fig. 7.2.9 Variation of net free charge density with position.

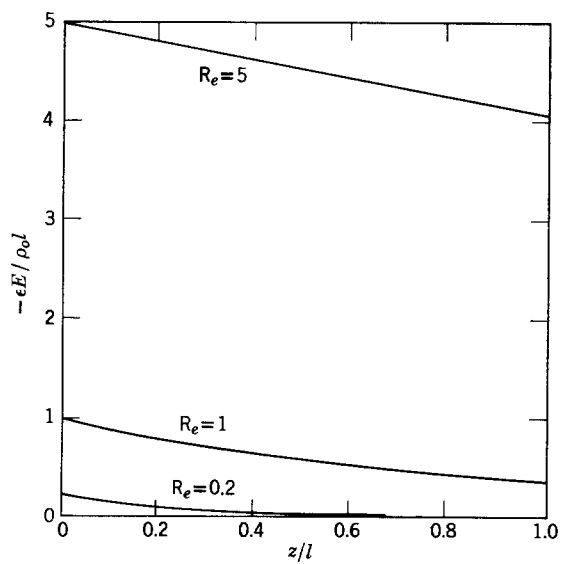


Fig. 7.2.10 Variation of electric field intensity with position.

Table 7.2.2

R_e	$\frac{\epsilon V_{oc}}{\rho_o l^2}$
0.2	0.0373
1.0	0.632
5.0	4.55

Thus for fixed geometry (l), fixed charge density at $z = 0(\rho_o)$, and fixed permittivity the open-circuit voltage is increased by increasing the velocity (V) of the belt or by decreasing the conductivity (σ) of the belt material. In either case the electric Reynolds number is increased. Note from (h) that the open-circuit voltage increases in proportion to R_e . Thus for the highest possible open-circuit voltage we must make the velocity as high as possible and the conductivity as low as possible.

In the previous example, we assumed that the electric field intensity and material motion were in the same direction. Practical generators are shown in Figs. 7.2.8*b-c*. In these devices, the component of the electric field intensity due to charge on the belt (as compared to that on the terminals) is essentially perpendicular to the direction of motion, and hence a somewhat different model is appropriate for quantitative purposes.

For conceptual simplicity we have so far in this section emphasized the effect of material motion on electric fields in situations in which the material could be essentially characterized by a single electric Reynolds number. Cases in which electric fields interact with continuous media are extremely diverse, both in terms of the mechanism basic to the interaction and in terms of possible applications. This point can be illustrated by considering an example that has some of the basic ingredients of a thunderstorm. This is a system that involves a highly insulating gas (the air, with a relaxation time that can be measured in hours) and water drops (with a relaxation time considerably less than a millisecond). The experiment is an electrohydrodynamic dynamo, in that the motion of the liquid drops is responsible for the generation of extremely high voltages with no external electrical excitations.

Example 7.2.6. The experiment is shown in Fig. 7.2.11. It can be simply constructed from two cans, two pieces of wire, and a pair of pipettes connected to a container of water. Flow of water through the pipettes is adjusted so that the streams break up into drops in the vicinity of the conducting rings. These rings encircle the streams but do not make electrical contact with the drops. The rings are connected, respectively, to the opposite cans, as shown in Fig. 7.2.11. It is extremely important that these cans be carefully insulated from each other. This can be done, as shown in Fig. 7.2.11*a*, by supporting the cans on a dry piece of plastic.

With no external excitations, the apparatus shown in Fig. 7.2.11*a* is capable of producing 10 to 20 kV. This potential difference, which appears between the cans, can be measured with an electrostatic voltmeter, not shown in Fig. 7.2.11*a*.

The formulation of a quantitative model for this experiment affords an opportunity not only to illustrate further the effect on electric fields of media in motion but also to undertake the modeling process itself. In all the examples found in this chapter mathematical models were involved. Here we are involved with the experiment itself and must translate the

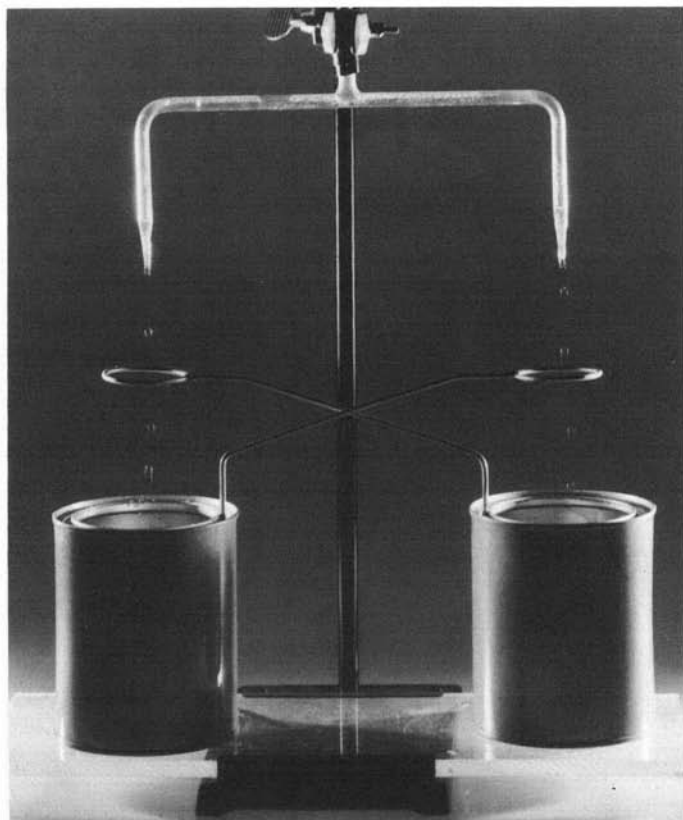


Fig. 7.2.11a Water drops fall into the cans through cross-connected wire loops. A potential difference of more than 20 kV between cans is spontaneously generated by the motion of the drops. For optimum operation the drops should form nearer to the rings than shown. This is accomplished by increasing the flow rate.

observed phenomena into a mathematical explanation. (Of course, actually envisioning the experiment is an even more demanding intellectual process for which we must credit Lord Kelvin.*)

First of all we require a plausible explanation for the generated voltage. Suppose that at a given instant the left ring in Fig. 7.2.11b has a slight excess charge (either due to random fluctuations or to an initial charge purposely placed on the ring). Then charges of the opposite polarity will be induced on the stream passing through the ring, as shown in Fig. 7.2.12. This stream breaks up into drops in the region of the ring; and, as a result, the fully formed drops carry a net induced charge into the container below. But this charge is communicated by means of the wire to the ring encircling the right-hand stream. Hence positive charges are induced on this stream as images of the negative charges on the ring. Now we have the mechanism for regenerative feedback because the positive charge on the right

* L. B. Loeb, *Static Electrification*, Springer, Berlin, 1958, p. 60.

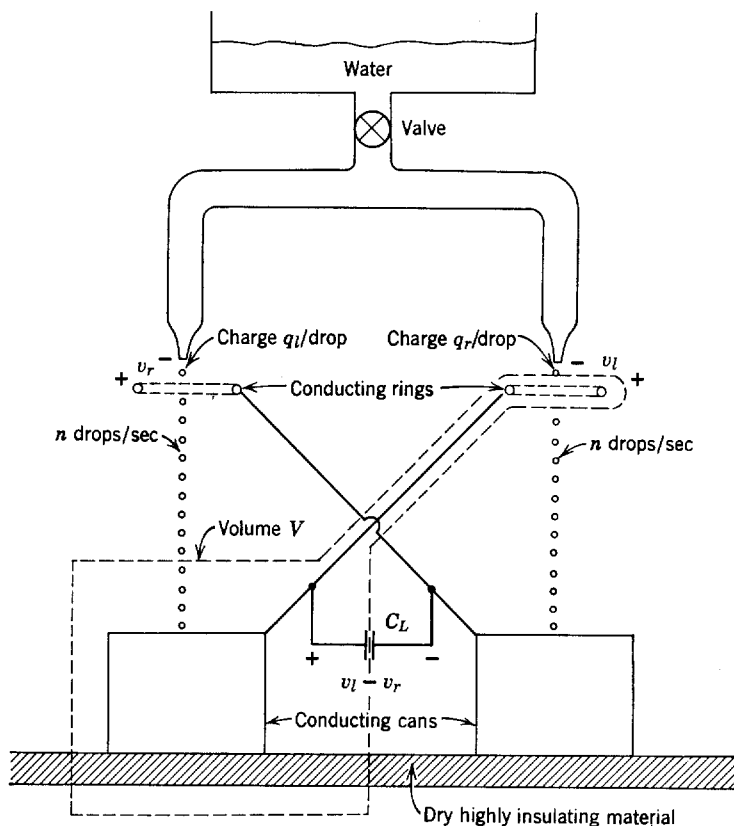


Fig. 7.2.11b Schematic representation of the experiment shown in Fig. 7.2.11a. The capacitance C_L represents the effect of an external electrostatic voltmeter connected between the cans. The volume V is used to define quantitatively the voltages v_r and v_l in Example 7.2.6.

stream falls into the right container, where it is communicated to the left ring by the connecting wire. On the left ring further negative charges are induced on the stream, with a resulting build up of voltage.

In terms of this explanation it is apparent why it is important that there be two relaxation times in the problem. The charge must relax to the surface of the stream before it breaks up. Once the drops are formed, however, the net charge on an individual drop is essentially conserved. These conditions are satisfied here because the water has a relaxation time that is much shorter than the time required for a drop to form, whereas the air has a relaxation time that is much longer than the time required for a drop to reach the container below.

To model the experiment quantitatively we assume that the charge on each drop is proportional to the voltage on the inducing ring with respect to the water in the pipette (which is at ground potential). Thus

$$q_l = -C_i v_r, \quad (a)$$

$$q_r = -C_i v_l, \quad (b)$$

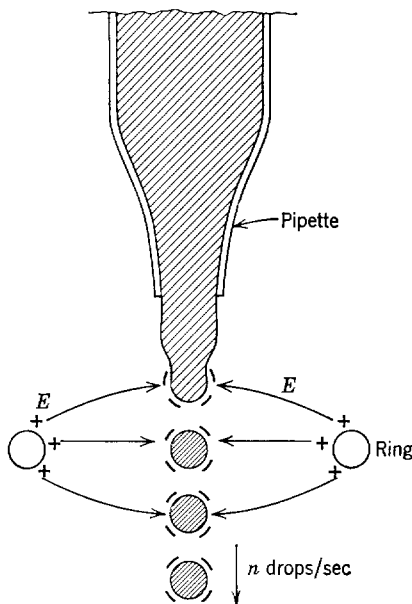


Fig. 7.2.12 As the drops form near the rings in Fig. 7.2.11, image charges are induced on the surface of the water. When the drops are completely formed (and insulated from one another) they carry a net charge, the sign of which is determined by the sign of the charge on the respective rings.

where C_i is the constant of proportionality which has the dimension of capacitance. Because there are two equipotentials in addition to ground (the base and the water in the pipettes), the system can be modeled by the equivalent circuit of Fig. 7.2.13. The two current sources represent the rate of charge transport to the cans by the drops. The capacitances C are capacitances to ground for each can and ring and C_L is the capacitance between cans plus a load such as an electrostatic voltmeter.

From Fig. 7.2.11b conservation of charge for the volume V requires

$$-nC_i v_r = C \frac{dv_l}{dt} + C_L \frac{d(v_l - v_r)}{dt}. \quad (c)$$

This is the node equation for the left node in Fig. 7.2.13. Similarly,

$$-nC_i v_l = C \frac{dv_r}{dt} + C_L \frac{d(v_r - v_l)}{dt}. \quad (d)$$

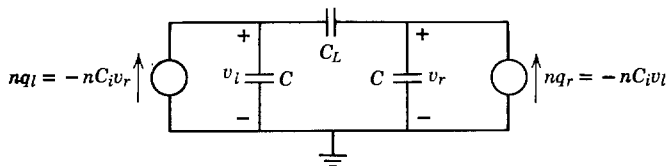


Fig. 7.2.13 Equivalent circuit of electrostatic generator of Fig. 7.2.11.

These expressions could be reduced to a single second-order equation for either of the dependent variables. Hence there are two solutions. By symmetry we expect that one of these solutions is odd, in the sense that $v_l = -v_r$. That this is, in fact, the case can be seen by substituting this condition into (c) and (d) and seeing that they *both* reduce to the form

$$(C + 2C_L) \frac{dv_l}{dt} - nC_l v_l = 0. \quad (e)$$

Similarly, if $v_r = v_l$, both (c) and (d) reduce to the form

$$C \frac{dv_l}{dt} + nC_l v_l = 0. \quad (f)$$

The general solutions are appropriate linear combinations of the solutions of (e) and (f):

$$v_l = A \exp\left(\frac{nC_l}{C + 2C_L} t\right) + B \exp\left(-\frac{nC_l}{C} t\right), \quad (g)$$

$$v_r = -A \exp\left(\frac{nC_l}{C + 2C_L} t\right) + B \exp\left(-\frac{nC_l}{C} t\right). \quad (h)$$

where the constants A and B are determined by initial conditions. Any slight unbalance in initial voltage (such as that supplied by natural noise) will lead to exponential growth of the odd part of the solution and decay of the even part. In practice the potential difference between the cans builds up until there is electrical breakdown between the cans or the electrical force deflects the drops until they hit the inducing rings. The example discussed here illustrates the effect of material motion on the electric field. The drop deflections represent the reverse effect of the fields on the motion.

7.2.3 Sinusoidal Excitation and Charge Relaxation with Motion

A comparison between the developments of Section 7.1, concerned with magnetic diffusion, and those found in this section (Section 7.2) shows a considerable analogy. As might be expected from Section 7.1.3, the introduction of an additional characteristic time by the agent of a sinusoidal excitation makes it possible to tailor the relaxation process to engineering requirements. In this section we undertake to contrast the relaxation process with magnetic diffusion in the presence of motion and with a sinusoidal excitation. The example considered illustrates how charge relaxation can be used to measure the velocity of a moving medium.

A thin slab of slightly conducting material moves to the right with velocity V , as shown in Fig. 7.2.14. This slab passes between plane parallel electrodes which are broken into three sections with different terminations (Fig. 7.2.14b). In the section to the left the plates are driven by a sinusoidally varying potential. From the preceding sections we know that because the slab is insulated from the plates by the intervening air, charges will relax to the surfaces of the slab in a way that tends to exclude the imposed electric field. Hence, as the slab leaves this exciter region at $z = 0$, symmetry requires that there be a sinusoidally varying surface charge density σ_o on the upper slab surface and $-\sigma_o$ on the lower surface.

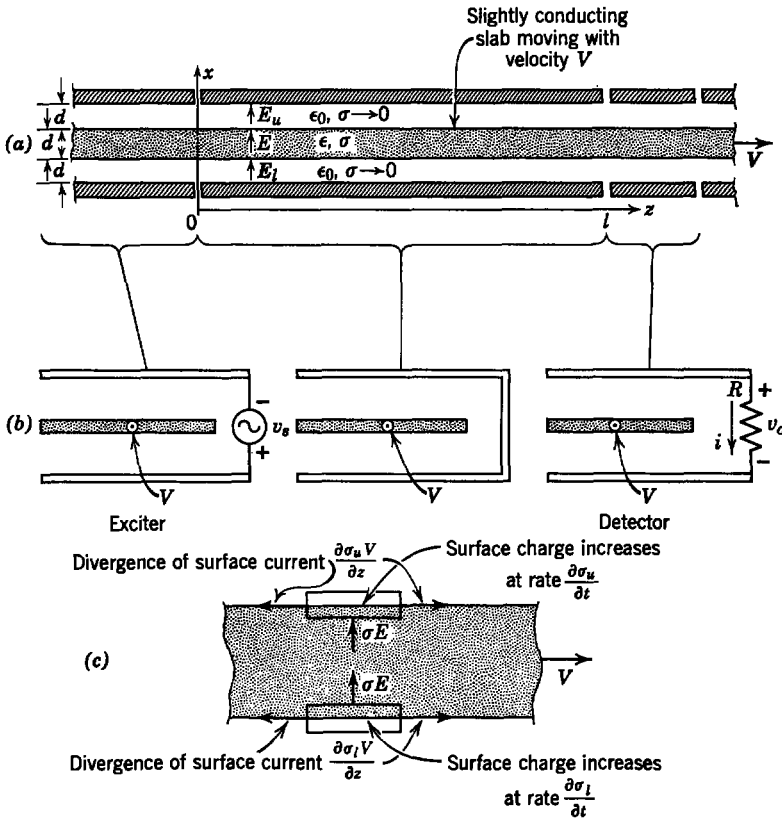


Fig. 7.2.14 (a) A slightly conducting slab moves to the right with velocity V through electric fields imposed or constrained as shown in (b); (c) conduction currents normal to the surfaces lead to an accumulation of surface charge that can be convected away.

As the slab passes through the region $0 < z < l$, the surface charges induced in the exciter region tend to relax so that by the time they reach the detector region to the right the surface charge densities can be considerably attenuated. Those charges that remain induce image charges on the electrodes with a resulting current through the resistance R and a proportionate generated output signal v_o . Because of the charge relaxation this signal can be a measure of the material velocity, as shown in the development that follows.

We know from preceding sections that there is no charge in the bulk of the moving slab. On the surfaces, however, the normal conduction current leads to an accumulation of charge that can be convected away by the motion of the slab. Using the electric field intensities defined in Fig. 7.2.14a,

conservation of charge on the upper surface requires that (see Fig. 7.2.14c)

$$\sigma E = \frac{\partial \sigma_u}{\partial t} + V \frac{\partial \sigma_u}{\partial z}, \quad (7.2.18)$$

where the surface charge density σ_u is related to the field intensities by

$$\sigma_u = \epsilon_0 E_u - \epsilon E. \quad (7.2.19)$$

Equation 7.2.18 is a particular application of boundary condition (6.2.36),* discussed in Section 6.2.2 (see Example 6.2.3).

Similarly, conservation of charge on the lower surface requires

$$-\sigma E = \frac{\partial \sigma_l}{\partial t} + V \frac{\partial \sigma_l}{\partial z}, \quad (7.2.20)$$

where

$$\sigma_l = \epsilon E - \epsilon_0 E_l. \quad (7.2.21)$$

Now, in general, the electric field in the slab, and in the regions above and below, must satisfy the electric field equations with $\rho_f = 0$. We can avoid introducing the complication of solving these equations by making a quasi one-dimensional model based on the distance d , shown in Fig. 7.2.14a, being much smaller than a characteristic (wave) length in the z -direction. Because the plates are short-circuited in the region $0 < z < l$, we not only assume that $\mathbf{E} = E(z, t)\mathbf{i}_z$ but write

$$d(E_u + E + E_l) = 0 \quad (7.2.22)$$

because the integral of the field along a line joining the plates must be zero.

The one-dimensional model implied by (7.2.18) to (7.2.22) ignores any effect introduced by a component of field intensity in the z -direction. This component however, does exist and can be computed from E_x through the condition that the field intensity be irrotational (7.2.1). This second-order effect of one of the field components characterizes "quasi" one-dimensional models as they are found in a variety of physical situations.

At $z = 0$ we assume that the exciter has induced the surface charges

$$\sigma_l = -\sigma_u = \sigma_0 \sin \omega t. \quad (7.2.23)$$

For this reason we look for solutions to (7.2.18) to (7.2.22) that satisfy the requirement that $E_u = E_l$. Then, from (7.2.22),

$$E = -(E_u + E_l) = -2E_u, \quad (7.2.24)$$

and it follows from (7.2.19) that the upper surface charge density is related to the field intensity within the moving slab by

$$E = \frac{-2\sigma_u}{\epsilon_0 + 2\epsilon}. \quad (7.2.25)$$

* See Table 6.1, Appendix E.

This relation, together with (7.2.18) then shows that

$$\frac{\partial \sigma_u}{\partial t} + V \frac{\partial \sigma_u}{\partial z} + \frac{2\sigma}{\epsilon_0 + 2\epsilon} \sigma_u = 0. \quad (7.2.26)$$

Once the upper surface charge density has been found from this expression the lower surface charge density and the three field intensities can be found; That is, $\sigma_u = -\sigma_l$ (for the excitation considered here), $E_u = E_l$, and E is given by (7.2.24).

Because the charge is varying sinusoidally at $z = 0$, we assume that the steady-state solution has the form

$$\sigma_u = \text{Re} [\hat{\sigma}_u(z)e^{j\omega t}]. \quad (7.2.27)$$

Substitution into (7.2.26) shows that

$$\frac{d\hat{\sigma}_u}{dz} + \hat{\sigma}_u \left[\frac{2\sigma}{(\epsilon_0 + 2\epsilon)V} + \frac{j\omega}{V} \right] = 0. \quad (7.2.28)$$

The solution to this equation, which satisfies the boundary condition of (7.2.23), is

$$\sigma(z, t) = \sigma_o \exp \left[-\frac{2\sigma l}{(\epsilon_0 + 2\epsilon)V l} z \right] \sin \omega \left(t - \frac{z}{V} \right). \quad (7.2.29)$$

The distribution of surface charge on the moving slab, as given by (7.2.29), is sketched in Fig. 7.2.15. The relaxing charge appears as a wave that propagates to the right with the phase velocity V and with an exponential envelope determined by the electric Reynolds number R_e , defined in this case as

$$R_e = \frac{(\epsilon_0 + 2\epsilon)V}{2\sigma l}. \quad (7.2.30)$$

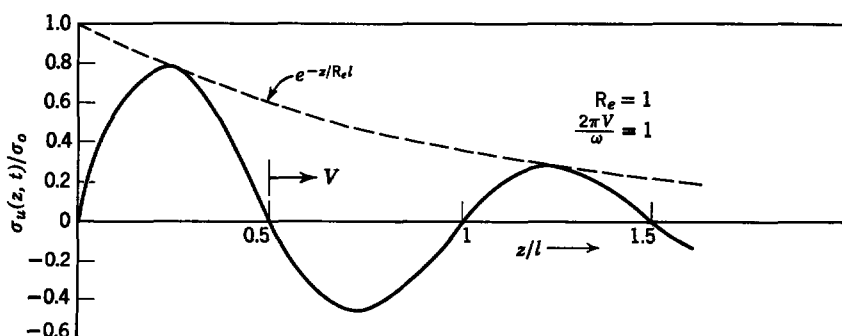


Fig. 7.2.15 Distribution of free charge density on the upper surface of the moving, slightly conducting slab of Fig. 7.2.14. Points of constant phase move with the velocity V of the material.

Note that by contrast with the magnetic diffusion waves of Section 7.1.3 the relaxation wave found here propagates only by virtue of the material motion. There are no waves that propagate upstream, as there were in Section 7.1.3.

We can make practical use of the relaxation interaction by using the section of plates to the right in Fig. 7.2.14*a, b* to detect the charge on the surface. This is done by attaching a resistance R between the plates and measuring the induced voltage. If we assume that this resistance is small enough to ensure that the drop in potential across the resistance is negligible compared with Ed in the section to the left, the induced current is simply

$$i = -A\epsilon_0 \frac{\partial E_u}{\partial t}, \quad (7.2.31)$$

where A is the area of one of the detector plates. Here we have assumed that the detector plates are of sufficiently small extent in the z -direction that the integral of the surface charge over the surface of one of the plates can be approximated by the product of A and the charge at $z = l$. It follows that the voltage v_o from the detector has the magnitude

$$|v_o| = |i| R = \frac{A\epsilon_0 \omega \sigma_o}{(\epsilon_0 + 2\epsilon)} e^{-1/R_e}. \quad (7.2.32)$$

The dependence of this signal on the electric Reynolds number is shown in Fig. 7.2.16. As we expected intuitively, the output signal increases with velocity and decreases with conductivity or length l . Unfortunately, the functional dependence is not so convenient as it was in the system discussed

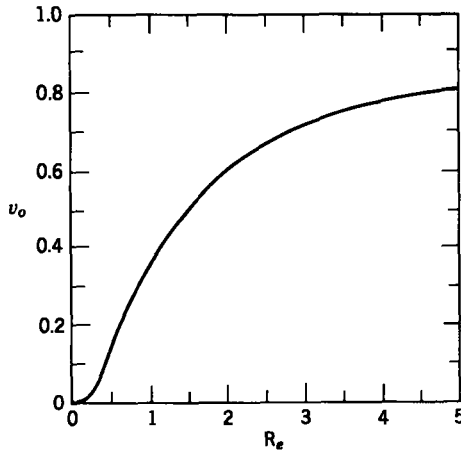


Fig. 7.2.16 Output voltage as a function of electric Reynolds number $R_e = [(\epsilon_0 + 2\epsilon)V]/2\sigma l$ for the system of Fig. 7.2.14.

in Section 7.1.3. It is important in this regard to note that systems in which R_m (magnetic Reynolds number) is on the order of unity and in which R_e is about unity are in entirely different velocity and conductivity ranges. This point is emphasized in Section 7.3.

7.2.4 Traveling-Wave Charge Relaxation in a Moving Conductor

As illustrated in Section 7.1.4, interactions between a periodic traveling wave and moving media provide considerable physical insight into a diversity of situations. In plane geometry the analytical difficulties involved in magnetic field systems were minimal and in the electric field system considered here the mathematical description is even simpler. Even so, as discussed in Section 7.3, a great deal can be learned that is applicable to more complicated situations.

The physical situation to be considered, which is analogous to that discussed in Section 7.1.4, is shown in Fig. 7.2.17. A slightly conducting material moves to the right with a velocity V just below a segmented electrode which supports a traveling wave of potential

$$v = \text{Re } \hat{v} e^{j(\omega t - kz)} \quad (7.2.33)$$

It is assumed that material (a) has negligible electrical conductivity. We wish to determine the effect of the motion on the distribution of potential (hence electric field) above and below the surface (at $x = 0$) of the moving conductor. We know in advance that because electrical conductivity σ is uniform in the moving conductor there is no bulk free charge there. Therefore the interactions between the fields and the media occur at the surface of the moving material where surface charges assume a distribution and magnitude determined by the motion and by the velocity of the imposed traveling wave of potential.

Because there is no free charge in either region (a) or (b), (7.2.8), which follows from (7.2.5), reduces to

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \quad (7.2.34)$$

Here we assume that the problem is two-dimensional so that $\partial \phi / \partial y = 0$. The traveling wave of potential given by (7.2.33) constitutes a boundary condition on the potential ϕ in the region (a) above the conductor. To match this condition we assume that the potential ϕ takes the form

$$\phi = \text{Re } \hat{\phi}(x) e^{j(\omega t - kz)} \quad (7.2.35)$$

in both regions (a) and (b). Substitution of this form into (7.2.34) shows that in region (a)

$$\hat{\phi}_a = A \sinh kx + B \cosh kx, \quad (7.2.36)$$

where A and B are arbitrary constants to be determined by the boundary

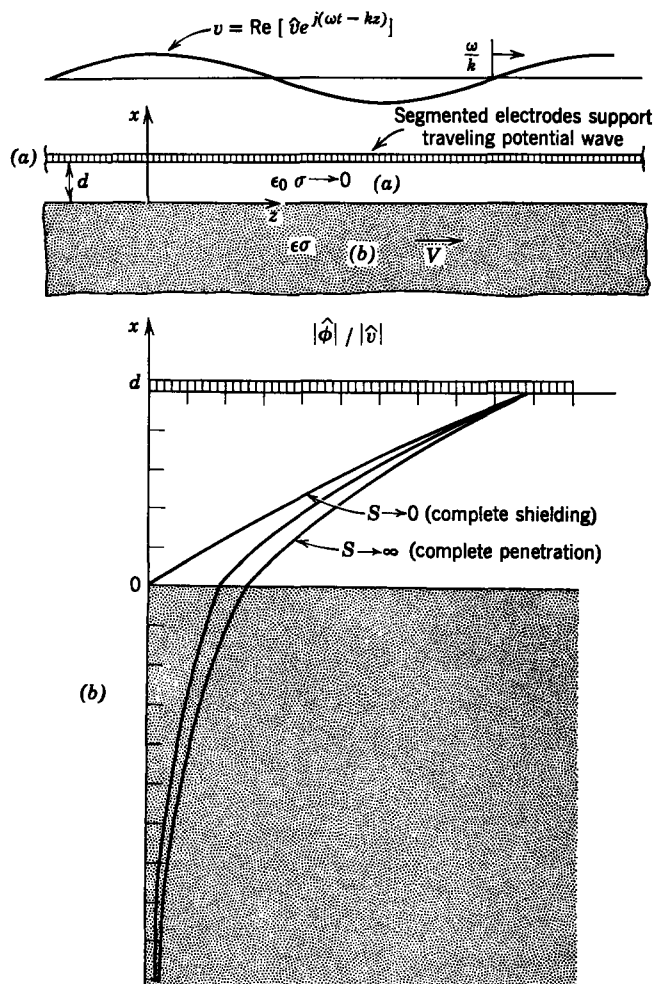


Fig. 7.2.17 (a) A slightly conducting material moves to the right with velocity V just below a segmented electrode which supports a traveling potential wave; (b) distribution of potential above and below the conductor for various values of $S = (\omega - kV)\epsilon_0/\sigma$.

conditions. In region (b) it is more convenient to write the two solutions in the form of exponentials. Then, if we assume that the material in region (b) extends many wavelengths $2\pi/k$ in the $-x$ -direction, one of these solutions becomes unbounded as $x \rightarrow -\infty$ and the appropriate solution is

$$\hat{\phi}_b = C e^{kx}. \quad (7.2.37)$$

In retaining this half of the solution in region (b) we are assuming that k is a positive number so that the potential will not go to infinity as $x \rightarrow -\infty$.

Three boundary conditions, one of which is (7.2.33), determine the constants A , B , and C . A second condition develops because the potential is continuous at $x = 0$.

$$\hat{\phi}_a(0) = \hat{\phi}_b(0). \quad (7.2.38)$$

The third condition requires conservation of charge on the interface. This equation has the same origins as (7.2.18) in Section 7.2.3. It can be written in terms of the potentials by recognizing that the surface charge density is

$$\sigma_f = \epsilon_0 E_x^a - \epsilon E_x^b = -\epsilon_0 \frac{\partial \phi_a}{\partial x} + \epsilon \frac{\partial \phi_b}{\partial x}. \quad (7.2.39)$$

Then the condition that the conduction current normal to the interface lead to an increase in the surface charge measured in a frame moving with the material is written as

$$-\sigma \frac{\partial \phi_b}{\partial x} = \left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial z} \right) \left(-\epsilon_0 \frac{\partial \phi_a}{\partial x} + \epsilon \frac{\partial \phi_b}{\partial x} \right). \quad (7.2.40)$$

This boundary condition is a particular case of (6.2.36) (see Table 6.1)*. For the assumed traveling-wave solution (7.2.35) it becomes

$$-\sigma \frac{d\hat{\phi}_b}{dx} = j(\omega - kV) \left(-\epsilon_0 \frac{d\hat{\phi}_a}{dx} + \epsilon \frac{d\hat{\phi}_b}{dx} \right). \quad (7.2.41)$$

Now we are in a position to find the constants A , B , and C by requiring that (7.2.36) and (7.2.37) satisfy the boundary conditions of (7.2.33), (7.2.38), and (7.2.41). Substitution gives

$$A \sinh kd + B \cosh kd = \hat{v}, \quad (7.2.42)$$

$$B - C = 0, \quad (7.2.43)$$

$$A(jS) - C \left(1 + jS \frac{\epsilon}{\epsilon_0} \right) = 0, \quad (7.2.44)$$

where S is the normalized frequency (measured in the moving frame of the material) normalized to the relaxation time

$$S = \frac{(\omega - kV)\epsilon_0}{\sigma}. \quad (7.2.45)$$

The constants A , B , and C follow from (7.2.42) to (7.2.44) with the result that the potential distributions above and below the moving medium are

$$\phi_a = \text{Re} \frac{\hat{v}}{\Delta} \left[\left(1 + jS \frac{\epsilon}{\epsilon_0} \right) \sinh kx + jS \cosh kx \right] e^{j(\omega t - kz)}, \quad (7.2.46)$$

$$\phi_b = \text{Re} \frac{\hat{v}}{\Delta} jS e^{kx} e^{j(\omega t - kz)}, \quad (7.2.47)$$

where $\Delta = [1 + jS(\epsilon/\epsilon_0)] \sinh kd + jS \cosh kd$.

* Appendix E.

The magnitude of the potential distribution predicted by these equations is shown in Fig. 7.2.17*b*. The potentials for an intermediate value and an extreme value of S are shown to emphasize two important points. When the frequency in the moving frame of the medium is small compared with the reciprocal relaxation time $S' \ll 1$, the fields are excluded from the material and confined to region (a). In this extreme charges have a sufficient amount of time [one period $2\pi/(\omega - kV)$, as measured in the frame of the medium] to relax to the surface, where they shield the electric field from the medium; that is, in the limit $S \rightarrow 0$ the potential in the medium is zero, whereas that just above is

$$\phi_a = \text{Re} \left[\hat{v} \frac{\sinh kx}{\sinh kd} e^{j(\omega t - kz)} \right]. \quad (7.2.48)$$

Note that this is simply the potential distribution found when the medium is modeled as a perfectly conducting material.

In the opposite extreme, in which the normalized frequency $|S|$ is very large, the potential distributions become

$$\phi_a = \text{Re} \left[\hat{v} \frac{\cosh kx + \epsilon/\epsilon_0 \sinh kx}{\cosh kd + \epsilon/\epsilon_0 \sinh kd} e^{j(\omega t - kz)} \right], \quad (7.2.49)$$

$$\phi_b = \text{Re} \left[\hat{v} \frac{e^{kx} e^{j(\omega t - kz)}}{\cosh kd + \epsilon/\epsilon_0 \sinh kd} \right]. \quad (7.2.50)$$

From Fig. 7.2.17*b* it is clear that in this limit the electric field completely penetrates the moving medium; that is, the potential distribution is just as it would be if the moving medium were modeled as a perfectly insulating material with no free charge at the surface. (Note that (7.2.49) and (7.2.50) show that there is no free charge at $x = 0$.) In the high-frequency limit one period of excitation (measured in the moving frame of the medium) is not sufficient time for appreciable free charge to relax to the surface. The parameter S can be large either because ω is large or because kV is large; that is, if the material is fixed ($V = 0$) and the imposed wave has a high frequency ω , the fields penetrate the medium with little shielding from free surface charges. Similarly, the frequency ω of the imposed spacially periodic wave could be zero and the same penetration would result by having the material move with a velocity V large enough that the frequency kV observed in the moving medium would be large compared with a reciprocal relaxation time.

Although we do not pursue this point here, the traveling wave of potential results in a time average electric shear on the material surface, hence gives rise to a z -directed force, much as described in Section 7.1.4 for a magnetic field system.*

* For a development of this subject see J. R. Melcher, "Traveling-Wave Induced Electroconvection," *Phys. Fluids* 9, 1548-1555 (1966).

7.3 CONCLUSION

The subjects of magnetic diffusion and charge relaxation concern themselves with the behavior of fields in the presence of dissipative media. In each case a characteristic time is associated with the dissipation mechanism. In the case in which currents (hence magnetic fields) diffuse in a conducting medium this is the magnetic diffusion time $\mu\sigma l^2$. Similarly, the relaxation time ϵ/σ for free charge governs the dynamical behavior of electric fields in slightly conducting materials.

The nature of the field solutions is determined by the ratio of these times to times that characterize the dynamics of the system; for example, in a magnetic field system excited at the frequency ω , field distributions are determined by the magnetic Reynolds number $\mu\sigma l^2\omega$ (the ratio of diffusion time to the period of excitation). Alternatively, this number can be thought of as the square of the ratio of a characteristic length in the system to the skin depth δ (see Section 7.1.3). A second dynamical time is introduced by motion. With a characteristic velocity V and length l , a second magnetic Reynolds number $\mu\sigma V l$ indicates the importance of material motion in determining the distribution of currents. As we saw in Section 7.1.3, there are situations in which the mechanisms of altering the fields represented by these characteristic numbers are present simultaneously.

In an electric field system excitation at the frequency ω leads to an electric Reynolds number $\omega\epsilon/\sigma$, whereas motion with the velocity V through a characteristic distance l has an effect on the electric field determined by $\epsilon V/\sigma l$. A simple summary example provides the opportunity to review these effects while contrasting the behavior of magnetic and electric field systems.

A cylinder of material with conductivity σ is placed between

the pole faces of a magnet as shown in Fig. 7.3.1a.

With the cylinder stationary, the initially uniform magnetic field completely penetrates the conducting material, as shown in Fig. 7.3.2a.

a pair of plane parallel electrodes, as shown in Fig. 7.3.1b.

With the cylinder stationary, the electric field is completely shielded from the interior of the conducting material by free charges that relax to the cylindrical surface, as shown in Fig. 7.3.2b.

The effect of material motion on the steady-state fields is demonstrated by considering the consequences of making the conductor rotate. If the cylinder has a radius R and angular velocity Ω , a characteristic velocity is ΩR . The

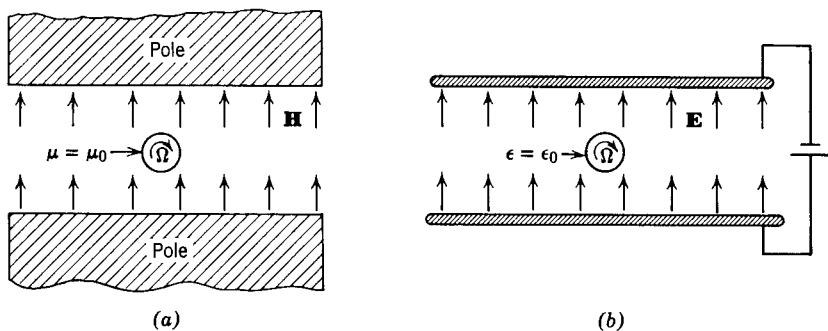


Fig. 7.3.1 (a) Magnetic field system; (b) electric field system.

motion alters the fields of Fig. 7.3.2 if

the magnetic Reynolds number R_m is significant compared with unity

$$R_m = \mu\sigma VR = \mu\sigma\Omega R^2.$$

the electric Reynolds number R_e is significant compared with unity

$$R_e = \frac{\epsilon V}{\sigma R} = \frac{\epsilon\Omega}{\sigma}.$$

The conditions under which R_m and R_e are on the order of unity are very different; for example, suppose that $\Omega = 10$ rad/sec and $R = 10$ cm. Then,

if we let $\mu = \mu_0$, the magnetic Reynolds number is unity if $\sigma = 7.95 \times 10^6$ mhos/m, which is about the conductivity of solder.

if we let $\epsilon = \epsilon_0$, the electric Reynolds number is unity if $\sigma = 8.85 \times 10^{-11}$ mhos/m, which is about the conductivity of corn oil.

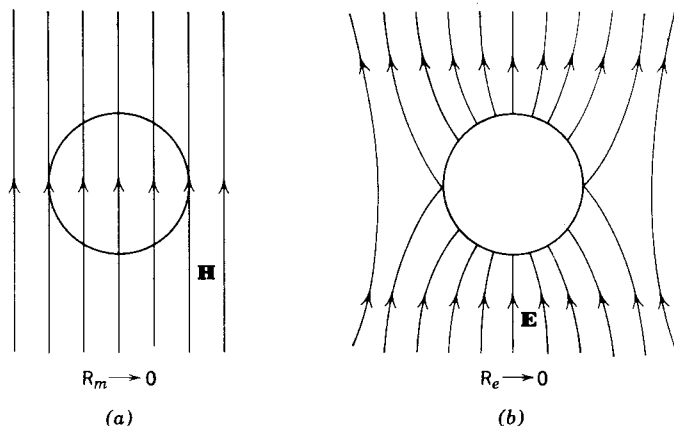


Fig. 7.3.2 (a) Magnetic fields penetrate stationary media; (b) electric fields are excluded from stationary media.

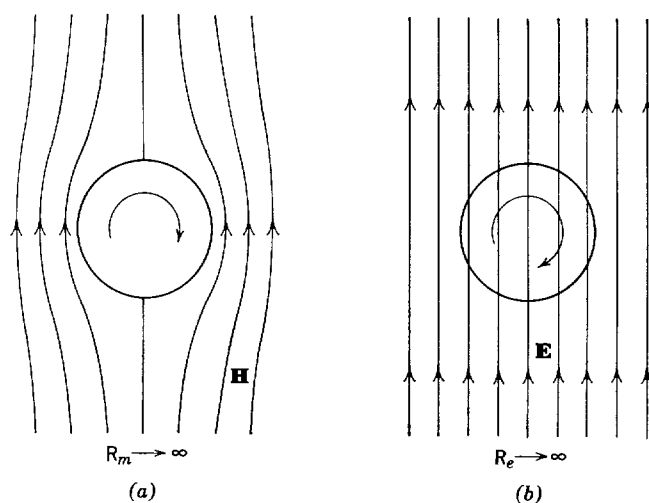


Fig. 7.3.3 (a) Magnetic field completely shielded from cylinder; (b) electric field completely penetrates cylinder.

By making the rotational velocity Ω very large there is a large motional effect on the imposed fields. Hence

for $R_m \gg 1$ the magnetic fields are completely excluded from the rotating cylinder by induced currents, as shown in Fig. 7.3.3a.

for $R_e \gg 1$ the electric field completely penetrates the rotating cylinder because charges are not able to relax to the cylindrical surface, as shown in Fig. 7.3.3b.

The observations that concern the effect of the rotating cylinders on the fields are evident from Sections 7.1.4 and 7.2.4. There we analyzed (a limiting case in which $\omega = 0$) the field effects of material motion in the presence of a temporally constant, spatially periodic wave. The cylinders considered in this summary are simply those problems with one wavelength along the cartesian coordinate z “wrapped around on itself.” Of course, the cylindrical geometry requires alterations in the mathematical details. The essential physical features of the problems in cartesian coordinates and those described here, however, are the same.

It should be apparent from the discussions of Sections 7.1.4 and 7.2.4 that we have also developed a picture of what would happen if the cylinders were to be fixed but the fields were to rotate. In the magnetic field case currents would be induced in the rotating cylinder which in turn would develop a torque. This is illustrated in Section 7.1.4 in plane geometry and is a distributed picture of the induction machine discussed in Section 4.1.6b. Situations in which a torque is produced by a rotating electric field are not so common but still exist.

PROBLEMS

7.1. If a short-circuited inductor is modeled by an ideal inductance (no resistance) in series with a resistance R equal to its internal resistance, the time-constant for decay of an initial flux $\lambda = \Lambda$ is $\tau = L/R$. Design an inductor with a time-constant L/R of 100 sec. Specify the dimensions and the material involved; for example, you may wish to use

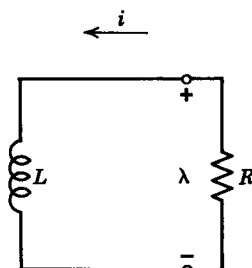


Fig. 7P.1

copper wire, in which case you should specify the size of the wire, the number of turns required, and the size of the winding. You are not allowed to use a superconducting winding.

7.2. This problem is a variation on the situation described in Section 7.1.1. A conducting slab of material with thickness d is positioned between the pole faces of a magnet, as shown in Fig. 7P.2. The switch S has been open for a long time so that the magnetic field in the gap

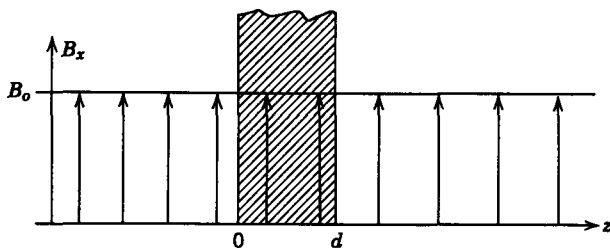


Fig. 7P.2

is as shown. When $t = 0$, the switch is closed and the magnetic field outside the material in the gap collapses.

- (a) Compute the flux density distribution as a function of time in the gap.
- (b) Find the current distribution from the result of (a).

7.3. A fixed block of material with conductivity σ makes electrical contact with a pair of perfectly conducting parallel plates, as shown in Fig. 7P.3. Dimensions are such that $w \ll L$ and $w \ll D$. A current source I (amperes) is distributed along the edge of the plates at $x = -L$. $I(t) = I_0 u_{-1}(t)$, as shown.

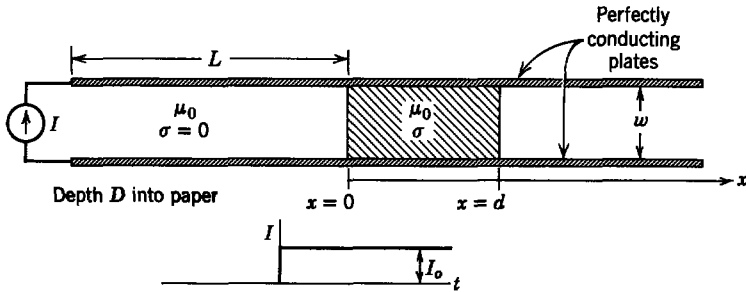


Fig. 7P.3

- Sketch and dimension the distribution of current in the block an instant after the step of current is applied.
- Sketch and dimension the distribution of current as $t \rightarrow \infty$.
- In terms of σ and the system dimensions, how long do you *estimate* that it takes the current to reach essentially the distribution of (b)? (This should be an order of magnitude calculation.)

7.4. A pair of perfectly conducting plates is short-circuited through a conducting block, as shown in Fig. 7P.4a. The block and plates extend a long distance to the right. A current excitation $i(t) = \text{Re}(te^{j\omega t})$ is applied uniformly to the plates along their left edges.

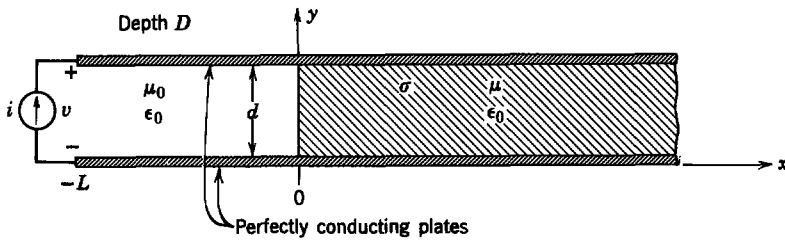


Fig. 7P.4a

- Find the magnetic flux density in the region between the plates.
- Find the current density in the block.
- Sketch the results of (a) and (b).
- Find the equivalent reactance seen at the current source. Using the equivalent circuit shown in Fig. 7P.4b, what are the values of L and R ? How do they depend on frequency?

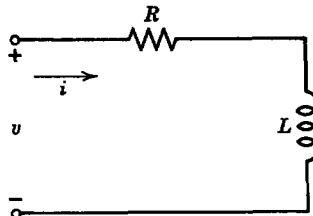


Fig. 7P.4b

7.5. A block of material with conductivity σ is bounded by plane-parallel, perfectly conducting electrodes, as shown in Fig. 7P.5. The plates are driven at the left by the current (amperes) $I_0(t) = \text{Re}(\tilde{I}e^{j\omega t})$, where $\tilde{I}$ is a given complex constant. This current is returned through the block of conductor.

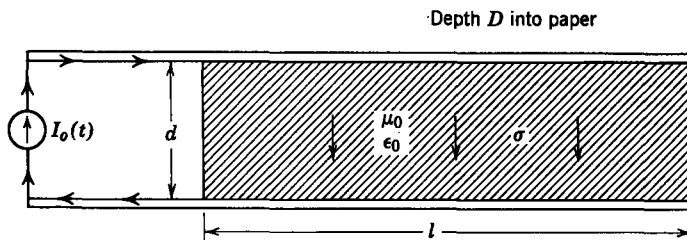


Fig. 7P.5

- Determine the distribution of magnetic field and current within the block.
- Outline how you would find the time-average magnetic force on the block.

7.6. As an example of electromagnetic phenomena that occur in conductors at rest, consider the system in Fig. 7.1.1 with the constant-current sources and switch replaced by an alternating current source, $i(t) = I \cos \omega t$. Make all of the assumptions of Section 7.1.1 and adopt the coordinate system of Fig. 7.1.2. Interest is confined to the sinusoidal steady-state problem.

- Find the magnetic flux density in the conducting slab.
- Find the current density in the slab.
- Sketch the results of parts (a) and (b).

7.7. A pair of plane-parallel, perfectly conducting plates is driven at the left, as shown in Fig. 7P.7a, by a current source $I(t)$, which is a step function. The plates are short-circuited by two conducting slabs spaced a distance l and of thickness Δ . An instant after the current I is established all the current is returned in the left slab; region (1) contains a uniform magnetic field, and region (2) is free of magnetic field. Then, with a characteristic time τ , the field diffuses into region (2).

- Use the "equivalent circuit model" shown in Fig. 7P.7b to establish this characteristic time.
- Physically, why is this characteristic time much longer than the diffusion time based on the thickness Δ of the conducting slabs [(7.1.28) with $d = \Delta$]?

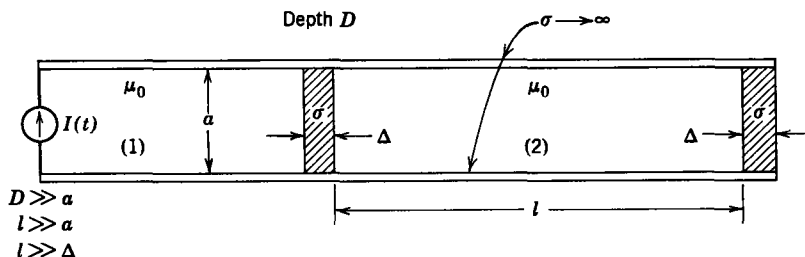


Fig. 7P.7a

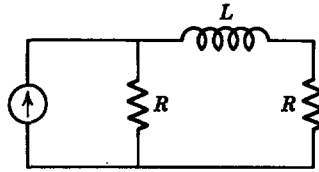


Fig. 7P.7b

- (c) Discuss how you would solve this problem by using a field description of the diffusion process.

7.8. The cross section of a cylindrical pair of conductors is shown in Fig. 7P.8a. The current I is a step function and flows azimuthally in the outer conductor. Hence, when $t = 0$, the magnetic field intensity has the distribution of Fig. 7P.8b. The inner conductor acts as a secondary of a transformer and supports an induced current that tends to shield the magnetic

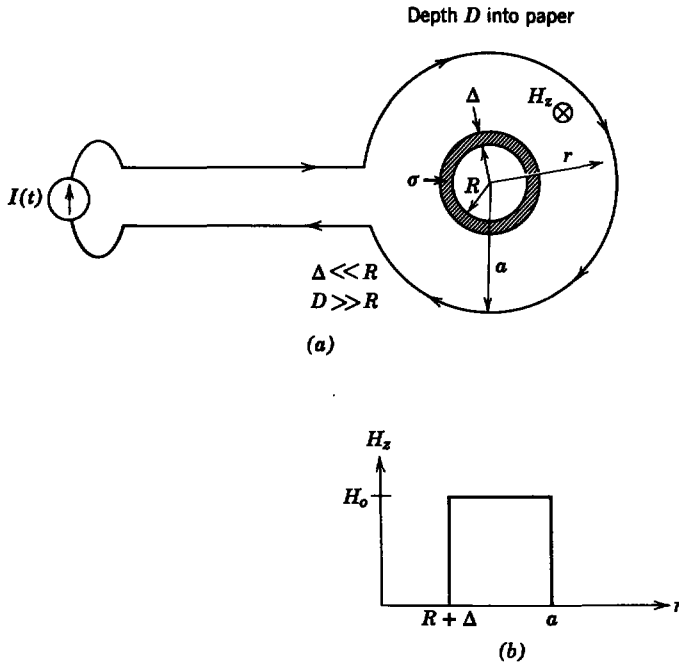


Fig. 7P.8a

field induced by I from the center region. Hence inside the center cylinder the magnetic field rises to its final value with some characteristic time. What is this characteristic time in terms of the system dimensions and σ ? *Hint.* See Problem 7.7.

7.9. As a possible plasma containment scheme, it has been proposed to create a cylindrical column of plasma within a long solenoid. The plasma is created by use of an arc discharge through a gas. After the plasma is formed the solenoid is excited by rapid discharge of a

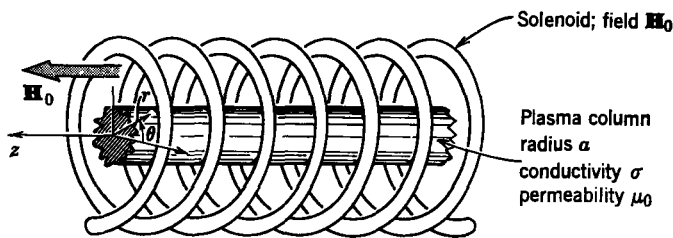


Fig. 7P.9a

capacitor bank connected to the terminals. The magnetic field of the solenoid is initially excluded from the interior of the plasma column by surface currents which flow in the θ direction on the plasma; for this reason the device is called a "Theta-pinch machine." Because the plasma has only a finite conductivity, the magnetic field diffuses into the interior. Therefore to design the machine it is necessary to determine the time associated with this field diffusion. In Fig. 7P.9a the plasma column is shown in the magnetic field. Neglect end effects by assuming an infinitely long system. Furthermore, assume that the plasma remains stationary during the diffusion process.

- Write an equation governing the magnetic flux density $\mathbf{B}$ inside the plasma.
- Because there are only θ currents independent of θ , assume that $\mathbf{B} = \hat{\mathbf{z}}B(r, t)$, where $B(r, t)$ has the form

$$B(r, t) = \mu_0 H_0 - \hat{B}(r)e^{-\alpha t}, \quad \alpha > 0,$$

which satisfies the condition that $B(r, t) \rightarrow \mu_0 H_0$ as $t \rightarrow \infty$. Using the results of part (a), write an equation for $\hat{B}(r)$.

- The equation obtained in part (b) for $\hat{B}(r)$ is called "Bessel's equation of zeroth order." The general solution to this equation is

$$\hat{B}(r) = C_1 J_0(\sqrt{\mu_0 \sigma \alpha} r) + C_2 N_0(\sqrt{\mu_0 \sigma \alpha} r),$$

where J_0 is the Bessel function of zeroth order and the first kind and N_0 is the Bessel

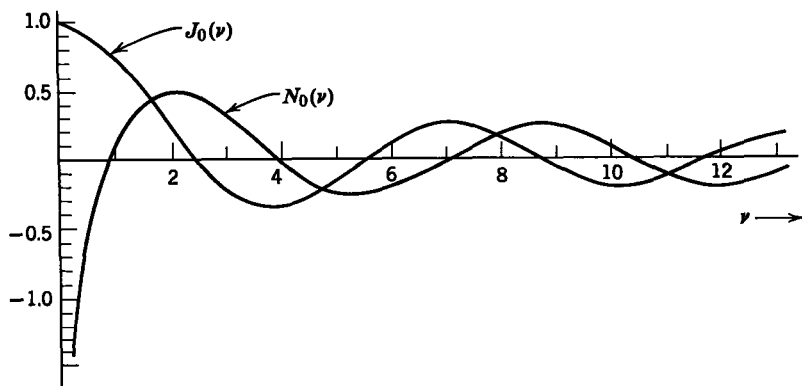


Fig. 7P.9b

function of zeroth order and the second kind (sometimes called a "Neumann function"). See Fig. 7P.9b.

Using the boundary condition at $r = 0$, argue that $C_2 = 0$, hence that $\hat{B}(r) = C_1 J_0(\sqrt{\mu_0 \sigma \alpha} r)$.

- (d) Now apply the boundary condition at $r = a$ that $B(r, t) = \mu_0 H_0$ for all $t > 0$ and show that $J_0(\sqrt{\mu_0 \sigma \alpha} a) = 0$. This transcendental equation determines the allowed values of α .

Roots of $J_0(v_i) = 0$ and the Corresponding Values of $J_1(v_i)$

i	v_i	$J_1(v_i)$
1	2.4048	0.5191
2	5.5201	-0.3403
3	8.6537	0.2715
4	11.7915	-0.2325
5	14.9309	0.2065
6	18.0711	-0.1877

The values of α may be obtained from the table as

$$\sqrt{\mu_0 \sigma \alpha} a = v_i; \quad \alpha = \frac{1}{\mu_0 \sigma} \left(\frac{v_i}{a} \right)^2,$$

where $J_0(v_i) = 0$.

- (e) On examination of the results, it is evident that $\hat{B}(r) = C_i J_0[v_i(r/a)]$ is a solution for any of the allowed values of v_i . The most general solution for $B(r, t)$ is obtained by using superposition; hence

$$B(r, t) = \mu_0 H_0 - \sum_{i=1}^{\infty} C_i J_0 \left(v_i \frac{r}{a} \right) \exp \left(- \frac{v_i^2 t}{\mu_0 \sigma a^2} \right),$$

and to evaluate the constants C_i the last boundary condition $B(r, t = 0) \equiv 0$ is used. This condition implies that

$$\sum_{i=1}^{\infty} C_i J_0 \left(v_i \frac{r}{a} \right) = \mu_0 H_0,$$

which is just a series in terms of the functions $J_0[v_i(r/a)]$.

Using the integrals

$$\int_0^a r J_0 \left(v_i \frac{r}{a} \right) J_0 \left(v_j \frac{r}{a} \right) dr = \begin{cases} \frac{a^2}{2} J_1^2(v_i), & v_i = v_j, \\ 0, & v_i \neq v_j, \end{cases}$$

and

$$\int_0^a r J_0 \left(v_i \frac{r}{a} \right) dr = \frac{a^2}{v_i} J_1(v_i),$$

evaluate C_1 , C_2 , and C_3 .

- (f) What is the fundamental time constant for the diffusion? Evaluate this time constant for $a = 5$ cm and $\sigma = 10^4/4\pi(\Omega\text{m})^{-1}$.

7.10. A slab of conducting metal (conductivity σ) moves in the x -direction with the constant velocity U . This slab makes contact with perfectly conducting electrodes which are driven at the left end by a constant current K A/m. This source is distributed along the z -axis but does not get in the way of the moving slab. Assume that there is no fringing and that

$$\mathbf{J} = \mathbf{i}_y J_y(x),$$

$$\mathbf{B} = \mathbf{i}_x B_x(x),$$

$$\mathbf{E} = \mathbf{i}_y E_y(x).$$

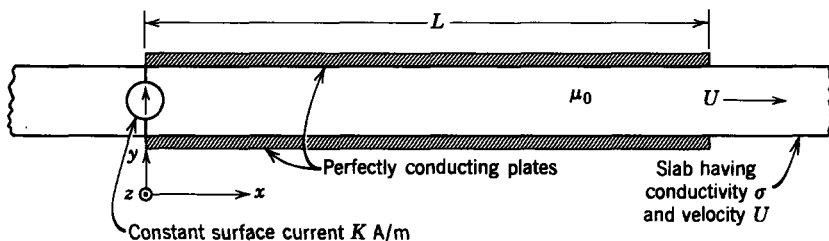


Fig. 7P.10

- Write a single differential equation for $B_x(x)$.
- Write the boundary conditions satisfied by B_x .
- Find an expression for $B_x(x)$ between the perfectly conducting plates.
(Reference: see Section 7.1.2a.)

7.11. In the example in Section 7.1.2a we found the flux density and current density given by (7.1.49) and (7.1.50) for the system in Fig. 7.1.9.

- Calculate the magnetic force density $\mathbf{J} \times \mathbf{B}$ applied to the moving conductor.
- Integrate this force density throughout the volume of the conductor to find the total force that must be supplied by the velocity source. Show that this force is independent of conductivity σ and velocity v .
- Evaluate the power supplied by the velocity source and specify how much power goes to J^2/σ losses in the moving conductor and how much power goes into the current source.

7.12. The electric and magnetic fields were found in Section 7.1.2b for a block of conductor moving between shortcircuited, perfectly conducting plates (see Figs 7.0.1 and 7.1.11). In the fixed frame these fields are functions of both z and t . Show that the fields expressed in the fixed frame satisfy the magnetic diffusion equation (7.1.9.).

7.13. A slab of conductor moves in the z -direction with a velocity V , as shown in Figs. 7.1.12 and 7P.13. A perfectly permeable magnetic circuit with the z length L has the same configuration as that in Fig. 7.1.12, except that it has no excitation coils. Instead, an external source is used to constrain $B_y(-L, t)$ to be $B_y(-L, t) = \text{Re } B_0 \exp j\omega t$.

- What physical arguments would you use to show that the condition at $x = 0$, where the moving slab leaves the region between the pole faces, is described approximately by $B_y(0, t) = 0$?
- Find an expression for $B_y(z, t)$, $-L < z < 0$.
- What is J_x in the range $-L < z < 0$?
- Sketch B_y and J_x in the limit at which $\omega \rightarrow 0$. How would you physically arrange the excitation to make these fields a reasonable approximation? (See Section 7.1.2a.)

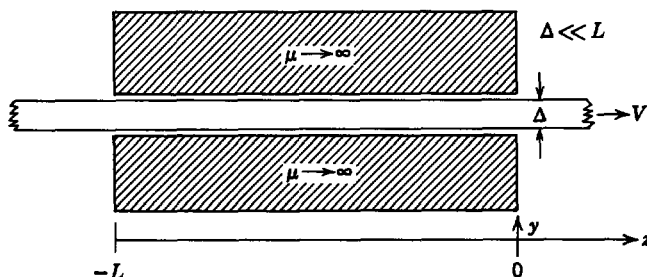


Fig. 7P.13

7.14. You are working on a transportation project and are asked to make an analysis of the following basic method of both levitating and propelling a train. The train rides just above a track which is composed of a slab with conductivity σ . Superconducting coils within the train are arranged to produce a current that can be represented by the current sheet $\mathbf{K}$ (see Fig. 7P.14). This current sheet is backed by a highly permeable magnetic shield ($\mu \rightarrow \infty$) which is also attached to the train. (The shield prevents magnetization of the passengers' watches and the attendant possibility that $t \neq t'$.) Here x' is the distance measured with respect to the moving train. Hence, because the train is moving in the

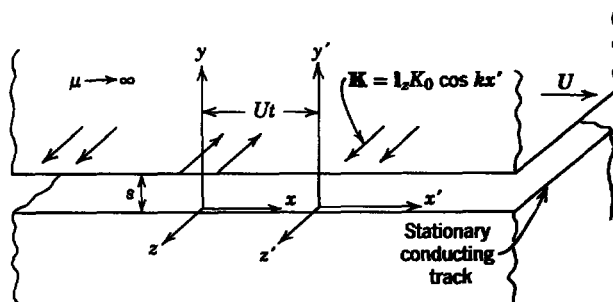


Fig. 7P.14

x -direction with velocity U , $x = Ut + x'$. We wish to compute the time average force per unit area that presumably holds the train a distance s ($ks \ll 1$) above the track.

- Express the surface current in the fixed frame $\mathbf{K}(x, t)$.
- Assume that the track is infinitely thick in the y -direction (under what conditions is this a good assumption?) and compute the magnetic field and current in the conducting track. Assume that $\partial/\partial z = 0$.
- Compute the time average force per unit area (in the x - z plane) that holds up the vehicle.
- Compute the force per unit (x - z) area that tends to propel the train in the x -direction. Do you see any basic problems with the proposed scheme of propulsion?

7.15. (a) Compute the fields for the example of Section 7.1.4, taking into account the effect of the spacing s between the moving conductor and the traveling current sheet.

- (b) Check to see that the results of Section 7.1.4 are obtained in the limit $ks \ll 1$.
 (c) Check to see that currents induced in the moving conductor approach zero as $ks \rightarrow \infty$.

7.16. A pair of perfectly conducting electrodes with area A are shown in Fig. 7P.16. Between the plates there is a fluid of depth b , conductivity σ , and permittivity ϵ , bounded from above by an insulating gas. When $t = 0$, there is no charge in the system and the switch S is closed.

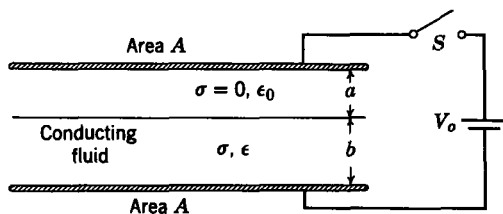


Fig. 7P.16

- (a) What are the electric fields in the liquid (E_l) and in the gas (E_g) an instant after $t = 0$?
 (b) What are these electric fields as $t \rightarrow \infty$?
 (c) Find the charge (q) on the fluid-gas interface as a function of time.

7.17. A spherically symmetric system of free charges and conductors is shown in Fig. 7P.17. A sphere of material of conductivity σ_i is embedded at the center of a second material with conductivity σ_o . When $t = 0$, there is a volume charge density $\rho_o(r)$ in the region $r < R_i$, but no charge density in the outer region. There are no surface charges when $t = 0$.

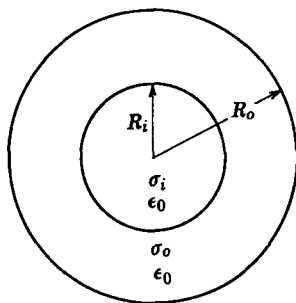


Fig. 7P.17

- (a) Find the charge density at points $r < R_i$ in the center sphere and for $R_i < r < R_o$ in the outer region for $t > 0$.
 (b) Find the surface charges at $r = R_i$ and $r = R_o$ for $t > 0$.
 (c) Sketch the results of (b) for $\sigma_i > \sigma_o$ and $\sigma_i < \sigma_o$.

Hint. You may wish to define the total charge in the inner sphere as

$$q(t) = 4\pi \int_0^{R_i} \rho_o(r) r^2 dr$$

7.18. Three long, cylindrical, highly conducting shells of radius a , b , and c , respectively, are aligned concentrically, as shown in Fig. 7P.18. The space between the cylinders is filled with a material of permittivity ϵ and conductivity σ . Initially, the cylinders of radii a and b are biased with a battery V_o and switch S in position (1) as shown. At a time $t = 0$, the switch is connected to terminal (2).

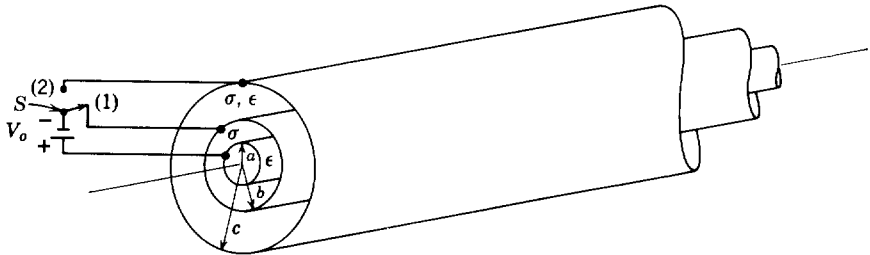


Fig. 7P.18

- Find expressions for the electric field intensity E in the regions between the cylinders.
- Find the expression for the charge per unit length on the cylinder of radius b .
- Find the equivalent circuit for a unit length of the cylinders.

7.19. A pair of metal spheres is suspended in a beaker of slightly conducting liquid, as shown in Fig. 7P.19. The lower sphere is fixed to the bottom, whereas the upper one is free to move (say, with deflection x). The fluid has a constant conductivity σ and permittivity ϵ . An external source of potential v is used to establish the charge q on the upper sphere.

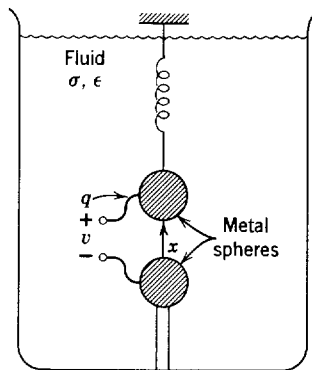


Fig. 7P.19

Then the source of potential is removed, leaving the sphere electrically isolated, except for conduction through the liquid, and free to move. Use the integral laws for an electric field system [(1.1.24) to (1.1.26) and definitions of Table 2.1]* to show that, if the initial charge on the sphere is Q , then as a function of time, the charge q on the sphere is $q = Qe^{-t/\tau}$; $\tau = \epsilon/\sigma$. Your proof should be valid even though the upper sphere is moving [$x = x(t)$].

* Appendix E.

7.20. A pair of parallel slightly conducting plates of conductivity σ and spacing d are insulated from each other, as shown in Fig. 7P.20. When $t = 0$, there is a charge $-Q$ uniformly distributed over the volume of the upper plate and a charge $+Q$ uniformly distributed over the volume of the lower plate. Assume that $d \ll D$:

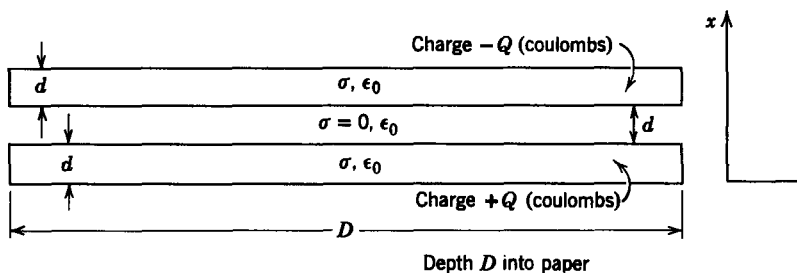


Fig. 7P.20

- Make a dimensioned sketch of the electric field E_x as a function of x when $t = 0$ and as $t \rightarrow \infty$.
- Compute the total force on the lower plate as a function of time.

7.21. Consider the problem of Example 7.2.3 with

$$\epsilon = \epsilon_1 + \frac{\epsilon_2}{l} x,$$

$$\sigma = \sigma_1 + \frac{\sigma_2}{l} x.$$

- Find $E_x(x, t)$ and $\rho_f(x, t)$.
- Discuss the effect of the nonuniform ϵ .

7.22. A fluid moves with the constant velocity $U\mathbf{i}_x$ between parallel plates, as shown in Fig. 7P.22a. The fluid is uncharged ($\rho_f = 0$) for $x < 0$. It has the uniform conductivity σ (which is small) and permittivity ϵ . At $x = 0$ charges are introduced by means of a two-dimensional source midway between the plates. Hence at $x = 0$ the charge has the distribution shown in Fig. 7P.22b, where $\hat{\rho}_0$ is a given complex constant and Δ is the initial thickness of the layer of charge that has been injected.

- Find the charge distribution across the channel at a downstream position $x = l$.
- Describe how you would use the potential induced by this charge on two downstream electrodes to measure the flow velocity U . (Assume that you have control over the frequency. There is more than one way to accomplish this and only a qualitative description is required.)

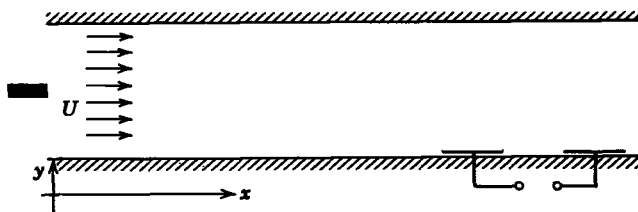


Fig. 7P.22a

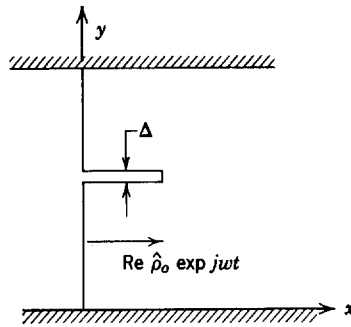


Fig. 7P.22b

7.23. A fluid with conductivity σ flows in the x -direction through a pipe with insulating walls and cross-sectional area A . The velocity of the fluid is approximated as constant (U). Ions are injected into the flow at $x = 0$ with a source that maintains the charge density at $x = 0$ equal to ρ_o (a constant).

- Find the charge density distribution between the screen electrodes.
- Find the distribution of electric field $E_x(x)$ between the electrodes.
- What is the voltage V generated across the load resistance R ?

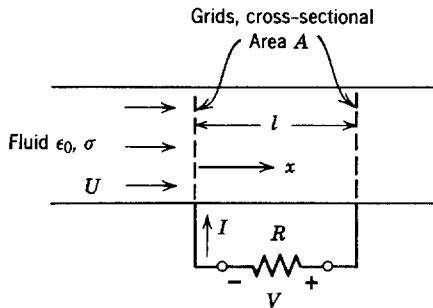


Fig. 7P.23

7.24. When Lord Kelvin invented his electrostatic generator (see Example 7.2.6), he had in mind the generation of dc high voltages. In 1966 William C. Euerle, a student at MIT, elaborated on Lord Kelvin's original idea to make an electrostatic ac generator. This device, with a Y-connected resistive load, is shown in Fig. 7P.24. You can assume that each of the "drippers" is characterized by the same parameters.

- Write three equations for the voltages v_1 , v_2 , and v_3 .
- Use the dynamic description of the potentials found in (a) to show that it is possible for the system to behave as an oscillator; that is, that any one of the voltages would behave as a sinusoid with an exponentially growing amplitude.
- For what values of the parameters will the load limit the oscillations to constant amplitude?
- Under the conditions of (c), what is the oscillation frequency?

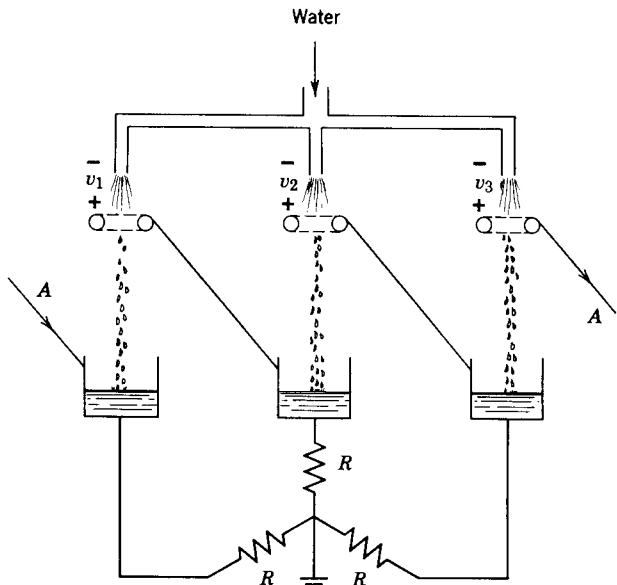


Fig. 7P.24

7.25. Two systems that are proposed to measure the depth d of water in a tank are shown in Fig. 7P.25. In system (a) the depth is indicated by measuring the “inductance” of a loop of copper conductor which encloses a portion of the water; in system (b) the “capacitance” of a pair of copper electrodes is to be used to indicate the depth. You are given that d is on the order of 1 cm., where the properties of the water are about $\sigma = 10^{-2}$ mho/m, $\epsilon = 81 \times \epsilon_0 = 81 \times 8.85 \times 10^{-12}$ F/m, $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m. Measurements of both the

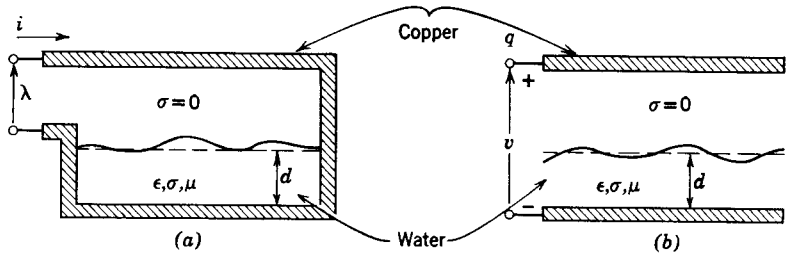


Fig. 7P.25

“capacitance” and “inductance” are to be made with a 100-kc bridge. Which of the two devices would work and why? For what values of σ would both devices be attractive for this application?

7.26. A pair of devices, proposed for measuring the level x of water, is shown in Fig. 7P.26. In case (a) an iron core magnet is driven by a 1-kc signal with the resulting voltage v_o . In case (b) a pair of plates is parallel to the surface and is driven by a voltage source

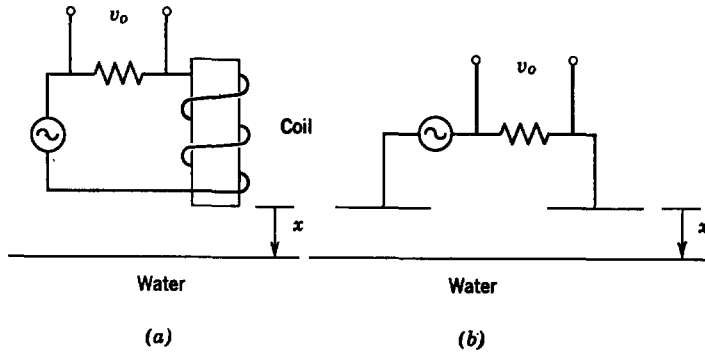


Fig. 7P.26

with a signal v_o across a series resistor. In each case it is desired that v_o be sensitive to the proximity of the water so that the device can be used to indicate the level of the water. Which of the two devices would you use to solve an engineering problem? Give your reasoning.

7.27. Consider the example in Section 7.2.4 but with the additional complication of a perfectly conducting electrode bounding the material (b) from below at $x = -f$.

- (a) Find the potential distribution in regions (a) and (b).
- (b) Check to see that the results of Section 7.2.4 are obtained if $|fk| \gg 1$.

7.28. A sheet of slightly conducting material moves in the z -direction with the constant velocity U . Above and below the sheet electrodes impose the traveling potential waves shown in Fig. 7P.28. The sheet thickness is small compared with c , so that both sheet surfaces can be considered to be at $x = 0$. The constitutive law for a fixed section of the

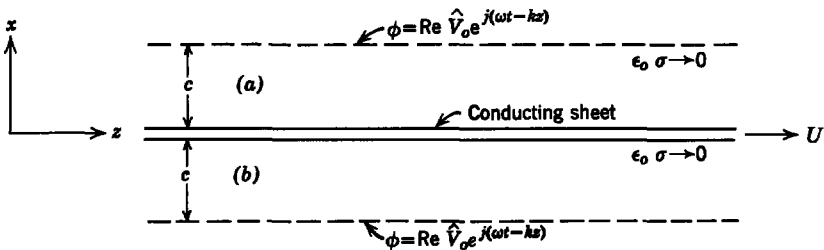


Fig. 7P.28

sheet is $K_f = \sigma_s E_z$, where σ_s is a surface conductivity, K_f is a surface conduction current density, and E_z is the electric field intensity tangential to the sheet.

- (a) Write differential equations and boundary conditions in terms of the potentials $\phi(x, z, t)$ above (a) and below (b) the sheet.
- (b) Determine the time average force/unit length z and unit width y in the z -direction. At what frequency will it have its largest value?
- (c) How would you adjust the traveling potential wave to produce a time average force as $\sigma_s \rightarrow 0$?

Chapter 8

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

8.0 INTRODUCTION

Chapter 7 is restricted to the effects of mechanical motion on magnetic and electric fields. In general, electromechanical interactions involve effects on the mechanical system from the electromagnetic fields as well. These arise from the mechanical forces of electrical origin.

In Chapters 3 through 6 we were concerned with total forces acting on rigid bodies. In systems in which the mechanical medium must be represented by a deformable continuum the details of the force distribution must be known. Hence in continuum electromechanics we are concerned with magnetic or electric force densities, which are, in general, functions of space and time.

Electromagnetic fields are defined by forces composed of two parts: those exerted on free charges by electric fields and those exerted on free currents (moving free charges) by magnetic fields. The relative importance of these forces depends on the type of system being considered. In magnetic field systems, as defined in Section 1.1, the important field excitation is provided by the free current density $\mathbf{J}_f$. Hence for magnetic field systems the only important forces arise from the interactions of the free current density $\mathbf{J}_f$ with magnetic fields. Similarly, the only forces of significance in electric field systems, as defined in Section 1.1, are the interactions of free charge density ρ_f with electric fields. The validity of these assumptions is checked in particular problems. Following the pattern established in earlier sections, we treat forces in magnetic field and electric field systems separately. Our object is to describe electromagnetic forces mathematically in alternative forms that will prove useful in work with continuum electromechanical systems.

Two other technically important electromagnetic forces are those resulting from the interactions of polarization density $\mathbf{P}$ with electric fields and magnetization density $\mathbf{M}$ with magnetic fields. In Chapters 3 to 5 we calculate total forces on polarizable and magnetizable bodies by using an energy method. We extend this method to account for force densities in polarized or magnetized media that are electrically linear, isotropic, and homogeneous. This limitation in our discussion of polarization and magnetization forces is imposed because use of an energy method requires a knowledge of the mechanical and thermodynamic properties of the material.

8.1 FORCES IN MAGNETIC-FIELD SYSTEMS

Consider first the force resulting from the interaction of moving free charge (i.e., $\mathbf{J}_f$) and a magnetic field. The Lorentz force (1.1.28) gives the total magnetic force on a charge q moving with velocity $\mathbf{v}$ as

$$\mathbf{f} = q\mathbf{v} \times \mathbf{B}. \quad (8.1.1)$$

The force density $\mathbf{F}$ (newtons per cubic meter) can be obtained from this expression by writing

$$\mathbf{F} = \lim_{\delta V \rightarrow 0} \frac{\sum_i \mathbf{f}_i}{\delta V} = \lim_{\delta V \rightarrow 0} \frac{\sum_i q_i \mathbf{v}_i \times \mathbf{B}_i}{\delta V}, \quad (8.1.2)$$

where $\mathbf{f}_i$, q_i , and $\mathbf{v}_i$ refer to all the particles in δV and $\mathbf{B}_i$ is the flux density experienced by q_i . If we can say that all particles within δV experience the same flux density $\mathbf{B}$, we can use the definition of free current density (see Section B.1.2)* to write (8.1.2) as

$$\mathbf{F} = \mathbf{J}_f \times \mathbf{B}. \quad (8.1.3)$$

The general definition of (8.1.2) requires the averaging of products, whereas the result of (8.1.3) is the product of averages. It is not, in general, true for variables x and y that

$$[xy]_{av} = [x]_{av}[y]_{av}.$$

The force density expressed by (8.1.3) however, agrees, to a high degree of accuracy, with all experimental results obtained with common conductors. The relation (8.1.3) is valid because the volume δV can be made small enough to enclose a region of essentially constant magnetic flux density, although still including many free charges.

In fact, we could have used (8.1.3) rather than (8.1.1) as the definition of $\mathbf{B}$, for the original experiments of Biot and Savart and later Ampère† concerned themselves with relating the force density to the free current density

* $\mathbf{J}_f = \lim_{\delta V \rightarrow 0} \left[\left(\sum_i q_i \mathbf{v}_i \right) / \delta V \right]$

† J. D. Jackson, *Classical Electrodynamics*, Wiley, New York 1962, p. 133.

$\mathbf{J}$. Some writers start with (8.1.3) as the basic definition of the magnetic force on moving free charge.* However, the averaging process used to make (8.1.2) and (8.1.3) consistent is then inherent to the definition.

It is important to remember that (8.1.3) represents the average of forces on the *charges*. This is equivalent to the force on a medium if there is some mechanism by which each charge transmits the Lorentz force to the material. For example, in a conductor, the charges can be thought of as particles moving through a viscous material—in which case the force that acts on each charge is transmitted to the medium by the viscous retarding force and (8.1.3) is the force density experienced by the medium.

There are situations in which the charges do not interact individually with the medium. For example, in a polarized medium, *pairs* of charges (dipoles) transmit a force to the medium—each pair being connected through the structure of an atom or molecule. For these cases it is the dipoles rather than the charges that transmit a force to the medium. Then it is appropriate to consider the average of the forces on individual dipoles as equivalent to the force density on the medium. This class of forces is developed in Section 8.5.

The force density given in (8.1.3) is expressed in terms of source and field quantities. It is useful to have the force expressed as a function of field quantities alone because we often solve field problems without calculating the free current density. We find it useful to define the *Maxwell stress tensor* as a function of the field quantities from which the force density can be obtained by space differentiation. The Maxwell stress tensor is particularly useful for finding electromechanical boundary conditions in a concise form. It is useful also for finding the total electromagnetic force on a body.

A tensor has particular properties that are useful in this and the chapters which follow. We therefore devote Section 8.2 to a discussion of the stress tensor, using magnetic field stresses as an example.

We can write (8.1.3) in terms of the magnetic field intensity and in a particularly useful form when the medium has a constant permeability, that is, with the constituent relation†

$$\mathbf{B} = \mu \mathbf{H}. \quad (8.1.4)$$

We can use (8.1.4) and Ampère's law for magnetic field systems (1.1.1)‡ to write (8.1.3) in the form

$$\mathbf{F} = \mu(\nabla \times \mathbf{H}) \times \mathbf{H}. \quad (8.1.5)$$

It is a vector identity that this expression can be written as

$$\mathbf{F} = \mu(\mathbf{H} \cdot \nabla)\mathbf{H} - \frac{\mu}{2} \nabla(\mathbf{H} \cdot \mathbf{H}), \quad (8.1.6)$$

* See, for example, Jackson, *ibid.*, p. 137.

† Arguments are given in Section 8.5 to show that (8.1.3) is the force density on free currents in the presence of a constant permeability μ . For now we assume that this is the case.

‡ Table 1.2, Appendix E.

There are three components to this vector equation, but we usually do not write them out unless specific situations are under consideration. There are manipulations, however, that become easier to perform when the equations are viewed component by component. They can be carried out without dealing with cumbersome expressions by using *index notation*.*

In what follows we assume a right-hand cartesian coordinate system x_1, x_2, x_3 . The component of a vector in the direction of an axis carries the subscript of that axis. When we write F_m we mean the m th component of the vector $\mathbf{F}$, where m can be 1, 2, or 3. The mathematical formalism is illustrated by using the force density of (8.1.6) as an example. When we write the differential operator $\partial/\partial x_n$, we mean $\partial/\partial x_1, \partial/\partial x_2$, or $\partial/\partial x_3$. When the index is repeated in a single term, it implies summation over the three values of the index

$$\frac{\partial H_n}{\partial x_n} = \frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial x_2} + \frac{\partial H_3}{\partial x_3} = \nabla \cdot \mathbf{H}$$

and

$$H_n \frac{\partial}{\partial x_n} = H_1 \frac{\partial}{\partial x_1} + H_2 \frac{\partial}{\partial x_2} + H_3 \frac{\partial}{\partial x_3} = \mathbf{H} \cdot \nabla.$$

This illustrates the *summation convention*. On the other hand, $\partial H_m/\partial x_n$ represents any one of the nine possible derivatives of components of $\mathbf{H}$ with respect to coordinates. We define the *Kronecker delta* δ_{mn} which has the values

$$\delta_{mn} = \begin{cases} 1, & \text{when } m = n, \\ 0, & \text{when } m \neq n. \end{cases} \quad (8.1.7)$$

The Kronecker delta has the property (remember to sum on an index that appears twice)

$$\delta_{mn} H_n = H_m$$

and

$$\delta_{mn} \frac{\partial}{\partial x_n} = \frac{\partial}{\partial x_m},$$

which can be verified by using the definition (8.1.7).

With these definitions we write the m th component of (8.1.6) as

$$F_m = \mu H_n \frac{\partial H_m}{\partial x_n} - \frac{\mu}{2} \frac{\partial}{\partial x_m} (H_k H_k). \quad (8.1.8)$$

* A. J. McConnell, *Applications of the Absolute Differential Calculus*, Blackie, London, 1951, Chapter 1.

We use the property of the Kronecker delta $[\partial/\partial x_m = \delta_{mn}(\partial/\partial x_n)]$ and some manipulation to write this expression as

$$F_m = \frac{\partial}{\partial x_n} \left(\mu H_n H_m - \frac{\mu}{2} \delta_{mn} H_k H_k \right) - H_m \frac{\partial \mu H_n}{\partial x_n}. \quad (8.1.9)$$

The last term on the right is

$$H_m (\nabla \cdot \mu \mathbf{H}) = H_m (\nabla \cdot \mathbf{B}) = 0;$$

thus we finally write (8.1.9) in the concise form

$$F_m = \frac{\partial T_{mn}}{\partial x_n}, \quad (8.1.10)$$

where the Maxwell stress tensor T_{mn} is given by

$$T_{mn} = \mu H_n H_m - \frac{\mu}{2} \delta_{mn} H_k H_k. \quad (8.1.11)$$

If we know the magnetic field intensity $\mathbf{H}$ in a region of space, we can calculate the components of the stress tensor T_{mn} . We need only to calculate at most six components because the stress tensor is symmetric:

$$T_{mn} = T_{nm}. \quad (8.1.12)$$

Differentiation of (8.1.11) with respect to the space coordinate according to (8.1.10) gives the force density on the current-carrying matter in that region of space. We should keep in mind that (8.1.10) is simply an alternative way of expressing the m th component of $\mathbf{J} \times \mathbf{B}$. Moreover, we must use the *total* $\mathbf{H}$ to obtain the correct answer from (8.1.10).

Now suppose we wish to find the m th component of the total force $\mathbf{f}$ on material contained within the volume V . We can find it by performing the volume integration:

$$f_m = \int_V F_m dV = \int_V \frac{\partial T_{mn}}{\partial x_n} dV. \quad (8.1.13)$$

When we define the components of a vector $\mathbf{A}$ as

$$A_1 = T_{m1}, \quad A_2 = T_{m2}, \quad A_3 = T_{m3}, \quad (8.1.14)$$

we can write (8.1.13) as

$$f_m = \int_V \frac{\partial A_n}{\partial x_n} dV = \int_V (\nabla \cdot \mathbf{A}) dV. \quad (8.1.15)$$

We now use the divergence theorem to change the volume integral to a surface integral,

$$f_m = \oint_S \mathbf{A} \cdot \mathbf{n} da = \oint_S A_n n_n da, \quad (8.1.16)$$

where n_n is the n th component of the outward-directed unit vector $\mathbf{n}$ normal to the surface S and the surface S encloses the volume V . Substitution from (8.1.14) back into this expression yields

$$f_m = \oint_S T_{mn} n_n da. \quad (8.1.17)$$

Hence we can find the total force of magnetic origin on the matter within a volume V by knowing only the fields along the surface of the volume. This is an important result.

8.2 THE STRESS TENSOR

In the preceding section we introduced the Maxwell stress tensor as an ordered array of nine functions of space and time $T_{mn}(\mathbf{r}, t)$ from which we can calculate magnetic force densities and total forces. The concept of a tensor will be useful to us in later chapters for describing mechanical stresses and deformations in elastic and fluid media. Consequently, we now digress from our study of electromagnetic forces to develop some tensor concepts.

We first consider the tensor representation of stresses with the object of attaching physical significance to the components of a stress tensor. Then mathematical techniques that are used with the stress tensor to find surface stresses (tractions) and volume force densities are introduced. Finally, we introduce some mathematical properties of tensors in general. These properties are introduced in a context in which physical interpretations can be made easily. It is important to remember that tensor analysis is a mathematical formalism that is particularly useful for analyzing a wide variety of physical systems.*

We have remarked that the Maxwell stress tensor is an ordered array of nine functions of space and time. It is conventional to write this array in matrix form as

$$T_{mn}(\mathbf{r}, t) = \begin{bmatrix} T_{11}(\mathbf{r}, t) & T_{12}(\mathbf{r}, t) & T_{13}(\mathbf{r}, t) \\ T_{21}(\mathbf{r}, t) & T_{22}(\mathbf{r}, t) & T_{23}(\mathbf{r}, t) \\ T_{31}(\mathbf{r}, t) & T_{32}(\mathbf{r}, t) & T_{33}(\mathbf{r}, t) \end{bmatrix}. \quad (8.2.1)$$

The first index marks the row and the second, the column in which the element appears. As indicated by (8.1.11) and (8.1.12), the Maxwell stress tensor is symmetric. In the matrix of (8.2.1) the symmetry is about the diagonal. Although the symmetry property has been established only for the Maxwell stress tensor, we find that all the tensors we use in this book are symmetric.

* For a more detailed discussion of tensor calculus than we need in this book see, for example, B. Spain, *Tensor Calculus*, Interscience, New York, 1960.

8.2.1 Stress and Traction

A physical interpretation of the stress tensor follows from (8.1.17) which relates the total force on matter within the volume V enclosed by the surface S to an integral over the surface S . The integrand $T_{mn}n_n$ has the dimension of a force per unit area and, in view of the summation convention with a repeated index, $T_{mn}n_n$ is the m th component of a vector. The vector whose components are $T_{mn}n_n$ has special significance and is therefore given the name *traction* and a symbol τ . Thus the m th component of the traction is written as*

$$\tau_m = T_{mn}n_n = T_{m1}n_1 + T_{m2}n_2 + T_{m3}n_3. \quad (8.2.2)$$

We show subsequently that the traction τ , defined by (8.2.2), is actually the vector force per unit area applied to a surface of arbitrary orientation. For the moment, however, we use (8.2.2) to attach some physical significance to the components of the stress tensor.

Assume that the surface integral of (8.1.17) is to be taken over the rectangular volume whose faces are perpendicular to the coordinate axes illustrated in Fig. 8.2.1. We can express (8.1.17) as the sum of six integrals taken over the six plane faces of the volume. As an example, consider the top face, which has the outward directed normal vector.

$$\mathbf{n} = \mathbf{i}_1.$$

The components of this normal vector are

$$n_1 = 1, \quad n_2 = n_3 = 0.$$

Consequently, the three components of the traction on the top surface are

$$\tau_1 = T_{11}, \quad \tau_2 = T_{21}, \quad \tau_3 = T_{31}.$$

These components and the vector τ are illustrated in Fig. 8.2.1. Next, consider the bottom face, which has the outward directed normal vector

$$\mathbf{n} = -\mathbf{i}_1.$$

The components of this normal vector are

$$n_1 = -1, \quad n_2 = n_3 = 0.$$

* Note that the subscript on the traction τ_m is the same as the *first* subscript on the stress tensor component T_{mn} . This choice for the order of subscripts on T_{mn} is a matter of convention. Although the convention used here is prevalent in the literature, the opposite convention is used. Therefore it is wise to identify the convention used in each case by inspecting equations of the form of (8.1.10) or (8.1.17).

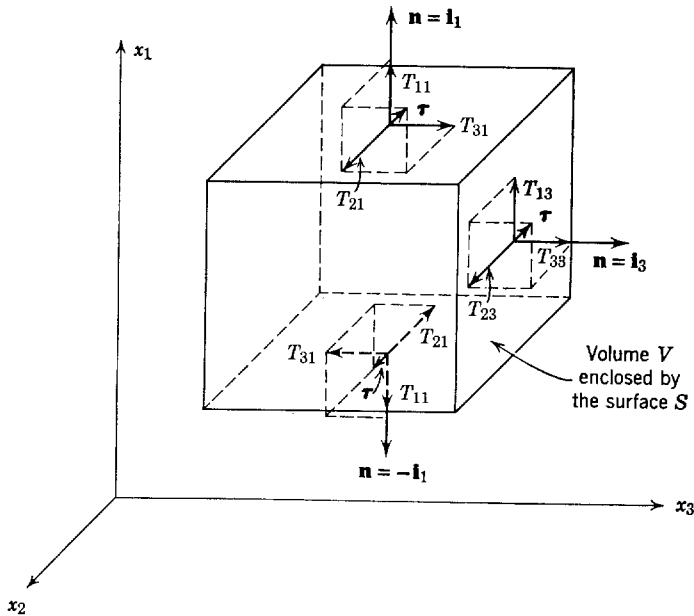


Fig. 8.2.1 A rectangular volume V , acted on by a stress T_{mn} .

Thus the three components of the traction on the bottom face are

$$\tau_1 = -T_{11}, \quad \tau_2 = -T_{21}, \quad \tau_3 = -T_{31}.$$

These components and the vector $\boldsymbol{\tau}$ are illustrated in Fig. 8.2.1.

A similar process can be followed to find the surface traction $\boldsymbol{\tau}$ on each of the other faces. The vector and its components for the face with outward directed normal vector $\mathbf{n} = \mathbf{i}_3$ are also shown in Fig. 8.2.1.

We have shown that *the component T_{mn} of the stress tensor can be physically interpreted as the m th component of the traction applied to a surface with a normal vector in the n -direction*. Thus T_{23} is the x_2 -directed component of the traction applied to a surface whose normal vector is $\mathbf{i}_3$.

We use the ideas developed with Fig. 8.2.1 to construct, in component form, the tractions on all six faces of a rectangular volume in Fig. 8.2.2. The faces are perpendicular to the three axes and the position of each face is defined. The corresponding stresses act in opposite directions on opposite faces. Consequently, if each component of the stress tensor is a constant over the whole volume, the stresses exactly oppose one another and no net force is applied to the material inside the volume. The stress tensor must vary with space to produce a net force.

To illustrate this mathematically we assume the dimensions of the volume

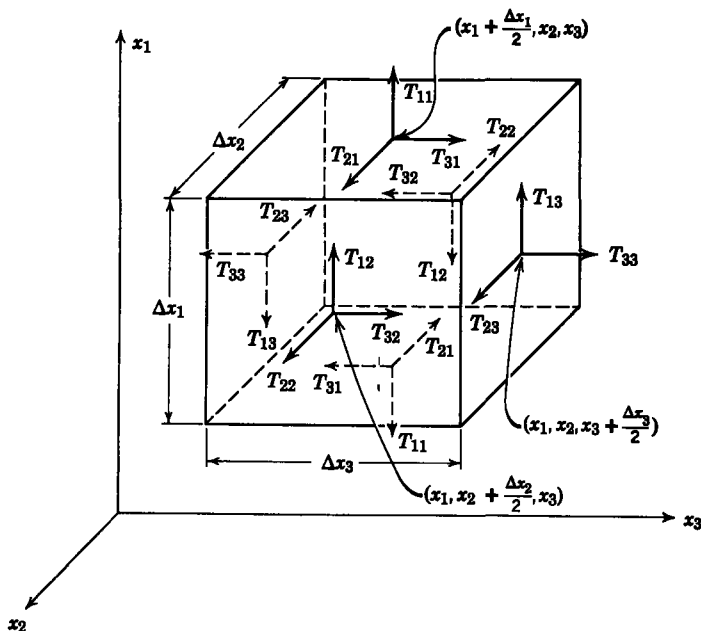


Fig. 8.2.2 Rectangular volume with center at (x_1, x_2, x_3) showing the surfaces and directions of the stresses T_{mn} .

to be small enough that components of the stress tensor do not vary appreciably over one face. We use (8.1.17) to evaluate the x_1 -component of the total force applied to the material within the volume as

$$\begin{aligned}
 f_1 = & T_{11} \left(x_1 + \frac{\Delta x_1}{2}, x_2, x_3 \right) \Delta x_2 \Delta x_3 - T_{11} \left(x_1 - \frac{\Delta x_1}{2}, x_2, x_3 \right) \Delta x_2 \Delta x_3 \\
 & + T_{12} \left(x_1, x_2 + \frac{\Delta x_2}{2}, x_3 \right) \Delta x_1 \Delta x_3 - T_{12} \left(x_1, x_2 - \frac{\Delta x_2}{2}, x_3 \right) \Delta x_1 \Delta x_3 \\
 & + T_{13} \left(x_1, x_2, x_3 + \frac{\Delta x_3}{2} \right) \Delta x_1 \Delta x_2 - T_{13} \left(x_1, x_2, x_3 - \frac{\Delta x_3}{2} \right) \Delta x_1 \Delta x_2.
 \end{aligned}
 \tag{8.2.3}$$

Here we have evaluated the components of the stress tensor at the centers of the surfaces on which they act; for example, the stress component T_{11} acting on the top surface is evaluated at a point having the same x_2 - and x_3 -coordinates as the center of the volume but an x_1 coordinate $\Delta x_1/2$ above the center.

The dimensions of the volume have already been specified as quite small. In fact, we are interested in the limit as the dimensions go to zero. Consequently, each component of the stress tensor is expanded in a Taylor series

about the value at the volume center with only linear terms in each series retained to write (8.2.3) as

$$\begin{aligned} f_1 = & \left(T_{11} + \frac{\Delta x_1}{2} \frac{\partial T_{11}}{\partial x_1} - T_{11} + \frac{\Delta x_1}{2} \frac{\partial T_{11}}{\partial x_1} \right) \Delta x_2 \Delta x_3 \\ & + \left(T_{12} + \frac{\Delta x_2}{2} \frac{\partial T_{12}}{\partial x_2} - T_{12} + \frac{\Delta x_2}{2} \frac{\partial T_{12}}{\partial x_2} \right) \Delta x_1 \Delta x_3 \\ & + \left(T_{13} + \frac{\Delta x_3}{2} \frac{\partial T_{13}}{\partial x_3} - T_{13} + \frac{\Delta x_3}{2} \frac{\partial T_{13}}{\partial x_3} \right) \Delta x_1 \Delta x_2 \end{aligned}$$

or

$$f_1 = \left(\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} \right) \Delta x_1 \Delta x_2 \Delta x_3. \quad (8.2.4)$$

All terms in this expression are to be evaluated at the center of the volume (x_1, x_2, x_3) . We have thus verified our physical intuition that space-varying stress tensor components are necessary to obtain a net force.

From (8.2.4) we can obtain the x_1 -component of the force density $\mathbf{F}$ at the point (x_1, x_2, x_3) by writing

$$F_1 = \lim_{\Delta x_1, \Delta x_2, \Delta x_3 \rightarrow 0} \frac{f_1}{\Delta x_1 \Delta x_2 \Delta x_3} = \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3}. \quad (8.2.5)$$

The limiting process makes the expansion of (8.2.4) exact. The summation convention is used to write (8.2.5) as

$$F_1 = \frac{\partial T_{1n}}{\partial x_n}. \quad (8.2.6)$$

A similar process for the other two components of the force and force density yields the general result that the m th component of the force density at a point is

$$F_m = \frac{\partial T_{mn}}{\partial x_n}. \quad (8.2.7)$$

This is the result obtained in (8.1.10), which was derived for magnetic forces. Thus we have made the transition from the integral in (8.1.17) to the derivative in (8.2.7)—the reverse of the process in which we used the divergence theorem to obtain (8.1.17) from (8.1.10).

Although the formalism presented in this section is based on a result derived with magnetic forces, the stress tensor has a more general significance, as we shall see in later chapters; for example, the rectangular volume in Fig. 8.2.2 can be a block of elastic material with mechanical stresses applied to the surfaces. Our derivation and interpretations are still valid with respect to mechanical forces and force densities. For the moment we restrict our

examples to consider only magnetic forces because they are the only ones we have introduced formally.

Example 8.2.1. To illustrate some properties of the stress tensor and the mathematical techniques used with it, consider the system illustrated schematically in Fig. 8.2.3. The system consists of a long, cylindrical, nonmagnetic ($\mu = \mu_0$) conductor whose axis coincides with the x_3 -axis. The conductor carries a uniform constant current density

$$\mathbf{J} = \mathbf{i}_3 J. \quad (\text{a})$$

An electromagnet, not shown, produces a uniform magnetic field intensity

$$\mathbf{H}_0 = \mathbf{i}_1 H_0, \quad (\text{b})$$

when $J = 0$. The conductor is long enough that we can ignore any variations with x_3 ; thus the problem is two-dimensional.

Because the field problem is linear, we can superimpose the field $\mathbf{H}_0$ with the field excited by the current density $\mathbf{J}$. To calculate the two nonzero components H'_1 and H'_2 , due to $\mathbf{J}$, we establish a cylindrical coordinate system as illustrated in Fig. 8.2.4 and use the integral form of Ampère's law to obtain

$$H_\theta = \frac{Jr}{2} \quad \text{for } r < R, \quad (\text{c})$$

$$H_\theta = \frac{JR^2}{2r} \quad \text{for } r > R. \quad (\text{d})$$

The transformation from cylindrical to cartesian coordinates* is used to find the cartesian components of this field. We then add the externally applied field H_0 to obtain the total field intensity as

$$\left. \begin{aligned} H_1 &= H_0 - \frac{J}{2} x_2 \\ H_2 &= \frac{J}{2} x_1 \end{aligned} \right\} \quad \text{for } x_1^2 + x_2^2 < R^2. \quad (\text{e})$$

$$\left. \begin{aligned} H_1 &= H_0 - \frac{JR^2}{2} \left(\frac{x_2}{x_1^2 + x_2^2} \right) \\ H_2 &= \frac{JR^2}{2} \left(\frac{x_1}{x_1^2 + x_2^2} \right) \end{aligned} \right\} \quad \text{for } x_1^2 + x_2^2 > R^2, \quad (\text{f})$$

The component H_3 is zero; thus we use (8.1.11) and (8.2.1) to write the stress tensor

$$(T_{mn}) = \begin{bmatrix} \frac{\mu_0}{2} (H_1^2 - H_2^2) & \mu_0 H_1 H_2 & 0 \\ \mu_0 H_1 H_2 & \frac{\mu_0}{2} (H_2^2 - H_1^2) & 0 \\ 0 & 0 & -\frac{\mu_0}{2} (H_1^2 + H_2^2) \end{bmatrix}. \quad (\text{g})$$

* H. B. Phillips, *Analytic Geometry and Calculus*, 2nd ed., Wiley, New York, 1946, p. 206.

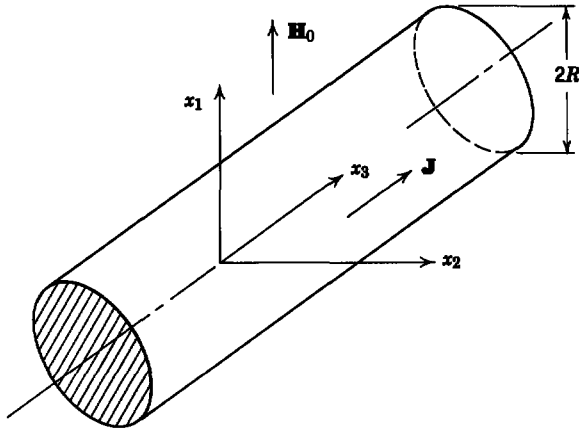


Fig. 8.2.3 A cylindrical conductor carrying uniform current density in the presence of a uniform applied field.

Now (e) or (f) can be used with this expression to find the components of the stress tensor both inside and outside the conductor. First the force density inside the conductor is calculated from (8.1.3):

$$\mathbf{F} = \mathbf{J}_f \times \mathbf{B} = -i_1 J \mu_0 H_2 + i_2 J \mu_0 H_1$$

or

$$\mathbf{F} = -i_1 \frac{\mu_0 J^2 x_1}{2} + i_2 \mu_0 J \left(H_0 - \frac{J x_2}{2} \right). \quad (\text{h})$$

Thus there is a force density term due to the interaction between the current density and the externally applied field and a term due to interaction of the current density with the field it produces.

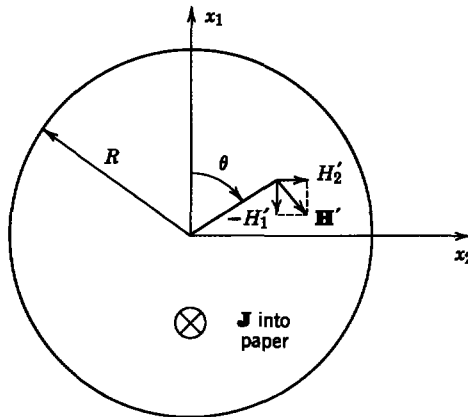


Fig. 8.2.4 Geometry for calculating fields excited by $\mathbf{J}$.

To calculate this same force density from the stress tensor we use (8.1.10) and write for the x_2 -component

$$F_2 = \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2}. \quad (i)$$

By substitution of (e) into (g) this expression becomes

$$F_2 = \frac{\partial}{\partial x_1} \left[\frac{\mu_0 J x_1}{2} \left(H_0 - \frac{J x_2}{2} \right) \right] + \frac{\partial}{\partial x_2} \left\{ \frac{\mu_0}{2} \left[\frac{J^2 x_1^2}{4} - \left(H_0 - \frac{J}{2} x_2 \right)^2 \right] \right\}. \quad (j)$$

Performance of the indicated differentiations yields the x_2 -component of (h), as it should. A similar process can be used to calculate the x_1 -component of the force density and also to show that the x_3 -component of the force density is zero.

It should be evident from a comparison of the effort required to obtain (h) and (j) that the stress tensor is not normally used to calculate force density in a system such as this. We present this example to illustrate the correspondence between the two methods and to illustrate the mathematical processes involved.

It is clear that outside the conductor the force density must be zero because the current density is zero; however, (f) and (g) show that the stress tensor has nonzero components in this region. To show that (8.1.10) yields a zero force density in this region we write the expression for the x_2 -component of (8.1.10) outside the conductor ($x_1^2 + x_2^2 > R^2$):

$$F_2 = \frac{\partial}{\partial x_1} \left\{ \frac{\mu_0 J R^2}{2} \left(\frac{x_1}{x_1^2 + x_2^2} \right) \left[H_0 - \frac{J R^2}{2} \left(\frac{x_2}{x_1^2 + x_2^2} \right) \right] \right\} + \frac{\partial}{\partial x_2} \left\{ \frac{\mu_0}{2} \left[\frac{J^2 R^4}{4} \left(\frac{x_1}{x_1^2 + x_2^2} \right)^2 - \left[H_0 - \frac{J R^2}{2} \left(\frac{x_2}{x_1^2 + x_2^2} \right) \right]^2 \right] \right\}. \quad (k)$$

The indicated differentiation can be carried out to verify that this component of force density is zero. A similar process can be used to show that F_1 and F_3 , calculated from the stress tensor, are zero outside the conductor.

We now turn to the problem of calculating the total magnetic force on a length l of the conductor. First a volume integration of the force density given by (h) is performed. Because there are no variations of the fields with x_3 , we use as a volume element

$$dV = l dx_1 dx_2$$

and use (h) and the geometry of Fig. 8.2.4 to write

$$\mathbf{f} = \int_{-R}^R \int_{-\sqrt{R^2-x_1^2}}^{\sqrt{R^2-x_1^2}} \left[-i_1 \frac{\mu_0 J^2 x_1}{2} + i_2 \mu_0 J \left(H_0 - \frac{J x_2}{2} \right) \right] l dx_1 dx_2.$$

We integrate this equation with respect to x_2 , evaluate the result at the limits, and obtain

$$\mathbf{f} = \int_{-R}^R [-i_1 \mu_0 J^2 x_1 \sqrt{R^2 - x_1^2} + i_2 2 \mu_0 J H_0 \sqrt{R^2 - x_1^2}] l dx_1$$

Evaluation of this integral with the specified limits yields the final result

$$\mathbf{f} = i_2 J \mu_0 H_0 \pi R^2 l. \quad (l)$$

This is simply the uniform force density due to the externally applied field H_0 multiplied by the volume $\pi R^2 l$. That the forces due to the self-field canceled out is a result of the cylindrical symmetry. Thus the force density due to the self-fields tends to deform the conductor but produces no net force that tends to move it.

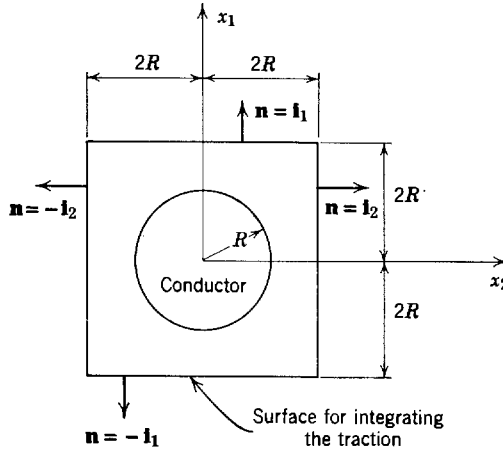


Fig. 8.2.5 Illustrating the surface for integrating the traction.

To use the stress tensor in calculating this same total force we use (8.1.17) with a surface that encloses a length l of the conductor. To make it quite clear that we can use a surface that is totally outside the body we choose a surface of length l and of square cross section with sides $4R$, as shown in end view in Fig. 8.2.5. Because this surface is completely outside the conductor we must use (f) with (g) to calculate the components of the stress tensor.

None of the quantities varies with x_3 ; consequently, we recognize that the contribution to (8.1.17) from the two ends perpendicular to the x_3 -axis is zero. The contribution from one end is the negative of that from the other end. We calculate only the x_2 -component of the force. A similar process can be carried out for the other two components and (l) indicates that they integrate to zero.

We use the four lateral surfaces whose normal vectors are defined in Fig. 8.2.5 to write (8.1.17) for the x_2 -component as

$$f_2 = \int_{-2R}^{2R} T_{21}(2R, x_2)l \, dx_2 - \int_{-2R}^{2R} T_{21}(-2R, x_2)l \, dx_2 \\ + \int_{-2R}^{2R} T_{22}(x_1, 2R)l \, dx_1 - \int_{-2R}^{2R} T_{22}(x_1, -2R)l \, dx_1. \quad (m)$$

The stress components in the integrands are given by (g) and can be evaluated in terms of the magnetic field components by using (f). Then integration yields the result

$$f_2 = J\mu_0 H_0 \pi R^2 l.$$

This is the same as (l) which was obtained by integrating the volume force density throughout the conductor.

We have verified in an example that we can obtain the total force on current-carrying material within a volume by integrating the traction over a surface enclosing the volume. It is illuminating to investigate the nature of the tractions involved in this integration. For this purpose we refer to Fig. 8.2.2 in which we interpreted the components of the stress tensor as being the components of the traction. Thus we recognize that the first two integrals in (m) involve the x_2 -component of the traction applied to surfaces whose normal vectors

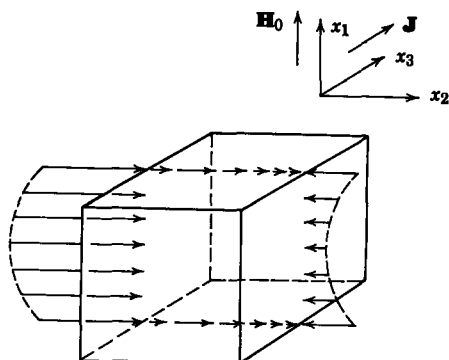


Fig. 8.2.6 Stress distribution.

are in the x_1 -direction. Because these tractions are applied along a surface they are referred to as *shear stresses*. The second two integrals in (m) involve components of the traction that are perpendicular to the surfaces to which they are applied. Such tractions are called *normal stresses*.

If we wish to carry our interpretation a step further and say that there are stresses transmitted through space by the magnetic field as indicated by the Maxwell stress tensor, we can interpret the integrands of (m) as being stresses applied to the four surfaces. We use the integrands to sketch these stresses in Fig. 8.2.6. The shear stresses are equal on top and bottom and are in the direction of the net force. The normal stresses are compressive and there is an excess of stress applied to the left side.

Although the interpretation of the Maxwell stress tensor as representing mechanical stresses transmitted by fields through empty space is often useful it must be employed with understanding; for example, we could add a constant to all components of the stress tensor and not change the results of our calculations of force density and total force. The stress pattern of Fig. 8.2.6, however, would be changed markedly.

In (8.2.2) we defined the m th component τ_m of the traction τ as

$$\tau_m = T_{mn}n_n. \quad (8.2.8)$$

The traction was interpreted as the vector force per unit area applied to a surface with components n_n of the normal vector $\mathbf{n}$. The integral force equation (8.1.17) suggests that τ represents the force per unit area for a surface of arbitrary orientation. This fact is emphasized by the discussion which follows.

Figure 8.2.7 is a tetrahedron with three of its edges parallel to x_1, x_2, x_3 -axes. One surface of the tetrahedron has a normal vector $\mathbf{n}$ and supports the traction τ (which, in general, is not in the direction of $\mathbf{n}$). Because three of the surfaces have normal vectors that are in the axis directions, the tractions on these surfaces can be written in terms of the components T_{mn} , whereas the traction on the fourth surface is the unknown τ . Although the surface tractions (and in particular T_{mn}) depend on the space coordinates, it has been

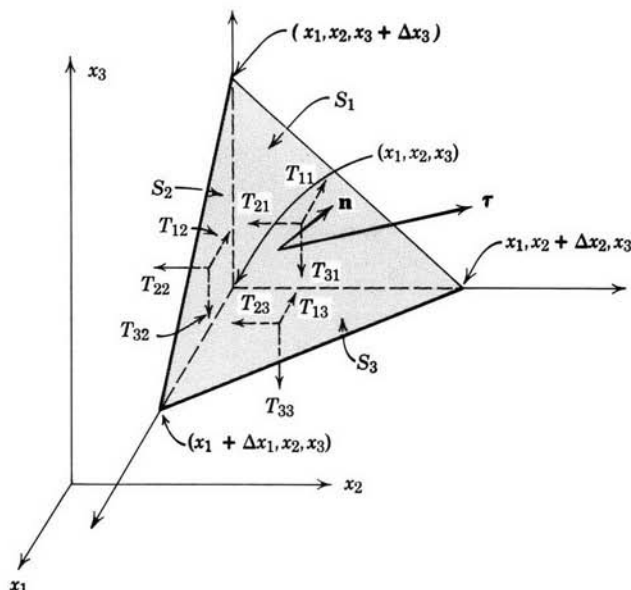


Fig. 8.2.7 The small tetrahedron used to find the surface traction τ on a surface with the normal vector $\mathbf{n}$ in terms of the components of the stress tensor T_{mn} .

implicitly assumed that T_{mn} is a continuous function. Hence, as $\Delta x_1, \Delta x_2, \Delta x_3 \rightarrow 0$, the traction τ must balance the stresses on the negative surfaces. Here we use the fact that the volume forces are proportional to the volume $\Delta x_1 \Delta x_2 \Delta x_3$, whereas the surface tractions produce forces proportional to areas, that is, $\Delta x_1 \Delta x_2$, $\Delta x_2 \Delta x_3$ or $\Delta x_3 \Delta x_1$. Hence in the limit in which $\Delta x_1, \Delta x_2, \Delta x_3 \rightarrow 0$, the prism of material is not in force equilibrium unless the surface forces balance.

If the surface with the normal $\mathbf{n}$ has the area S , the negative surfaces have the areas $S n_1, S n_2, S n_3$,* respectively, and continuity of the stresses which act in the x_1 -direction gives rise to the equation

$$\tau_1 S \cong T_{11} S n_1 + T_{12} S n_2 + T_{13} S n_3. \quad (8.2.9)$$

In the limit in which the dimensions of the tetrahedron become small (8.2.9) becomes exact. Since the equation can also be written for the other components of the stress, (8.2.8) follows.

* A proof of this geometric relation can be made by using Gauss's theorem $\oint_S \mathbf{A} \cdot \mathbf{n} \, da = \int_V (\nabla \cdot \mathbf{A}) \, dV$ with $\mathbf{A} = \mathbf{i}_1$. The volume integral vanishes and the surface integral (integrated over the surface of the tetrahedron) becomes $-S_1 + S n_1 = 0$, where S_1 is the area of the back surface with the normal $-\mathbf{i}_1$. Similar arguments hold using $\mathbf{A} = \mathbf{i}_2$ and $\mathbf{A} = \mathbf{i}_3$.

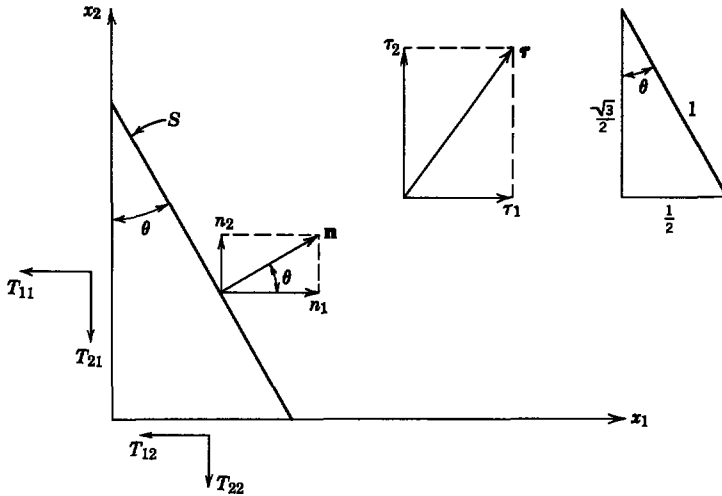


Fig. 8.2.8 Example of surface traction τ acting on a particular surface S .

Example 8.2.2. A brief example will help to fix the meaning of (8.2.8). We wish to derive the traction τ on the surface S shown in Fig. 8.2.8, given the stresses T_{11} , T_{12} , etc.

It is assumed that $\mathbf{n}$ lies in the x_1 - x_2 plane, so that from the figure the normal vector is

$$\mathbf{n} = \mathbf{i}_1 \frac{\sqrt{3}}{2} + \mathbf{i}_2 \frac{1}{2}. \quad (\text{a})$$

Note that the components of $\mathbf{n}$ are *not* the unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$. According to (8.2.8), the components of τ acting on the surface S are

$$\begin{aligned} \tau_1 &= T_{11} \frac{\sqrt{3}}{2} + T_{12} \frac{1}{2}, \\ \tau_2 &= T_{21} \frac{\sqrt{3}}{2} + T_{22} \frac{1}{2}, \\ \tau_3 &= 0, \end{aligned} \quad (\text{b})$$

where we have assumed that T_{31} , T_{32} , and T_{33} are zero or that there are no components of the stress acting in the x_3 -direction. This example should make it clear that all we have done in writing (8.2.8) is to formalize our interpretation of the stress components as forces per unit area acting on surfaces that are perpendicular to the axis directions. The results could be derived from inspection of Fig. 8.2.8 without making use of (8.2.8). Try it!

8.2.2 Vector and Tensor Transformations

In our discussion so far we have interpreted the physical properties of the stress tensor in terms of the vector traction τ whose components are defined by (8.2.2). We now use the mathematical properties of the vector τ to describe some mathematical properties of the stress tensor.

The traction τ is a vector. The components of this vector depend on the coordinate system in which τ is expressed; for example, the vector might be directed in one of the coordinate directions (x_1, x_2, x_3), in which case there would be only one nonzero component of τ . In a second coordinate system (x'_1, x'_2, x'_3), this same vector might have components in all of the coordinate directions. Analyzing a vector into orthogonal components along the coordinate axes is a familiar process. The components in a cartesian coordinate system (x'_1, x'_2, x'_3) are related to those in the cartesian coordinate system (x_1, x_2, x_3) by the three equations

$$\tau'_p = a_{pr}\tau_r, \quad (8.2.10)$$

where a_{pr} is the cosine of the angle between the x'_p -axis and the x_r -axis.

Example 8.2.3. Suppose that we wish to use (8.2.10) to compute the components τ'_m of the vector τ' in the primed coordinate system shown in Fig. 8.2.9, in terms of the known components τ_m of τ in the unprimed coordinate system. (It should be recognized that the x'_1 axis in this figure is in the direction of the normal in Fig. 8.2.8, so that we can consider this example as an extension of the preceding one.) From the geometry the cosine of the angle between

$$x'_1 \text{ and } x_1 = a_{11} = \frac{\sqrt{3}}{2}, \quad x'_1 \text{ and } x_2 = a_{21} = \frac{\sqrt{3}}{2},$$

$$x'_1 \text{ and } x_3 = a_{31} = \frac{1}{2}, \quad x'_2 \text{ and } x_3 = a_{32} = 1,$$

$$x'_2 \text{ and } x_1 = a_{21} = -\frac{1}{2}, \quad \text{all others} = 0.$$

Hence by definition

$$[a_{mn}] = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

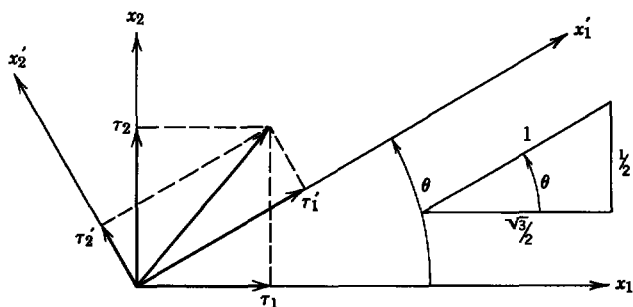


Fig. 8.2.9 Geometrical relationship between the primed and unprimed coordinate systems for Example 8.2.3.

Then (8.2.10) gives

$$\begin{aligned}\tau'_1 &= \frac{\sqrt{3}}{2} \tau_1 + \frac{1}{2} \tau_2, \\ \tau'_2 &= -\frac{1}{2} \tau_1 + \frac{\sqrt{3}}{2} \tau_2.\end{aligned}$$

From this example (8.2.10) should be recognized as a simple statement of vector addition. Again, we could have obtained the result from Fig. 8.2.9 without the formalism of (8.2.10).

Equation 8.2.10 forms the basis for determining how to transform components of the stress tensor from one coordinate system to another.

According to (8.2.2), the components of τ are

$$\tau_r = T_{rs} n_s. \quad (8.2.11)$$

Now we consider a particular cartesian coordinate system (x'_1, x'_2, x'_3) established in such a way that one of the axes (say x'_1) has the same direction as $\mathbf{n}$. A pictorial representation of the two coordinate systems is given in Fig. 8.2.10. The components (n_1, n_2, n_3) of the normal vector are the cosines of the angles between the (x_1, x_2, x_3) axes and the normal direction, which is also the direction of x'_1 . Hence from the definition following (8.2.10) $(n_1, n_2, n_3) = (a_{11}, a_{12}, a_{13})$ and (8.2.11) can also be written as

$$\tau_r = T_{rs} a_{1s}. \quad (8.2.12)$$

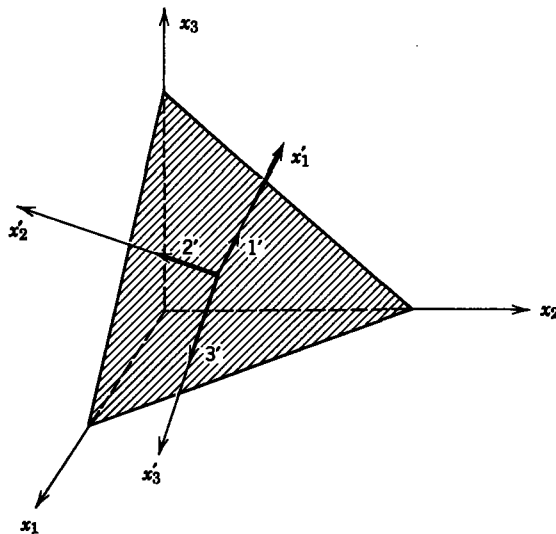


Fig. 8.2.10 Relationship between the primed and unprimed coordinates showing the x'_1 -axis coincident with the normal vector.

Because x'_1 is perpendicular to the surface, x'_2 and x'_3 lie in the surface. We see that $(\tau'_1, \tau'_2, \tau'_3)$ are just the components of the stress acting on a surface with a normal in the direction of the x'_1 -axis, that is,

$$\tau'_p = T'_{p1}, \quad (8.2.13)$$

but we can also use (8.2.10) to express τ'_p as

$$\tau'_p = a_{pr}\tau_r, \quad (8.2.14)$$

which by (8.2.12) gives a relation for τ'_p in terms of the stress components in the unprimed coordinates.

$$\tau'_p = a_{pr}(T_{rs}a_{1s}) \quad (8.2.15)$$

Then from 8.2.13

$$T'_{p1} = a_{pr}a_{1s}T_{rs}. \quad (8.2.16)$$

Finally, the designation of the normal direction by the x'_1 -axis is arbitrary, and the preceding arguments could be repeated with 1 replaced by 2 or 1 replaced by 3. Hence we have shown that

$$T'_{pq} = a_{pr}a_{qs}T_{rs}. \quad (8.2.17)$$

This relation provides the rule for finding the components of the stress in the primed coordinates, given the components in the unprimed coordinates. It serves the same purpose in dealing with tensors that (8.2.10) serves in dealing with vectors. In much of the literature a vector or *first-order tensor* is defined as an array of three numbers that transforms according to an equation in the form of (8.2.10). In the same way, a *second-order tensor* is defined as an array of numbers that transforms according to an equation in the form of (8.2.17).*

Example 8.2.4. Suppose we wish to find the stress component T'_{11} expressed in the primed coordinate system of Fig. 8.2.9 in terms of the components T_{mn} in the unprimed system. Then (8.2.17) gives

$$T'_{11} = a_{11}a_{11}T_{11} + a_{11}a_{12}T_{12} + a_{11}a_{13}T_{13} + a_{12}a_{11}T_{21} + a_{12}a_{12}T_{22} + a_{12}a_{13}T_{23} \\ + a_{13}a_{11}T_{31} + a_{13}a_{12}T_{32} + a_{13}a_{13}T_{33}$$

or, in particular, from the values of a_{mn} given in Example 8.2.3,

$$T'_{11} = \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{2} T_{11} + \frac{\sqrt{3}}{2} \left(\frac{1}{2}\right) T_{12} + \left(\frac{1}{2}\right) \frac{\sqrt{3}}{2} T_{21} + \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) T_{22}.$$

A second example provides a useful result.

Example 8.2.5. Given the stress components T_{mn} expressed in a cylindrical coordinate system with the coordinates r , θ , and z , what are the components of the stress tensor

* See, for example, Spain, *op. cit.*, pp. 6-9.

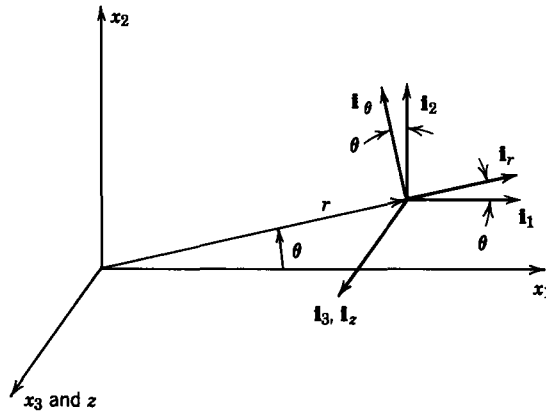


Fig. 8.2.11 Geometrical relationship between cartesian and cylindrical coordinate systems.

expressed in a cartesian coordinate system with axes x_1 , x_2 , and x_3 , as illustrated in Fig. 8.2.11.*

The relationship between the unit vectors is shown in Fig. 8.2.11. The cartesian coordinate system plays the role of the "primed" system. We can see by inspection that the cosine of the angle between

$$\begin{aligned} \mathbf{i}_1 \text{ and } \mathbf{i}_r &= \cos \theta, \\ \mathbf{i}_1 \text{ and } \mathbf{i}_\theta &= \cos(\theta + 90^\circ) = -\sin \theta, \\ \mathbf{i}_2 \text{ and } \mathbf{i}_r &= \cos(90^\circ - \theta) = \sin \theta, \\ \mathbf{i}_2 \text{ and } \mathbf{i}_\theta &= \cos \theta, \\ \mathbf{i}_3 \text{ and } \mathbf{i}_z &= 1, \\ \text{all others} &= 0. \end{aligned}$$

Therefore we can write

$$(a_{mn}) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The components of the stress now follow directly by making use of (8.2.17):

$$\begin{aligned} T_{11} &= T_{rr} \cos^2 \theta - 2T_{r\theta} \sin \theta \cos \theta + T_{\theta\theta} \sin^2 \theta, \\ T_{12} &= T_{rr} \sin \theta \cos \theta + T_{r\theta}(\cos^2 \theta - \sin^2 \theta) - T_{\theta\theta} \sin \theta \cos \theta, \\ T_{13} &= T_{rz} \cos \theta - T_{z\theta} \sin \theta, \\ T_{22} &= T_{rr} \sin^2 \theta + 2T_{r\theta} \sin \theta \cos \theta + T_{\theta\theta} \cos^2 \theta, \\ T_{23} &= T_{rz} \sin \theta + T_{z\theta} \cos \theta, \\ T_{33} &= T_{zz}. \end{aligned}$$

* When the components of a stress tensor are expressed in polar coordinates or any other curvilinear coordinates, care must be exercised in taking space derivatives. This is analogous to taking derivatives of vectors in curvilinear coordinates.

Before we leave the subject of tensor transformations we must make a final important observation. The direction cosines a_{mn} which transformed the vector in (8.2.10) were defined with the understanding that the components of τ were expressed in an *orthogonal* coordinate system. There were therefore implicit trigonometric relations between these direction cosines. If we state them formally, it is possible to extend the concept of a tensor to situations in which the transformations (8.2.10) and (8.2.17) are not geometrical in origin.* These relations are easily established by means of (8.2.10).

Equation 8.2.10 is the transformation of a vector τ from an unprimed to a primed coordinate system. There is, in general, nothing to distinguish the two coordinate systems. We could just as well define a transformation from the primed to the unprimed coordinates by

$$\tau_s = b_{sp}\tau'_p, \quad (8.2.18)$$

where b_{sp} is the cosine of the angle between the x_s -axis and the x'_p -axis. But b_{sp} , from the definition following (8.2.10), is then also

$$b_{sp} \equiv a_{ps}; \quad (8.2.19)$$

that is, the transformation which reverses the transformation (8.2.10) is

$$\tau_s = a_{ps}\tau'_p. \quad (8.2.20)$$

Now we can establish an important property of the direction cosines a_{ps} by transforming the vector τ to an arbitrary primed coordinate system and then transforming the components τ'_m back to the unprimed system in which they must be the same as those we started with. Equation 8.2.10 provides the first transformation, whereas (8.2.20) provides the second; that is, we substitute (8.2.10) into (8.2.20) to obtain

$$\tau_s = a_{ps}a_{pr}\tau_r. \quad (8.2.21)$$

Remember that we are required to sum on both p and r ; for example, consider the case in which $s = 1$:

$$\begin{aligned} \tau_1 &= (a_{11}a_{11} + a_{21}a_{21} + a_{31}a_{31})\tau_1 \\ &\quad + (a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32})\tau_2 \\ &\quad + (a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33})\tau_3. \end{aligned} \quad (8.2.22)$$

This relation must hold in general. We have not specified either a_{ps} or τ_m . Hence the second two bracketed quantities must vanish and the first must be unity. We can express this fact much more concisely by stating that in general

$$a_{ps}a_{pr} = \delta_{sr} \quad (8.2.23)$$

* J. C. Slater and N. H. Frank, *Mechanics*, 1st ed., McGraw-Hill, New York, 1947, Appendix V.

[this is the Kronecker delta as defined in (8.1.7)], for then (8.2.21) is reduced to the identity $\tau_s = \tau_s$.

8.3 FORCES IN ELECTRIC FIELD SYSTEMS

We now consider the forces that develop in electric field systems. The Lorentz force (1.1.28) gives the force on a charge q in an electric field $\mathbf{E}$ as

$$\mathbf{f} = q\mathbf{E}. \quad (8.3.1)$$

The force density $\mathbf{F}$ can be found by averaging (8.3.1) over a small volume:

$$\mathbf{F} = \lim_{\delta V \rightarrow 0} \frac{\sum_i \mathbf{f}_i}{\delta V} = \lim_{\delta V \rightarrow 0} \frac{\sum_i q_i \mathbf{E}_i}{\delta V}, \quad (8.3.2)$$

where q_i represents all the charges in δV , $\mathbf{E}_i$ is the electric field acting on the i th charge, and $\mathbf{f}_i$ is the force on the i th charge. It is found experimentally that free charges are almost never dense enough to make the microscopic field $\mathbf{E}_i$ seen by a charge appreciably different from the average (macroscopic) field $\mathbf{E}$. Consequently, because all charges in the volume δV experience the same electric field $\mathbf{E}$, we use the definition $\rho_f = \lim_{\delta V \rightarrow 0} \sum_i q_i / \delta V$ to write (8.3.2) as

$$\mathbf{F} = \rho_f \mathbf{E}. \quad (8.3.3)$$

Once again, remember that this is the force density on the charges and (as for the magnetic force density (8.1.3)) can be construed as the material force density only if each of the charges transmits its force to the medium.

The constituent relation is

$$\mathbf{D} = \epsilon \mathbf{E}. \quad (8.3.4)$$

For this development we assume that ϵ is a constant, but this restriction is relaxed in Section 8.5*. In this case we write (8.3.3) in terms of the electric field intensity by using Gauss's law (1.1.12)†:

$$\mathbf{F} = (\nabla \cdot \epsilon \mathbf{E}) \mathbf{E}. \quad (8.3.5)$$

We now express (8.3.5) as the space derivative of a stress tensor by recognizing that for electric field systems $\nabla \times \mathbf{E} = 0$. Hence (8.3.5) can be written as

$$\mathbf{F} = (\nabla \cdot \epsilon \mathbf{E}) \mathbf{E} + (\nabla \times \mathbf{E}) \times \epsilon \mathbf{E}. \quad (8.3.6)$$

* Arguments are given in Section 8.5 to show that this is the force density on free charges embedded in a material with a constant permittivity. For now we *assume* that the only effect of a uniform linear dielectric on the free charge force density is to replace $\epsilon_0 \rightarrow \epsilon$.

† Table 1.2, Appendix E.

We now use a vector identity on the last term to obtain*

$$\mathbf{F} = (\nabla \cdot \epsilon \mathbf{E})\mathbf{E} + \epsilon(\mathbf{E} \cdot \nabla)\mathbf{E} - \frac{1}{2}\epsilon \nabla(\mathbf{E} \cdot \mathbf{E}). \quad (8.3.7)$$

Using the index notation introduced in Section 8.1, we combine the first two terms and write the m th component of this equation as

$$F_m = \frac{\partial}{\partial x_n} (\epsilon E_m E_n) - \frac{\epsilon}{2} \frac{\partial}{\partial x_m} (E_k E_k). \quad (8.3.8)$$

The Kronecker delta is now used to write

$$\frac{\partial}{\partial x_m} = \delta_{mn} \frac{\partial}{\partial x_n}$$

and to put (8.3.8) in the desired form,

$$F_m = \frac{\partial T_{mn}}{\partial x_n}, \quad (8.3.9)$$

where the Maxwell stress tensor T_{mn} for electric field systems is given by

$$T_{mn} = \epsilon E_m E_n - \frac{\epsilon}{2} \delta_{mn} E_k E_k. \quad (8.3.10)$$

Note that this expression has the same form as (8.1.11) if we replace ϵ with μ and $\mathbf{E}$ with $\mathbf{H}$. The stress tensor here has all the general properties discussed in Section 8.2.2.

Both electric and magnetic forces are usually included in the Maxwell stress tensor†; however, we have not combined these forces because they usually do not occur in appreciable amounts in the same system. We use the term Maxwell stress tensor to denote that function from which electromagnetic force densities can be obtained by differentiation, as in (8.3.9). In different systems the Maxwell stress tensor represents different functions.

Example 8.3.1. To illustrate the use of the different expressions for force density and total force consider the electrostatic problem defined in Fig. 8.3.1.

The system consists of two regions of vacuum separated by a nonpolarizable ($\epsilon = \epsilon_0$) slab of thickness δ in the x_1 -direction and of infinite extent in the other two directions. The slab contains a volume charge density

$$\rho_f = \rho_f^0 \left(1 - \frac{x_1}{\delta}\right) \quad (a)$$

for $0 < x_1 < \delta$. The electric field in the region

$$x_1 < 0$$

is constrained to be

$$\mathbf{E} = \mathbf{i}_1 E_1^0 + \mathbf{i}_2 E_2^0 + \mathbf{i}_3 E_3^0. \quad (b)$$

* $(\nabla \times \mathbf{A}) \times \mathbf{A} = (\mathbf{A} \cdot \nabla)\mathbf{A} - \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A})$.

† J. A. Stratton, *Electromagnetic Theory*, McGraw-Hill, New York, 1941, pp. 97–103.

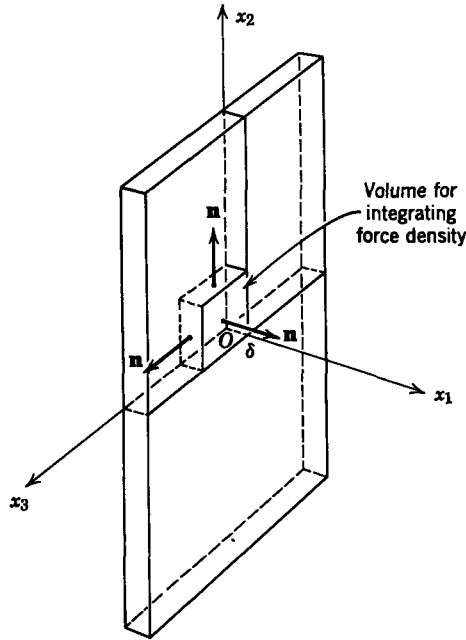


Fig. 8.3.1 Slab of material supporting a volume charge density.

After finding the electric field in the remainder of the system, we wish to compute in two ways the total force per unit area on the slab, first by doing a volume integration of the force density and then by doing a surface integration of the stress tensor.

To find the fields in the system we use the differential equations

$$\nabla \times \mathbf{E} = 0, \quad (c)$$

$$\nabla \cdot \epsilon_0 \mathbf{E} = \rho_f. \quad (d)$$

Because the slab has infinite extent in the x_2 - x_3 plane, we can (for purposes of illustration) assume no variation of $\mathbf{E}$ in the x_2 - and x_3 -directions.

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_3} = 0.$$

Then (c) shows that everywhere

$$E_2 = E_2^0,$$

$$E_3 = E_3^0.$$

Equations a and d give

$$\frac{\partial E_1}{\partial x_1} = \frac{\rho_f^0}{\epsilon_0} \left(1 - \frac{x_1}{\delta} \right).$$

Integration of this expression yields

$$E_1 = \frac{\rho_f^0}{\epsilon_0} \left(x_1 - \frac{x_1^2}{2\delta} \right) + C_1.$$

We use the boundary condition on the normal component of $\mathbf{E}$ at $x_1 = 0$ to evaluate the constant of integration.

$$C_1 = E_1^0.$$

Thus

$$E_1 = E_1^0 + \frac{\rho_f^0}{\epsilon_0} \left(x_1 - \frac{x_1^2}{2\delta} \right), \quad \text{for } 0 < x_1 < \delta,$$

and use of the boundary condition on the normal component of $\mathbf{E}$ at $x_1 = \delta$ yields

$$E_1 = E_1^0 + \frac{\rho_f^0 \delta}{2\epsilon_0}, \quad \text{for } x_1 > \delta.$$

The only region in which free charge exists is for $0 < x_1 < \delta$, where we can write

$$\mathbf{F} = \rho_f \mathbf{E} = \rho_f^0 \left(1 - \frac{x_1}{\delta} \right) \left\{ \mathbf{i}_1 \left[E_1^0 + \frac{\rho_f^0}{\epsilon_0} \left(x_1 - \frac{x_1^2}{2\delta} \right) \right] + \mathbf{i}_2 E_2^0 + \mathbf{i}_3 E_3^0 \right\}.$$

Taking a volume with unit dimension in the x_2 - and x_3 -directions, we write for the total force per unit area in an x_2 - x_3 plane:

$$\mathbf{f} = \int_0^\delta \mathbf{F} dx_1.$$

Performance of the indicated integration yields

$$\mathbf{f} = \mathbf{i}_1 \rho_f^0 \left(\frac{E_1^0 \delta}{2} + \frac{\rho_f^0 \delta^2}{8\epsilon_0} \right) + \mathbf{i}_2 \frac{\rho_f^0 E_2^0 \delta}{2} + \mathbf{i}_3 \frac{\rho_f^0 E_3^0 \delta}{2}.$$

We can obtain this same result by using the stress tensor, which we need only along the surface that encloses the slab.

The surface selected for integrating (8.1.17) is the one shown in Fig. 8.3.1 which has unit area in the x_2 - x_3 plane and thickness δ in the x_1 -direction. Because the fields are independent of x_2 and x_3 , the contributions from the surfaces perpendicular to x_2 and x_3 add to zero. We need only consider the surfaces of unit area perpendicular to x_1 . Thus we have

$$f_1 = T_{11}(\delta) - T_{11}(0),$$

$$f_2 = T_{21}(\delta) - T_{21}(0),$$

$$f_3 = T_{31}(\delta) - T_{31}(0).$$

Using the components T_{mn} defined with the fields derived earlier, we have [remember that $E_2(0) = E_2(\delta)$ and $E_3(0) = E_3(\delta)$]

$$f_1 = \frac{\epsilon_0}{2} [E_1^2(\delta) - E_1^2(0)] = \frac{\epsilon_0}{2} \left[\frac{E_1^0 \rho_f^0 \delta}{\epsilon_0} + \frac{(\rho_f^0)^2 \delta^2}{4\epsilon_0} \right],$$

$$f_2 = \epsilon_0 [E_1(\delta) E_2(\delta) - E_1(0) E_2(0)] = \epsilon_0 E_2^0 \frac{\rho_f^0 \delta}{2\epsilon_0},$$

$$f_3 = \epsilon_0 [E_1(\delta) E_3(\delta) - E_1(0) E_3(0)] = \epsilon_0 E_3^0 \frac{\rho_f^0 \delta}{2\epsilon_0}.$$

Thus the result is the same as that obtained by the volume integration. Note that in the surface integration we needed only the fields outside the space occupied by the charge. The fields, of course, are affected by the presence of the charge.

The most significant advantages of a formulation that uses the stress tensor arise because forces on the material within a volume can be determined without knowing the details of the volume force distribution (i.e., the distribution of currents or charges). Moreover, in many problems we are at liberty to choose the surface of integration and this can further simplify the computation. The next example illustrates how the choice of the surface of integration that is most convenient (or makes the integration possible) depends on symmetry and boundary conditions and further shows how the stress tensor can be used to obtain a total force in a situation in which a more direct approach would be difficult if not impossible.

Example 8.3.2. A pair of perfectly conducting plates at the potential difference V_0 is shown in Fig. 8.3.2. One of these plates is flat and the other has a step at the middle, as shown. Both plates extend far enough in the x_3 -direction that we can consider the problem as two-dimensional ($\partial/\partial x_3 = 0$). We wish to find the force in the x_1 -direction on a section of length l (in the x_3 -direction) of the bottom plate, including the effect of the fringing fields. To do this it is assumed that both $c \gg a$ and $d \gg b$, so that the regions of nonuniform electric field near the ends and near the step are separated by regions of essentially uniform electric field intensity.

To carry out the surface integration [the x_1 -component of (8.1.17)]

$$f_1 = \oint_S T_{1n} n_n da \quad (a)$$

we choose the surface shown in Fig. 8.3.2. Surfaces (1), (3) and (5) have the normal vector $\mathbf{n} = \mathbf{i}_2$, whereas surfaces (2) and (4) have the normal $\mathbf{n} = \mp \mathbf{i}_1$, respectively. Hence we can write (a) as

$$f_1 = \int_{(1)(5)} T_{12} da + \int_{(3)} T_{12} da - \int_{(2)} T_{11} da + \int_{(4)} T_{11} da + \int_{(6)} T_{1n} n_n da. \quad (b)$$

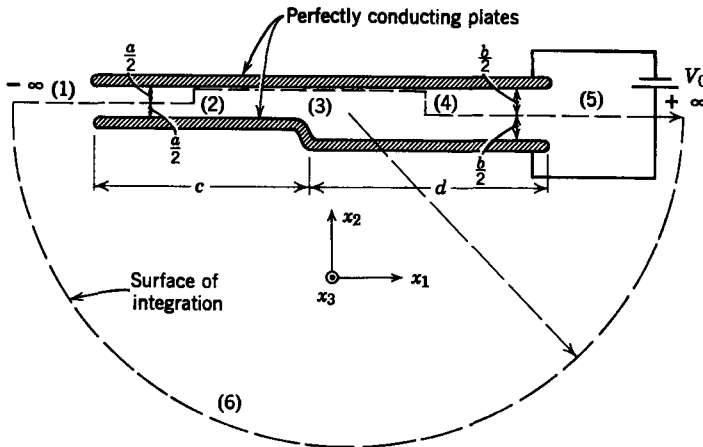


Fig. 8.3.2 Conducting plates at the potential difference V_0 .

The contributions from the surfaces with normals $\pm i_3$ have been ignored, for they cancel. Because the surface of integration (6) is far from the plates (at infinity), we expect the contribution of the last integral to be zero. We can argue that this is the case by making (6) the surface of a cylinder of radius R , with the plates at the origin. Far from the plates the electric field distribution is essentially that of a dipole.* Hence $|E| \approx 1/R^2$ and $|T_{1n}| \approx 1/R^4$. It follows that although the surface area of integration is proportional to R^2 the integral (6) decreases as $1/R^2$ and vanishes as $R \rightarrow \infty$. From (8.3.10)

$$T_{12} = \epsilon E_1 E_2. \quad (c)$$

Surfaces (1) and (5) are half way between the plates where by symmetry $E_1 = 0$. Hence the first term in (b) is also zero. Moreover, because $E_1 = 0$ along the perfectly conducting plate where surface (3) is located, the second integral vanishes also.

From (8.3.10)

$$T_{11} = \frac{1}{2}\epsilon(E_1^2 - E_2^2 - E_3^2). \quad (d)$$

Surfaces (2) and (4) are in regions of uniform electric field intensity. Hence

$$\begin{aligned} E &= \frac{V_0}{a} i_2; \quad \text{on surface (2),} \\ E &= \frac{V_0}{b} i_2; \quad \text{on surface (4).} \end{aligned} \quad (e)$$

Because these surfaces make the only contribution to the surface integral, (b), (d), and (e) become

$$f_1 = \int_{(2)} \frac{1}{2}\epsilon \left(\frac{V_0}{a}\right)^2 da - \int_{(4)} \frac{1}{2}\epsilon \left(\frac{V_0}{b}\right)^2 da. \quad (f)$$

The stresses are constant over the surfaces of integration and therefore the integral is performed by multiplication of the appropriate areas:

$$f_1 = \frac{1}{2}\epsilon \left(\frac{V_0}{a}\right)^2 \left(\frac{la}{2}\right) - \frac{1}{2}\epsilon \left(\frac{V_0}{b}\right)^2 \left(\frac{lb}{2}\right) \quad (g)$$

or

$$f_1 = \frac{\epsilon V_0^2 l}{4} \left(\frac{1}{a} - \frac{1}{b}\right). \quad (h)$$

The electric force on the lower plate (for $a < b$, as shown in Fig. 8.3.2) tends to pull in the x_1 -direction. If we had closed the surface above the top plate, the signs of the normal vectors involved would have been reversed to give an equal and opposite force on the top plate.

8.4 THE SURFACE FORCE DENSITY

Magnetic and electric fields in many situations are found by modeling the current or charge distributions by surface currents or surface charges. In these systems surface forces of electrical origin must be considered; for

* R. M. Fano, L. J. Chu, and R. B. Adler, *Electromagnetic Fields, Energy, and Forces*, Wiley, New York, 1960, p. 92.

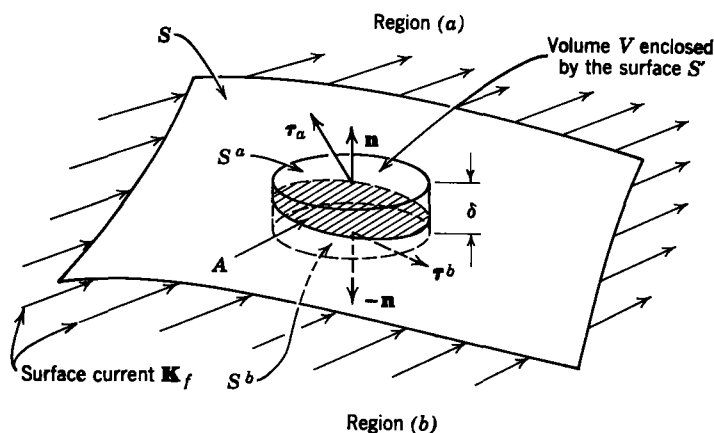


Fig. 8.4.1 Small thin volume V which encloses a section A of a surface S supporting the surface current $\mathbf{K}_f$.

example, if the surface S supports a surface current density $\mathbf{K}_f$, as shown in Fig. 8.4.1, and is immersed in a magnetic field $\mathbf{H}$, we expect a surface force similar in form to (8.1.3) with $\mathbf{K}_f$ playing the role of $\mathbf{J}_f$.

The surface force density $\mathbf{T}$ (newtons per square meter) is defined in terms of the force $\mathbf{f}$ on the material within the small thin volume shown in Fig. 8.4.1. It is the force per unit area A on the surface S intersected by the volume V in the limit in which first the thickness δ and then the area A become small. The stress tensor provides a convenient means of evaluating $\mathbf{T}$, for the total force can be written as a surface integral (8.1.17). In the limit in which $\delta \rightarrow 0$ the contribution to this integral along the sides (of height δ) of the volume V becomes vanishingly small and the only contributions come from the tractions acting on the surfaces S^a and S^b . In the limit $S^a \rightarrow S^b \rightarrow A$ we have

$$T_m = \tau_m^a + \tau_m^b, \quad (8.4.1)$$

where $\boldsymbol{\tau}^a$ and $\boldsymbol{\tau}^b$ are the tractions acting on the surfaces S^a and S^b , respectively.

If we define $\mathbf{n}$ as the unit vector *normal to the surface S* and directed from region (b) to region (a), the surface tractions can be evaluated by using (8.2.8):

$$T_m = (T_{mn}^a - T_{mn}^b)n_n. \quad (8.4.2)$$

Remember that in (8.2.8) $\mathbf{n}$ is the unit vector normal to a surface that *encloses* the volume of integration in Fig. 8.4.1. Over the top surface the normal vector is $\mathbf{n}$, but over the bottom surface it is $-\mathbf{n}$. Hence the minus sign in (8.4.2).

The Maxwell stresses T_{mn} are functions of either the magnetic or electric fields. Therefore (8.4.2) is a convenient expression for the surface force density on either a surface current or a surface charge.

8.4.1 Magnetic Surface Forces

As already pointed out, the magnetic surface force should be equivalent to the cross product of the surface current with a magnetic field. In this cross product, however, do we use the value of $\mathbf{H}$ from region (a) or from region (b)? In fact the average value of $\mathbf{H}$ should be used and the force per unit area $\mathbf{T}$ acting on a surface current $\mathbf{K}_f$ is

$$\mathbf{T} = \mu \mathbf{K}_f \times \frac{(\mathbf{H}^a + \mathbf{H}^b)}{2}. \quad (8.4.3)$$

We can prove that this relation is, in fact, valid by showing that it is equivalent to (8.4.2).

The surface current density from (6.2.14)* is

$$\mathbf{K}_f = \mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) \quad (8.4.4)$$

and (8.4.3) becomes

$$\mathbf{T} = \mu [\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b)] \times \frac{(\mathbf{H}^a + \mathbf{H}^b)}{2}. \quad (8.4.5)$$

We now use a vector identity† to rewrite this expression in component form as

$$T_m = \mu (H_m^a - H_m^b) \mathbf{n} \cdot \frac{(\mathbf{H}^a + \mathbf{H}^b)}{2} - \frac{\mu n_m}{2} (\mathbf{H}^a \cdot \mathbf{H}^a - \mathbf{H}^b \cdot \mathbf{H}^b). \quad (8.4.6)$$

The first term of this equation can be simplified by using (6.2.7)* $\mathbf{n} \cdot \mu \mathbf{H}^a = \mathbf{n} \cdot \mu \mathbf{H}^b$, whereas we replace n_m with $n_n \delta_{mn}$ in the second term.

$$T_m = (\mu H_m^a H_n^a - \frac{1}{2} \delta_{mn} \mu H_k^a H_k^a) n_n - (\mu H_m^b H_n^b - \frac{1}{2} \delta_{mn} \mu H_k^b H_k^b) n_n. \quad (8.4.7)$$

Our expression is now identical with (8.4.2), if we note that the magnetic stress is given by (8.1.11). We can alternatively write the surface force in terms of the fields alone (stresses) by using (8.4.2) or in terms of surface currents and an average magnetic field (8.4.3).

8.4.2 Electric Surface Forces

The surface force in an electric field system can be expressed as the product of the surface charge density σ_f and the average electric field intensity.

$$\mathbf{T} = \sigma_f \frac{(\mathbf{E}^a + \mathbf{E}^b)}{2}. \quad (8.4.8)$$

* Table 6.1, Appendix E.

† $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{C} \cdot \mathbf{B})$.

8.1 Electromagnetic Force Densities, Stress Tensors, and Surface Force Densities for Quasi-Static Magnetic and Electric Field Systems*

Description	Force Density $\mathbf{F}$	Stress Tensor T_{mn} $F_m = \frac{\partial T_{mn}}{\partial x_n}$ (8.1.10)	Surface Force Density $T_m = [T_{mn}]n_n$
Force on media carrying free current density $\mathbf{J}_f$, μ constant	$\mathbf{J}_f \times \mathbf{B}$ (8.1.3)	$T_{mn} = \mu H_m H_n - \delta_{mn} \frac{1}{2} \mu H_k H_k$ (8.1.11)	$\mathbf{T} = \mathbf{K}_f \times \mathbf{l}$ $\mathbf{K}_f = \mathbf{n} \times [\mathbf{J}]$ (8.4.3)
Force on media supporting free charge density ρ_f , ϵ constant	$\rho_f \mathbf{E}$ (8.3.3)	$T_{mn} = \epsilon E_m E_n - \delta_{mn} \frac{1}{2} \epsilon E_k E_k$ (8.3.10)	$\mathbf{T} = \sigma_f \langle \mathbf{E} \rangle$ $\sigma_f = \mathbf{n} \cdot [\epsilon \mathbf{E}]$ (8.4.8)
Force on free current plus magnetization force in which $\mathbf{B} = \mu \mathbf{H}$ both before and after media are deformed	$\mathbf{J}_f \times \mathbf{B} - \frac{1}{2} \mathbf{H} \cdot \mathbf{H} \nabla \mu$ $+ \frac{1}{2} \nabla \left(\mathbf{H} \cdot \mathbf{H} \rho \frac{\partial \mu}{\partial \rho} \right)$ (8.5.38)	$T_{mn} = \mu H_m H_n$ $- \frac{1}{2} \delta_{mn} \left(\mu - \rho \frac{\partial \mu}{\partial \rho} \right) H_k H_k$ (8.5.41)	
Force on free charge plus polarization force in which $\mathbf{D} = \epsilon \mathbf{E}$ both before and after media are deformed	$\rho_f \mathbf{E} - \frac{1}{2} \mathbf{E} \cdot \mathbf{E} \nabla \epsilon$ $+ \frac{1}{2} \nabla \left(\mathbf{E} \cdot \mathbf{E} \rho \frac{\partial \epsilon}{\partial \rho} \right)$ (8.5.45)	$T_{mn} = \epsilon E_m E_n$ $- \frac{1}{2} \delta_{mn} \left(\epsilon - \rho \frac{\partial \epsilon}{\partial \rho} \right) E_k E_k$ (8.5.46)	

$$\equiv \frac{\mathbf{A}^a + \mathbf{A}^b}{2}$$

$$\equiv \mathbf{A}^a - \mathbf{A}^b$$

This result expresses the surface force in a form that is similar to that of the force density (8.3.3). We can show that this equation is correct by demonstrating that it is equivalent to (8.4.2). First, we write (8.4.8) in terms of the electric fields, using Gauss's law to express σ_f (6.2.33)*

$$\mathbf{T} = \epsilon \mathbf{n} \cdot (\mathbf{E}^a - \mathbf{E}^b) \frac{(\mathbf{E}^a + \mathbf{E}^b)}{2}. \quad (8.4.9)$$

By use of a vector identity,† this becomes

$$\mathbf{T} = \epsilon \mathbf{n} (\mathbf{E}^a - \mathbf{E}^b) \cdot \frac{(\mathbf{E}^a + \mathbf{E}^b)}{2} - \epsilon (\mathbf{E}^a - \mathbf{E}^b) \times \left[\mathbf{n} \times \frac{(\mathbf{E}^a + \mathbf{E}^b)}{2} \right]. \quad (8.4.10)$$

This looks like the long way to go about it, but in this form the expression can be factored by using the condition (6.2.31)*, $\mathbf{n} \times \mathbf{E}^a = \mathbf{n} \times \mathbf{E}^b$.

$$\mathbf{T} = \frac{1}{2} \epsilon \mathbf{n} (\mathbf{E}^a \cdot \mathbf{E}^a - \mathbf{E}^b \cdot \mathbf{E}^b) - \epsilon [\mathbf{E}^a \times (\mathbf{n} \times \mathbf{E}^a) - \mathbf{E}^b \times (\mathbf{n} \times \mathbf{E}^b)]. \quad (8.4.11)$$

If we now use this same vector identity again,‡

$$\mathbf{T} = \epsilon [\mathbf{E}^a (\mathbf{n} \cdot \mathbf{E}^a) - \mathbf{E}^b (\mathbf{n} \cdot \mathbf{E}^b)] - \frac{1}{2} \epsilon \mathbf{n} (\mathbf{E}^a \cdot \mathbf{E}^a - \mathbf{E}^b \cdot \mathbf{E}^b), \quad (8.4.12)$$

and this equation is equivalent to the traction in terms of the stress (8.4.2), as can be seen by writing the m th component of $\mathbf{T}$ from (8.4.12)

$$T_m = [(\epsilon E_m^a E_n^a - \frac{1}{2} \delta_{mn} \epsilon E_k^a E_k^a) - (\epsilon E_m^b E_n^b - \frac{1}{2} \delta_{mn} \epsilon E_k^b E_k^b)] n_n \quad (8.4.13)$$

and using (8.3.10). Surface forces and their corresponding stresses are summarized in Table 8.1.

Example 8.4.1. The three plane parallel electrodes of Fig. 8.4.2. provide an example of a force on a surface charge. The plates are assumed to be perfectly conducting, with the outer plates connected together. If we ignore the fringing fields, we have

$$\begin{aligned} \mathbf{E}^a &= \frac{v}{d-x} \mathbf{i}_1, \\ \mathbf{E}^b &= \frac{-v}{x} \mathbf{i}_1, \end{aligned} \quad (a)$$

for the fields between the plates.

We now use several methods to compute the force acting on the middle plate.

First, we use the stress, as given in (8.4.2). The force of electrical origin on the middle plate in the $\mathbf{i}_1$ -direction is

$$f^e = AT_1 = A(T_{11}^a - T_{11}^b), \quad (b)$$

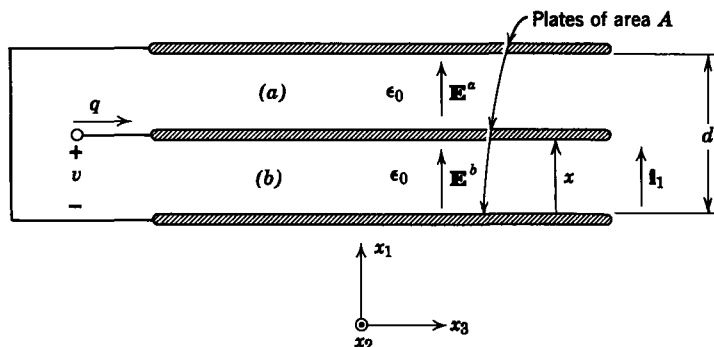
which in view of (a) becomes

$$f^e = \frac{A\epsilon_0}{2} [(E_1^a)^2 - (E_1^b)^2] = \frac{A\epsilon_0}{2} \left[\frac{v^2}{(d-x)^2} - \frac{v^2}{x^2} \right]. \quad (c)$$

* Table 6.1, Appendix E.

† $(\mathbf{C} \cdot \mathbf{A})\mathbf{B} = \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) - \mathbf{A} \times (\mathbf{C} \times \mathbf{B})$.

‡ $\mathbf{A} \times (\mathbf{B} \times \mathbf{A}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{A}) - \mathbf{A}(\mathbf{A} \cdot \mathbf{B})$.

Fig. 8.4.2 Plane-parallel electrodes with Area A .

This same force can be calculated from (8.4.8). We first compute the surface charge density as

$$\sigma_f = \epsilon_0(\mathbf{E}^a - \mathbf{E}^b) \cdot \mathbf{i}_1 = \epsilon_0 \left(\frac{v}{d-x} + \frac{v}{x} \right) \quad (d)$$

and then use (8.4.8)

$$f^e = AT_1 = \frac{A\epsilon_0}{2} \left(\frac{v}{d-x} + \frac{v}{x} \right) \left(\frac{v}{d-x} - \frac{v}{x} \right) = \frac{A\epsilon_0}{2} \left[\frac{v^2}{(d-x)^2} - \frac{v^2}{x^2} \right]. \quad (e)$$

This is the same expression as in (c).

Finally, we use the energy method introduced in Chapter 3* to find the force on the middle plate by noting that the system has one mechanical terminal pair (f^e, x) . The capacitance of the electrical terminal pair (v, q) (Fig. 8.4.2) is

$$C = \frac{A\epsilon_0}{x} + \frac{A\epsilon_0}{d-x}. \quad (f)$$

Hence the stored coenergy (which is the same as the energy, since the system is electrically linear) is

$$W'(v, x) = \frac{1}{2} C v^2 = \frac{1}{2} A \epsilon_0 \left(\frac{1}{x} + \frac{1}{d-x} \right) v^2, \quad (g)$$

and we have

$$f^e = \frac{\partial W'}{\partial x} = \frac{1}{2} A \epsilon_0 \left[\frac{1}{(d-x)^2} - \frac{1}{x^2} \right] v^2, \quad (h)$$

where we have used (h) of Table 3.1; of course, this result is also the same as given by (c).

8.5 THE MAGNETIZATION AND POLARIZATION FORCE DENSITIES

So far in this chapter the discussion has been limited to electric and magnetic forces on media that support free charges and free currents. In Chapter 3 examples often involve forces on magnetized or polarized media. In these examples the electric or magnetic fields are excited by means of free charges

* See Table 3.1, Appendix E.

or free currents. However, in many cases, the media subjected to the forces of electric origin do not themselves support free charges or free currents. Such forces, which are found by means of the energy method, must be attributed to the magnetization or polarization of the media.

When an atom or molecule of a substance is subjected to an external electric or magnetic field, the physical microscopic structure is distorted. Although the medium may be electrically neutral on a macroscopic scale, on an atomic scale it is composed in part of charged particles. It is the reaction of these charged particles to the Lorentz force that gives rise to the distortion of the microscopic structure. On a macroscopic level these effects are observed as a magnetization or polarization of the medium. For a wide range of substances it is possible to characterize the magnetization or polarization by simple constitutive laws, such as those introduced in Section 1.1.1; for example, certain isotropic materials can be characterized by a linear relation between the magnetic flux density $\mathbf{B}$ and the field intensity $\mathbf{H}$, $\mathbf{B} = \mu\mathbf{H}$. Similarly, for many isotropic dielectrics, $\mathbf{D} = \epsilon\mathbf{E}$.

In Chapter 3 we found forces of electrical origin by first establishing the electrical terminal relations for the system, then computing the electrical energy (or coenergy) stored in the system and finally using the energy function and the principle of conservation of energy to find the force of electric origin. In problems involving magnetization or polarization the first step in this procedure is made possible by knowing the appropriate electrical constitutive law.

In this section we wish to derive the force density by using the energy approach introduced in Chapter 3. Hence the derivation begins with the constitutive laws. Because these laws hold only for particular classes of material, the resulting force expressions are also restricted in validity. In particular, we consider media that are isotropic both before and after the magnetic or electric field is applied. Liquids and gases are most clearly in this category, some types of interaction with solids can be so modeled. Attention is given first to magnetization forces. The derivation is then easily revised to account for polarization forces.

8.5.1 Examples with One Degree of Freedom

Two simple examples help to establish the nature of the magnetization force density and show how its derivation relates to the energy method of Chapter 3. Figure 8.5.1 shows a slab of magnetizable material that is free to slide between the pole faces of a magnetic yoke. The force of electric origin tends to make the slab move into the region between the pole faces. This problem involves the rigid body motion of the material. By contrast, a second example (in Fig. 8.5.2) involves a medium that has an interface at ξ

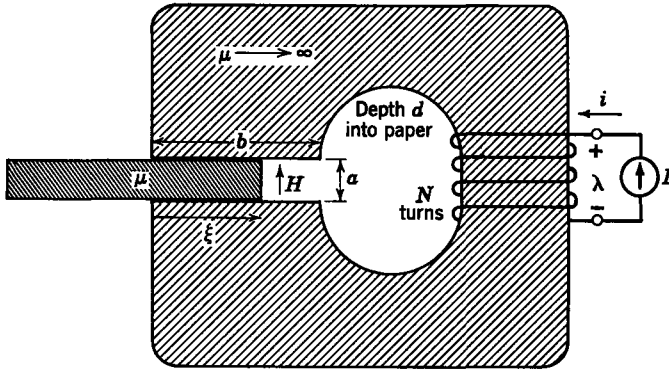


Fig. 8.5.1 A magnetic field intensity H is produced in the gap of the magnetic yoke. As a result, a magnetization force tends to pull the slab of magnetic material into the region between the pole faces.

but is otherwise surrounded by rigid walls. Hence a deflection of the interface must lead to a change in the volume occupied by the material. For the present purposes we assume that the material can deform only in the x_1 -direction.

These two examples have been selected for discussion because they characterize situations in which the magnetization force density is commonly operative. In the first case the force arises because the region occupied by the magnetic field includes a magnetically inhomogeneous material (the air and the magnetic solid). In the example of Fig. 8.5.2 there is an additional contribution to the force, caused by the change in volume of the material. This contribution is called the magnetostriction force.

The force in these examples can be computed by using the energy method of Chapter 3, since in each case there is only one degree of freedom. We first

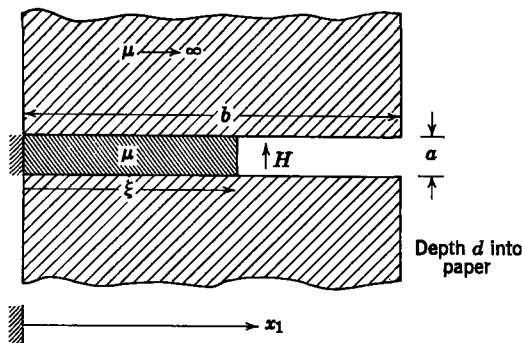


Fig. 8.5.2 The gap in the magnetic yoke of Fig. 8.5.1, with an experiment that demonstrates the magnetostrictive force. The surface at ξ is free to move. Because the material is otherwise surrounded by rigid walls, the motions of the surface must involve a compression or expansion of the medium.

review the energy approach by finding the force in each case associated with the displacement ξ . Then in the next section the same technique is extended to find the continuum force density and we return to these examples to illustrate its significance.

In each of the examples conservation of energy in the electromechanical coupling requires that

$$\lambda \delta i = \delta W' - f \delta \xi, \quad (8.5.1)$$

where W' is the total coenergy, as defined in Chapter 3*, and f is the total force of electrical origin associated with the displacement ξ . The symbol δ is used to indicate incremental changes in the independent variables i and ξ . It has the same significance as d in Chapter 3 and is introduced to avoid confusion with integration symbols such as da and dV , which indicate surface and volume elements.

We can establish the coenergy W' by integrating (8.5.1) in such a way that it is not necessary to know f . First we integrate on ξ (put the system together mechanically) with $i = 0$, but because the force of electrical origin is then zero, this integration makes no contribution. Here, of course, we preclude the possibility that the material is initially magnetized. The remaining integration takes the familiar form

$$W' = \int \lambda \delta i. \quad (8.5.2)$$

If the magnetic material is electrically linear, λ and i are related by the inductance and (8.5.2) yields

$$W' = \frac{1}{2} i^2 L(\xi). \quad (8.5.3)$$

Hence we have established the function W' from information about the electrical system, essentially the $\lambda-i$ relation.

We now hold the independent electrical variable i fixed (say by means of a constant current source). Then the left-hand side of (8.5.1) makes no contribution to the energy balance and this equation becomes,

$$\left(\frac{1}{2} i^2 \frac{\partial L}{\partial \xi} - f \right) \delta \xi = 0. \quad (8.5.4)$$

Here it is important to recognize that in this context ξ is an independent variable. Incremental displacements $\delta \xi$ are arbitrary. It therefore follows from (8.5.4) that the quantity in parentheses is zero.

$$f = \frac{1}{2} i^2 \frac{\partial L}{\partial \xi}. \quad (8.5.5)$$

In the next section we use this procedure to find the continuum force density at each point in the movable medium. Before embarking on that

* See Tables 2.1 and 3.1, Appendix E.

development we consider the specific examples shown in Figs. 8.5.1. and 8.5.2.

In the example of Fig. 8.5.1 the yoke is assumed to be perfectly permeable, hence in the gap $H = Ni/a$. In addition, $B = \mu H$ in the movable slab. It follows that the inductance L is

$$L = \frac{N^2 d}{a} [b\mu_0 + \xi(\mu - \mu_0)]. \quad (8.5.6)$$

Then from (8.5.5) the force is

$$f = (da)\frac{1}{2}H^2(\mu - \mu_0). \quad (8.5.7)$$

In deriving this expression we have assumed that μ in the movable slab is independent of the displacement ξ . This is reasonable as long as the material moves as a rigid body. In the example in Fig. 8.5.2 a displacement of the interface at ξ clearly is accompanied by a change in the density of the material. We expect that there is an associated change in the permeability which can be expressed as

$$\mu = \mu(\rho), \quad (8.5.8)$$

where ρ is the density (mass per unit volume) of the material. It is clear that ρ is in turn a function of ξ , for conservation of mass requires that

$$\rho \xi ad = \text{total mass of material} = \text{constant}. \quad (8.5.9)$$

Differentiation of this expression with respect to ξ gives

$$\frac{\partial \rho}{\partial \xi} = -\frac{\rho}{\xi}, \quad (8.5.10)$$

which shows how changes in density arise from motions of the surface at ξ .

We now use (8.5.6) to find the total force, including the dependence of μ on ξ (through the density ρ).

$$f = da\frac{1}{2}H^2 \left[(\mu - \mu_0) + \xi \frac{\partial \mu}{\partial \rho} \frac{\partial \rho}{\partial \xi} \right]. \quad (8.5.11)$$

From (8.5.10) this force can also be written as

$$f = da\frac{1}{2}H^2 \left[(\mu - \mu_0) - \rho \frac{\partial \mu}{\partial \rho} \right]. \quad (8.5.12)$$

The compressibility of the material gives rise to an additional term, as can be seen by comparing (8.5.7) and (8.5.12). This magnetostrictive force is significant when material deformations that lead to changes in the density are important.

We have considered these special cases to make it clear that the basic thermodynamic techniques introduced in Chapter 3 provide the fundamental means by which magnetization and polarization forces can be derived. The derivations of the next two sections are something new in our development only because the objective is a force density rather than a finite number of total forces. One way to consider the continuum situation is shown schematically in Fig. 8.5.3. A magnet is excited by a current i and a magnetizable material is subjected to the resulting magnetic field. Now, if we divide the material into small volume elements, deformations can be described by simply indicating the displacement ξ^i of each element.

There are three degrees of freedom for each volume element and therefore the i th displacement must be represented by three terminal pairs, which are summarized by vector terminal variables ξ^i and $\mathbf{f}^i$. Say that the medium has been divided into m regions. Then conservation of energy for the electromechanical coupling requires that

$$i \delta \lambda = \delta W + \sum_{i=1}^m \mathbf{f}^i \cdot \delta \xi^i; \quad (8.5.13)$$

that is, an increment of energy $i \delta \lambda$ introduced through the electrical terminals either increases the magnetic energy stored by the amount δW or does work $\mathbf{f}^i \cdot \delta \xi^i$ on one or more of the elements of volume.

In terms of the coenergy (as discussed in Section 3.1.2b)* this statement of conservation of energy becomes

$$\lambda \delta i = \delta W' - \sum_{i=1}^m \mathbf{f}^i \cdot \delta \xi^i. \quad (8.5.14)$$

Now, if we used a large number of elements m , the force $\mathbf{f}^i$ divided by the volume of the i th element would constitute the force density acting in the neighborhood of the i th element. Hence it should be clear that in principle we can find the force density by using this familiar energy method. Rather than using the summation, we take the limit in which $m \rightarrow \infty$ at the outset and represent the summation by an integration.

One significant point can be made without further mathematical developments. In Section 8.1 we claim (without proof) that the force density $\mathbf{J}_f \times \mathbf{B}$ on free currents remains valid even if the currents are immersed in a material

* See Table 3.1, Appendix E.

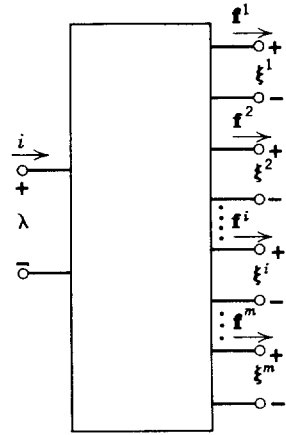


Fig. 8.5.3 Schematic representation of electromechanical coupling in which deformations of a continuum are represented by m vector displacements.

with a uniform and constant μ . We can use the energy method of finding forces to see that this must be true.

Suppose for a moment that we had a system that did not involve magnetization and used the energy method to find the force density $\mathbf{F} = \mathbf{J}_f \times \mathbf{B}$. Now, we do not need to carry out this formalism because we already know the answer. The point is this. Consider the same system, but with a uniform permeability μ . The energy function W' , from which the force density is found, is computed from an electrical terminal relation that in turn is found by using Maxwell's equations. The only change in the laws governing the fields is that $\mu_0 \rightarrow \mu$. Hence the only change in the energy function W' is that $\mu_0 \rightarrow \mu$, and the force density remains the same as in the form without the uniform $\mu: \mathbf{J}_f \times \mathbf{B}$. Of course, $\mathbf{B}$ is computed by using Maxwell's equations with $\mu_0 \rightarrow \mu$.

Note that this conclusion is consistent with the derivation of the Maxwell stress tensor given in Section 8.1. In what follows we concern ourselves with finding the magnetization force on material in which $\mathbf{J}_f = 0$.

8.5.2 The Magnetization Force Density

The force density on magnetizable material can be found by following the same procedure outlined in Section 8.5.1. For this purpose we consider the experiment shown in Fig. 8.5.4. A perfectly permeable magnetic yoke is excited by the current i . Our experiment is carried out in the region between the rigid pole faces, where the magnetic field is concentrated (just as it was in Fig. 8.5.1). In this region a deformable magnetic material has a displacement from the coordinate position $\mathbf{r}$ given by $\delta \boldsymbol{\xi}(\mathbf{r})$.

We define the force per unit volume $\mathbf{F}(\mathbf{r})$ as acting on the material at $\mathbf{r} + \delta \boldsymbol{\xi}$. This makes it possible to write (8.5.14) as

$$\lambda \delta i = \int_V \delta w' dV - \int_V \mathbf{F} \cdot \delta \boldsymbol{\xi} dV. \quad (8.5.15)$$

The function w' is defined as the *coenergy density* and can be integrated over the volume V to find the total coenergy W' . Although the specific geometry of the magnetic circuit is superfluous, it does help to fix attention on a physically reasonable system. For convenience we have included only a single one-turn electrical excitation in the system, with the magnetic circuit arranged to concentrate the magnetic field in the volume V . It is convenient to define this volume V as being enclosed by three surfaces S' , S'' , and S''' , shown in Fig. 8.5.4. The surface S' is bounded by the current path for i and encloses the magnetic circuit to the left. (It covers the pole face to the left like a sock on a foot with the current path i as the garter.) The surface S'' plays a similar role for the remaining section of magnetic circuit to the right (the other foot). Finally, the surface S''' encloses the entire system, with a slit

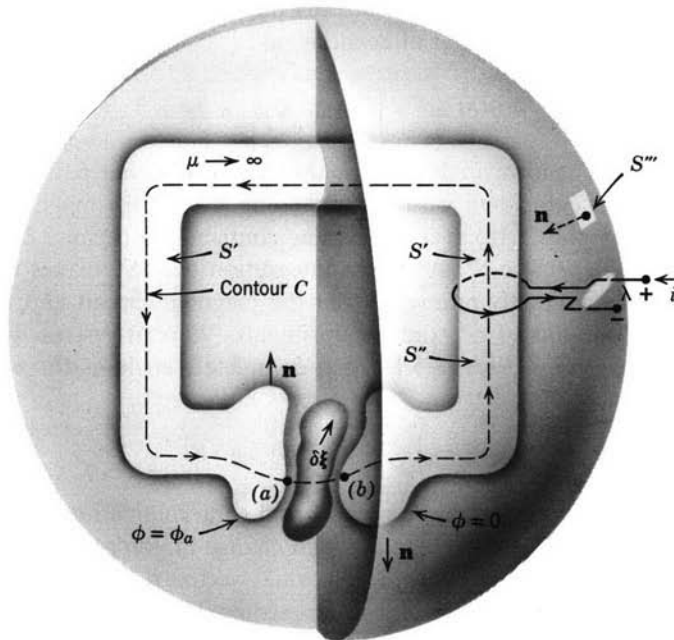


Fig. 8.5.4 Perfectly permeable magnetic circuit excited by the current i . A deformable magnetic material with permeability $\mu(\mathbf{r})$ is subjected to the concentration of magnetic field intensity between the pole faces.

left open for the electrical terminals. These three surfaces, taken together, enclose the volume V occupied by the deformable magnetizable material. Note that the normal vector $\mathbf{n}$ is directed *into* the volume V .

We require a field representation of the electrical input $\lambda \delta i$ for the purpose of writing all the terms in (8.5.15) as volume integrals. There is no free current $\mathbf{J}_f$ in the volume V , hence $\nabla \times \mathbf{H} = 0$. It is therefore convenient to write $\mathbf{H} = -\nabla\phi$. We are free to define ϕ as zero on the right-hand pole face in Fig. 8.5.4; hence since $\mathbf{H} = 0$ inside the magnetic yoke

$$i = \oint \mathbf{H} \cdot d\mathbf{l} = -\oint \nabla \phi \cdot d\mathbf{l} = -\phi_b + \phi_a = \phi_a, \quad (8.5.16)$$

where a is any point on the left pole face, as shown in Fig. 8.5.4. Remember that by definition

$$\lambda = \int_{S'} \mathbf{B} \cdot \mathbf{n} \, da = - \int_{S'} \mu \nabla \phi \cdot \mathbf{n} \, da. \quad (8.5.17)$$

The surface S' used to compute the flux λ is bounded by the conductor carrying the current i . In Fig. 8.5.4 this surface is defined so that it is coincident with the surface of the perfectly permeable magnetic circuit. Since ϕ_a is

the potential evaluated on the surface S' , these last two equations are used to write the incremental input of coenergy as

$$\lambda \delta i = - \int_{S'} \mu \delta \phi \nabla \phi \cdot \mathbf{n} da. \quad (8.5.18)$$

The surface S'' shown in Fig. 8.5.4 also has the current path for i as its periphery but is coincident with the remaining part of the magnetic circuit. The integration of (8.5.18) over S'' gives no contribution because ϕ is defined as zero over the surface S'' . A similar integration over S''' makes no contribution, for S''' is greatly removed from the magnetic circuit and $\mathbf{n} \cdot \mathbf{B} = 0$ in the neighborhood of S''' and the terminals. We can just as well use a surface of integration S in (8.5.18) that completely encloses the volume V . It follows that

$$\lambda \delta i = \int_V \nabla \cdot [(\mu \nabla \phi)(\delta \phi)] dV. \quad (8.5.19)$$

Here the surface integral has been converted to a volume integral by using Gauss's theorem. Note that there is a sign change in going from the surface integral to the volume integral. The normal vector $\mathbf{n}$ in Fig. 8.5.4 points into the volume V rather than outward as required in the usual statement of Gauss's theorem.

Because $\nabla \cdot \mathbf{B} = -\nabla \cdot \mu \nabla \phi = 0$, we can use an identity* to convert (8.5.19) to

$$\lambda \delta i = \int_V \frac{1}{2} \mu \delta (\nabla \phi)^2 dV. \quad (8.5.20)$$

It is now possible to write all the terms of (8.5.15) as volume integrals and express conservation of energy as

$$\int_V \frac{1}{2} \mu \delta (\nabla \phi)^2 dV = \int_V \delta w' dV - \int_V \mathbf{F} \cdot \delta \boldsymbol{\xi} dV. \quad (8.5.21)$$

We now put the system together, first mechanically and then electrically to find w' . As in Section 8.5.1, the last term in (8.5.21) makes no contribution to the coenergy stored during this process. We must remember that because the material is deformable the permeability at a given point is a function of deflection (e.g., the permeability at a given point in the gap of the magnetic circuit shown in Fig. 8.5.1 could be μ or μ_0 , depending on the position of the slab); that is, $\mu = \mu(\boldsymbol{\xi})$. Once the system is assembled mechanically, however, μ is constant and the remaining integration of (8.5.21) becomes

$$\int_V \delta [\frac{1}{2} \mu (\nabla \phi)^2] dV = \int_V \delta w' dV. \quad (8.5.22)$$

* $\nabla \cdot \mathbf{A} \psi = \psi \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \psi$.

Rather than carrying out the integration, it serves our purpose to recognize that if we integrate the quantity

$$\delta w' = \delta[\tfrac{1}{2}\mu(\nabla\phi)^2] \quad (8.5.23)$$

over the volume V , the incremental change in total coenergy will have been computed. This completes the first step in finding the force density $\mathbf{F}$ in that the coenergy has been found from the electrical properties of the material. Note that we have assumed that $\mathbf{B} = \mu\mathbf{H}$ both before and after the material is deformed.

As in the preceding section, the next step uses the coenergy density to determine the force density. This is done by constraining the current i to be a constant so that the left-hand side of (8.5.15), and hence (8.5.21), is zero. From (8.5.16) this means that ϕ is held constant on S' . Changes in coenergy now occur because of changes $\delta\boldsymbol{\xi}$ in the material displacement.

A few manipulations on $\delta w'$ make the remaining terms assume a familiar form. From (8.5.23)

$$\delta w' = \tfrac{1}{2}(\nabla\phi)^2 \delta\mu + \tfrac{1}{2}\mu \delta(\nabla\phi)^2. \quad (8.5.24)$$

The integral over the volume of the second term in this equation vanishes, as can be seen by first using an identity to write it as

$$\tfrac{1}{2}\mu \delta(\nabla\phi)^2 = \mu \nabla\phi \cdot \nabla(\delta\phi) = \nabla \cdot (\delta\phi\mu \nabla\phi) - \delta\phi \nabla \cdot (\mu \nabla\phi). \quad (8.5.25)$$

Because $\nabla \cdot \mathbf{B} = 0$, the last term is zero, whereas the integral over the volume of the remaining term can be transformed by Gauss's theorem to an integral over S , where ϕ is constant, hence $\delta\phi = 0$.

Because the last term in (8.5.24) makes no contribution, the conservation of energy equation (8.5.15) becomes (remember, $\mathbf{H} = -\nabla\phi$)

$$\int_V (\tfrac{1}{2}\mathbf{H} \cdot \mathbf{H} \delta\mu - \mathbf{F} \cdot \delta\boldsymbol{\xi}) dV = 0. \quad (8.5.26)$$

This equation is the generalization of (8.5.4). Note that the permeability plays the same role as the inductance in determining the dependence of the coenergy on the displacement of the material. To determine the force density we must relate the permeability μ to the material displacement. (This is analogous to finding the inductance L of (8.5.6) as a function of ξ .)

There are two ways in which the permeability at a point $\mathbf{r}$ can change. Either the material is initially inhomogeneous, in which case a displacement can transport material of different permeability into the region of $\mathbf{r}$, or the density of the material can change with a resulting change in permeability.

Consider first the effect of inhomogeneities. After the displacement $\delta\boldsymbol{\xi}$, the permeability at $\mathbf{r}$ is that of the material that was at $\mathbf{r} - \delta\boldsymbol{\xi}$ before the displacement. Hence

$$\delta\mu = \lim_{\delta\boldsymbol{\xi} \rightarrow 0} [\mu(\mathbf{r} - \delta\boldsymbol{\xi}) - \mu(\mathbf{r})]. \quad (8.5.27)$$

Taylor's expansion makes it possible to write this as

$$\delta\mu = \lim_{\delta\xi \rightarrow 0} \left[\mu(\mathbf{r}) - \delta\xi_i \frac{\partial\mu}{\partial x_i}(\mathbf{r}) + \cdots - \mu(\mathbf{r}) \right]. \quad (8.5.28)$$

In the limit

$$\delta\mu = -\delta\xi \cdot \nabla\mu. \quad (8.5.29)$$

If this is the only mechanism by which the permeability can change, (8.5.26) becomes

$$\int_V (-\frac{1}{2}\mathbf{H} \cdot \mathbf{H} \nabla\mu - \mathbf{F}) \cdot \delta\xi dV = 0. \quad (8.5.30)$$

It is now crucial to recognize that the displacement $\delta\xi$ is arbitrary, in that ξ is an independent variable in the same sense as in the analogous lumped parameter derivation [see (8.5.4)]. Hence to satisfy (8.5.30) the quantity in parentheses must vanish.

$$\mathbf{F} = -\frac{1}{2}\mathbf{H} \cdot \mathbf{H} \nabla\mu. \quad (8.5.31)$$

This contribution to the magnetization force density results because of inhomogeneities in the magnetic material. An example involving a force of this type, considered in Section 8.5.1 (Fig. 8.5.1), serves to illustrate the significance of (8.5.31).

Example 8.5.1. The slab of magnetic material and adjacent pole faces for the problem of Fig. 8.5.1 is shown in Fig. 8.5.5. Here the distribution of μ in the gap is plotted as a function of x_1 with the transition from μ to μ_0 at the surface (s) expanded over a thickness Δ . The magnetic field intensity H is uniform throughout the gap. The gradient of μ in the x_1 -direction is zero in the bulk of the slab but has the value

$$\nabla\mu = \frac{\mu_0 - \mu}{\Delta} \mathbf{i}_1 \quad (a)$$

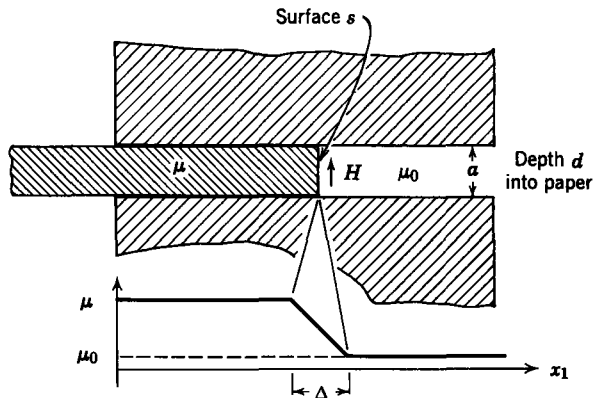


Fig. 8.5.5 Magnetic slab of Fig. 8.5.1 free to slide in the x_1 -direction. Near the surface s the permeability undergoes a rapid change. This region is shown (expanded) to have a thickness Δ over which the permeability varies linearly from μ to μ_0 .

in the expanded region of the surface (s). Hence the force per unit volume acting near the surface s is constant and given by

$$F_1 = -\frac{1}{2}H^2 \frac{(\mu_0 - \mu)}{\Delta} \mathbf{i}_1. \quad (b)$$

The total force is the integral of the force density over the volume of the slab. Because $\nabla\mu$ is constant over the volume Δad and zero elsewhere, this integration reduces to

$$f_1 = -\frac{1}{2}H^2 \frac{(\mu_0 - \mu)}{\Delta} (\Delta ad) \mathbf{i}_1. \quad (c)$$

Note that this result is in agreement with (8.5.7). We may view the force density given by (8.5.31) as the generalization of (8.5.5). The force on the slab does not depend on the thickness Δ , as can be seen from (c). We could have used a distribution of $\mu(x_1)$ other than that shown in Fig. 8.5.5 to arrive at the same answer. Certainly the answer holds in the limit in which $\Delta \rightarrow 0$. This point is easily seen if the force density is represented in terms of a stress tensor, a point to which we return in the next section.

There remains the task of computing the force density that results from changes in the density of the material. As we saw in Section 8.5.1, compression of the material leads to a magnetostriction force. There it was accounted for by including the effect of changes in density on the inductance L . Here it is incorporated as it leads to a change in μ ; that is, in addition to the change in μ given by (8.5.29), there is a change

$$\delta\mu = \frac{\partial\mu}{\partial\rho} \delta\rho. \quad (8.5.32)$$

The decrease in density $-\delta\rho$ is proportional to the density ρ and the increase in the volume occupied by the material $\nabla \cdot \delta\boldsymbol{\xi}$. Hence*

$$-\delta\rho = \rho \nabla \cdot \delta\boldsymbol{\xi}. \quad (8.5.33)$$

Now, if we combine these last two equations, the first term in (8.5.26) can be written as

$$\int_V \frac{1}{2} \mathbf{H} \cdot \mathbf{H} \delta\mu \, dV = \int_V -\frac{1}{2} \mathbf{H} \cdot \mathbf{H} \frac{\partial\mu}{\partial\rho} \rho \nabla \cdot \delta\boldsymbol{\xi} \, dV. \quad (8.5.34)$$

In order to find the force density, we must write the integrand of this expression in the form $(\quad) \cdot \delta\boldsymbol{\xi}$. With this end in mind, we use an identity† to write

* A statement that the decrease in mass within the volume V_e is equal to the mass transported out of the volume through the surface S_e is given by $-\int_{V_e} \delta\rho \, dV = \oint_{S_e} \rho \delta\boldsymbol{\xi} \cdot \mathbf{n} \, da$.

Gauss's theorem converts the surface integral to a volume integral. To first order in $\delta\rho$ and $\delta\boldsymbol{\xi}$, $\nabla \cdot \rho \delta\boldsymbol{\xi} = \rho \nabla \cdot \delta\boldsymbol{\xi}$ and (8.5.33) follows.

† $\psi \nabla \cdot \mathbf{A} = \nabla \cdot \psi \mathbf{A} - \mathbf{A} \cdot \nabla \psi$.

(8.5.34) as

$$\int_V \frac{1}{2} \mathbf{H} \cdot \mathbf{H} \delta\mu dV = - \int_V \nabla \cdot \left(\frac{1}{2} \mathbf{H} \cdot \mathbf{H} \frac{\partial\mu}{\partial\rho} \rho \delta\boldsymbol{\xi} \right) dV \\ + \int_V \nabla \left(\frac{1}{2} \mathbf{H} \cdot \mathbf{H} \frac{\partial\mu}{\partial\rho} \rho \right) \cdot \delta\boldsymbol{\xi} dV. \quad (8.5.35)$$

The first integral is in a form in which by Gauss's theorem it can be written as an integral over the surface S of the volume V . On the surface S either the fields $\mathbf{H}$ are zero (the surface is outside the field region in Fig. 8.5.4) or $\delta\boldsymbol{\xi} \cdot \mathbf{n}$ is zero (because the surface is adjacent to the rigid pole faces). Hence the first term in (8.5.35) makes no contribution and the last term is in the desired form.

It is now possible to write (8.5.26) with the effects of inhomogeneity and changes in density included. There is a contribution to $\delta\mu$ from (8.5.29) (due to inhomogeneity) and from (8.5.32) which has already been incorporated into (8.5.35):

$$\int_V \left[-\frac{1}{2} \mathbf{H} \cdot \mathbf{H} \nabla\mu + \nabla \left(\frac{1}{2} \mathbf{H} \cdot \mathbf{H} \frac{\partial\mu}{\partial\rho} \rho \right) - \mathbf{F} \right] \cdot \delta\boldsymbol{\xi} dV = 0. \quad (8.5.36)$$

As before, we use the arbitrary nature of $\delta\boldsymbol{\xi}$ to conclude that the force density is

$$\mathbf{F} = -\frac{1}{2} \mathbf{H} \cdot \mathbf{H} \nabla\mu + \nabla \left(\frac{1}{2} \mathbf{H} \cdot \mathbf{H} \frac{\partial\mu}{\partial\rho} \rho \right). \quad (8.5.37)$$

Of course, the first term is the same as that given by (8.5.31). The second term is added to account for forces that accompany (or cause) changes in the density of the material and is referred to as the *magnetostriction force density*.

8.5.3 The Stress Tensor

It is often convenient to express the force density in terms of a stress tensor. This is done in this section, with both forces on free currents and magnetization forces included. Thus the appropriate force density is the superposition of (8.1.3) and (8.5.37):

$$\mathbf{F} = \mathbf{J}_f \times \mu\mathbf{H} - \frac{1}{2} \mathbf{H} \cdot \mathbf{H} \nabla\mu + \nabla \left(\frac{1}{2} \mathbf{H} \cdot \mathbf{H} \frac{\partial\mu}{\partial\rho} \rho \right). \quad (8.5.38)$$

The m th component of this equation can be written [using (8.1.8) to express $\mathbf{J}_f \times \mathbf{B}$] as

$$F_m = \mu H_n \frac{\partial H_m}{\partial x_n} - \frac{\mu}{2} \frac{\partial}{\partial x_m} H_k H_k - \frac{1}{2} H_k H_k \frac{\partial\mu}{\partial x_m} + \frac{\partial}{\partial x_m} \left(\frac{1}{2} H_k H_k \frac{\partial\mu}{\partial\rho} \rho \right). \quad (8.5.39)$$

Because $\partial \mu H_n / \partial x_n = 0$, the first term in this equation can be written as $\partial \mu H_n H_m / \partial x_n$. The second and third terms combine. Then, by introducing the Kronecker delta δ_{mn} (8.1.7), (8.5.39) can be written as

$$F_m = \frac{\partial T_{mn}}{\partial x_n}, \quad (8.5.40)$$

where

$$T_{mn} = \mu H_n H_m - \frac{1}{2} \delta_{mn} H_k H_k \left(\mu - \frac{\partial \mu}{\partial \rho} \rho \right). \quad (8.5.41)$$

Note that except for the magnetostriction term the stress tensor takes the same form as it did in Section 8.1, in which only the force on the free current $\mathbf{J}_f$ is considered. This similarity is deceptive unless it is remembered that the magnetic field intensity $\mathbf{H}$ [in (8.5.41)] is not the same with and without the current density $\mathbf{J}_f$. Moreover, μ in (8.5.41) is a function of position rather than a constant, as it was in (8.1.11).

Example 8.5.2. The problem shown in Fig. 8.5.2 serves as an illustration for the application of the stress tensor. In this example the slab of magnetic material is free to slide in the x_1 -direction but is constrained at $x_1 = 0$ so that the left end of the slab is fixed. The magnetic field intensity $\mathbf{H}$ is uniform throughout the slab and adjacent region of free space. It is therefore apparent from (8.5.38) that the force density in the x_1 -direction is present only at the left and right extremes of the slab in which the permeability μ and magnetostriction constant $\rho \partial \mu / \partial \rho$ undergo rapid variations. Deformations are independent of forces at the left end because it is fixed. At the right end there is a surface force that can be found by using the stress tensor. From (8.4.2) the force per unit area acting on the right end of the slab is

$$T_1 = T_{11}^a - T_{11}^b, \quad (a)$$

where (a) and (b) indicate the regions to the right and left of the surface. From (8.5.41) (a) becomes

$$T_1 = -\frac{1}{2} \mu_0 H^2 + \frac{1}{2} \left(\mu - \frac{\partial \mu}{\partial \rho} \rho \right) H^2. \quad (b)$$

Here we have taken the $\partial \mu / \partial \rho$ as zero in the free-space region (a). The total force on the end of the slab is (b) multiplied by the area ad , and this result agrees with that found in Section 8.5.1 (8.5.12) by using a model with a single degree of freedom.

Force densities, stress tensors, and surface force densities in magnetic field systems are summarized in Table 8.1. Note that the superposition of the free current force density and the magnetization force density leads to the same stress tensor as for the magnetization force density alone.

8.5.4 Polarization Force Density and Stress Tensor

So far we have limited our discussion to forces induced in magnetic materials by magnetic fields. Polarization forces, induced in dielectric materials

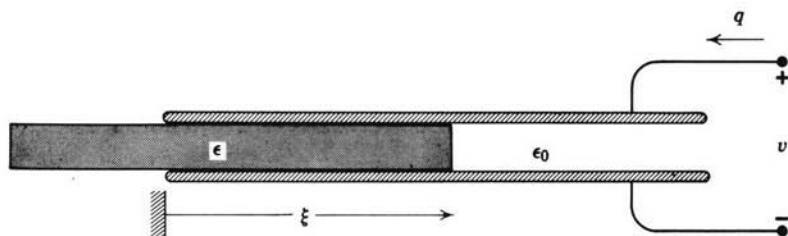


Fig. 8.5.6 A voltage v applied to perfectly conducting electrodes leads to a polarization force that tends to draw the block of dielectric into the region between the plates.

subjected to electric fields, are similar to magnetization forces. The lumped-parameter example shown in Fig. 8.5.6 is analogous to that shown in Fig. 8.5.1. A potential between the perfectly conducting electrodes induces a polarization force in the dielectric material which tends to draw it into the region between the plates. We could find this force by again using the energy methods introduced in Chapter 3* and writing a conservation of energy equation analogous to (8.5.1).

$$q \delta v = \delta W' - f \delta \xi. \quad (8.5.42)$$

Now, W' is the electric coenergy, q is the charge on the upper electrode, and v is the potential of the upper electrode with the potential of the lower electrode defined as zero.

We are interested here in finding the polarization force density and so generalize (8.5.42) to write a conservation of energy expression analogous to (8.5.15) (the physical system is shown in Fig. 8.5.7).

$$q \delta v = \int_V \delta w' dV - \int_V \mathbf{F} \cdot \delta \boldsymbol{\xi} dV. \quad (8.5.43)$$

If we define the potential $\phi (\mathbf{E} = -\nabla \phi)$ as being zero on the lower electrode, then $v = \phi$ evaluated on the upper electrode, whereas the total charge q is the integral of the surface charge $-\epsilon \mathbf{n} \cdot \nabla \phi$ over the surface of the upper electrode. Hence (8.5.43) becomes

$$-\int_{S'} \epsilon \nabla \phi \cdot \mathbf{n} \delta \phi da = \int_V \delta w' dV - \int_V \mathbf{F} \cdot \delta \boldsymbol{\xi} dV. \quad (8.5.44)$$

As in Section 8.5.2, the surface of integration S' can be extended to enclose a volume V that includes all the deformable material with no further contribution to the integral. This is true because ϕ is zero on the lower electrode which is enclosed by a surface S'' . Moreover, there is no contribution to an integration over a surface S''' that encloses the entire system, for this surface

* Table 3.1, Appendix E.

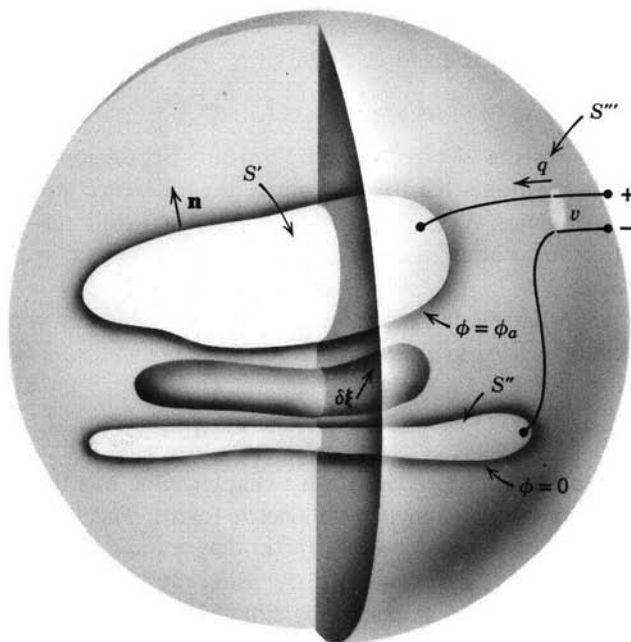


Fig. 8.5.7 A pair of perfectly conducting rigid electrodes imposes an electric field $\mathbf{E} = -\nabla\phi$ on a polarizable deformable dielectric medium. The surface S' encloses the upper electrode which is at the potential $\phi = v$.

is either far from the system or arranged (in the neighborhood of the terminals) so that $\mathbf{n} \cdot \mathbf{D} = 0$. The surface $S = S' + S'' + S'''$ completely encloses the volume V , and we can make use of Gauss's theorem to convert the left-hand side of (8.5.44) to a volume integral over V .

As a consequence of these manipulations, the left-hand side of (8.5.44) takes the same form as (8.5.19) with $\mu \rightarrow \epsilon$. Of course, ϕ now has the physical significance of being the potential for the electric field rather than for the magnetic field. All of the mathematical steps following (8.5.19) are valid, however, and we are led to a polarization force density with the same form as (8.5.37), with $\mathbf{H} \rightarrow \mathbf{E}$ and $\mu \rightarrow \epsilon$. If we superimpose on this force density, the force density on free charges ρ_f , the force density is

$$\mathbf{F} = \rho_f \mathbf{E} - \frac{1}{2} \mathbf{E} \cdot \mathbf{E} \nabla \epsilon + \nabla \left(\frac{1}{2} \mathbf{E} \cdot \mathbf{E} \frac{\partial \epsilon}{\partial \rho} \rho \right). \quad (8.5.45)$$

The first term is the free charge force density, the second is due to inhomogeneities in the dielectric, and the last results from changes in the material density. This last term is called the *electrostriction force density*.

Manipulations of (8.5.45) that incorporate the irrotational nature of the electric field intensity show that the stress tensor representation of combined free charge and polarization force densities is

$$T_{mn} = \epsilon E_m E_n - \frac{1}{2} \delta_{mn} E_k E_k [\epsilon - (\partial \epsilon / \partial \rho) \rho]. \quad (8.5.46)$$

Note that without the electrostriction term this expression is as obtained for the free charge alone (8.3.10). Of course, the difference now is that ϵ can be a function of space. At the same time the electric field intensity that must be used in (8.5.46) is affected by the presence of free charge in the material, for $\nabla \cdot \epsilon \mathbf{E} = \rho_f$.

Force densities, stress tensors, and surface force densities in electric field systems are summarized in Table 8.1.

8.6 DISCUSSION

There have been two objectives in this chapter. One was the development of a field description of magnetic and electric forces. This led to the concept of a stress tensor, which was convenient in determining total forces from a knowledge of the fields over a surface enclosing the volume of interest. The stress tensor is also useful in describing singular force distributions such as surface forces. The stress tensor, as developed here, is of interest as a basic mathematical representation. As illustrated in the chapters that follow, it can be used to represent a variety of physical quantities.

Our second objective has been to develop a picture of the distribution of forces due to magnetization and polarization. This was done while illustrating the important fact that the energy methods which form the theme of Chapter 3 are of equal significance in formulating a continuum description of electro-mechanical interactions.

From our derivations and discussion it should be clear that attention has been confined to a simple class of materials but that similar techniques can be used to determine force densities in more complicated media; for example, extensions of the energy method should allow us to find the force density in materials that are electrically nonlinear. Certainly the energy methods of Chapter 3 are not confined to electrically linear systems. Most solids do not exhibit the simple linear isotropic constitutive laws used here. Nonetheless, energy methods can be employed to find the force density in such situations,* although the formalisms used may be somewhat different from those used here.†

* J. A. Stratton, *Electromagnetic Theory*, McGraw-Hill, New York, 1941, p. 140.

† P. Penfield and H. Haus, *Electrodynamics of Moving Media*, M.I.T. Press, Cambridge, Mass., 1967; W. F. Brown, Jr., "Theory of Magnetoelastic Effects in Ferromagnetism," *J. Appl. Phys.*, **36**, 994 (1965).

PROBLEMS

8.1. An identity $\nabla \cdot (\psi \mathbf{A}) = \psi \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \psi$ is given, where ψ = scalar, $\mathbf{A}$ = vector. Show by means of index notation that this identity is valid.

8.2. Show, by means of index notation, that the following vector equation is valid: $\mathbf{B} \cdot \nabla (\psi \mathbf{A}) = \psi \mathbf{B} \cdot \nabla \mathbf{A} + \mathbf{A} \mathbf{B} \cdot \nabla \psi$; ψ = scalar and $\mathbf{A}$ and $\mathbf{B}$ are vectors.

8.3. Consider two orthogonal coordinate systems (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) . The primed coordinate system is related to the unprimed system as follows: $x'_3 = x_3$; the x'_1 -axis makes an angle of 60° with the x_1 -axis as shown in Fig. 8P.3.

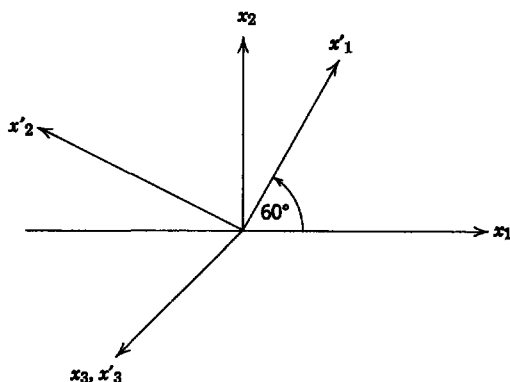


Fig. 8P.3

- (a) You are given the components of a vector $\mathbf{A}$ in the (x_1, x_2, x_3) system: $A_1 = 1$; $A_2 = 2$; $A_3 = -1$. Find the components of $\mathbf{A}$ in the (x'_1, x'_2, x'_3) system by using the transformation law for vectors, $A'_i = a_{ik} A_k$, where a_{ik} is the rotation matrix between (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) .
- (b) A tensor T_{mn} in the (x_1, x_2, x_3) system has elements $T_{11} = 1$, $T_{22} = 2$, $T_{12} = T_{21} = 3$, $T_{33} = 1$, and $T_{13} = T_{31} = T_{23} = T_{32} = 0$. Find the elements of T_{mn} in the (x'_1, x'_2, x'_3) system by using $T'_{ij} = a_{ik} a_{jl} T_{kl}$.

8.4. A system has a stress tensor

$$T_{ij} = \begin{bmatrix} \frac{P_0^2}{2a^2} (x_1^2 - x_2^2) & -\frac{P_0^2}{a^2} x_1 x_2 & 0 \\ -\frac{P_0^2}{a^2} x_1 x_2 & \frac{P_0^2}{2a^2} (x_2^2 - x_1^2) & 0 \\ 0 & 0 & \frac{P_0^2}{2a^2} (-x_1^2 - x_2^2) \end{bmatrix}.$$

Find the volume force density that results from this stress tensor.

8.5. A flat plate of infinite extent is parallel to the x_3 -axis and intersects the x_1 and x_2 -axes, as shown in Fig. 8P.5. In region (2) $\mathbf{E} = 0$, whereas in region (1) the electric field is given by $\mathbf{E} = E_0(\frac{2}{3}\mathbf{i}_1 + \mathbf{i}_2)$. Find the x_1 , x_2 , and x_3 components of the force on the section of the

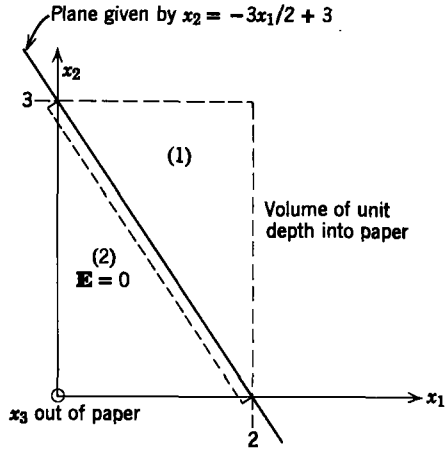


Fig. 8P.5

plate (per unit depth in the x_3 -direction) that extends from the x_1 - to the x_2 -axes. Do this by integrating the Maxwell stress tensor over the surface of the volume shown in Fig. 8P.5, which encloses this section of the plate.

8.6. A pair of parallel insulating sheets is shown in Fig. 8P.6. The sheet at $y = d$ supports a surface charge density $-\sigma_f$, whereas the sheet at $y = 0$ supports the image surface charge density σ_f . Hence the electric field between the plates *due to the charges* is $(\sigma_f/\epsilon_0)\mathbf{i}_y$. External

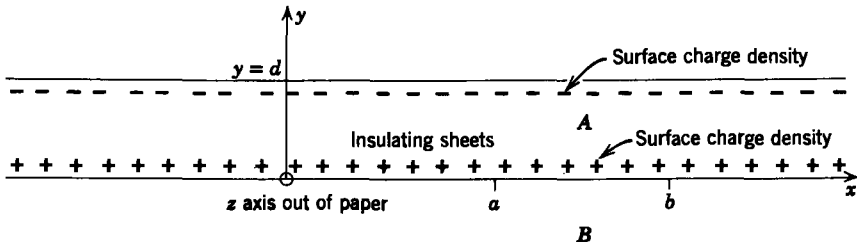


Fig. 8P.6

electrodes are used to impose an additional uniform electric field given everywhere by $\mathbf{E} = E_0\mathbf{i}_x + E_0\mathbf{i}_y$, where E_0 is a constant.

- Write the components of the Maxwell stress tensor at points A and B in terms of σ_f and E_0 .
- Use the Maxwell stress tensor to find the total electric force in each of the coordinate directions on the section of the lower sheet between $x = a$ and $x = b$ having depth D in the z -direction.

8.7. Two perfectly conducting plates are arranged as shown in Fig. 8P.7. A magnetic field trapped between the plates is established in such a way that it does not penetrate the perfectly conducting plates. Also $H_3 = 0$ and $\partial/\partial x_3 = 0$. Under the assumption that $b \ll L$, find the x_1 -component of the force per unit x_3 on the section of the lower plate between $x_1 = L$ and $x_1 = -L$. You may assume that, when $x_1 = -L$, $\mathbf{H} = H_0\mathbf{i}_1$, where H_0 is a known constant.

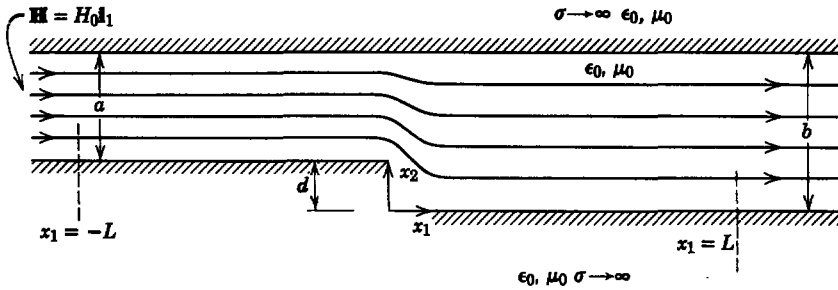


Fig. 8P.7

8.8. Three perfectly conducting plates are arranged as shown in Fig. 8P.8. A potential difference V_0 between the middle electrode and the outer electrodes is shown. Under the assumption that $a \ll l$, $b \ll l$, use the Maxwell stress tensor to find the force on the middle plate in the x -direction. Be sure to give all of your arguments.

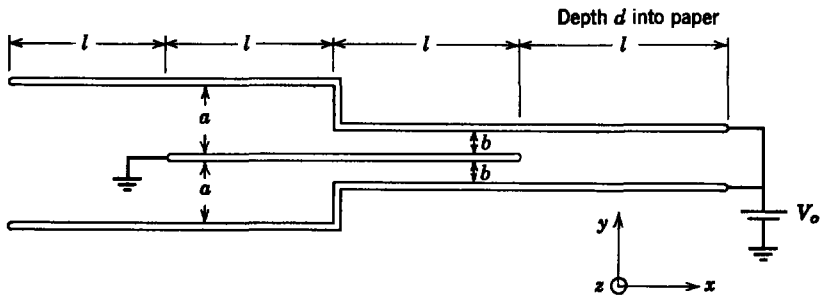


Fig. 8P.8

8.9. Capacitor plates with depth d (into the paper), length l , and spacing s are arranged as shown in Fig. 8P.9. Many of the plates are distributed along the x_1 -axis. The plates have, alternately, the potentials $+V_0$ and $-V_0$, as shown, so that an electric field exists between each pair of them. You are to find the force in the x_2 -direction on the section of plate enclosed by the volume V , which has a depth $w \ll d$ into the paper and encloses a section of the plate centered between its x_3 extremes.

8.10. Figure 8P.10 shows an electromechanical electrostatic voltmeter for measuring the relaxation time in liquids with very long relaxation times. The two outer conducting plates are fixed. The middle plate is constrained by a spring that is relaxed when $x = a$ but otherwise free to move in the x -direction. This plate (mass M) moves in a liquid dielectric of uniform conductivity σ and permittivity ϵ (the ϵ/σ to be measured). The liquid fills the region between the plates.

- Use Maxwell's stress tensor to find the total electric force on the middle plate in the x -direction as a function of the potential v of the middle plate and the position x . (Your answer should be exact, as $[s/(a-x)] \rightarrow 0$.)
- Use the energy method to check the result of part (a).
- The switch S has been closed for a long enough time to establish the middle plate in static equilibrium. Write the equations of motion for the plate position $x(t)$ (as many equations as unknowns) after the switch is opened.

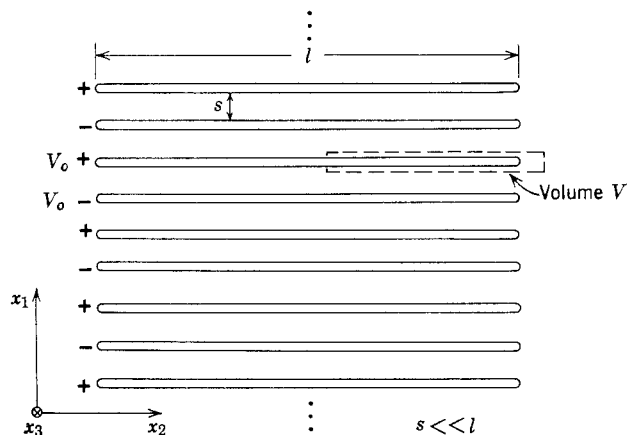


Fig. 8P.9

- (d) Assume that the inertial force on the plate can be ignored (that the plate moves very slowly) and find $x(t)$. Is your assumption that the inertial force can be ignored consistent with the liquid having a very long relaxation time?
- (e) How would you use this device to measure the relaxation time of the liquid?

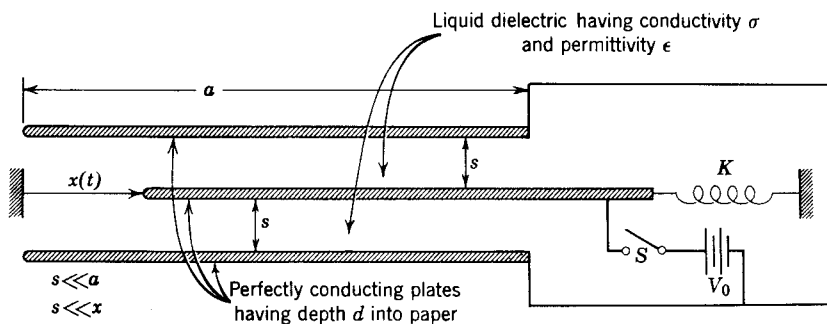


Fig. 8P.10

8.11. Two parallel conducting plates with a potential difference V_0 are shown in Fig. 8P.11. Assuming that $c < b < a \ll l \ll D$ and that the fringing fields are zero at the extreme points A and B , find the force in the x_1 -direction on the lower plate.

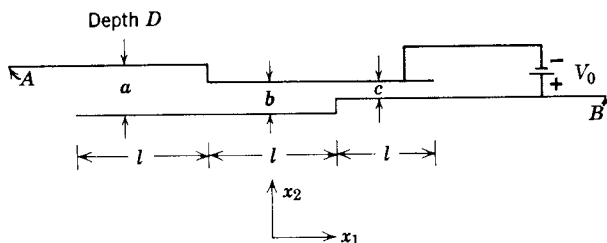


Fig. 8P.11

8.12. In Fig. 8P.12 two parallel perfectly conducting electrodes extend from $x_1 = 0$ to $x_1 = \infty$ and are infinite in the x_3 -direction. The separation of the electrodes in the x_2 -direction is a . A potential $\phi = \phi_0 \sin(\pi/a)x_2$ is established along the x_2 -axis at $x_1 = 0$.

- Find the electric field intensity E everywhere between the plates and sketch.
- Find the total force on the bottom plate per unit depth in the x_3 -direction.
- Find the total force on the top plate per unit depth in the x_3 -direction.

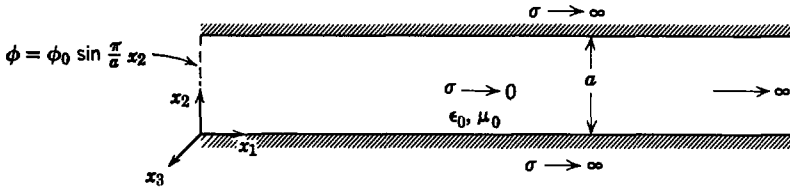


Fig. 8P.12

8.13. In the system in Fig. 8P.13 the geometry of two equipotentials is defined. These equipotentials are maintained at a potential difference V_0 by the battery, and the upper conductor has a movable section ($-a < x_1 < a$), as indicated. The system has a large width w in the x_3 -direction; thus we neglect any variations with x_3 and approximate the potential in the region between the conductors with the expression

$$\phi = \frac{V_0}{3a^2} (x_2^2 - x_1^2 - a^2);$$

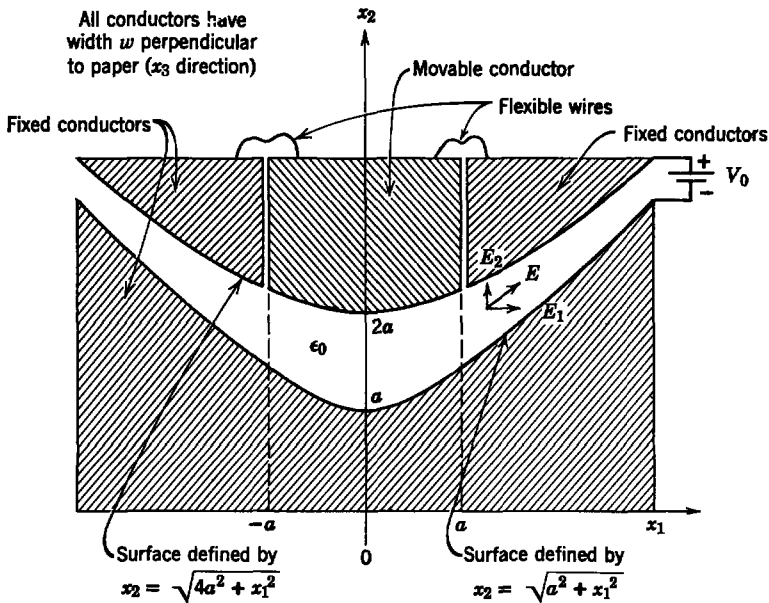


Fig. 8P.13

the two nonzero components of electric field intensity are then

$$E_1 = \frac{2V_0x_1}{3a^2}; \quad E_2 = -\frac{2V_0x_2}{3a^2}.$$

- Find the components T_{22} and T_{21} of the Maxwell stress tensor between the conductors in terms of ϵ_0 , V_0 , a , x_1 , and x_2 .
- Use the stress tensor to find the component f_2 of the force applied to the movable section of the upper conductor ($-a < x_1 < a$) by the electric field. Assume that the movable conductor is held in equilibrium in the position shown by externally applied forces.
- Prove that $f_1 = 0$ by using the stress tensor.
- Find f_2 by using the surface force density written in terms of the surface charge density σ_r (see Section 8.4.2).

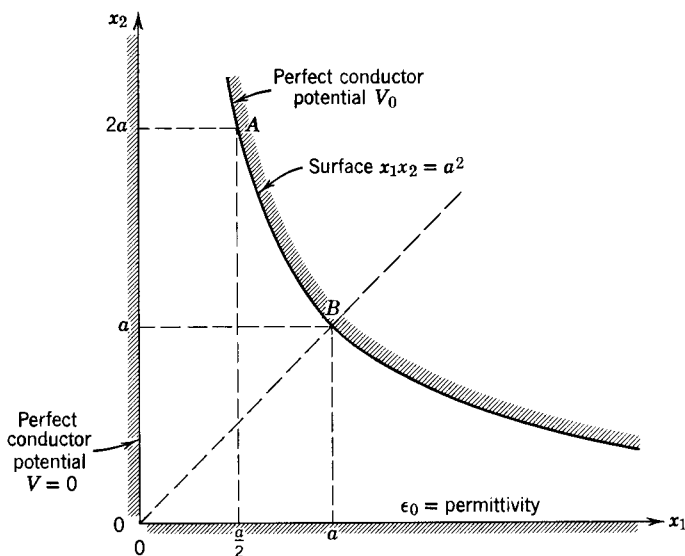


Fig. 8P.14

8.14. Figure 8P.14 shows two equipotential surfaces that are very long in the x_3 -direction. The electric potential is

$$\phi = \frac{V_0}{a^2} x_1 x_2,$$

where $\mathbf{E} = -\nabla\phi$.

- Evaluate all elements of the stress tensor for the region between the perfect conductors.
- Find the total force applied by the field to the segment of the curved conductor between points A and B and having depth D in the x_3 -direction.

8.15. A conducting block moves with the velocity V between plane-parallel, perfectly conducting electrodes, short-circuited as shown in Fig. 8P.15. A uniform magnetic field $\mathbf{H}_0$ is imposed. Ignore the magnetic field induced by currents flowing in the block,

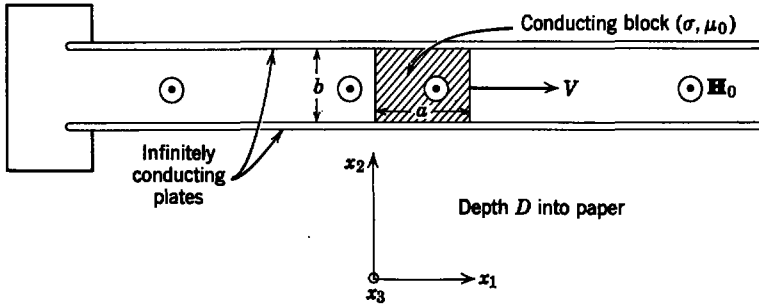


Fig. 8P.15

- (a) Compute the total force on the block using $\mathbf{J} \times \mu_0 \mathbf{H}$.
 (b) Show that in this case the Maxwell stress tensor gives zero force on the block.
 (c) Why do the results of (a) and (b) differ?

8.16. Figure 8P.16 shows a block of conducting material free to slide between two perfectly conducting plates that extend to infinity on the right. The conductivity of the block may be taken as $\sigma = \sigma_0(1 + \sin \pi x/2L)$ and the permeability as $\mu = \mu_0$. The conductivity σ_0 is a

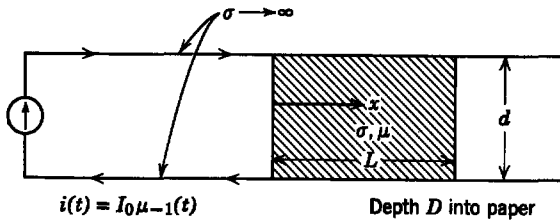


Fig. 8P.16

positive constant and x is the distance from the left-hand edge of the block. Find the total force of electromagnetic origin on the block as a function of time. Assume $d \ll D$, $d \ll L$.

8.17. A slab of conducting material (e.g., graphite) is sandwiched between perfectly conducting plates, as shown in Fig. 8P.17. The dimension a is much smaller than D and the x -dimension is much larger than the skin depth at the frequency ω .

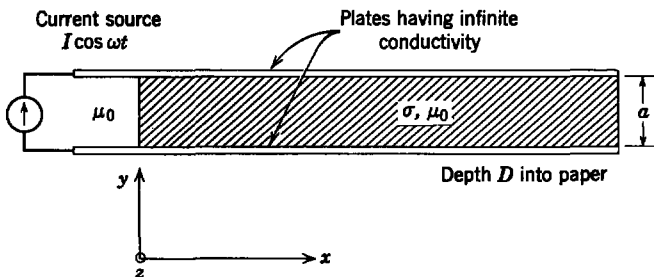


Fig. 8P.17

- (a) Find the steady-state magnetic field $\mathbf{H}$ and current density $\mathbf{J}$ in the slab.
- (b) Compute the total force on the slab in the x -direction by integrating $\mathbf{J} \times \mu_0 \mathbf{H}$ over a volume.
- (c) Compute the total force on the slab in the x -direction by integrating the Maxwell stress tensor over a surface.

8.18. A rigid, perfectly conducting body of arbitrary shape is positioned between two perfectly conducting infinite plates, as shown in Fig. 8P.18. The plates are at a potential

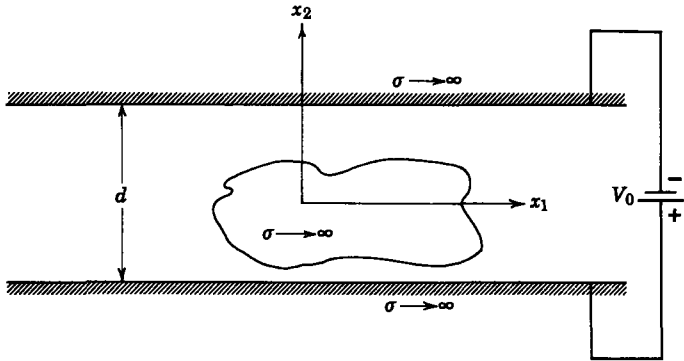


Fig. 8P.18

difference V_0 . Take advantage of the fact that far from the body $\mathbf{E} = i_2(V_0/d)$ to calculate the x_1 -directed force on the body.

8.19. A pair of wires carries the constant current I as shown in Fig. 8P.19. The spacing $2a$ of the wires is much larger than the radius of either wire.

- (a) Use the force density $\mathbf{J} \times \mathbf{B}$ to determine the force on a unit length of the right wire in the x_1 -direction.

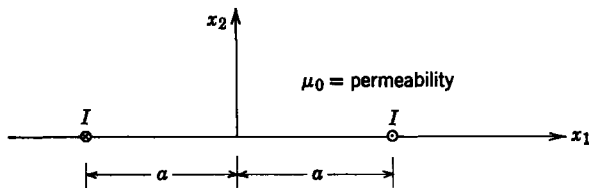


Fig. 8P.19

- (b) Now enclose this section of wire with a convenient surface and integrate the Maxwell stress tensor over the surface to find the force in the x_1 -direction. Compare your answer with that found in (a).
Hint. A "convenient" surface might take advantage of the fact that the fields go to zero as x_1 and $x_2 \rightarrow \infty$ and that $x_1 = 0$ is a plane of symmetry.

8.20. Two line charges of strength $\pm\lambda$ per unit x_3 are located at $x_2 = +a$ and $x_2 = -a$ (see Fig. 8P.20). The line charges extend to $\pm\infty$ in the x_3 -direction.

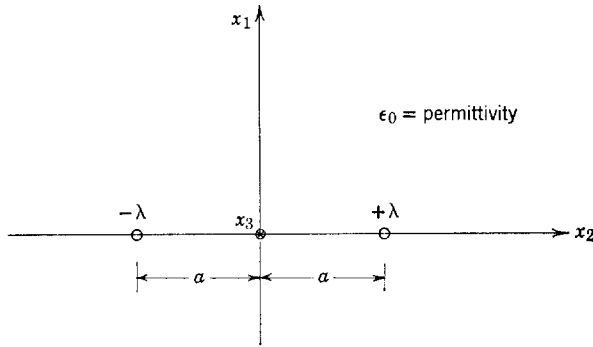


Fig. 8P.20

- (a) Use the Maxwell stress tensor to find the force in the x_2 -direction per unit depth in the x_3 -direction exerted by the electric field on the line charge at $x_2 = +a$.
- (b) Can you think of any other way of computing this force? If so, check it with part (a).

8.21. In Problem 7.14 a vehicle system was proposed in which a magnetic field provided both suspension (i.e., levitation) and propulsion forces. There it was assumed that the condition $ks \ll 1$ is valid and, to calculate the volume force density, $\mathbf{J} \times \mathbf{B}$ was applied. The Maxwell stress tensor provides an alternate and useful method for the calculation of

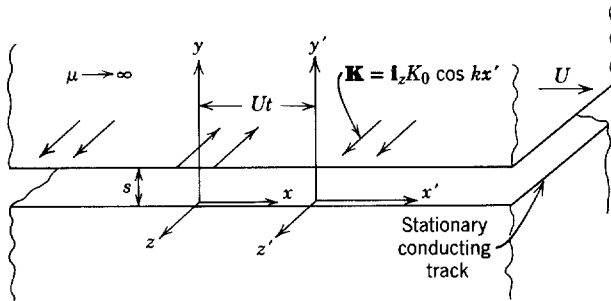


Fig. 8P.21

the forces per unit area (Fig. 8P.21). The solution for the magnetic field in the region $-\infty < y < 0$ is

$$B_x = \text{Re} [\mu_0 K_0 e^{\alpha y} e^{jk(x-Ut)}]$$

and

$$B_y = \text{Re} \left[\frac{-jk\mu_0 K_0}{\alpha} e^{\alpha y} e^{jk(x-Ut)} \right],$$

where

$$\alpha = k \left(1 - j \frac{\mu_0 \sigma U}{k} \right)^{1/2}.$$

- Write the components of the Maxwell stress tensor explicitly in terms of B_x and B_y . Present your results in matrix form.
- Using the stress tensor, compute the time average force per unit area (in the x - z plane) that holds the vehicle up. Take advantage of the periodic variation with x to define a suitable surface.
- Again using the stress tensor, compute the time average force per unit area (x - z) that tends to propel the train.

8.22. A pair of perfectly conducting plane-parallel electrodes is shorted by a bar of metal with conductivity σ (a constant) (Fig. 8P.22). A source of constant current I_0 (amperes) is

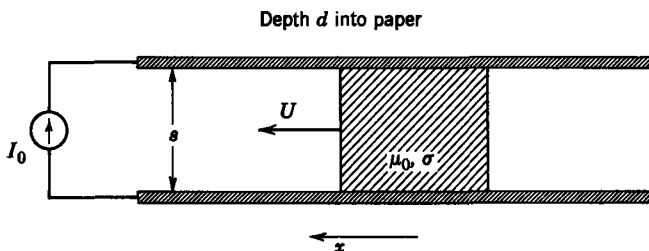


Fig. 8P.22

distributed along the left edges of the plates, and the block moves with the velocity U in the x -direction. What is the magnetic force on the block in the x -direction? Your answer should include the possibility that the magnetic Reynolds number is large or small.

8.23. A pair of perfectly conducting plane parallel electrodes “sandwich” a slab of lossy dielectric of thickness b and a region of free space of thickness $(a - b)$, as shown in Fig. 8P.23. The conductivity of the slab varies in the x -direction, and σ_0 and σ_1 are constants.

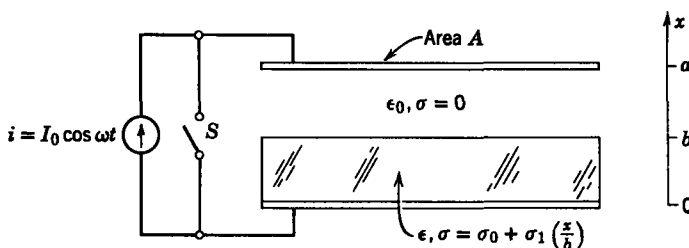


Fig. 8P.23

When $t < 0$, the switch S is closed and no electric fields exist between the plates. When $t = 0$, the switch S is opened. Neglect fringing fields and find the force in the x -direction on the upper plate as a function of time.

8.24. A pair of planar, diverging, perfectly conducting plates has a constant potential difference V_0 and the dimensions shown in Fig. 8P.24. What is the total electrical force on the lower plate in the x -direction? (Note that x is the radial direction half-way between the plates.)

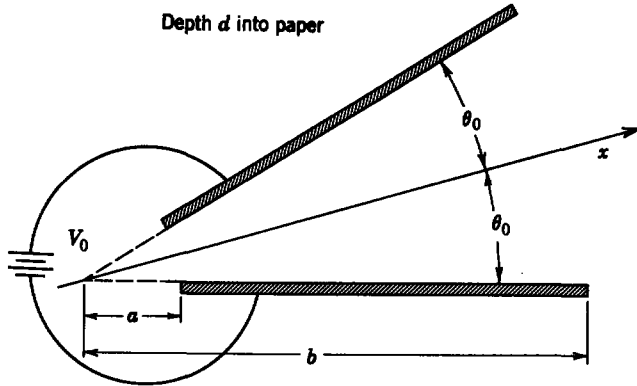


Fig. 8P.24

8.25. The dielectric slab shown in Fig. 8P.25 is free to slide in the x_1 -direction. The upper and lower surfaces of the slab are in contact with perfectly conducting plates. The remaining volume is free space. Find the x_1 -component of force on the slab. Use the Maxwell stress tensor.

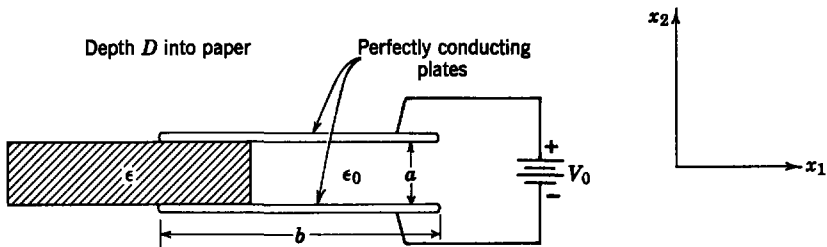


Fig. 8P.25

8.26. An elastic material is placed between two equipotential surfaces with its left-hand edge fixed to a rigid insulating wall, as shown in Fig. 8P.26. The right-hand edge of the elastic bulk is free and the permittivity of the material is a function of its mass density $\epsilon_1 = \epsilon_1(\rho)$. Free space fills the remaining volume. A potential difference (V_0) exists between the two plates.

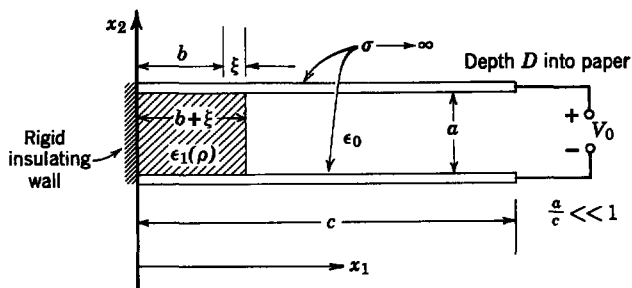


Fig. 8P.26

- (a) Using the Maxwell stress tensor for polarizable material, find the force on the right-hand edge of the elastic bulk.
 (b) Using energy methods, find the force on the right-hand edge of the elastic bulk.
 (c) Compare the answers of parts (a) and (b).

8.27. The force density on a polarized fluid with permittivity $\epsilon(x_1, x_2, x_3, t)$ is $\mathbf{F} = -\frac{1}{2}\mathbf{E} \cdot \mathbf{E} \nabla \epsilon + \frac{1}{2} \nabla(\epsilon \mathbf{E} \cdot \mathbf{E} b)$, where the free charge $\rho_f = 0$, $\nabla \times \mathbf{E} = 0$, and $b = (\rho/\epsilon) (\partial \epsilon / \partial \rho)$ is a parameter that accounts for the electrostriction of the fluid (Fig. 8P.27). The planar

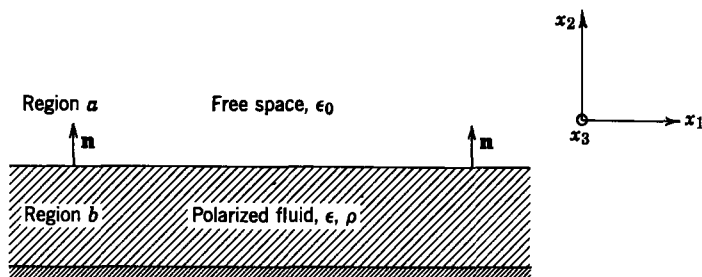


Fig. 8P.27

surface between dielectrics ϵ_0 and ϵ has a normal vector $\mathbf{n}$. Show that the polarization forces alone cannot exert a traction $\boldsymbol{\tau}$ on the interface between the two dielectrics which has a shear component. Remember that $T_m = [T_{mn}^a - T_{mn}^b]n_n$.

8.28. Use the electric force density of (8.5.45) to obtain the stress tensor of (8.5.46).

Chapter 9

SIMPLE ELASTIC CONTINUA

9.0 INTRODUCTION

The study of the effects of motion on electric and magnetic fields (Chapter 7) and of electromagnetic force densities (Chapter 8) provides the background necessary for an introduction to the electromechanics of continuous media. To someone familiar with the dynamics of continuous media this is a pretentious statement, for it implies that the description of distributed mechanical systems requires only a minor addition to the largely electromagnetic considerations so far introduced. In general, this is far from the case; for example, does the mechanical medium consist of a solid or a fluid? In either case the equations of motion vary considerably with the particular fluid or solid under study. These equations generally involve three-dimensional deformations, hence are likely to be at least as complicated as the electromagnetic field equations if not more so.

Fortunately, many of the most significant and practical interactions with continuous media can be modeled in terms of one or two-dimensional structures that not only retain the salient features of the three-dimensional dynamics but represent idealizations that we should like to approach in practice. In this and the next chapter attention is confined to situations in which the mechanical side of the electromechanical problem takes the form of one of two simple models: the thin rod subject to longitudinal motions and wires and membranes undergoing transverse motions. The derivation of the one- and two-dimensional equations of motion for these simple cases serves to illustrate the essential steps required to write the more general expressions for elastic media and fluids, as undertaken in Chapters 11 and 12. At the same time the continuum electromechanical dynamics studied in this and the next chapter give a preview of types of dynamics found in acoustics, fluid dynamics, electron beam-plasma dynamics, magnetohydrodynamics, electrohydrodynamics, and microwave magnetics.

In this chapter the discussion is limited to electromechanical interactions

with continuous media that occur through boundary conditions representable by terminal pairs. In Chapter 10 we consider physical situations in which the electromechanical coupling is itself distributed and in which our lumped parameter concept of a terminal pair can no longer account for the coupling.

9.1 LONGITUDINAL MOTION OF A THIN ROD

Longitudinal motion of a thin elastic rod provides a logical first topic in discussing the dynamics of elastic continua. This is true because we emphasize the wavelike nature of the dynamics; and in a thin rod longitudinal waves have a particularly simple form. As we shall see, waves in a thin rod can propagate without changing their shapes; hence they can be understood by means of comparatively simple mathematical techniques. This distortion-free behavior of the thin rod is used in applications such as acoustic delay lines and electromechanical filters in which the properties of the electromechanical system are especially attractive. We discuss some applications later in this section.

To describe longitudinal motion in an elastic rod we must make a mathematical model. This process consists essentially of two steps: (a) a mathematical description of force equilibrium for a small element of the rod and (b) a description of the elastic property of the rod.

We consider the long thin rod shown in Fig. 9.1.1*a*. The rod has a uniform cross section of area A perpendicular to the longitudinal (x_1)-direction. We apply forces in the x_1 -direction and observe motion in the x_1 -direction. By "thin" we mean that the dimensions of the rod perpendicular to x_1 are small enough that effects of any transverse motion are negligible. The

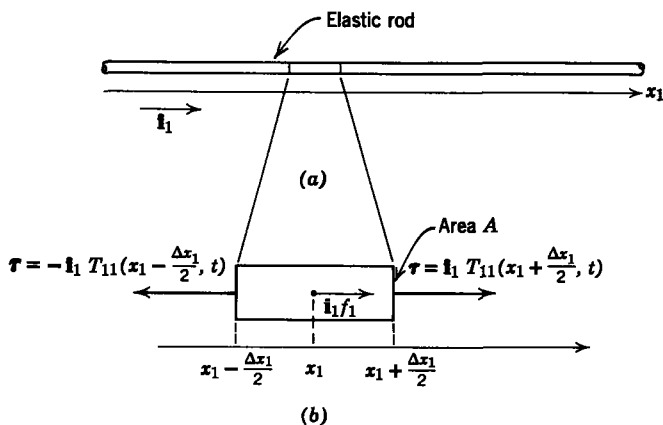


Fig. 9.1.1 Thin elastic rod with axis in the x_1 -direction and uniform cross section of area A : (a) the rod; (b) force and tractions applied to an element of length Δx_1 centered at x_1 .

The traction applied to the right-hand segment by the left-hand segment is

$$\tau = -\mathbf{i}_1 \frac{f}{A}. \quad (9.1.3)$$

We define the stress T_{11} (as in Section 8.2.1) transmitted by the rod as

$$T_{11} = \frac{f}{A}. \quad (9.1.4)$$

Then we obtain the x_1 -component of the mechanical traction τ_1 as

$$\tau_1 = T_{11}n_1, \quad (9.1.5)$$

where n_1 is the magnitude of the x_1 -component of the *outward* directed unit normal vector for the segment of rod to which the traction is applied. For this one-dimensional case, illustrated in Fig. 9.1.2, $n_1 = +1$ for the left-hand segment and $n_1 = -1$ for the right-hand segment. Equation 9.1.5 will be recognized as a simple special case of (8.2.8). Positive stress ($T_{11} > 0$) indicates tension and negative stress ($T_{11} < 0$) indicates compression.

Although our arguments have been based on a static experiment with no body forces applied, we can extend these definitions to the general case in which there are body forces that vary with space (x_1) and time. It is still true that a transverse cut must indicate force equilibrium, but the force transmitted at the cut will not be equal to the force applied at the ends. In this case we specify that the stress T_{11} is a function of space and time $T_{11}(x_1, t)$ and use (9.1.5) to calculate the surface traction applied to an element of material by the adjacent material. Thus the surface tractions are represented in Fig. 9.1.1*b*, and the net force due to the surface tractions, correct to first-order terms in Δx_1 , is

$$\mathbf{i}_1 A \left[T_{11} \left(x_1 + \frac{\Delta x_1}{2}, t \right) - T_{11} \left(x_1 - \frac{\Delta x_1}{2}, t \right) \right] = \mathbf{i}_1 \frac{\partial T_{11}}{\partial x_1} A \Delta x_1. \quad (9.1.6)$$

Note that the right side of (9.1.6) can be interpreted as the mechanical body force density ($\partial T_{11}/\partial x_1$) acting throughout the volume $A \Delta x_1$. The force density $\partial T_{11}/\partial x_1$ is a simple case of the general expression in (8.2.7). Here the stress T_{11} has a mechanical origin.

One of the forces applied to the small element of the rod illustrated in Fig. 9.1.1*b* is the acceleration force. To find this force we need to describe the instantaneous position of the element with respect to the inertial coordinate system (x_1). This is done conventionally by describing the displacement of the element with respect to its position in static equilibrium and with no applied forces. We illustrate it in Fig. 9.1.3. In Fig. 9.1.3*a* the rod is in static equilibrium with no forces applied. Then the element of material labeled *a*

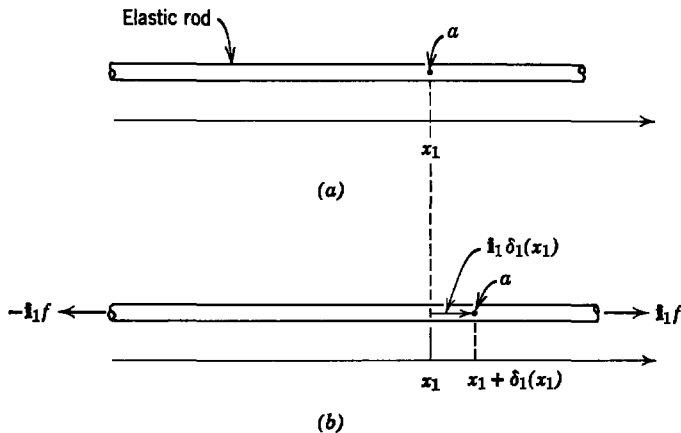


Fig. 9.1.3 Definition of displacement: (a) unstressed rod in static equilibrium; (b) stressed rod indicating definition of displacement δ_1 .

has the position x_1 . In Fig. 9.1.3b we apply forces of magnitude f at the ends of the rod, and it is stretched, thus moving the element a of material to the point $x_1 + \delta_1(x_1)$. With time-varying forces the displacement δ_1 from equilibrium will be the function of both space and time, $\delta_1(x_1, t)$.

Referring back to the element of the rod in Fig. 9.1.1b, we can describe the instantaneous displacement of the element as

$$\delta_1(x_1 - \delta_1, t),$$

that is, the equilibrium position of the matter that is instantaneously at position x_1 is $x_1 - \delta_1$. We now make the assumption, to be justified in Example 9.1.1, that the displacement δ_1 in an elastic material is usually small enough that we can use small-signal, linear differential equations with constant coefficients to describe the motion. Thus we expand the displacement δ_1 in a Taylor series about the value at x_1 and obtain

$$\delta_1(x_1 - \delta_1, t) = \delta_1(x_1, t) - \frac{\partial \delta_1(x_1, t)}{\partial x_1} \delta_1(x_1, t) + \cdots \quad (9.1.7)$$

Usually, δ_1 and its space derivatives are small enough to allow us to neglect all but the first term on the right of (9.1.7). Thus the acceleration of the element centered at x_1 in Fig. 9.1.1b is

$$\frac{\partial^2 \delta_1(x_1, t)}{\partial t^2}.$$

Because the local displacement of the material is small, the fractional change in mass density will be small. Consequently, in the spirit of the

linearized theory we assume that the mass density ρ of the rod is constant and write the x_1 -component of Newton's second law for the element in Fig. 9.1.1*b* as

$$\rho A \Delta x_1 \frac{\partial^2 \delta_1}{\partial t^2} = \frac{\partial T_{11}}{\partial x_1} A \Delta x_1 + F_1 A \Delta x_1; \quad (9.1.8)$$

that is, the inertial force is equal to the mechanical force on the element from adjacent material plus any externally applied body forces. We divide this expression by the volume $A \Delta x_1$ of the element and obtain the desired equation of force equilibrium:

$$\rho \frac{\partial^2 \delta_1}{\partial t^2} = \frac{\partial T_{11}}{\partial x_1} + F_1. \quad (9.1.9)$$

Note that each term in this equation is a force density.

As the second step in finding the equations of motion for a thin elastic rod we introduce the elastic property of the material to relate stress T_{11} and displacement δ_1 . The form of this relation results from a mathematical description of experimental results obtained for a wide variety of elastic materials.

It is found experimentally that the elastic stress depends on how much the material is deformed, the stress increasing as the deformation increases. This is a statement for continuous media, analogous to the statement for lumped-parameter systems, that for an ideal spring the force is dependent on the relative displacement of the ends or on the deformation of the spring (see Section 2.2.1*a*).

To calculate the local deformation in a thin elastic rod we consider two grains of matter labeled *a* and *b* in Fig. 9.1.4. With no applied forces and static equilibrium these grains of matter are at positions x_1 and $x_1 + \Delta x_1$, as indicated in Fig. 9.1.4*a*. When forces are acting on the rod, the two grains of matter will have the positions indicated in Fig. 9.1.4*b*. Our objective is to find a unique relationship between the stress T_{11} and the displacement δ_1 . We expect that the *change* in the distance between the points *a* and *b*,

$$\{[\delta_1(x_1 + \Delta x_1) + x_1 + \Delta x_1] - [\delta_1(x_1) + x_1]\} - \Delta x_1,$$

will be proportional to the applied stress. This change, however, is also proportional to the original distance Δx_1 between points *a* and *b*. To obtain a measure of the elongation that is independent of Δx_1 , we normalize the change in length to the unstretched length and take the limit $\Delta x_1 \rightarrow 0$. The resulting function e_{11} is called the *normal strain* and is

$$e_{11} = \lim_{\Delta x_1 \rightarrow 0} \frac{[\delta_1(x_1 + \Delta x_1) - \delta_1(x_1)]}{\Delta x_1} = \frac{\partial \delta_1}{\partial x_1}. \quad (9.1.10)$$

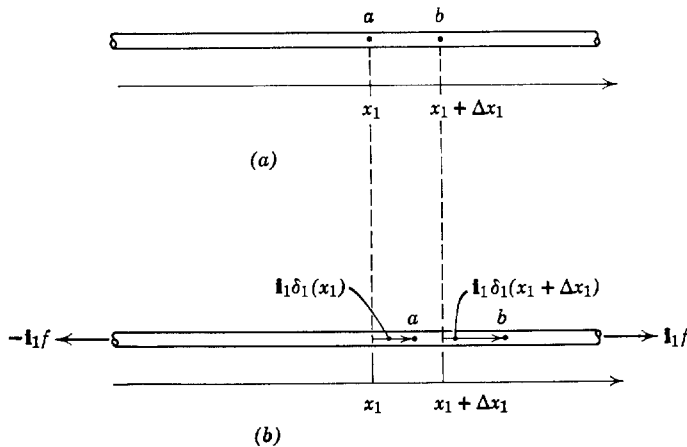


Fig. 9.1.4 Displacements of two adjacent particles: (a) unstretched; (b) stretched.

This geometrical relation is often called the *strain-displacement* relation and has its three-dimensional counterpart derived in Chapter 11.

We define an *ideal* elastic material as one in which the stress T_{11} is a function only of the strain e_{11} . This is analogous to the definition of an ideal spring in Section 2.2.1a. An ideal *linear* elastic material has a linear relation between stress and strain conventionally written as

$$T_{11} = E e_{11}. \quad (9.1.11)$$

The constant of proportionality E is called *Young's modulus* or the *modulus of elasticity* and (9.1.11) is often referred to as *Hooke's law*, or as the *stress-strain relation*. Equation 9.1.11, which introduces the physical properties of the material, is analogous to the constituent relations of electromagnetic theory, as discussed in Section 6.3. The modulus of elasticity E , like ϵ or σ , is found by laboratory measurements.

In our treatment we consider only ideal linear elastic media as described by (9.1.11). It is well to remember that the linear model holds over a limited range of strain and that some materials do not behave linearly.

We can now summarize the equations of motion that we use to describe longitudinal motion in a thin rod of linear elastic material. Force equilibrium is described by (9.1.9) and the stress-displacement equation is obtained by using (9.1.10) in (9.1.11).

$$\rho \frac{\partial^2 \delta_1}{\partial t^2} = \frac{\partial T_{11}}{\partial x_1} + F_1, \quad (9.1.9)$$

$$T_{11} = E \frac{\partial \delta_1}{\partial x_1}. \quad (9.1.12)$$

Table 9.1 Modulus of Elasticity E and Density ρ for Representative Materials*

Material	E -units of 10^{11} N/m^2	ρ -units of 10^3 kg/m^3	v_p -units† of m/sec
Aluminum (pure and alloy)	0.68–0.79	2.66–2.89	5100
Brass (60–70% Cu, 40–30% Zn)	1.0–1.1	8.36–8.51	3500
Copper	1.17–1.24	8.95–8.98	3700
Iron, cast (2.7–3.6% C)	0.89–1.45	6.96–7.35	4000
Steel (carbon and low alloy)	1.93–2.20	7.73–7.87	5100
Stainless steel (18% Cr, 8% Ni)	1.93–2.06	7.65–7.93	5100
Titanium (pure and alloy)	1.06–1.14	4.52	4900
Glass	0.49–0.79	2.38–3.88	4500
Methyl methacrylate	0.024–0.034	1.16	1600
Polyethylene	$1.38\text{--}3.8 \times 10^{-3}$	0.915	530
Rubber	$0.79\text{--}4.1 \times 10^{-5}$	0.99–1.245	46

* See S. H. Crandall, and N. C. Dahl, *An Introduction to the Mechanics of Solids*, McGraw-Hill, New York, 1959, for a list of references for these constants and a list of these constants in English units.

† Computed from average values of E and ρ .

The modulus of elasticity E and density ρ for common solids which can have elastic behavior (for small deformations) are given in Table 9.1. The equations of motion for a thin rod are summarized in Table 9.2 at the end of the chapter. The magnitude of the displacement resulting from a moderate applied stress is small, as illustrated in the following example.

Example 9.1.1. A metal rod is supported at one end by a rigid structure and subjected to a force $f = 100 \text{ lb}$ at the other end (Fig. 9.1.5). Using a rod made of aluminum ($E = 0.7 \times 10^{11} \text{ N/m}^2$) in the dimensions shown, we wish to find the increase in length caused by the force f . The weight of the rod is small compared with f and can be neglected.

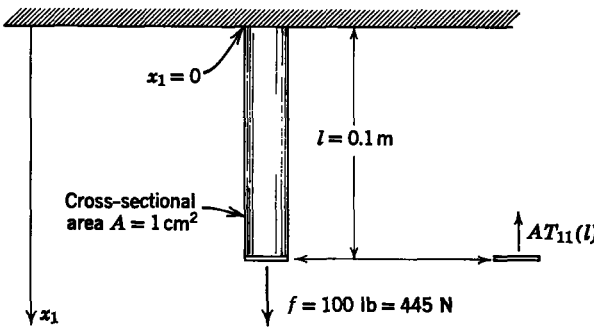


Fig. 9.1.5 Metal rod fixed at $x_1 = 0$ and subject to the force f at $x_1 = l$.

The rod is static. Hence (9.1.9) and (9.1.12) yield, with $F_1 = 0$,

$$\frac{d^2\delta_1}{dx_1^2} = 0. \quad (a)$$

with the solution

$$\delta_1 = Cx_1 + D. \quad (b)$$

Because $\delta(0) = 0$, the constant $D = 0$. The remaining constant is found from the boundary condition on the stress at $x_1 = l$; that is, the force equilibrium of a thin slice of the rod at $x_1 = l$ (see Fig. 9.1.5) requires that

$$AT_{11}(l) = f = AE \frac{d\delta_1}{dx_1}(l), \quad (c)$$

where A is the cross sectional area of the rod. Equations b and c show that $C = f/AE$ and that the displacement evaluated at $x_1 = l$, where it has its largest value, is

$$\delta_1 = \frac{fl}{EA} = \frac{(445)(0.1)}{(0.7 \times 10^{11})(10^{-4})} = 6.36 \times 10^{-6} \text{ m}, \quad (d)$$

or about 2.5×10^{-4} in. Note that although this displacement is extremely small it can be made arbitrarily large by increasing the length of the rod. It is the rate of change of the displacement, or the stress, that must be small if the linear stress-strain relation is to remain valid.*

9.1.1 Wave Propagation Without Dispersion

We consider a case in which the body force density F_1 in (9.1.9) is zero. Then (9.1.9) and (9.1.12) yield†

$$\frac{\partial^2 \delta}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 \delta}{\partial x^2}. \quad (9.1.13)$$

This is called the *wave equation* because it has solutions of the general form

$$\delta = \delta_+(x - v_p t) + \delta_-(x + v_p t), \quad (9.1.14)$$

where

$$v_p = \left(\frac{E}{\rho} \right)^{1/2}, \quad (9.1.15)$$

which can be verified by substituting (9.1.14) into (9.1.13). The function δ_+ represents a wave traveling in the $+x$ -direction and the function δ_- represents a wave traveling in the $-x$ -direction.

To an observer traveling with a velocity such that the phase (or argument) of δ_+ is constant the function δ_+ will have a space variation that does not vary with time. The required velocity is found by setting

$$x - v_p t = \text{constant} \quad (9.1.16)$$

* A discussion of inelastic behavior is given in most texts on the mechanics of solids; for example, G. Murphy, *Mechanics of Materials*, Ronald, New York, 1948, p. 23.

† In what follows the subscripts used in the preceding section are dropped. In the one-dimensional problems to be considered the subscripts are not needed.

and differentiating with respect to time to obtain

$$\frac{dx}{dt} = v_p. \quad (9.1.17)$$

Thus the observer must be traveling in the positive x -direction at the *phase velocity* v_p . Note that the phase velocity is the same for all x and all t . This justifies the interpretation of the function δ_+ as a wave traveling in the positive x -direction.

Similar reasoning shows that an observer must travel in the negative x -direction with speed v_p to observe a constant spatial distribution of δ_- .

Phase velocities for rods made of representative elastic materials are given in Table 9.1.

Because the waves δ_+ and δ_- propagate with constant speed and do not change their shape (or disperse) with time, they are referred to as non-dispersive. For any given problem the functions δ_+ and δ_- are determined by initial conditions and boundary conditions. This is illustrated with simple examples. In the process we introduce techniques that will prove useful in later sections in which the wave propagation will not be so simple as in the thin rod.*

9.1.1a Wave Propagation and Characteristics

We first consider the dynamics of a thin elastic rod of infinite length with general initial conditions given by

$$v(x, 0) = \frac{\partial \delta}{\partial t}(x, 0) = v_o(x), \quad (9.1.18)$$

$$T(x, 0) = E \frac{\partial \delta}{\partial x}(x, 0) = T_o(x), \quad (9.1.19)$$

* The reader may be familiar with waves in transmission lines, which are fully analogous to those considered here. To see this, note that (9.1.9) and (9.1.12) can be written (with $F_1 = 0$) as

$$\rho \frac{\partial v}{\partial t} = \frac{\partial T}{\partial x}; \quad \frac{\partial T}{\partial t} = E \frac{\partial v}{\partial x},$$

which are to be compared with the equations

$$L \frac{\partial I}{\partial t} = - \frac{\partial V}{\partial x}; \quad \frac{\partial V}{\partial t} = - \frac{1}{C} \frac{\partial I}{\partial x},$$

where I and V are the transmission line voltage and current and L and C are the inductance and capacitance per unit length. A discussion of wave transients on transmission lines is given in R. B. Adler, L. J. Chu, and R. M. Fano, *Electromagnetic Energy Transmission and Radiation*, Wiley, New York, 1960, p. 127.

where v is the local velocity $\partial\delta/\partial t$. Note that the two independent initial conditions necessary for the solution of the second-order differential equation (9.1.13) are specified as two independent derivatives of δ . The initial conditions can be specified in other ways.

In what follows we find it convenient to replace the two independent variables x and t by two new independent variables α and β defined by

$$\alpha = x - v_p t, \quad (9.1.20)$$

$$\beta = x + v_p t. \quad (9.1.21)$$

Thus we write (9.1.14) as

$$\delta = \delta_+(\alpha) + \delta_-(\beta), \quad (9.1.22)$$

and we use the definition of velocity v as $\partial\delta/\partial t$ and (9.1.12) to write the velocity and stress in terms of δ_+ and δ_- as

$$v(x, t) = -v_p \left[\frac{d\delta_+}{d\alpha} - \frac{d\delta_-}{d\beta} \right], \quad (9.1.23)$$

$$T(x, t) = E \left[\frac{d\delta_+}{d\alpha} + \frac{d\delta_-}{d\beta} \right]. \quad (9.1.24)$$

It is useful to view the behavior on an x - t plane, as illustrated in Fig. 9.1.6. Formally, we wish to find the values of v and T at any point (x, t) for $t > 0$, given the values of v and T at $t = 0$ (along the x -axis in Fig. 9.1.6). To achieve

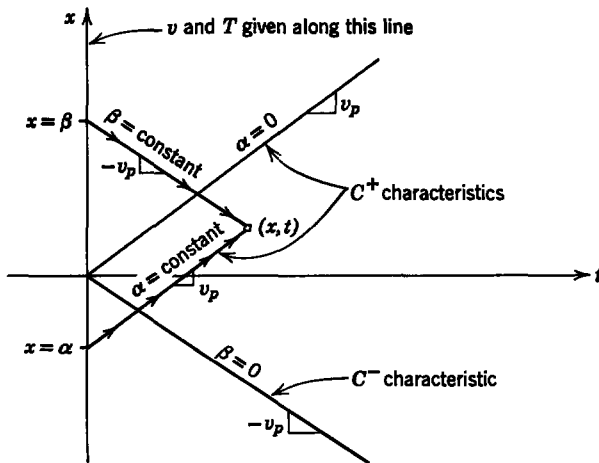


Fig. 9.1.6 The characteristic lines (9.1.20) and (9.1.21) in the x - t plane showing the C^+ and C^- characteristics that intersect at the point (x, t) .

this we solve (9.1.23) and (9.1.24) to obtain

$$\frac{d\delta_+}{d\alpha}(\alpha) = \frac{1}{2} \left[\frac{T(x, t)}{E} - \frac{v(x, t)}{v_p} \right], \quad (9.1.25)$$

$$\frac{d\delta_-}{d\beta}(\beta) = \frac{1}{2} \left[\frac{T(x, t)}{E} + \frac{v(x, t)}{v_p} \right]. \quad (9.1.26)$$

The left side of (9.1.25) is a function of α alone; consequently, for a particular value of α , $d\delta_+/d\alpha$ is constant. We find the value of the constant by recognizing that at $t = 0$, $x = \alpha$ (9.1.20), and the constant value of (9.1.25) is determined by using the initial conditions of (9.1.18) and (9.1.19) thus:

$$\frac{d\delta_+}{d\alpha}(\alpha) = \frac{1}{2} \left[\frac{T_o(\alpha)}{E} - \frac{v_o(\alpha)}{v_p} \right]. \quad (9.1.27)$$

In a similar manner we note that the left side of (9.1.26) is a function of β alone. For any value of β we determine $d\delta_-/d\beta$ by noting that at $t = 0$, $x = \beta$ (9.1.21) and using the initial conditions of (9.1.18) and (9.1.19) to obtain

$$\frac{d\delta_-}{d\beta}(\beta) = \frac{1}{2} \left[\frac{T_o(\beta)}{E} + \frac{v_o(\beta)}{v_p} \right]. \quad (9.1.28)$$

The value of T (or v) can now be found at any point (x, t) in the plane of Fig. 9.1.6 by using the facts that $d\delta_+/d\alpha$ is constant along a path of constant α and $d\delta_-/d\beta$ is constant along a path of constant β . As indicated by (9.1.20) and (9.1.21), $\alpha = \text{constant}$ and $\beta = \text{constant}$ are straight lines in the x - t plane. All lines of constant α are parallel with a positive slope of v_p and the line for $\alpha = 0$ passes through the origin as indicated. All lines of constant β are parallel with a slope $-v_p$ and the $\beta = 0$ line passes through the origin.

The lines $\alpha = \text{constant}$ and $\beta = \text{constant}$ in the x - t plane are called *characteristics*.* Because α is the argument of δ_+ , we refer to the family of lines $\alpha = \text{constant}$ as the C^+ characteristics. Similarly, the family of lines representing $\beta = \text{constant}$ are called the C^- characteristics.

A particular point (x, t) is the intersection of one C^+ and one C^- characteristic, as illustrated in Fig. 9.1.6. The particular values of α and β are given by (9.1.20) and (9.1.21) for the values of x and t at the point in question. Hence we can find the value of T or v at any point in the x - t plane by using these values of α and β in (9.1.27) and (9.1.28) and those results in (9.1.23) and (9.1.24) to find the stress $T(x, t)$ and the velocity $v(x, t)$; for example, the stress is found to be

$$T(x, t) = \frac{E}{2} \left\{ \left[\frac{T_o(x - v_p t)}{E} - \frac{v_o(x - v_p t)}{v_p} \right] + \left[\frac{T_o(x + v_p t)}{E} + \frac{v_o(x + v_p t)}{v_p} \right] \right\}. \quad (9.1.29)$$

* R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves*, Interscience, New York, 1948, Chapter II.

Physically we have found that the instantaneous value of the stress T (or velocity v) at the point (x, t) is determined by the initial ($t = 0$) values of stress and velocity at the positions $x = \alpha$ and $x = \beta$ along the rod. The initial conditions at $x = \alpha$ propagate (along the C^+ characteristic) in the positive x -direction with the velocity v_p and reach the point x under observation at the time t at which the measurement is to be made. Similarly, initial conditions at $x = \beta$ propagate (along the C^- characteristic) in the negative x -direction with velocity v_p and reach the point x at time t . Thus the values of T and v at (x, t) depend on the initial conditions at only two points. This is a property of nondispersive waves.

Before we consider a particular example we make one further observation. There is no mathematical reason why we could not find a solution to (9.1.13) for points to the left of the x -axis in Fig. 9.1.6. We could make an argument similar to the one just given to find the values of $d\delta_+/d\alpha$ and $d\delta_-/d\beta$ at a point $(x, t < 0)$ by following the characteristics from the x -axis to the point in question. In doing so, however, we would have assumed that the data at $t = 0$ can determine the dynamics before $t = 0$; that is, we would have made the present depend on the future. Implicit to our solution is the assumption, based on independent physical reasoning, that in terms of time the cause must come before the effect. When this physical reasoning is used to discriminate between solutions, we invoke the condition of causality.* We shall find it necessary to make further use of this condition to provide physically meaningful initial conditions and boundary conditions.

Example 9.1.2. As a special case, we consider the motions of the thin rod shown in Fig. 9.1.7. External forces are applied at the cross sections $x = a$ and $x = -a$ to produce an initial stress T_m over the length $-a < x < a$. With the rod in a static condition ($v = 0$), these forces are removed to give the initial stress distribution shown in Fig. 9.1.8. In this figure the x - t plane forms the "floor" of a three-dimensional plot, where the stress T provides the vertical axis. Hence at $t = 0$ the stress distribution is uniform along the x -axis between the points $x = \pm a$ and zero elsewhere. Because the initial conditions and

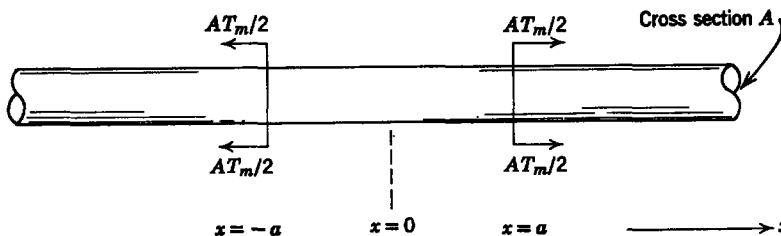


Fig. 9.1.7 Thin rod subject to an initial uniform static stress T_m over the section $-a < x < a$. At $t = 0$ the external forces $AT_m/2$ are removed.

* See, for example, P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, New York, 1953, p. 834.

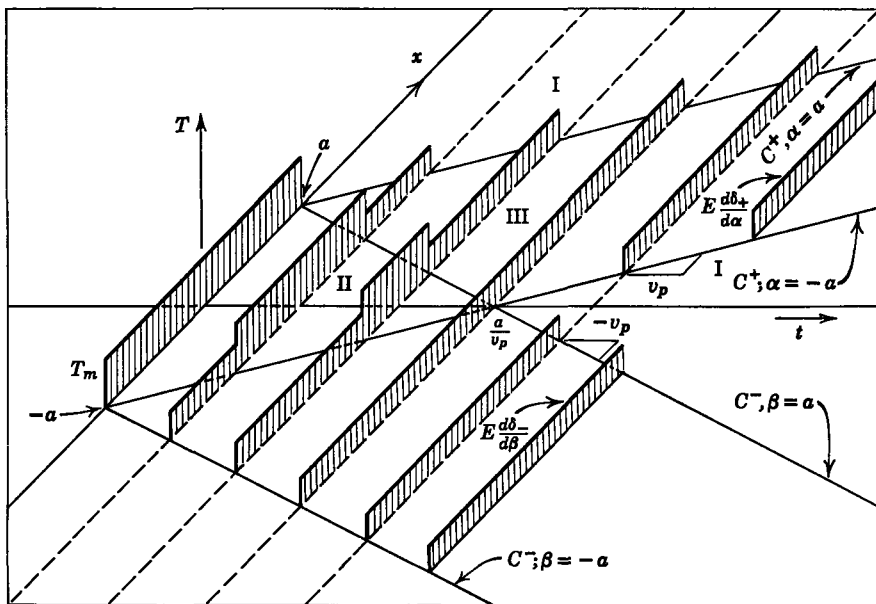


Fig. 9.1.8 Stress distribution T as a function of x at succeeding instants in time. When $t = 0$, the stress is uniform between $x = a$ and $x = -a$ and zero elsewhere; the velocity is zero.

the characteristics are symmetrical about the t -axis, we confine our attention to the half of the x - t plane in which $x \geq 0$.

The C^+ and C^- characteristics intersecting the x -axis at $x = \pm a$ are shown plotted in the x - t plane. We see that these particular characteristics divide the $x > 0$ half of the x - t plane into three types of regions, labeled I, II, and III in Fig. 9.1.8. These regions have the following properties:

- I The characteristics that cross at the point (x, t) originate on the x -axis, where $T = 0$ ($x > a, x < -a$). There are two of these regions.
- II The characteristics that cross at the point (x, t) originate on the x -axis, where $T = T_m$ ($-a < x < a$).
- III Of the characteristics that cross at the point (x, t) , C^+ originates when $t = 0$ where $T = T_m$ ($-a < x < a$), and C^- originates where $T = 0$ ($x > a$).

From (9.1.27) and (9.1.28) it follows that in

Region I

$$\frac{d\delta_+}{d\alpha} = 0, \quad \frac{d\delta_-}{d\beta} = 0.$$

Region II

$$\frac{d\delta_+}{d\alpha} = \frac{T_m}{2E}, \quad \frac{d\delta_-}{d\beta} = \frac{T_m}{2E}.$$

Region III

$$\frac{d\delta_+}{d\alpha} = \frac{T_m}{2E} \quad \frac{d\delta_-}{d\beta} = 0.$$

The stress distribution now follows from (9.1.24), and this is plotted at succeeding instants of time in Fig. 9.1.8. The edge of the initial stress distribution at $x = a$ propagates along the C^+ and C^- characteristics originating at $x = a$, and similarly the edge at $x = -a$ propagates along the C^+ and C^- characteristics originating at $x = -a$. In Region I the front edge of the forward traveling wave has not had time to arrive from $x = a$, hence the stress is still zero. In Region II the backward traveling wave from $x = a$ and the forward traveling wave from $x = -a$ have not had time to arrive and the stress is still T_m . In Region III, however, the forward wave has arrived from $x = a$ but the backward wave from $x = -a$ has not yet arrived. For time $t > a/v_p$ the waves are two separate pulses propagating with the velocity v_p in the $+x$ and $-x$ directions, respectively.

We have followed the development of the waves graphically to encourage a physical understanding of the relationship between the characteristics and the wave propagation. If we required an analytical result only, (9.1.29) could be used with the initial conditions

$$\begin{aligned} T_0(x) &= T_m[u_{-1}(x+a) - u_{-1}(x-a)], \\ v_0(x) &= 0, \end{aligned}$$

where $u_{-1}(x+a)$ is a unit step function defined as

$$\begin{aligned} 1 & \text{ for } x > -a, \\ 0 & \text{ for } x < -a, \end{aligned}$$

to obtain the result

$$T = \frac{T_m}{2} [u_{-1}(x - v_p t + a) - u_{-1}(x - v_p t - a) + u_{-1}(x + v_p t + a) - u_{-1}(x + v_p t - a)].$$

This expression is the same as that found with our graphical solution. When the initial conditions are given as complicated analytical functions of x , the analytical approach is more convenient than the graphical approach.

Attention has so far been confined to the dynamics near the center of a very long rod. In an actual rod the waves shown in Fig. 9.1.8 will eventually encounter ends or boundaries. The resulting dynamics are the subject of the next subsection.

9.1.1b Wave Reflection at a Boundary

Constraints imposed on the ends of the rod enter the mathematical description as boundary conditions; for example, an end may be free, as shown in Fig. 9.1.9a, in which case force equilibrium (for a thin slice of the material at the very end) requires that the instantaneous stress be zero. More obviously, if the end is fixed (Fig. 9.1.9b), the velocity must always be zero. In general, the ends can be attached to springs, masses, and dampers, or, as we shall see in Section 9.1.2, they can be excited by electromechanical transducers which act essentially as dependent mechanical sources.

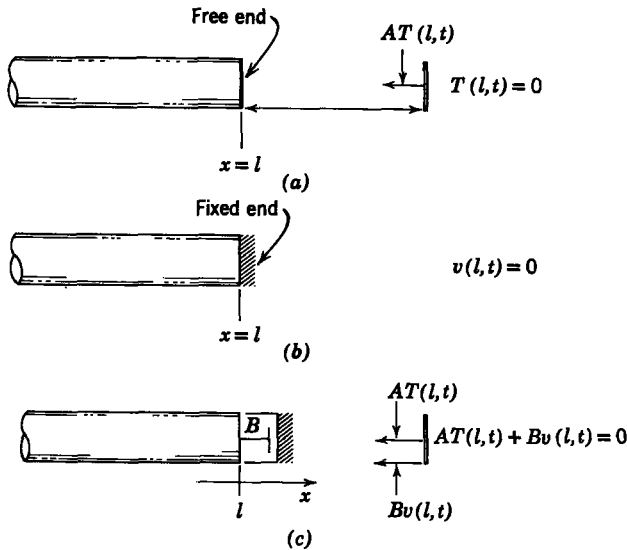


Fig. 9.1.9 Simple boundary conditions on the end of a thin rod: (a) free end; (b) fixed end; (c) end attached to a damper producing a total force Bv .

Force equilibrium on the end of an elastic rod attached to a linear damper, as illustrated in Fig. 9.1.9c, yields a boundary condition of the form

$$v(l, t) + CT(l, t) = 0, \quad (9.1.30)$$

where C is a constant ($C = A/B$ in Fig. 9.1.9c). This expression can also be used in its limiting forms to represent fixed and free end conditions; for example, if $C = 0$ (an infinitely stiff damper), we have the fixed end condition ($v = 0$) in Fig. 9.1.9b, whereas if $C \rightarrow \infty$ (limit of zero damping constant B) the free end condition ($T = 0$) in Fig. 9.1.9a results. A boundary condition of the form of (9.1.30) is used to illustrate the influence of boundary conditions on the dynamic behavior of a thin elastic rod.

We indicated in (9.1.14) that the motion in the rod is specified at any point by two waves, δ_+ propagating in the $+x$ -direction and δ_- which propagates in the $-x$ -direction. We further pointed out that the functions δ_+ and δ_- are determined by initial and boundary conditions. A wave that encounters a boundary is reflected; thus a forward wave δ_+ becomes a backward wave δ_- at a boundary. The relation between the incident and reflected waves depends on the boundary condition, as expressed by (9.1.30).

In Section 9.1.1a we learned that the δ_+ and δ_- waves propagate with constant amplitude along the C^+ and C^- characteristics. Hence a point in the x - t plane, such as A , shown in Fig. 9.1.10, is unaffected by the boundaries because it is the intersection of characteristics that do not originate on the

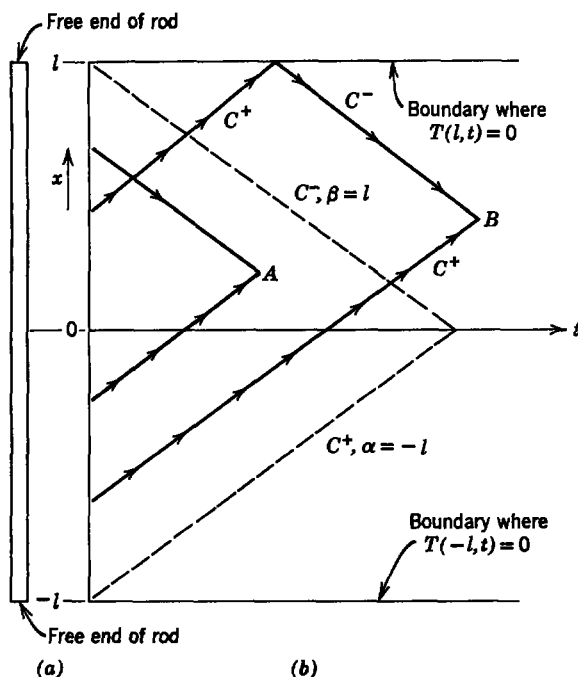


Fig. 9.1.10 (a) Thin rod of length $2l$ centered at $x = 0$; (b) an x - t plot showing the characteristics relevant to the effect of the ends on the dynamics.

boundaries. At points such as B , however, outside the cone formed by the C^- characteristic $\beta = l$ and the C^+ characteristic $\alpha = -l$, one or both of the intersecting characteristics C^+ and C^- originates on a boundary; for example, the values of T and v at the point B shown in Fig. 9.1.10 are determined by a C^+ characteristic originating on the initial conditions at $t = 0$ and a C^- characteristic originating on the boundary at $x = l$. Hence we must use the boundary condition to determine the value of $(d\delta_-/d\beta)(\beta)$ along the C^- characteristic. To do this we set $x = l$ and substitute (9.1.23) and (9.1.24) into (9.1.30) and solve for $d\delta_-/d\beta$:

$$\frac{d\delta_-}{d\beta}(\beta) = -\frac{d\delta_+}{d\alpha}(\alpha) \left(\frac{CE - v_p}{CE + v_p} \right). \quad (9.1.31)$$

In this equation $d\delta_+/d\alpha$ is the value for the incident wave and thus is determined for this problem by the initial condition at $t = 0$, $x = \alpha$. As indicated by (9.1.31), the boundary condition and the incident wave determine completely the value of the reflected wave that propagates along the C^- characteristic. Analogous arguments can be made at the boundary $x = -l$, where the δ_- wave reflects as a δ_+ wave.

and the boundary condition 9.1.30 is

$$v(l, t) + \frac{1}{\sqrt{E\rho}} T(l, t) = 0. \quad (9.1.33)$$

One way in which this boundary condition can be obtained is shown in Fig. 9.1.9c, in which the rod is terminated in a viscous damper with a constant B . Force equilibrium for the end of the rod is the same as condition (9.1.33) if

$$B = A\sqrt{\rho E}; \quad (9.1.34)$$

that is, if the viscous damper has this coefficient, an incident wave will not be reflected*

Example 9.1.4. We can illustrate the significance of the boundary condition given by (9.1.33) by considering the dynamics that result if the end of a static rod is given the excitation $T(0, t) = T_0(t)$, as shown in Fig. 9.1.12. Because all the C^- characteristics either originate on the x -axis (at a time when there is no motion and no stress in the rod) or on the boundary at $x = l$, where no reflected waves can arise because of the boundary condition, we conclude that $d\delta_-/d\beta$ is zero everywhere in the portion of the x - t plane pertinent to the problem (Fig. 9.1.12). We can evaluate $d\delta_+/d\alpha$ at $x = 0$ from the excitation condition and (9.1.24); that is, the C^+ characteristic originating at $t = t'$ is given by [see (9.1.20)] $\alpha = -v_p t'$, hence we can write

$$\left. \frac{d\delta_+}{d\alpha} \right|_{\alpha = -v_p t'} = \frac{T_0(t')}{E}. \quad (a)$$

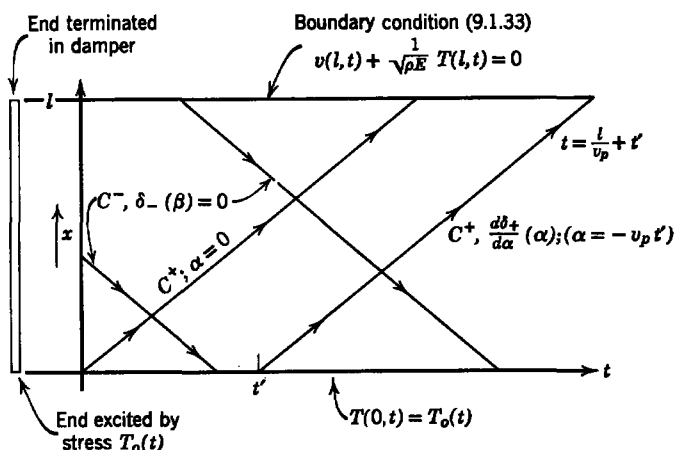


Fig. 9.1.12 Excitation $T_0(t)$ at one end of a thin rod transmitted to a matched end at $x = l$.

* In the terminology of transmission line theory we say that the termination is "matched" to the rod or that the rod is terminated in its "characteristic impedance." See R. B. Adler, L. J. Chu, and R. M. Fano, *Electromagnetic Energy Transmission and Radiation*, Wiley, New York, 1960, pp. 88-90.

It follows that along the characteristic

$$\alpha = -v_p t' = x - v_p t \quad (b)$$

we have

$$T(x, t) = T_o(t'). \quad (c)$$

Equation b relates t' to t and allows us to write (c) as

$$T(x, t) = T_o\left(t - \frac{x}{v_p}\right). \quad (d)$$

In particular, at the end of the rod where $x = l$,

$$T(l, t) = T_o\left(t - \frac{l}{v_p}\right). \quad (e)$$

As we expected, we have found that a signal $T_o(t)$, introduced on the rod at the end where $x = 0$, appears at the opposite end delayed by the time l/v_p , or the time required for the signal (d) to travel the length of the rod. With the boundary condition of (9.1.33), a pulse introduced at one end will travel the length of the rod and leave no after effects in the form of reflections.

Wave propagation on a thin rod with a boundary condition in the form of (9.1.33) will play a basic role in the electromechanical delay line described in Section 9.1.2.

9.1.2 Electromechanical Coupling at Terminal Pairs

One of the most important ways in which coupling occurs between electric or magnetic fields and continuous media is through the boundary conditions. In the one-dimensional motions considered in this section the boundaries can be described in terms of the displacement (or velocity) and the stress evaluated at a fixed point in space (x). Because these boundary variables are only functions of time, they form a mechanical terminal pair; for example, if the end of the rod is at $x = 0$, the terminal pair of Fig. 9.1.13*b* can be used to describe the boundary condition applied to the thin rod in Fig. 9.1.13*a*.

Lumped-parameter electromechanical devices are often coupled to mechanical terminal pairs formed from boundary variables in much the same way as discussed in Chapters 2 and 5. As an example, Fig. 9.1.13*a* shows a plunger attached to the end of the rod (at $x = 0$, say). This plunger is subject to a force of electrical origin, as shown, and has the position $y(t)$. Other forces acting on the plunger are the forces $AT(0, t)$ from the attached rod and an inertial force. Within an arbitrarily defined constant, the displacement at the end of the rod is y or $y(t) = \delta(0, t)$.

Figure 9.1.13*b* formalizes the mechanical terminal pair. We write the force equilibrium equation as

$$M \frac{d^2 \delta(0, t)}{dt^2} = AT(0, t) + f^e. \quad (9.1.35)$$

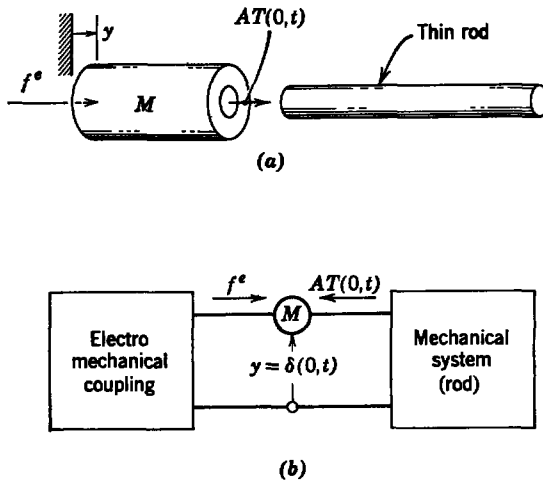


Fig. 9.1.13 Electromechanical coupling at the end of a thin rod: (a) physical system; end of rod attached to mass M acted on by the force of electric origin f^e ; (b) formal representation.

Note that, in general, f^e will involve the displacement $\delta(0, t)$ and electrical variables such as currents. The force equation (9.1.35) is the boundary condition presented by the coupling network to the distributed mechanical system. Its significance is demonstrated in the following example.

Example 9.1.5. Transmission systems that support nearly nondispersive waves are required to transmit a signal with a minimum of distortion. As we have pointed out, electromagnetic transmission lines have much the same dynamical behavior as the elastic rod that is the subject of this section. Because it takes a finite time for waves to propagate from one end of these systems to the other, a common application is to the production of time delays.

Acoustic waves propagate with velocities that are on the order of 4000 m/sec, as shown for various materials in Table 9.1. By contrast, electromagnetic waves propagate with velocities on the order of the speed of light in free space (3×10^8 m/sec). Hence the mechanical waves are useful in producing long time delays* (on the order of 10^{-3} sec). If, however, an electrical signal is required, it is necessary to use electromechanical coupling at the input and output of the mechanical structure. One system is shown in Fig. 9.1.14a. The input signal is the current $i_i(t)$ applied to the terminals of the transducer to the left. By proper design this current produces an electrical force on the left end of the elastic rod that is essentially proportional to the current i_i . This force is transmitted in the form of a stress wave to the right end of the rod, where it produces motion of the magnetic plunger in the output transducer hence an induced voltage $v_o(t)$. The conductance G and inductance L of the terminal pair (i_2, λ_2) are adjusted to absorb the transmitted wave without producing a reflected wave traveling in the $-x$ -direction. In this way the system is designed so that v_o is proportional to $i_i(t)$ delayed by l/v_p sec.

* See, for example, W. P. Mason, ed., *Physical Acoustics*, Academic, New York, 1964, Vol. 1, Part A, Chapters 6 and 7.

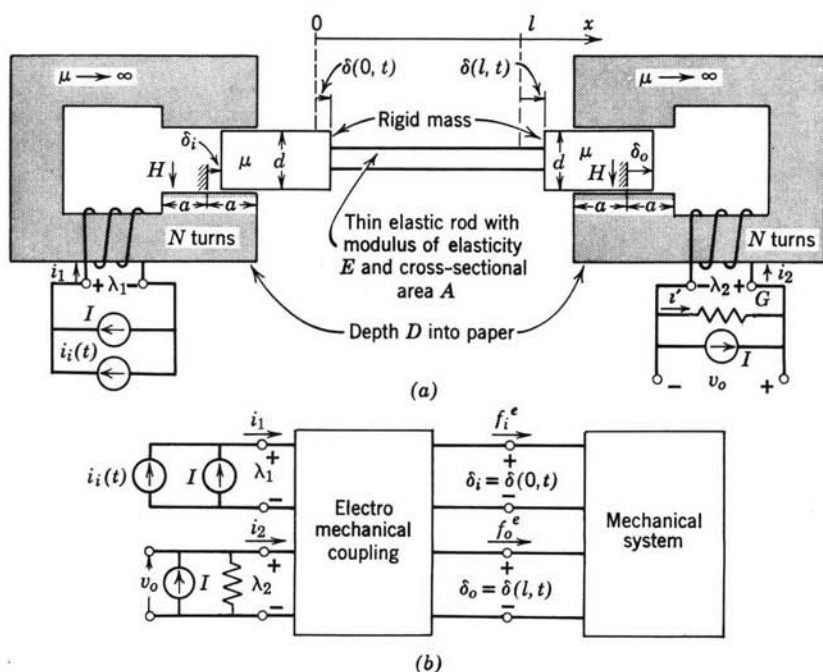


Fig. 9.1.14 (a) Electromechanical delay line designed to give an output signal $v_o(t)$ which is proportional to $i_i(t)$ delayed by l/v_p sec; (b) circuit representation of (a).

We begin by finding the force f_i^e of electrical origin on the plunger of the input transducer. This is a simple application of the ideas introduced in Chapters 2 and 3*. The magnetic field intensity in the gap of the magnetic circuit is assumed to be uniform so that in the gap

$$H = \frac{Ni_1}{d}. \quad (a)$$

Hence the flux density through the plunger is B_1 and through the air gap, B_2 , where

$$B_1 = \frac{\mu Ni_1}{d}, \quad (b)$$

$$B_2 = \frac{\mu_0 Ni_1}{d}.$$

The terminals (i_1, λ_1) link the total flux through the gap N times and we can write

$$\lambda_1 = N[(a - \delta_i)DB_1 + (a + \delta_i)DB_2], \quad (c)$$

where

$$\delta_i = \delta(0, t).$$

* Essential equations summarized in Tables 2.1 and 3.1, Appendix E.

We substitute (a) and (b) into (c) and arrange the result in the form

$$\lambda_1 = L_0 \left[\left(\frac{\mu + \mu_0}{\mu - \mu_0} \right) - \left(\frac{\delta_i}{a} \right) \right] i_1, \quad (d)$$

where

$$L_0 = \frac{N^2 D a (\mu - \mu_0)}{d} \quad (e)$$

The transducer is electrically linear; hence*

$$W'_m = \frac{1}{2} L_0 \left(\frac{\mu + \mu_0}{\mu - \mu_0} - \frac{\delta_i}{a} \right) i_1^2, \quad (f)$$

and†

$$f_i^e = \frac{\partial W'_m}{\partial \delta_i} = - \frac{L_0 i_1^2}{2a}. \quad (g)$$

Now, we want this force to be proportional to the driving current i_i , and this is the purpose of the biasing current I . From the circuit in Fig. 9.1.14 we write

$$(i_1)^2 = (I + i_i)^2 \approx I^2 + 2Ii_i, \quad (h)$$

where we are assuming that the bias current I is large enough to justify dropping the term i_i^2 . In practice, the bias field produced by I may be obtained from a permanent magnet placed in the magnetic circuit. The equivalence of the current excitation and the permanent magnet is discussed in Section 2.1.1.

We are now able to write the force equation for the plunger, which we recognize as (9.1.35) with $\delta = \delta_i$. A further approximation, justified by our design requirements, is made at this point. If we wish to make the stress $T(0, t)$ in (9.1.35) proportional to the applied force (hence to the input current), we must design the system with the mass M small enough to make the inertia force negligible under the desired operating conditions. This approximation becomes less accurate as the frequency is raised. The inertia force is a factor to be considered if the fidelity of the delay line is to be explored in detail.

With the assumption of *negligible inertia force*, (9.1.35) becomes

$$AT(0, t) - \frac{L_0}{2a} (I^2 + 2Ii_i) = 0, \quad (i)$$

where we have used (g) and (h) to write the force f_i^e . From this expression it is clear that the stress $T(0, t)$ will have a constant part due to the bias current I and a time-varying part due to the signal current i_i . Thus we write

$$T(0, t) = T_s + T'(0, t), \quad (j)$$

where

$$T_s = \frac{L_0 I^2}{2aA} \quad (k)$$

$$T'(0, t) = \frac{L_0 I}{aA} i_i(t). \quad (l)$$

The output transducer is identical to the input transducer and has the same bias current I . Consequently, under equilibrium conditions the output transducer applies a force AT_s to the end of the rod at $x = l$ equal in magnitude and opposite in direction to the force applied

* Equation k, Table 3.1, Appendix E.

† Equation g, Table 3.1, Appendix E.

at $x = 0$ by the input transducer. The result of this equilibrium stress T_s is a slight elongation of the rod, very much as described in Example 9.1.1. Our equations of motion are linear; thus we can superimpose the displacements due to T_s and T' . Because we are interested only in the response to T' , we ignore the equilibrium elongation due to T_s^* and assume that displacements δ_i and δ_o are the increments of displacement due to the driving signal $T'(0, t)$.

As stated at the outset, we wish to have no reflected waves at $x = l$; consequently, we must make (9.1.33) the boundary condition at $x = l$. We now specify the properties of the output transducer that are necessary to achieve this end. Because

$$v(l, t) = \frac{d\delta_o}{dt},$$

the desired form of the boundary condition is

$$\frac{d\delta_o}{dt} + \frac{1}{\sqrt{\rho E}} T(l, t) = 0. \quad (m)$$

We write the equation of motion for the plunger of the output transducer

$$M \frac{d^2\delta_o}{dt^2} = f_o^e - AT(l, t). \quad (n)$$

The inertia force must be negligible under the desired operating conditions to achieve the boundary condition of (m). For this case (n) reduces to

$$f_o^e - AT(l, t) = 0. \quad (o)$$

Next, we recognize that the two transducers are identical except for the definition of the plunger displacement. Thus we obtain the properties of the output transducer from (d) to (g) by replacing δ_i with $-\delta_o$ and i_1 with i_2 . The force f_o^e is

$$f_o^e = \frac{L_0 i_2^2}{2a}. \quad (p)$$

We write

$$i_2(t) = I + i'(t), \quad (q)$$

with $|i'(t)| \ll I$. Then, dropping equilibrium terms from (o), we can write the incremental (time-varying) boundary condition as

$$\frac{L_0 I}{a} i' - AT'(l, t) = 0. \quad (r)$$

From Fig. 9.1.14 we recognize that the current i' is that flowing through the conductance G and is therefore given by

$$i' = -G \frac{d\lambda_2}{dt} \quad (s)$$

y analogy with λ_1 (d), λ_2 is

$$\lambda_2 = L_0 \left(\frac{\mu + \mu_0}{\mu - \mu_0} + \frac{\delta_o}{a} \right) (I + i'),$$

*Care must be exercised in generalizing this assumption; for example, if the force f^e is dependent on δ_i (as it is not in this example) and the rod is very long, the equilibrium displacement can affect the behavior markedly. In any such case, however, a correct analysis can be obtained by exercising care in linearizing the force f^e in terms of equilibrium and perturbation variables.

and (s) can be written, correct to linear terms in time-varying quantities, as

$$i' = -G \left[L_0 \left(\frac{\mu + \mu_0}{\mu - \mu_0} \right) \frac{di'}{dt} + \frac{L_0 I}{a} \frac{d\delta_o}{dt} \right]. \quad (t)$$

It is clear from (m) and (r) that the current i' must be proportional to $d\delta_o/dt$ if (m) is to be satisfied. Consequently, the output transducer must be operated in a regime such that

$$\frac{L_0 I}{a} \frac{d\delta_o}{dt} \gg L_0 \left(\frac{\mu + \mu_0}{\mu - \mu_0} \right) \frac{di'}{dt}$$

Assuming that this condition is satisfied, (t) becomes

$$i' = - \frac{GL_0 I}{a} \frac{d\delta_o}{dt} \quad (u)$$

and (m) and (r) become identical when

$$\frac{Aa^2}{GL_0^2 I^2} = \frac{1}{\sqrt{\rho E}}. \quad (v)$$

With the parameters thus adjusted, the conductance G absorbs the incident wave in the same way that the mechanical damper absorbed the incident wave in Section 9.1.1b.

With the driving stress $T'(0, t)$ given by (l) and with no reflected waves at $x = l$, the stress $T'(l, t)$ is

$$T'(l, t) = \frac{L_0 I}{aA} i_i \left(t - \frac{l}{v_p} \right), \quad (w)$$

where we have used the relation $T'(l, t) = T'(0, t - l/v_p)$ as shown in Example 9.1.4. The use of this result in r yields

$$i'(t) = i_i \left(t - \frac{l}{v_p} \right) \quad (x)$$

and the output voltage is

$$v_o = - \frac{i'(t)}{G} = - \frac{i_i(t - l/v_p)}{G} \quad (y)$$

Thus, with identical transducers and no reflected waves at $x = l$, the current i' is simply the driving current delayed by a time interval l/v_p and the output voltage $v_o(t)$ is a delayed replica of the input current $i_i(t)$.

In a practical device that uses wave propagation in an elastic material to obtain a time delay both electrical and mechanical damping are normally needed to obtain a matched condition and no reflections. Also, most practical electromechanical delay lines use magnetostrictive or piezoelectric transducers rather than the simple ones of our example.

9.1.3 Quasi-statics of Elastic Media

In the example of Fig. 9.1.14 the ends of the elastic rod are attached to plungers. In the analysis of Example 9.1.5 it is assumed that the plungers can be modeled as rigid masses but that the rod is deformable. Presumably, both the rod and the plungers are constructed of materials that exhibit elastic properties; consequently, the assumption is justified when signal transmission

(elastic wave propagation) through the plungers requires a time that is short compared with the time of transmission through the rod. In an intuitive way we recognize this as the condition that the plungers must be made of "stiffer" material than the rod; or, if both plungers and rod are made of the same material, the rod must be much longer than the plungers.

In this section we use the thin rod to illustrate the criteria that must be met in order to use lumped parameter models (see Section 2.2) for bodies made of elastic materials. The justification for lumped-parameter mechanical models is similar to the justification for using lumped-parameter electric circuit models. Hence our arguments in this section are similar to those presented in Section B.2.2.

Equations 9.1.9 and 9.1.12 are the equations of motion for the rod, which we write here in terms of the velocity $v(x, t) = \partial\delta/\partial t$ (with the body force density $F_1 = 0$):

$$\frac{\partial T}{\partial x} = \rho \frac{\partial v}{\partial t}, \quad (9.1.36)$$

$$\frac{\partial v}{\partial x} = \frac{1}{E} \frac{\partial T}{\partial t}. \quad (9.1.37)$$

If we have truly static solutions (a better name is time-independent solutions), we set the time derivatives equal to zero in (9.1.36) and (9.1.37) and obtain

$$\frac{\partial T}{\partial x} = 0, \quad (9.1.38)$$

$$\frac{\partial v}{\partial x} = 0. \quad (9.1.39)$$

Thus for static or steady systems the velocity v and stress T are independent of space x and time t , the values of v and T being determined by the boundary conditions.

The essence of a quasi-static analysis is the assumption that the static solutions are still valid with a time-varying excitation. The steady solutions are then used with the time derivatives on the right of (9.1.36) and (9.1.37) to calculate correction terms for T and v or to evaluate the accuracy of the approximation.

The quasi-static behavior of the thin rod is highly dependent on constraints imposed by boundary conditions. Two limiting cases (boundary conditions required by a fixed or a free end) result in systems in which the static solution for v or T is zero. In these cases single lumped-parameter elements can be used to represent the rod dynamics.

There is a complete analogy between the quasi-static behavior of the thin rod and the electromagnetic quasi-statics of plane-parallel electrodes driven

at one end and terminated in either an open circuit or a short circuit at the other. These electromagnetic problems are discussed in Appendix B (Section B.2.2), in which they are used to show the relationship between the quasi-static magnetic and electric field systems.

9.1.3a The Spring

Figure 9.1.15a shows a thin rod of cross-sectional area A , modulus of elasticity E , mass density ρ , and unstretched length l attached to a fixed support at $x = 0$ and driven by a force $f(t)$ at $x = l$. It is clear that for a static system ($f = \text{constant}$) the velocity v is zero and the stress T is uniform and given by

$$T(x) = \frac{f}{A}. \quad (9.1.40)$$

We now assume that this solution is still valid when the force is time varying; thus

$$T(x, t) = \frac{f(t)}{A} \quad (9.1.41)$$

To calculate the velocity v that results from this time-varying force we must use (9.1.41) in (9.1.37) to obtain

$$\frac{\partial v}{\partial x} = \frac{1}{EA} \frac{df}{dt}. \quad (9.1.42)$$

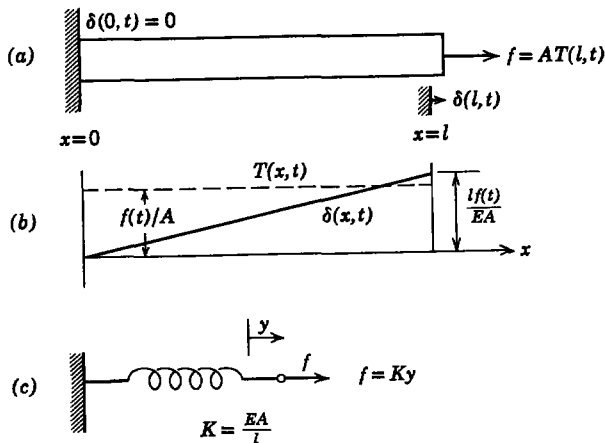


Fig. 9.1.15 (a) Thin elastic rod fixed at $x = 0$ and driven by $f(t)$ at $x = l$ showing (b) the quasi-static distribution of stress and displacement along the rod and (c) the equivalent lumped-parameter element.

Integration of this expression with respect to x and use of the boundary condition

$$v = 0 \quad \text{at} \quad x = 0$$

yields

$$v = \frac{x}{EA} \frac{df}{dt}. \quad (9.1.43)$$

We integrate this expression with respect to time and recognize that with $f = 0$, $\delta = 0$ to obtain

$$\delta(x, t) = \frac{x}{EA} f(t). \quad (9.1.44)$$

Thus, when we make the quasi-static approximation, the stress T and displacement δ are distributed along the rod, as illustrated in Fig. 9.1.15*b*.

We set

$$y(t) = \delta(l, t)$$

and write (9.1.44) as

$$y = \frac{1}{K} f, \quad (9.1.45)$$

where

$$K = \frac{EA}{l}.$$

This is the terminal relation of the spring illustrated in Fig. 9.1.15*c*. Thus we conclude that in the quasi-static approximation an elastic rod with a fixed end appears to a driving force at the other end as a massless spring.

It is worthwhile to explore the limitations on this ideal lumped-parameter model by evaluating correction terms that result from variations in stress caused by the time-varying velocity (9.1.36). This process is analogous to the evaluation of correction terms in the examples of Section B.2.2. We define the correction term for the stress as $T'(x, t)$ and write (9.1.36), using (9.1.43), as

$$\frac{\partial T'}{\partial x} = \frac{\rho x}{EA} \frac{d^2 f}{dt^2}. \quad (9.1.46)$$

We integrate with respect to x and use the boundary condition that $T' = 0$ at $x = l$ because the static solution for T accounts for the applied force. The result is

$$T' = \frac{\rho}{2EA} (x^2 - l^2) \frac{d^2 f}{dt^2}. \quad (9.1.47)$$

This correction term has a maximum magnitude at $x = 0$. Using this maximum value, we conclude that the quasi-static solution is valid, provided that

$$\left| \frac{T'}{T} \right| = \frac{\rho l^2}{2E} \frac{|d^2 f/dt^2|}{|f|} \ll 1. \quad (9.1.48)$$

We can interpret this result more effectively if we assume that

$$f = F_0 \cos \omega t.$$

Then (9.1.48) becomes

$$\left| \frac{T'}{T} \right| = \frac{l^2 \omega^2}{2v_p^2} \ll 1, \quad (9.1.49)$$

where the phase velocity $v_p = \sqrt{E/\rho}$. The wavelength of a longitudinal elastic wave of frequency ω and phase velocity v_p is

$$\lambda = \frac{2\pi v_p}{\omega}.$$

Thus we write (9.1.49) as

$$\frac{l^2 \omega^2}{2v_p^2} = 2\pi^2 \frac{l^2}{\lambda^2} \ll 1 \quad (9.1.50)$$

and conclude that the quasi-static approximation is valid, provided that the length of the rod is much shorter than an elastic wavelength at the frequency of interest. The condition of (9.1.48) can also be interpreted for transient systems by saying that the time of transmission of an elastic wave over the rod length l must be short compared with the shortest characteristic time of the driving force if the quasi-static approximation is to be valid.

9.1.3b The Mass

When the elastic rod is not fixed at $x = 0$, as it was in Fig. 9.1.15, but has a free end at $x = 0$ as shown in Fig. 9.1.16a, the quasi-static model is a

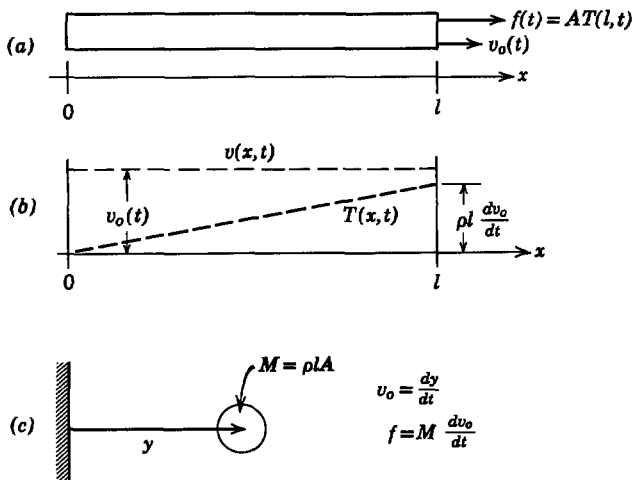


Fig. 9.1.16 (a) Thin elastic rod with free end at $x = 0$ and with end at $x = l$ driven by $v_o(t)$; (b) quasi-static distribution of stress and velocity; (c) equivalent lumped element.

rigid mass. This can be shown by specifying that at $x = l$ the rod is driven by a velocity source

$$v(l, t) = v_o(t) \quad (9.1.51)$$

For a steady solution with $v_o = \text{constant}$, the stress T is zero and the velocity is constant along the rod

$$v(x) = v_o \quad (9.1.52)$$

In a manner analogous to that of the preceding section, we now assume that v_o is time-varying but that (9.1.51) still describes the velocity distribution in the rod

$$v(x, t) = v_o(t). \quad (9.1.53)$$

We now use this velocity in (9.1.36) to write

$$\frac{\partial T}{\partial x} = \rho \frac{dv_o}{dt}, \quad (9.1.54)$$

which determines the stress. Integration of this expression and use of the free end condition ($T = 0$ at $x = 0$) yields

$$T(x, t) = \rho x \frac{dv_o}{dt} \quad (9.1.55)$$

The resulting quasi-static stress and velocity distributions are shown in Fig. 9.1.16b.

Evaluation of the total force supplied by the velocity source yields

$$f(t) = M \frac{dv_o}{dt}, \quad (9.1.56)$$

where $M = \rho l A$ is the total mass of the rod. This is the equation of motion for an ideal rigid mass for which the lumped element is given in Fig. 9.1.16c.

We could use (9.1.37) to evaluate a correction term in velocity and find the limit of accuracy of the quasi-static model. The process, however, is the same as that illustrated in the preceding section and the result, for an excitation frequency ω , is that given by (9.1.50). Thus we conclude that to model an elastic rod as a rigid mass the characteristic time of the motion must be long compared with the time taken for an elastic wave to travel from one end of the rod to the other.

Note that, because elastic waves propagate much less rapidly than electromagnetic waves, lumped-parameter mechanical models are likely to be inadequate at frequencies at which lumped electrical elements are an excellent approximation.

9.2 TRANSVERSE MOTIONS OF WIRES AND MEMBRANES

Among the most common structures used in connection with electromechanical systems are those that can be modeled as thin sheets or wires of elastic material subject to a large equilibrium tension. Acoustic devices are often characterized by lumped-parameter transducers coupled to wires or membranes (diaphragms). Current-carrying conductors under tension (and especially in the presence of large external magnetic fields) present continuum electromechanical problems that assume practical significance. These models also provide attractive vehicles for demonstrating many basic concepts, techniques, and phenomena of continuum electromechanics which have found application in more sophisticated configurations than are appropriate in our treatment. These applications are pointed out in the development.

The system to be considered is shown for equilibrium conditions in Fig. 9.2.1. The elastic sheet or membrane lies in the x - y plane ($z = 0$) and is assumed to be very thin in the z -direction*. It is stressed by a constant tension S (newtons per meter) applied along all four edges in the x - y plane. Thus the total force applied in the y -direction at the right-hand edge of the membrane is $S \Delta x$. The membrane has a surface mass density σ_m (kilograms per square meter).

We now wish to constrain the membrane of Fig. 9.2.1 in arbitrary ways along the edges, apply an arbitrary transverse force per unit area T_z , and describe the resulting motion. We assume that this transverse motion is small enough in amplitude that we can use a linear mathematical model. For such

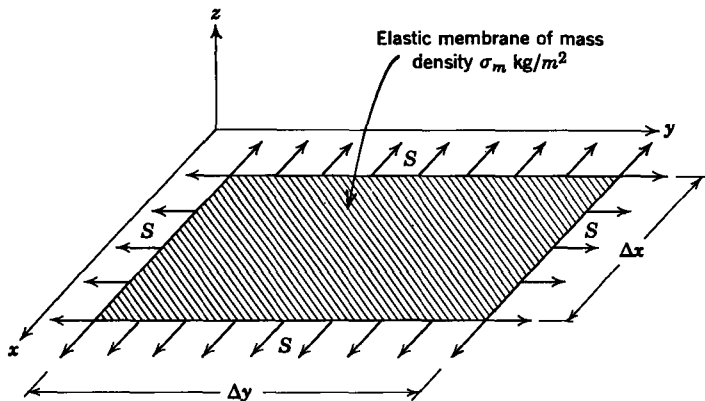


Fig. 9.2.1 A plane-elastic membrane in equilibrium subject to a tension S N/m along its edges.

* The mathematical model to be developed also describes accurately the motion of a thin sheet of fluid (bubble) whose dynamics are affected strongly by surface tension.

a case we find that the motion is independent of the elastic properties of the membrane but depends on the equilibrium tension S .

When the membrane is subjected to transverse (z -directed) excitations, it will undergo transverse motion. This motion is described by the transverse displacement $\xi(x, y, t)$ from equilibrium ($z = 0$). Thus to write the equation of motion we consider a rectangular section of membrane, with sides Δx and Δy and whose center is at position (x, y) , as illustrated in Fig. 9.2.2. We write the z -component of Newton's second law for this section and take the limit as Δx and Δy go to zero.

As stated earlier, the mathematical model is linear; consequently, we assume that the transverse displacement and its derivatives are small enough to justify the following assumptions:

1. The tension S is locally parallel to the surface of the membrane and constant in magnitude, independent of deformation.
2. The surface mass density σ_m is constant, independent of deformation.

With these assumptions we refer to Fig. 9.2.2 and write the z -component of Newton's second law as

$$\begin{aligned} \Delta x \Delta y \sigma_m \frac{\partial^2 \xi}{\partial t^2} = & S \Delta x \left[\frac{\partial \xi}{\partial y} \left(x, y + \frac{\Delta y}{2}, t \right) - \frac{\partial \xi}{\partial y} \left(x, y - \frac{\Delta y}{2}, t \right) \right] \\ & + S \Delta y \left[\frac{\partial \xi}{\partial x} \left(x + \frac{\Delta x}{2}, y, t \right) - \frac{\partial \xi}{\partial x} \left(x - \frac{\Delta x}{2}, y, t \right) \right] + T_z \Delta x \Delta y. \end{aligned} \quad (9.2.1)$$

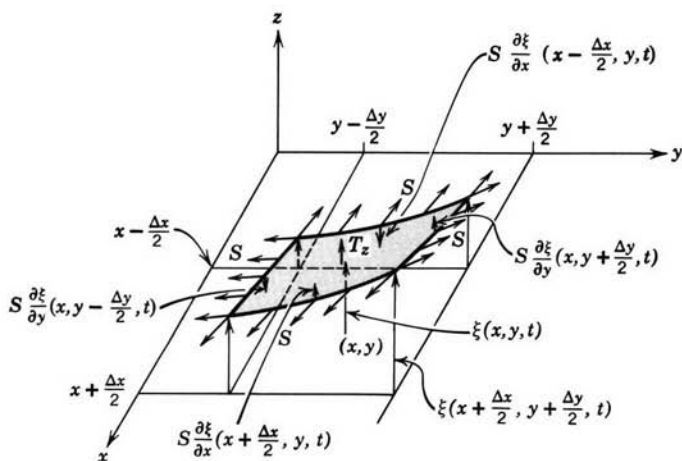


Fig. 9.2.2 Section of membrane having area $(\Delta x \Delta y)$ and subject to the uniform tension S . The displacement at the center of the section (x, y) is $\xi(x, y, t)$.

Division of this equation by the element of area $\Delta x \Delta y$ and taking the limit as $\Delta x \rightarrow 0$ and $\Delta y \rightarrow 0$ yield the desired result:

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right) + T_z. \quad (9.2.2)$$

Note that we have used essentially the same steps in deriving the equation of motion for the membrane as we used in Section 9.1 for the thin rod.

We recognize that the membrane can be excited by discrete terminal pairs (boundary conditions) or over the whole surface by the surface force density $T_z(x, y, t)$. In most cases considered in this book the surface force density T_z is of electrical origin and described mathematically as in Section 8.4. Attention is confined in this chapter to the case in which $T_z = 0$ and the membrane is excited through boundary conditions.

In the case in which the membrane is very thin in the y -direction or in which the deflection ξ does not depend on y , (9.2.2) becomes

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \frac{\partial^2 \xi}{\partial x^2} + T_z. \quad (9.2.3a)$$

If we multiply this equation by a y -dimension l_y , $l_y \sigma_m$ is the mass per unit length, Sl_y is the total tension (newtons), and $T_z l_y$ is the z -component of an externally applied force per unit length. Written in this way, (9.2.3a) is also the equation of motion for a wire (or a "string") under large tension and constrained to move in only one transverse direction. To avoid problems with nomenclature we write the equation of motion of a string as

$$m \frac{\partial^2 \xi}{\partial t^2} = f \frac{\partial^2 \xi}{\partial x^2} + S_z, \quad (9.2.3b)$$

where m = mass per unit length (kilograms per meter),

f = total tension (newtons),

S_z = transverse force per unit length (newtons per meter).

The equations of motion for a membrane and for a string are summarized at the end of the chapter in Table 9.2.

9.2.1 Driven and Transient Response, Normal Modes

In the absence of an external force per unit length, (9.2.3a) and (9.2.3b) state that the deflections $\xi(x, t)$ of a membrane or a wire satisfy the wave equation

$$\frac{\partial^2 \xi}{\partial t^2} = v_p^2 \frac{\partial^2 \xi}{\partial x^2}, \quad (9.2.4)$$

where the phase velocity is

$$v_p = \left(\frac{S}{\sigma_m} \right)^{1/2} \text{ for a membrane,} \quad (9.2.5a)$$

$$v_p = \left(\frac{f}{m} \right)^{1/2} \text{ for a wire.} \quad (9.2.5b)$$

Hence the discussion of waves given in Section 9.1 applies equally well to the deflections of the wire shown by Fig. 9.2.3.

The sinusoidal steady-state response of physical systems is of general interest. This has been illustrated many times in the preceding chapters, both in the context of lumped-parameter systems (Chapters 4 and 5) and distributed systems (Chapter 7). The simple wire, described by (9.2.4), gives an opportunity to develop the basic relationship between the driven response of a continuous medium and its transient response. The insights afforded by the discussion that follows form a necessary prelude to understanding the continuum electromechanical examples undertaken in Chapter 10.

A wide class of problems is illustrated by considering the situation in which the wire is driven at one end ($x = -l$) by a sinusoidal excitation

$$\xi(-l, t) = \xi_a \sin \omega_a t \quad (9.2.6)$$

and fixed at the other end

$$\xi(0, t) = 0. \quad (9.2.7)$$

Physically, the excitation must be turned on at some time. For convenience we assume that this happens when $t = 0$, at which time the wire has the initial conditions

$$\xi(x, 0) = \xi_0(x), \quad (9.2.8)$$

$$\frac{\partial \xi}{\partial t}(x, 0) = \dot{\xi}_0(x). \quad (9.2.9)$$

The initial and boundary conditions are imposed along the contours shown in Fig. 9.2.4.

Now, we wish to determine the deflections $\xi(x, t)$ which satisfy these initial and boundary conditions. By analogy with the solution of lumped

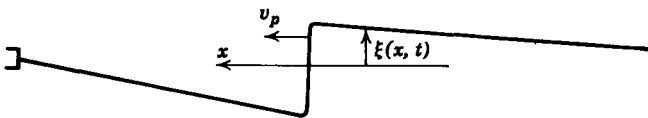


Fig. 9.2.3 Elastic wire or tightly wound helical spring under tension and plucked at one end. The wave is seen as it propagates to the left. The deflections of the spring provide a clear picture of the dynamics predicted by (9.2.4).

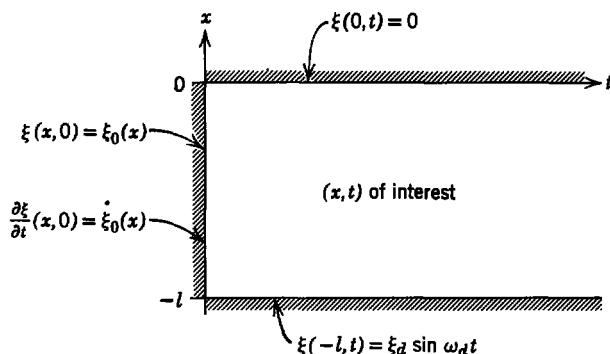


Fig. 9.2.4 Initial and boundary conditions in the x - t plane for wire fixed at $x = 0$, sinusoidally excited at $x = -l$, and having given initial conditions over the length of interest when $t = 0$.

parameter problems, discussed in Section 5.1.2, we divide the response into a part with the same sinusoidal steady-state character as the excitation and a transient part that is necessary to satisfy the initial conditions. In the discussion that follows we see a close connection between these two types of solution and their lumped-parameter counterparts.

In the analysis of lumped-parameter systems, defined by constant-coefficient ordinary differential equations, solutions take the form e^{st} . Similarly, distributed systems, defined by constant coefficient partial differential equations, have solutions that take the form*

$$\xi = \text{Re} [\hat{\xi} e^{j(\omega t - kx)}], \quad (9.2.10)$$

where the (angular) frequency ω and wavenumber k can, in general, be complex. This is shown by substituting (9.2.10) into (9.2.4), which requires that

$$\omega = \pm v_p k. \quad (9.2.11)$$

This relation between ω and k plays a role in continuum systems similar to that of the characteristic equation in lumped systems [see (5.1.6)]. Given the value of k (which represents the dependence of the deflection ξ on x), we obtain the possible frequencies of the solutions to (9.2.4). The relation between ω and k , given by (9.2.11), is referred to as the *dispersion equation*. We shall now see that it plays a fundamental role in determining both the sinusoidal steady-state and transient responses of continuous media.

* The general form of this solution could have been written as $e^{st} e^{\beta x}$, where s and β can be complex, to indicate the similarity to the e^{st} solution for total differential equations; however, (9.2.10) with ω and k real, represents a nondispersive wave which is our point of departure for studying continuum electromechanical dynamics in this context.

9.2.1a Sinusoidal Steady-State Response

It is assumed at the outset that the effects of initiating the excitation have died away,* hence it is appropriate to look for solutions with the same real frequency as the excitation. A plot of the dispersion equation (9.2.11) is shown in Fig. 9.2.5, in which it is made evident graphically that for a given frequency $\omega = \omega_d$, the dispersion equation will give two values of k , one the negative of the other ($k = \pm \omega_d/v_p$). Hence there are two possible solutions to (9.2.4) in the form of (9.2.10). A linear combination of these solutions is

$$\xi = \text{Re} \left\{ \xi_+ \exp \left[j\omega_d \left(t - \frac{x}{v_p} \right) \right] + \xi_- \exp \left[j\omega_d \left(t + \frac{x}{v_p} \right) \right] \right\}, \quad (9.2.12)$$

where ξ_+ and ξ_- are complex constants. Here it is evident that the response is composed of two waves propagating in opposite directions along the wire with equal phase velocities v_p .

For the particular problem at hand deflections are zero at $x = 0$ (9.2.7). This requires that the coefficients in (9.2.12) be negatives, so that solutions take the form

$$\xi = \text{Re} \left[\xi_- 2j \sin \frac{\omega_d x}{v_p} e^{j\omega_d t} \right]. \quad (9.2.13)$$

The coefficient ξ_- is, in turn, determined by the driving condition at $x = -l$ (9.2.6) (note that here we require the same frequency $\omega = \omega_d$ in the response as in the driving deflection).

$$\xi = -\xi_d \frac{\sin(\omega_d/v_p)x}{\sin(\omega_d/v_p)l} \sin \omega_d t. \quad (9.2.14)$$

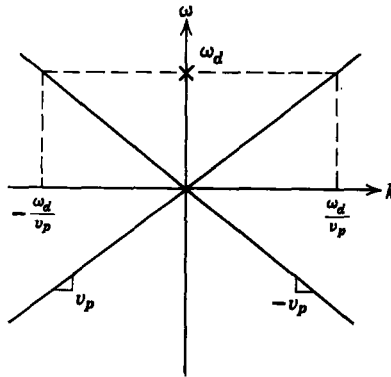


Fig. 9.2.5 Dispersion equation for ordinary waves on a wire.

* We return to this point later because, in fact, they may not “die away,” but rather grow with time.

This expression is the required sinusoidal steady-state response $\xi(x, t)$. It takes the form of a simple standing wave, as might be expected from the fact that it was obtained by superimposing two traveling waves of equal magnitude.

Remember that k is a linear function of ω_a , as shown in Fig. 9.2.5. Hence the shape of the deflection varies as the frequency is changed. At very low frequencies $\sin kx \rightarrow kx$, and (9.2.14) becomes

$$\xi = -\xi_a \left(\frac{x}{l} \right) \sin \omega_a t. \quad (9.2.15)$$

At any instant the low frequency deflections take the form of a straight line joining the fixed end to the instantaneous position of the sinusoidally varying deflection at $x = -l$. As the frequency is raised, the inertial effects of the wire come into play, and there is a tendency for it to bow outward. The response at low frequencies given by (9.2.15) would be found if the left-hand side of (9.2.4) (the inertial force on the wire) were ignored. This quasi-static behavior is completely analogous to the response of the elastic rod as described in Section 9.1.3a.

At frequencies such that

$$k = \pm \frac{\omega_a}{v_p} = \pm \frac{n\pi}{l}; \quad n = 1, 2, 3, \dots, \quad (9.2.16)$$

the denominator of (9.2.14) goes to zero and the response becomes infinite. This is an example of resonance, much as it is found in lumped-parameter systems. The salient feature of the continuum system is the infinite number of these resonances, each with a corresponding characteristic frequency and distribution of ξ in space. The relationship between the resonance frequencies and deflections is shown in Fig. 9.2.6. In this figure an experiment is sketched, wherein a taut spring is fixed at one end and excited at the other by attaching it to a rod with a sinusoidally varying position. In Fig. 9.2.6a the driving amplitude is very large to make evident the essentially linear distribution of the spring displacement at low frequencies. In Fig. 9.2.6b, c, d the excitation amplitude is kept the same and the resonances in the response are made evident. Of course, in the physical situation the finite mechanical losses limit the resonance amplitude to a finite value rather than the infinite value predicted by (9.2.14).

From the dynamics of lumped-parameter systems we know that a resonance peak indicates a driving frequency in the neighborhood of a natural frequency. In the actual experiment of Fig. 9.2.6 these natural frequencies are not purely real because mechanical damping adds an imaginary term; hence excitation at the purely real frequency ω_a gives rise to a bounded response. We see next that the natural frequencies predicted by our theory, which ignores the effects of damping, are indeed purely real. This we expect, in

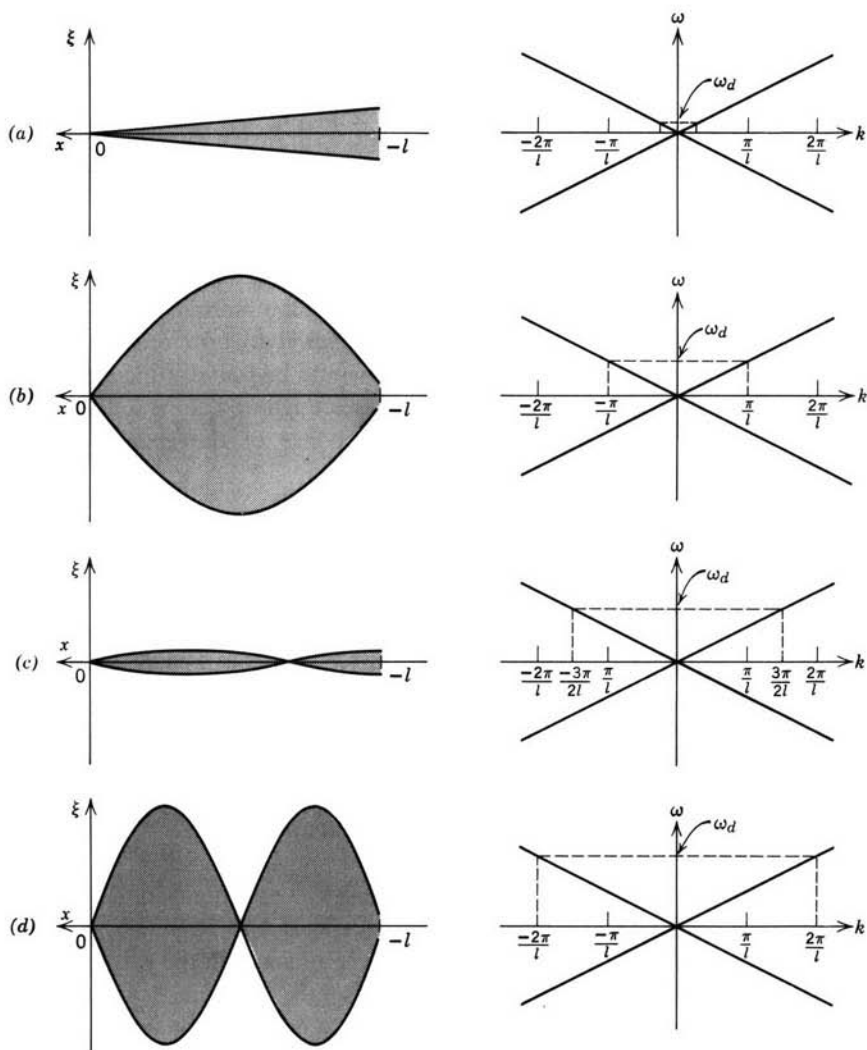


Fig. 9.2.6 Sketch of experiment in which a taut spring is fixed at the left end and deflected sinusoidally at the right end. (a) Deflections in the quasi-static limit at which the frequency is low compared with the reciprocal of the time required for a disturbance to propagate from one end of the spring to the other; (b) to (d) deflection as frequency is varied from value at which $k = \pi/l$ to $k = 2\pi/l$. The excitation amplitude is kept the same in going from (b) to (d). Actual experiment can be seen in film, "Complex Waves I" produced by Education Development Center for National Committee on Electrical Engineering Films.

view of the theoretically predicted resonances found in the response to the sinusoidal driving condition (9.2.14). Our ideal lossless model is accurate for predicting resonance frequencies, but not for calculating deflections at frequencies near resonance. The adequacy of our idealized model, which depends on the relative damping, must be ascertained for each physical situation and the purpose for which the model is to be used.

9.2.1b Transient Response

The steady-state solution given by (9.2.14) does not in general satisfy the initial conditions of (9.2.8-9). To satisfy these conditions, we require further solutions to (9.2.4) that can be added to the steady-state solution. Since the steady-state solution already satisfies the boundary conditions at $x = 0$ and $x = -l$, we require that these solutions satisfy the boundary conditions

$$\xi(0, t) = 0, \quad (9.2.17)$$

$$\xi(-l, t) = 0. \quad (9.2.18)$$

Again we resort to solutions in the form of (9.2.10). Now, however, ω is at the outset an unknown frequency to be determined from the boundary conditions. As in Section 9.2.1a, we take a linear combination of solutions that satisfy (9.2.17). This gives a solution in the form of (9.2.13) with $\omega_a \rightarrow \omega$.

$$\xi(x, t) = \text{Re}(A \sin kx e^{j\omega t}), \quad (9.2.19)$$

where A is a complex constant. The second boundary condition (9.2.18) is satisfied if

$$\sin kl = 0. \quad (9.2.20)$$

This is possible if

$$k = k_n = \frac{n\pi}{l}; \quad n = 1, 2, 3, \dots \quad (9.2.21)$$

Recall the procedure for finding the driven response. We used the dispersion equation to find the wavenumbers (the spatial behavior) by requiring that $\omega = \omega_a$. Now, to find the transient response we have used the boundary conditions to find the wavenumbers (9.2.21) and then used the dispersion equation (9.2.11) to find the possible frequencies of vibration.

$$\omega = \omega_n = \pm v_p k_n. \quad (9.2.22)$$

This relationship is shown graphically in Fig. 9.2.7.

We have found two solutions in the form of (9.2.19) for each value of k_n . Hence we write the n th *eigenmode**

$$\xi_n(x, t) = [A_n^+ e^{j\omega_n t} + A_n^- e^{-j\omega_n t}] \xi_n(x), \quad (9.2.23)$$

* C. R. Wylie, Jr., *Advanced Engineering Mathematics*, McGraw-Hill, New York, 1951, p. 234.

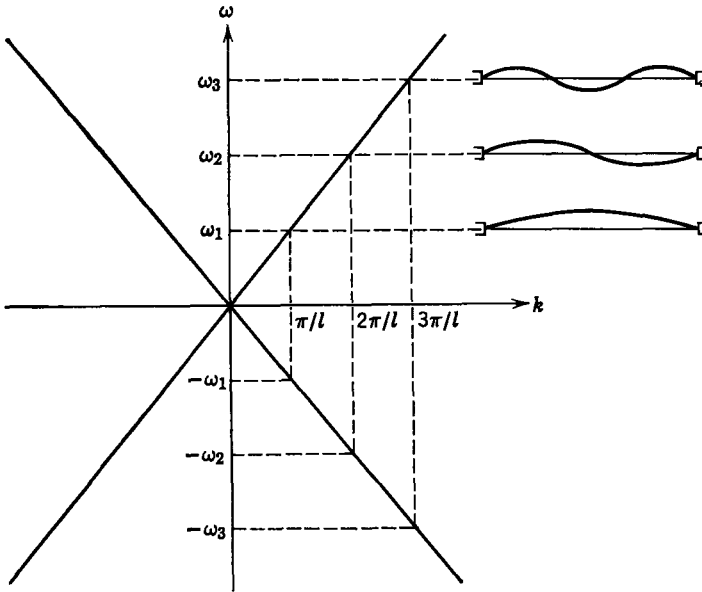


Fig. 9.2.7 Allowed wavenumbers (eigenvalues) $k = k_n$ as they are related to the eigenfrequencies ω_n by the dispersion equation.

where the “Re” has been dropped by requiring that the constant A_n^+ , if complex, be the complex conjugate of A_n^- and the *eigenfunction* $\xi_n(x)$ is

$$\xi_n(x) = \sin k_n x. \quad (9.2.24)$$

The constants k_n are called the *eigenvalues* of the problem, and the frequencies $\pm\omega_n$ are the *eigenfrequencies*. Specification of n fixes k_n , ω_n and the spatial dependence of the deflection mode.

The eigenfrequencies $\pm\omega_n$ are the natural frequencies of the distributed system, in the sense discussed for lumped systems in Section 5.1.1. In a distributed system the number of natural frequencies is infinite in contrast to the finite number that characterizes a finite number of interconnected lumped-parameter elements. The eigenfrequencies are placed in a familiar context by considering a time dependence of the form $e^{s_n t}$ [see (5.1.4)], where, in view of (9.2.10), $s_n = \pm j\omega_n$.

It has now been determined that the general deflection $\xi(x, t)$ is the sum of the transient solution, given by superimposing the modes of (9.2.23) and the driven response (9.2.14)*

$$\xi(x, t) = \sum_{n=1}^{\infty} (A_n^+ e^{j\omega_n t} + A_n^- e^{-j\omega_n t}) \xi_n(x) - \xi_d \frac{\sin(\omega_d/v_p)x}{\sin(\omega_d/v_p)l} \sin \omega_d t. \quad (9.2.25)$$

* It should be evident that this superposition is valid only when $\omega_d \neq \omega_n$.

The constants A_n^+ and A_n^- are determined by the initial conditions. To evaluate these constants observe that the eigenfunctions are *orthogonal*, in the sense that

$$\int_{-l}^0 \xi_n \xi_m dx = \begin{cases} 0; & n \neq m, \\ \frac{l}{2}; & n = m, \end{cases} \quad (9.2.26)$$

as can be seen by carrying out the integration for the particular functions at hand (9.2.24). (More general comments concerning the orthogonality relation are made in Section 9.2.1c.)

The orthogonality condition makes it possible to evaluate the constants A_n^+ and A_n^- , for when $t = 0$ (9.2.25) must give the initial deflection of (9.2.8):

$$\xi_0(x) = \sum_{n=1}^{\infty} (A_n^+ + A_n^-) \xi_n(x). \quad (9.2.27)$$

Now, if we multiply both sides of this expression by $\xi_m(x) = \sin k_m x$ and integrate over the length of the wire, it follows from (9.2.26) that

$$A_m^+ + A_m^- = \frac{2}{l} \int_{-l}^0 \xi_0(x) \sin k_m x dx, \quad (9.2.28)$$

for only one term in the infinite series is nonzero and that is the one in which $n = m$.

A second equation for the A_n 's is the result of the initial velocity condition in (9.2.9), which imposes the condition on the time derivative of (9.2.25):

$$\dot{\xi}_0(x) = \sum_{n=1}^{\infty} j\omega_n (A_n^+ - A_n^-) \xi_n(x) - \xi_a \omega_a \frac{\sin(\omega_a/v_p)x}{\sin(\omega_a/v_p)l}. \quad (9.2.29)$$

Now, when this equation is multiplied by $\xi_m(x)$ and integrated from $-l$ to 0, it follows that

$$A_m^+ - A_m^- = \frac{2}{j\omega_m l} \int_{-l}^0 \left[\dot{\xi}_0(x) + \xi_a \omega_a \frac{\sin(\omega_a/v_p)x}{\sin(\omega_a/v_p)l} \right] \sin k_m x dx. \quad (9.2.30)$$

The two expressions for the sum and difference of the A_n 's (9.2.28) and (9.2.30) are added and subtracted to obtain the explicit expressions

$$A_m^+ = \frac{1}{l} \int_{-l}^0 \left\{ \xi_0(x) + \frac{1}{j\omega_m} \left[\dot{\xi}_0(x) + \xi_a \omega_a \frac{\sin(\omega_a/v_p)x}{\sin(\omega_a/v_p)l} \right] \right\} \sin k_m x dx, \quad (9.2.31)$$

$$A_m^- = \frac{1}{l} \int_{-l}^0 \left\{ \xi_0(x) - \frac{1}{j\omega_m} \left[\dot{\xi}_0(x) + \xi_a \omega_a \frac{\sin(\omega_a/v_p)x}{\sin(\omega_a/v_p)l} \right] \right\} \sin k_m x dx. \quad (9.2.32)$$

For a given set of initial conditions we can now compute the constants $A_n^{\pm}$; hence the solution given by (9.2.25) is now completed.

It is only in unusual situations that we become interested in the detailed transient response of a continuum system. Here we are primarily interested in the fact that the eigenmodes given by (9.2.23) can be used to represent the consequences of arbitrary initial conditions. Hence the development demonstrates that the eigenmodes play the same role in the distributed system as the homogeneous solutions of (5.1.11) played in lumped-parameter systems. For this reason it is not surprising that in Chapter 10 we use the eigenmodes to study the stability of continuum systems. If one of the eigenfrequencies has a negative imaginary part, the corresponding eigenmode is unstable and becomes unbounded in time. This is a case in which the transient solution does not die away but rather dominates the driven response. Even when all the eigenmodes are stable, as described here, the theoretical transient solution [the series in (9.2.25)] does not die away but continues to execute oscillatory motions. Of course, in any real system that involves the vibrations of a wire these transient modes would decay (due to damping), thus leaving just the driven response.

Example 9.2.1. It is important to recognize that the eigenmodes represent those oscillations of the continuous medium that can be independent of one another; that is, with appropriate initial conditions, we can initiate any one of the eigenmodes at $t = 0$ and the ensuing oscillations will involve it alone. This can be illustrated by considering the following situation:

1. There is no drive, $\xi_d = 0$.
2. When $t = 0$, the string is static, $\dot{\xi}_0(x) = 0$.
3. When $t = 0$, the string has the deflection $\xi_0 = \xi_m \sin 3\pi x/l$. Using these initial conditions, it follows from (9.2.31) and (9.2.32) that the constants A_m^+ and A_m^- are

$$A_m^+ = \frac{1}{l} \int_{-l}^0 \xi_m \sin \frac{3\pi x}{l} \sin k_m x \, dx, \quad (a)$$

$$A_m^- = A_m^+, \quad (b)$$

and the solution (9.2.25) becomes

$$\xi(x, t) = \xi_m \cos \omega_3 t \sin \frac{3\pi x}{l}. \quad (c)$$

From this result it is clear that because the initial deflection has the same spatial distribution as the $n = 3$ eigenmode the motion persists as the single eigenmode $n = 3$, with the frequencies $\pm \omega_3$.

9.2.1c Orthogonality of Eigenmodes

A vector can be decomposed into three perpendicular components. We say that these components are orthogonal, in the sense that no part of one component is imbedded in another (i.e., the dot product of any pair of vectors is zero). It is in a sense analogous to this that eigenmodes are orthogonal. The orthogonality condition of (9.2.26) expresses the fact that there is no part of one of the eigenmodes imbedded in another. The decomposition of initial conditions into these modes is illustrated in Section 9.2.1b.

We were able to show that the orthogonality condition of (9.2.26) held by simply substituting the eigenfunctions of (9.2.24) into the integral, which could then be carried out. These functions are not always so easily integrated, and it is often necessary to use a less direct method of finding the orthogonality condition.

Because the solutions were found by using the differential equation, we expect that the orthogonality of two solutions is, in fact, a property of the differential equation and the boundary conditions. To show this observe that in terms of the eigenfunction ξ_n , solutions, as given by (9.2.23), take the form

$$\xi = \sum_{n=1}^{\infty} \hat{\xi}_n(x) e^{\pm j\omega_n t}. \quad (9.2.33)$$

It then follows from (9.2.4) that the eigenfunctions must satisfy the ordinary differential equation

$$\frac{d^2 \hat{\xi}_n}{dx^2} + k_n^2 \hat{\xi}_n = 0, \quad (9.2.34)$$

where k_n is introduced instead of the eigenfrequency ω_n because of (9.2.22), the dispersion equation.

In view of the form taken by the orthogonality condition (9.2.26), we now multiply (9.2.34) by another eigenmode $\hat{\xi}_m$ and integrate the expression over the length of the wire,

$$\int_{-l}^0 \hat{\xi}_m \frac{d^2 \hat{\xi}_n}{dx^2} dx + k_n^2 \int_{-l}^0 \hat{\xi}_m \hat{\xi}_n dx = 0. \quad (9.2.35)$$

The last term in this equation takes the form of the orthogonality condition. Further manipulations have the objective of eliminating the first term, which we integrate by parts* to obtain

$$\left[\hat{\xi}_m \frac{d \hat{\xi}_n}{dx} \right]_{-l}^0 - \int_{-l}^0 \frac{d \hat{\xi}_m}{dx} \frac{d \hat{\xi}_n}{dx} dx + k_n^2 \int_{-l}^0 \hat{\xi}_m \hat{\xi}_n dx = 0. \quad (9.2.36)$$

The second term in this equation is symmetric in m and n , which suggests that we now rederive this expression with the roles of m and n reversed to obtain (9.2.36) with $m \rightarrow n$. If this expression is then subtracted from (9.2.36), the terms that are symmetric in m and n subtract to zero and we obtain

$$\left[\frac{d \hat{\xi}_n}{dx} \hat{\xi}_m - \frac{d \hat{\xi}_m}{dx} \hat{\xi}_n \right]_{-l}^0 + (k_n^2 - k_m^2) \int_{-l}^0 \hat{\xi}_m \hat{\xi}_n dx = 0. \quad (9.2.37)$$

For the particular example undertaken in Section 9.2.1b the eigenfunctions were required to be zero at $x = 0$ and $x = -l$; hence the first term in

* $\int u dv = uv - \int v du$.

(9.2.37) vanishes to leave the desired orthogonality condition

$$(k_n^2 - k_m^2) \int_{-1}^0 \xi_m \xi_n dx = 0. \quad (9.2.38)$$

For differing eigenvalues ($m \neq n$) it is clear that the integral must vanish. This type of orthogonality proof is useful when the eigenfunctions are too complicated to make the performance of a direct integration easy.* Even when the continuum dynamics are governed by the simple wave equation, as illustrated here, the boundary conditions may be sufficiently complicated to warrant a proof of orthogonality in terms of the differential equation. The effect of boundary conditions on the normal modes is illustrated in the next section.

9.2.2 Boundary Conditions and Coupling at Terminal Pairs

The most common type of electromechanical coupling to continuous media can be modeled in terms of terminal pairs. The delay line analyzed in Example 9.1.5 illustrated this point. Mathematically, this class of situations is characterized by partial differential equations for the continuous media that do not involve electromechanical forces. Then the coupling is accounted for by means of boundary conditions, and when these boundary conditions can be formulated in terms of a finite set of variables (say forces and displacements) we can think of the problem formally in terms of coupling at terminal pairs.

The acoustic response of an auditorium to a public address system or the sonar sounding of the ocean floor exemplify this class of problem, in which electromechanical coupling occurs through the boundary conditions. In these examples the time required for an acoustic wave to propagate from one extreme to another in the continuous medium is significant compared with other times of interest, such as the period of excitation. Hence it is clear that in these cases many of the most significant aspects of the mechanics must be accounted for in terms of a continuum model. It is important to recognize, however, that the electromechanical aspects of the problem can often be modeled in terms of lumped parameters. We reserve the discussion of compressible fluids as a continuous medium and acoustic waves in fluids for Chapter 13 and use the simple wire and membrane models here to illustrate the basic considerations.

We have three objectives in this section: first, to see how the boundary conditions are written in the case of wires and membranes, including the possibility of electromechanical coupling; second, to see how the notions

* Our analysis lacks a proof that we have not left out a mode. The completeness of normal modes is discussed in R. V. Churchill, *Fourier Series and Boundary Value Problems*, McGraw-Hill, New York, 1941, p. 43.

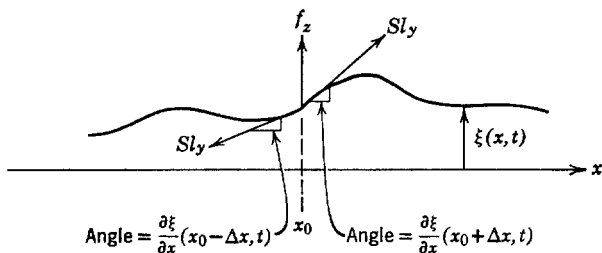


Fig. 9.2.8 Membrane with deflection ξ that depends only on (x, t) and is acted on by an externally applied force f_z at $x = x_0$. The force is distributed over the width l_y of the membrane.

introduced in Section 9.2.1 apply when the normal modes are more complicated than in a simple wire fixed at the ends; and finally, to compute the sinusoidal steady-state response of an electromechanical system, using as an example a simplified model of a loud speaker.

The one-dimensional motions of the membrane are defined by (9.2.3a), which we write as

$$\sigma_m l_y \frac{\partial^2 \xi}{\partial t^2} = S l_y \frac{\partial^2 \xi}{\partial x^2} + f_z(t) u_0(x - x_0). \quad (9.2.39)$$

Here the force acting on the membrane in the z -direction is concentrated at $x = x_0$; hence the force is the product of $f_z(t)$ (newtons) and a unit impulse at $x = x_0$.^{*} This physical situation is shown in Fig. 9.2.8.

A boundary condition for the effect of the force f_z on the membrane can be derived in two ways. The more physical method consists in writing a force balance equation for the mechanical node formed by the strip of membrane at $x = x_0$. In addition to the force f_z , other forces are due to the sections of membrane on either side of the node. Remember that the membrane is under the longitudinal tension S ; and when the membrane tilts there is a component of force in the z -direction which for small amplitude deflections is proportional to the slope of the deflection evaluated just to the right and left of the node. Hence the forces are as shown in Fig. 9.2.9.

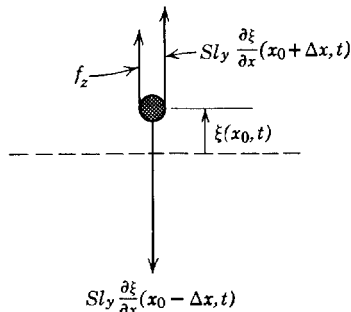


Fig. 9.2.9 Force balance for the strip of membrane at $x = x_0$ in Fig. 9.2.8.

^{*} The unit impulse function is most commonly used in circuit theory, in which its argument is time t rather than space x . See, for example, A. G. Bose, and K. N. Stevens, *Introductory Network Theory*, Harper & Row, New York, 1965.

If we designate

$$\lim_{\Delta x \rightarrow 0} (x_0 + \Delta x) = x^+ \quad \text{and} \quad \lim_{\Delta x \rightarrow 0} (x_0 - \Delta x) = x^-$$

the force balance equation becomes

$$f_z + Sl_y \left[\frac{\partial \xi}{\partial x}(x^+) - \frac{\partial \xi}{\partial x}(x^-) \right] = 0. \quad (9.2.40)$$

This equation represents a boundary condition for the membrane at $x = x_0$. Note the additional boundary condition on the deflection implied by the fact that ξ must be a continuous function at $x = x_0$.

An alternative procedure for finding the boundary condition is familiar because it is commonly used in connection with the electromagnetic field equations.* By use of this approach the equation of motion (9.2.39) is integrated from just to the left of x_0 to a point just to the right of x_0 .

$$\sigma_m l_y \int_{x_0 - \Delta x}^{x_0 + \Delta x} \frac{\partial^2 \xi}{\partial t^2} dx = Sl_y \int_{x_0 - \Delta x}^{x_0 + \Delta x} \frac{\partial}{\partial x} \left(\frac{\partial \xi}{\partial x} \right) dx + f_z(t) \int_{x_0 - \Delta x}^{x_0 + \Delta x} u_0(x - x_0) dx. \quad (9.2.41)$$

Because the deflection is a continuous function of x , the first term goes to zero in the limit where $\Delta x \rightarrow 0$. The first term on the right can be integrated to obtain the first derivative of ξ , whereas the last term is by definition simply $f_z(t)$. Hence we obtain

$$0 = Sl_y \left[\frac{\partial \xi}{\partial x}(x^+) - \frac{\partial \xi}{\partial x}(x^-) \right] + f_z(t). \quad (9.2.42)$$

in the limit in which $\Delta x \rightarrow 0$, which is the same as (9.2.40). Redefinition of Sl_y as the total tension f makes this expression useful for wires under tension.

The fact that the derivative is discontinuous at the point at which the force is applied is not surprising, in view of what we would expect to find if a taut wire were fixed at two ends and pulled upward at the center by a concentrated force. Force equilibrium for a concentrated force necessitates an abrupt change in the slope of the membrane or wire deflection.

The following example illustrates how normal modes and their eigenfrequencies are found in a case in which the boundary condition of (9.2.40) comes into play.

Example 9.2.2. A wire, with the tension f and mass per unit length m , is fixed at one end ($x = 0$) and attached to a pair of springs at the other end ($x = -l$), as shown in Fig.

* See, for example, Section 6.2.2, in which Gauss's theorem was used to derive the relationship between a singularity in volume charge density (the surface charge density) and the electric displacement vector.

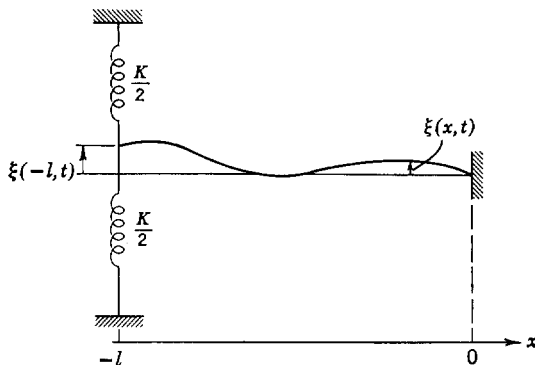


Fig. 9.2.10 A wire is fixed at one end and attached to a pair of springs; the combined spring constant is K .

9.2.10. We assume that the longitudinal tension on the wire is held in equilibrium at $x = -l$ by a constant longitudinal force, so that the end executes a purely transverse motion. The springs then exert a total force

$$f_z = -K\xi(-l, t). \quad (a)$$

Since there is no wire to the left of $x = -l$ to exert a transverse force on the node to which the springs are attached, boundary condition (9.2.40) becomes (here Sl_y is replaced by the wire tension f)

$$-K\xi(-l, t) + f \frac{\partial \xi}{\partial x}(-l, t) = 0 \quad (b)$$

for this particular case. Of course, the other boundary condition is

$$\xi(0, t) = 0. \quad (c)$$

Now, to find the eigenfrequencies we assume that solutions take the form

$$\xi = \text{Re} [\hat{\xi}(x)e^{j\omega t}], \quad (d)$$

where ω is an unknown frequency. Then the equation of motion [(9.2.3b) with $S_z = 0$] and boundary conditions require

$$\frac{d^2 \hat{\xi}}{dx^2} + k^2 \hat{\xi} = 0; \quad k^2 = \frac{\omega^2}{v_p^2}, \quad (e)$$

$$-K\hat{\xi}(-l) + f \frac{d\hat{\xi}}{dx}(-l) = 0, \quad (f)$$

$$\hat{\xi}(0) = 0. \quad (g)$$

The solution to (e) which satisfies boundary condition (g) is

$$\hat{\xi} = A \sin kx, \quad (h)$$

where A is an arbitrary constant. For this solution to satisfy (f)

$$KA \sin kl + fkA \cos kl = 0. \quad (i)$$

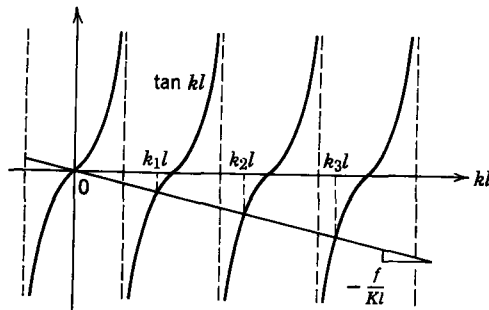


Fig. 9.2.11 A graphical solution of an eigenvalue equation [Example 9.2.2, (j)] shows the lowest three roots k_n .

Unless A is non-zero, there is no solution. Hence it follows from (i) that the eigenvalues k_n satisfy the equation

$$\tan kl = -\frac{f}{Kl}(kl), \quad (j)$$

which results from rearranging (i). This equation has an infinite number of solutions $k = k_n$, which can be designated by the index n and found graphically, as shown in Fig. 9.2.11. Once the eigenvalues k_n have been determined from the eigenvalue equation (j), the eigenfrequencies follow from the dispersion equation (e)

$$\omega_n = \pm k_n v_p. \quad (k)$$

Once again we have found modes in the form of (9.2.23). Note, however, that because of the boundary condition at $x = -l$ the eigenfrequencies ω_n are not harmonically related. Nevertheless, the modes are orthogonal in the sense of (9.2.38), as can be seen by evaluating (9.2.37) for the case at hand. It follows from (f) that each of the eigenfunctions satisfies a condition of the form

$$\frac{d\xi_n}{dx}(-l) = \frac{K}{f} \xi_n(-l). \quad (l)$$

Using this fact, the first term in (9.2.37) becomes

$$\frac{d\xi_n}{dx}(0)\xi_m(0) - \frac{d\xi_m}{dx}(0)\xi_n(0) - \frac{K}{f} [\xi_n(-l)\xi_m(-l) - \xi_m(-l)\xi_n(-l)]. \quad (m)$$

In view of boundary condition (g), it follows that this expression is zero, and from (9.2.37), that the modes (m) and (n) are orthogonal.

The purpose of the preceding example was to show how the boundary conditions can revise the character of the normal modes. As we saw in Section 9.2.1, these modes can be used to represent the response of a system to initial conditions. Also, the eigenfrequencies of the system, including any coupling as it occurs through the boundary conditions, play an important role in determining the nature of the sinusoidal steady-state driven response. This is illustrated by an example that involves electromechanical coupling to a membrane through the boundary condition.

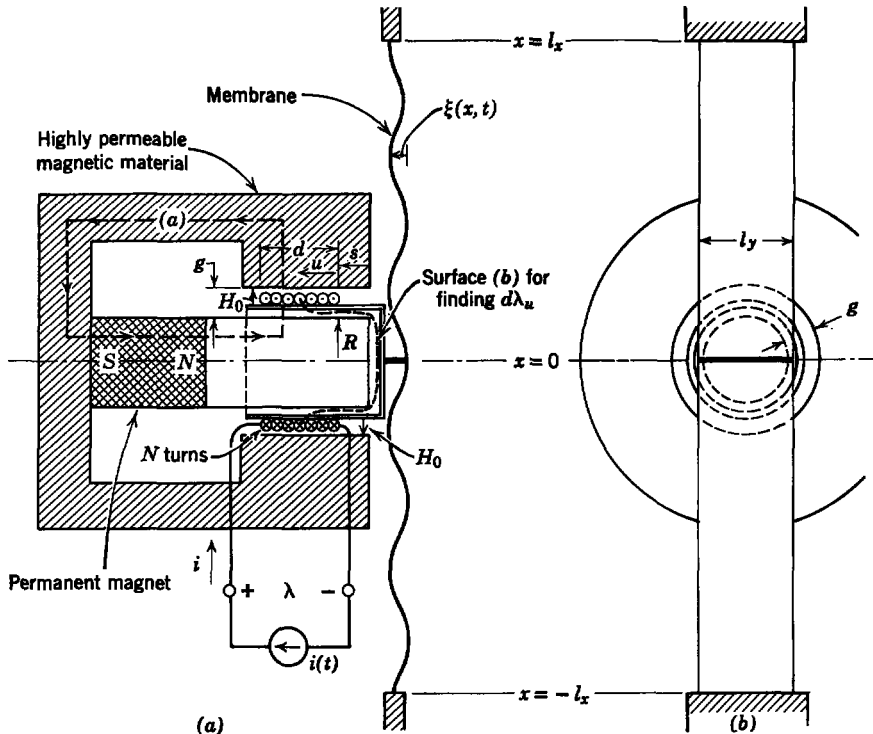


Fig. 9.2.12 One-dimensional model of a loudspeaker used in Example 9.2.3. The N turns are attached to the plunger, which in turn is attached to the membrane at $x = 0$; (a) side view showing a cross section of the transducer; (b) end view of the membrane showing the width l_y .

Example 9.2.3. The system shown in Fig. 9.2.12 illustrates the basic construction of a loudspeaker in which the lumped-parameter transducer excites the one-dimensional membrane as a diaphragm. Of course, circular or elliptical diaphragms (or cones) are commonly used in practical speakers. The simple system of Fig. 9.2.12, however, illustrates much of the basic dynamics and pertinent techniques without the use of Bessel functions. Our objective here is to study the multiple-mode dynamics of the membrane and to ascertain how the motion of the membrane is reflected in such terminal characteristics as the input impedance of the transducer.

The transducer shown is characteristic of those commonly used in low-frequency speakers.* It is constructed coaxially about the center line and a permanent magnet provides a radial magnetic field in the gap g . The N -turn voice coil is attached to a cylindrical plunger which has the displacement s as shown. The plunger and voice coil assembly is treated as a rigid body of mass M and is attached to the membrane at $x = 0$. The membrane is fixed at the ends $x = \pm l_x$.

* F. E. Terman, *Radio Engineering*, McGraw-Hill, New York, 1947, pp. 870–877.

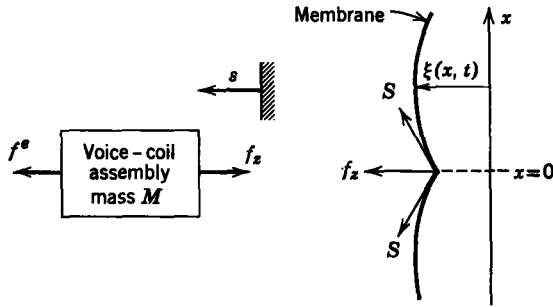


Fig. 9.2.13 Force equilibrium for the transducer plunger attached to the membrane at $x = 0$.

We can establish the boundary condition on the membrane at $x = 0$ by reference to Fig. 9.2.13 which shows the free-body diagram of the voice coil assembly and the membrane in the vicinity of $x = 0$. The force f^e is the force of electric origin applied to the voice coil and f_z is the force applied to the plunger by the membrane. The equation of motion for the mass M is then

$$M \frac{d^2 s}{dt^2} = f^e - f_z. \quad (a)$$

Recognizing that

$$\frac{ds}{dt} = \frac{\partial \xi}{\partial t}(0, t)$$

and using the boundary condition (9.2.40) to eliminate f_z , we obtain

$$M \frac{\partial^2 \xi}{\partial t^2}(0, t) = f^e + Sl_y \left[\frac{\partial \xi}{\partial x}(0^+, t) - \frac{\partial \xi}{\partial x}(0^-, t) \right]. \quad (b)$$

In addition the boundaries are fixed at $x = \pm l_x$; thus

$$\xi(l_x, t) = \xi(-l_x, t) = 0. \quad (c)$$

The problem can be simplified by recognizing that when the membrane is excited in the middle ($x = 0$) with both ends fixed, the response is symmetrical in x :

$$\xi(x, t) = \xi(-x, t).$$

Consequently, we recognize that

$$\frac{\partial \xi}{\partial x}(0^-, t) = -\frac{\partial \xi}{\partial x}(0^+, t)$$

and write (b) as

$$M \frac{\partial^2 \xi}{\partial t^2}(0, t) = f^e + 2Sl_y \frac{\partial \xi}{\partial x}(0^+, t). \quad (d)$$

Attention can now be confined to $0 < x < l_x$.

To complete the description of the system the force f^e must be related to the current $i(t)$ at the input terminals of the transducer. Thus we digress and use the techniques of Chapters 3, 6*, and 8 to make a mathematical model for the transducer.

* See Tables 3.1 and 6.1, Appendix E.

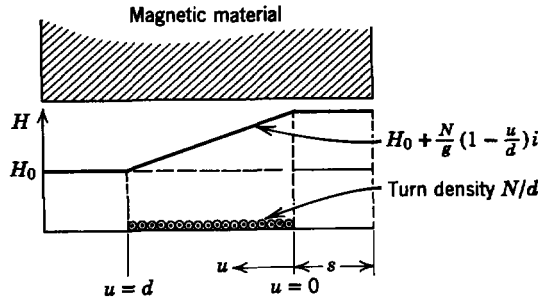


Fig. 9.2.14 Magnetic field intensity H in the gap g shows the uniform field H_0 that results from the permanent magnet and the distribution of field from the current i .

With reference to Fig. 9.2.12, the dimension g is assumed to be small enough that the radial variation of fields in the gap can be neglected. The magnetic field intensity in the gap is radial, independent of radius, and composed of a part H_0 due to the permanent magnet and a part due to current in the voice coil. The total field intensity in the gap can be found by using the integral form of Ampère's law with the contour (a) in Fig. 9.2.12 and assuming that the N voice-coil turns are uniformly distributed over the distance d . The field distribution is illustrated in Fig. 9.2.14.

To find the force of electric origin f^e we use the volume integration of $\mathbf{J} \times \mathbf{B}$ described in Section 8.1. The assumptions of no radial variation of magnetic field intensity and of a uniformly distributed voice coil with many turns N lead to the result that the contribution to f^e from a length (du) of the voice coil located at u is

$$df_u^e = 2\pi R \frac{Ni}{d} (du) \mu_0 \left[H_0 + \frac{N}{g} \left(1 - \frac{u}{d} \right) i \right].$$

Note that $\mathbf{J} \times \mathbf{B}$ is everywhere in the positive s -direction (see Fig. 9.2.12). Integration of this expression over the length d of the voice coil yields

$$f^e = 2\pi RN \left(\mu_0 H_0 i + \frac{\mu_0 N i^2}{2g} \right). \quad (e)$$

The first term represents the interaction between voice coil current and the field applied by the permanent magnet. The second term results from the current in one turn interacting with the field generated by current in all the other turns and is nonlinear in i . For good fidelity of sound reproduction a speaker is designed so that f^e is proportional to i ; consequently, good design results in the inequality

$$|H_0| \gg \left| \frac{Ni}{2g} \right|. \quad (f)$$

We assume in what follows that this inequality is satisfied and we can write f^e as

$$f^e = B_0 l i, \quad (g)$$

where

$$B_0 = \mu_0 H_0,$$

$$l = 2\pi RN.$$

The force produced by a voice coil is often written as in (g)* by ignoring the nonlinear term in (e) which can be eliminated only when condition (f) is satisfied.

The flux $d\lambda_u$ linked by an elemental coil at position u which has axial length du and therefore the number of turns $(N/d)du$ is found by using the surface of integration (b) shown in Fig. 9.2.12. This surface has the elemental coil as its boundary and extends through the air gap and over the end of the center pole piece. There is no contribution to the flux from that part of the surface at the end of the pole piece; consequently, the flux linked by the elemental coil at position u is

$$d\lambda_u = \left(\frac{N}{d}\right) du(2\pi R)\mu_0 \left\{ s \left(H_0 + \frac{Ni}{g} \right) + \int_0^u \left[H_0 + \frac{Ni}{g} \left(1 - \frac{u'}{d} \right) \right] du' \right\}$$

or

$$d\lambda_u = \left(\frac{N}{d}\right) du(2\pi R)\mu_0 \left[s \left(H_0 + \frac{Ni}{g} \right) + H_0 u + \frac{Ni}{g} \left(u - \frac{u^2}{2d} \right) \right]. \quad (h)$$

The total flux linked by the voice coil is found by summing the contributions $d\lambda_u$ of each coil. This amounts to integrating (h) over the length (d) of the coil

$$\lambda = 2\pi R\mu_0 \frac{N}{d} \int_0^d \left[H_0(s+u) + \frac{Ni}{g} \left(s+u - \frac{u^2}{2d} \right) \right] du$$

or

$$\lambda = 2\pi R\mu_0 N \left[H_0 \left(s + \frac{d}{2} \right) + \frac{Ni}{g} \left(s + \frac{d}{3} \right) \right]. \quad (i)$$

When we use condition (f), we can neglect the second term in (i) and write the terminal voltage of the voice coil as

$$v = \frac{d\lambda}{dt} = B_0 l \frac{ds}{dt}. \quad (j)$$

Recognizing that

$$\frac{ds}{dt} = \frac{\partial \xi}{\partial t}(0, t)$$

we write (j) as

$$v = B_0 l \frac{\partial \xi}{\partial t}(0, t). \quad (k)$$

Equations c, d, and g specify the boundary conditions applied to the membrane and (k) expresses the voltage seen by the input current source due to motion of the membrane.

We are interested in the steady-state response of the membrane to a sinusoidal voice-coil current

$$i(t) = \text{Re} (\hat{I} e^{j\omega t}) \quad (l)$$

where $\hat{I}$ is a complex number that determines the amplitude and phase of the input current and ω is real and the angular frequency of the input signal. The equations of motion for the membrane and transducer are linear with constant coefficients; consequently, all dependent variables have the same time dependence as (l). Hence we assume that

$$\xi(x, t) = \text{Re} [\hat{\xi}(x) e^{j\omega t}], \quad (m)$$

where $\hat{\xi}(x)$ is the complex amplitude.

* H. H. Skilling, *Electromechanics*, Wiley, New York, 1962, p. 19.

The wave equation is satisfied by ξ everywhere on the membrane. Hence substitution of (m) into (9.2.3a), with $T_z = 0$, and cancellation of the exponential yields

$$\frac{d^2 \xi}{dx^2} + k^2 \xi = 0, \quad (n)$$

where the wavenumber k is obtained from

$$k^2 = \frac{\omega^2 \sigma_m}{S}. \quad (o)$$

The general solution for (n) can be written as

$$\xi(x) = A \sin kx + B \cos kx, \quad (p)$$

where we have taken k to be the positive root of (o).

We first use the boundary condition of (c) to eliminate B from (p) with the result that

$$\xi(x) = A(\sin kx - \tan kl_x \cos kx). \quad (q)$$

Next, we use (l) and (m) in (d), cancel out the time-dependent factor $e^{j\omega t}$, and obtain

$$-\omega^2 M \xi(0) = B_0 I \hat{f} + 2S l_y \frac{d\xi}{dx}(0). \quad (r)$$

Substitution of (q) into this expression and evaluation of the constant A yields

$$A = \frac{B_0 I \hat{f}}{\omega^2 M \tan kl_x - 2kS l_y}, \quad (s)$$

and it follows from (q) that

$$\xi(x) = \frac{(\sin kx - \tan kl_x \cos kx)}{(\omega^2 M \tan kl_x - 2kS l_y)} B_0 I \hat{f}. \quad (t)$$

This result can be used with (m) to describe the motion at any point on the membrane.

When we write the terminal voltage of the voice coil as

$$v = \text{Re}(\hat{V} e^{j\omega t}),$$

we can use (k) and (t) to write the input admittance as

$$\frac{\hat{f}}{\hat{V}} = j \left[\frac{\omega M}{(B_0 I)^2} - \frac{2\sqrt{\sigma_m S} l_y}{(B_0 I)^2} \cot kl_x \right]. \quad (u)$$

Note that the first term depends only on the lumped-parameter loading of the transducer by the voice-coil assembly mass M and that the second term depends on the properties of the membrane. Note further that the admittance is imaginary, which indicates a lossless system.

Equation u can be written in terms of two susceptances S_1 and S_2 ; thus

$$\frac{\hat{f}}{\hat{V}} = jS_1 + jS_2, \quad (v)$$

where

$$S_1 = \frac{\omega M}{(B_0 I)^2},$$

$$S_2 = -\frac{2\sqrt{\sigma_m S} l_y}{(B_0 I)^2} \cot kl_x.$$

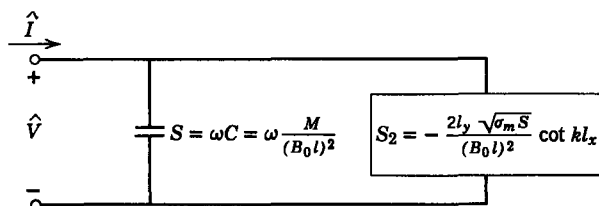


Fig. 9.2.15 Equivalent circuit seen by the current source $i(t)$ for the transducer shown in Fig. 9.2.12.

The susceptance S_1 appears capacitive with equivalent capacitance

$$C = \frac{M}{(B_0 l)^2}.$$

Thus we can draw the equivalent circuit of Fig. 9.2.15 to show separately the effects of the mass M and the membrane on the excitation source.

To study the effects of the membrane it is convenient to define a normalized frequency kl_x by

$$kl_x = \omega l_x \left(\frac{\sigma_m}{S} \right)^{1/4}.$$

The susceptances S_1 , S_2 and the total susceptance are plotted as functions of the normalized frequency in Fig. 9.2.16. The susceptance S_1 is linear with frequency, as indicated, but S_2

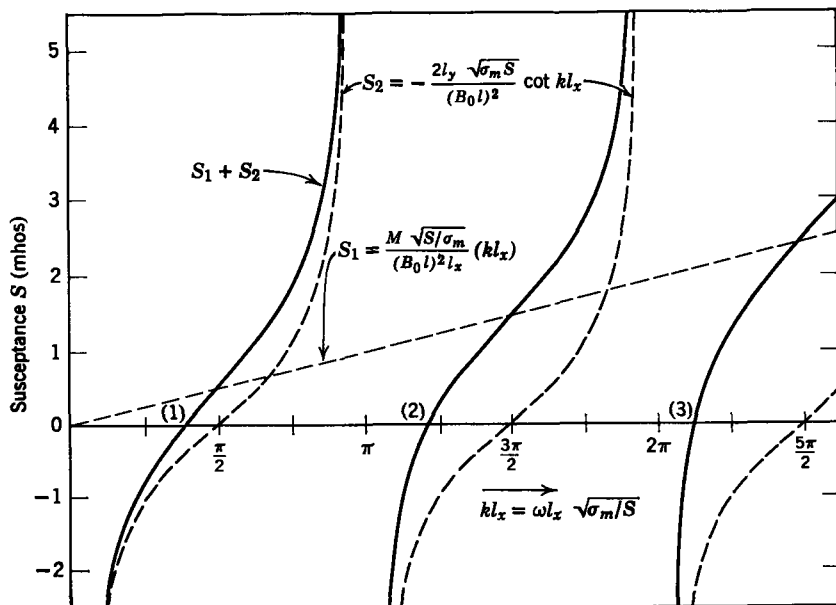


Fig. 9.2.16 The susceptance S of the system shown in Fig. 9.2.12 as a function of normalized frequency.

varies periodically with frequency, having zeros at $kl_x = \pi/2, 3\pi/2, 5\pi/2, \dots$, and poles at $kl_x = 0, \pi, 2\pi, 3\pi, \dots$. The addition of S_1 and S_2 to obtain the total input susceptance shows that the positions of the poles are the same as for S_2 but the zeros are reduced in frequency by the mass to values of kl_x indicated by points (1), (2), and (3) in Fig. 9.2.16. At a pole of the total susceptance the voltage is zero, regardless of the amplitude of the current, whereas at a zero of $(S_1 + S_2)$ zero current is required to obtain a finite voltage amplitude. This reflects our assumptions made initially that the membrane and transducer are not affected by electrical damping, viscous damping, or acoustic radiation. Of course, in loudspeakers the objective is to transfer as much energy as possible to an acoustic load (usually the air). The energy transfer, however, is usually comparatively small; consequently, our analysis gives a reasonably good first approximation to the behavior viewed from the electrical terminals. The resulting acoustic radiation can in many cases be computed under the assumption that the membrane motion derived here is not affected appreciably by the loading.* As we expect, the model is most in error at frequencies close to the singularities (poles and zeros) of admittance. The approximation can thus be improved by assuming small losses and treating the admittance in the vicinity of a singularity by the standard techniques of circuit theory.†

As indicated by the curves of Fig. 9.2.16, the system has multiple resonances due to the susceptance S_2 . This is a property of systems involving continuous media. The resonance of interest here results when a wave initiated at $x = 0$ travels to $x = l_x$, where it is reflected, and arrives back at $x = 0$ with a phase that reinforces the driving signal. Without the mass M , this occurs when the length l_x is an odd number of quarter wavelengths

$$l_x = \frac{1}{4}\lambda, \frac{3}{4}\lambda, \frac{5}{4}\lambda, \dots \quad (w)$$

We have defined the wavelength λ in the usual way as the phase velocity divided by the frequency

$$\lambda = \left(\frac{S}{\sigma_m} \right)^{1/2} \left(\frac{2\pi}{\omega} \right).$$

Each resonance described by (v) corresponds to a natural mode, each mode being characterized by a particular frequency. The natural modes have real frequencies and thus they appear as resonances in the transfer function.

With the mass M included, the resonances as given by points (1), (2), and (3) in Fig. 9.2.16 correspond to different modes on the membrane. We use the values of kl_x at each of these points (Fig. 9.2.16) in (t) to find the amplitude of the membrane displacement for the first three modes. The results are shown plotted in Fig. 9.2.17. Remember that from (m) the displacement is a sinusoidal function of time at any position x . Thus Fig. 9.2.17 represents what would be seen in a sideview snapshot taken at the instant of maximum deflection of the membrane. Alternatively, each curve of Fig. 9.2.17 can be interpreted as the envelope of a standing wave. The curves are plotted under the assumption that A is finite, as it is if the excitation I is decreased to zero as the resonance frequency is approached.

It should be clear from a study of Fig. 9.2.17 that the higher order modes will tend to excite acoustic signals that interfere with each other; for example, with the membrane in mode (3) an acoustic signal leaving the membrane at $x/l_x \approx 0.8$ is opposite in phase to the signal leaving the membrane at $x/l_x \approx 0.3$. When these two signals combine after having traveled the same distance, they interfere destructively. Such a property sets a design

* The applications of membranes to acoustic systems are discussed in W. P. Mason, *Electromechanical Transducers and Wave Filters*, Van Nostrand, 1942, p. 158.

† E. A. Guillemin, *Introductory Circuit Theory*, Wiley, New York, 1953, pp. 297–306.

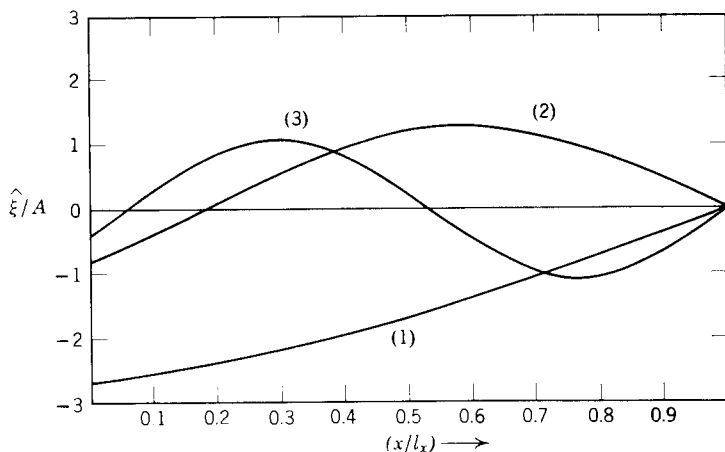


Fig. 9.2.17 Envelope of membrane deflection at the first three resonances of Fig. 9.2.16.

limitation on acoustic devices of this type. Moreover, when operation is in a higher order mode, the response function $[\hat{\xi}(x)/\hat{f}]$ varies violently with frequency. This effect is undesirable for most acoustic applications.

When the device is operated at low frequencies such that

$$kl_x \ll 1,$$

(t) shows that the response function is

$$\frac{\hat{\xi}}{\hat{f}} = \frac{(l_x - x)(B_0 l)}{2Sl_y - \omega^2 M l_x}. \quad (x)$$

In this case all points on the membrane have the same relative phase indicated by the low-frequency limit in Fig. 9.2.18. The membrane acts quasi-statically or as a massless spring (in the sense of Section 9.1.3a) and resonates with the mass M at approximately the frequency given by point (1) in Fig. 9.2.16. Most loudspeakers are operated above this first resonance but below the second resonance. The effect of acoustic radiation resistance and cone geometry are important factors in the design of a high-fidelity device.*

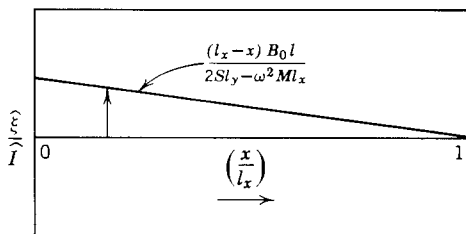


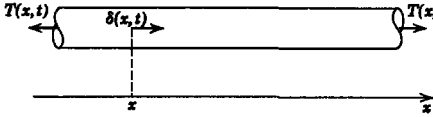
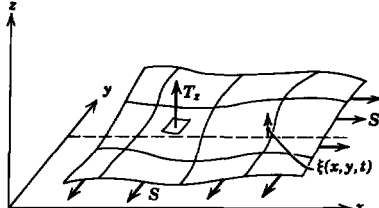
Fig. 9.2.18 Membrane deflection in the low-frequency limit at which the membrane behaves as a massless spring.

* F. E. Terman, *Radio Engineering*, McGraw-Hill, New York 1947, p. 872.

9.3 SUMMARY

Three practical, one-dimensional models introduced to illustrate the significance of continuum mechanical equations of motion are summarized in Table 9.2. The wire and membrane are used extensively in Chapter 10 to illustrate important types of continuum electromechanical behavior as they are found when fluids, plasmas, electron beams, or elastic media interact with electromagnetic fields. The wave dynamics studied in this chapter provide a background for understanding the more complicated dynamics that result in the presence of material convection and electromechanical coupling. The space-time behavior of waves, described in Section 9.1, is important in Chapter 10 for determining appropriate boundary conditions and visualizing

Table 9.2 Summary of One-Dimensional Mechanical Continua Introduced in Chapter 9

Thin Elastic Rod	
	$\rho \frac{\partial^2 \delta}{\partial t^2} = E \frac{\partial^2 \delta}{\partial x^2} + F_x$
	$T = E \frac{\partial \delta}{\partial x}$ δ—longitudinal (x) displacement T—normal stress ρ—mass density E—modulus of elasticity F_x—longitudinal body force density
Wire or "String"	
	$m \frac{\partial^2 \xi}{\partial t^2} = f \frac{\partial^2 \xi}{\partial x^2} + S_z$ ξ—transverse displacement m—mass/unit length f—tension (constant force) S_z—transverse force/unit length
Membrane	
	$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right) + T_z$ ξ—transverse displacement σ_m—surface mass density S—tension in y- and z-directions (constant force per unit length) T_z—z-directed force per unit area

transient situations. At the same time the frequency-wavenumber picture of the dynamics, represented by the dispersion equation introduced in Section 9.2, provides the unifying theme for Chapter 10.

PROBLEMS

9.1. A long thin steel cable of unstressed length l is hanging from a fixed support, as illustrated in Fig. 9P.1. Assume that the origin of coordinates is at the support and that x measures positive as shown. Assume that the steel cable has the following constants:

Cross-sectional area	$A = 10^{-4} \text{ m}^2$
Young's modulus	$E = 2.0 \times 10^{11} \text{ N/m}^2$
Mass density	$\rho = 7.8 \times 10^3 \text{ kg/m}^3$
Maximum allowable stress	$T_{\max} = 2 \times 10^9 \text{ N/m}^2$

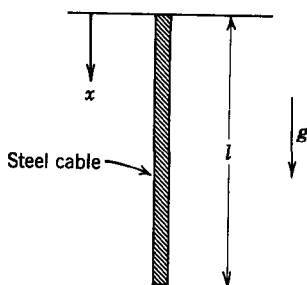


Fig. 9P.1

- Find the length of cable l for which the maximum stress in the cable just equals the maximum allowable stress.
- Find the displacement δ and stress T in the cable as functions of x .
- Find the total elongation of the cable.

9.2. Two thin elastic rods are arranged as shown in Fig. 9P.2. The first rod has modulus of elasticity E_1 , density ρ_1 , and cross-sectional area A_1 . It is attached at one end to a rigid wall and at the other to a very thin rigid plate of mass m and area A_m . On the other side of

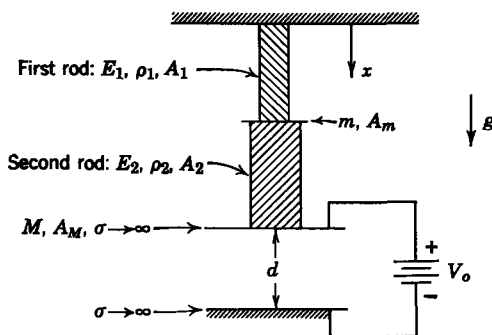


Fig. 9P.2

this plate is attached a second thin elastic rod with elastic modulus E_2 , density ρ_2 , and cross-sectional area A_2 . The other end of the second rod is fixed to a perfectly conducting thin plate with mass M and area A_M . This plate is held at a potential V_0 with respect to a second capacitor plate a distance d away. In the absence of gravity and with $V_0 = 0$, the length of the first rod is L_1 and the length of the second is L_2 . Assuming now that the system is immersed in a gravitational field g and that $V_0 \neq 0$, find the following:

- The stress in the first rod $T^{(1)}(x)$ and the displacement in the first rod $\delta^{(1)}(x)$.
- The stress in the second rod $T^{(2)}(x)$ and the displacement in the second rod $\delta^{(2)}(x)$.

9.3. In Fig. 9P.3 a thin elastic rod of cross-sectional area A , equilibrium length l , elastic modulus E , and mass density ρ is fixed at one end ($x = 0$) and attached to a rigid mass M

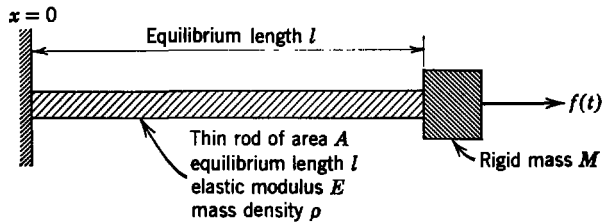


Fig. 9P.3

at the other ($x = l$). The mass is driven by a force source $f(t)$. The system constants are such that the mass M is much greater than the mass of the elastic rod; that is,

$$M \gg \rho Al.$$

The force source is constrained to be

$$f(t) = \text{Re} (f_0 e^{j\omega t}),$$

where f_0 and ω are positive real constants. The system is operating in the sinusoidal steady state. Neglect gravity.

- Find the displacement $\delta(x, t)$ and stress $T(x, t)$ in the elastic rod.
- Show a lumped-parameter mechanical system that represents the behavior of the system in Fig. 9P.3 for low frequencies (from $\omega = 0$ up to and including the lowest resonance frequency). Evaluate the equivalent elements in terms of the given parameters.

9.4. A long thin elastic rod with cross-sectional area A , unstressed length l , modulus of elasticity E , and mass density ρ is constrained at one end by the three ideal, lumped elements

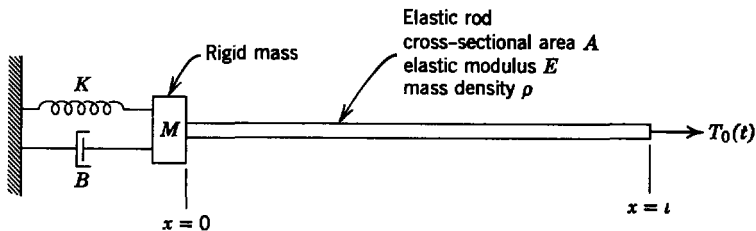


Fig. 9P.4

B , K , and M and at the other by a stress source $T_0(t)$, as shown in Fig. 9P.4. Determine the values of B , K , and M that are necessary for the response of the rod to $T_0(t)$ to be representable purely as a wave traveling in the negative x -direction. (Zero is an acceptable value for an element.)

9.5. A long thin rod of elastic modulus E , mass density ρ , and cross-sectional area A is fixed at one end ($x = 0$) and constrained at the other ($x = l$) by a force source $f(t)$ and a

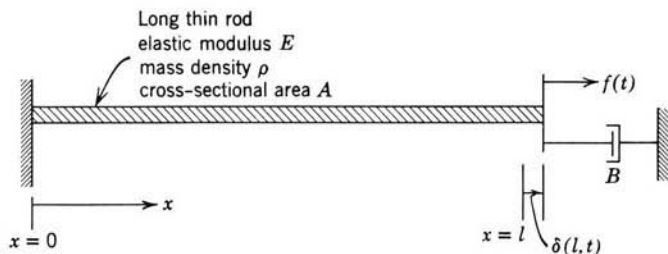


Fig. 9P.5

lumped linear damper of coefficient B , as illustrated in Fig. 9P.5. The applied force is sinusoidal $f(t) = \text{Re} (F_0 e^{j\omega t})$, where F_0 and ω are positive constants. The system is operating in the sinusoidal steady state.

- Write the boundary condition at $x = l$ in terms of the stress $T(l, t)$, the displacement $\delta(l, t)$, and the applied force $f(t)$.
- Assume that the displacement has the form $\delta(x, t) = \text{Re} [\delta(x) e^{j\omega t}]$. Find the complex amplitude $\delta(x)$ in terms of given data.
- If the damper coefficient B is positive and the frequency ω is real, can the system exhibit a resonance? That is, can the displacement δ be infinite with a zero applied force? Give justification for your answer.

9.6. A thin, circular magnetic rod is fixed at one end and constrained at the other end by a transducer (Fig. 9P.6). In the absence of an excitation, the transducer is simply biased by

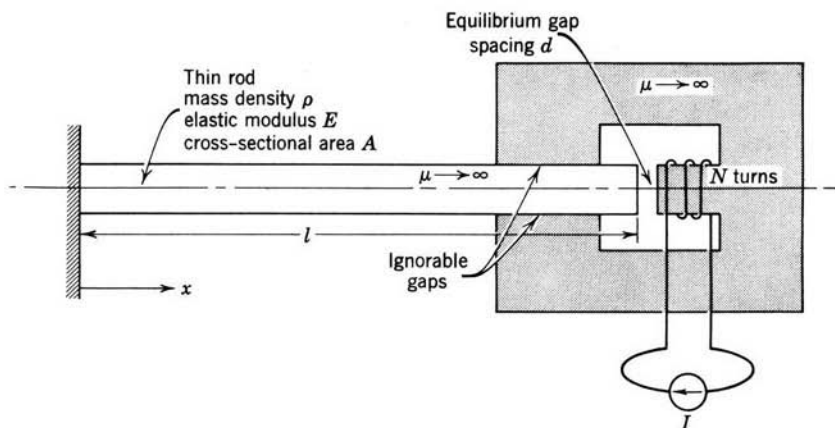


Fig. 9P.6

the constant current source I . When the rod is in static equilibrium, its length is l and the gap spacing is d . Compute the natural frequencies of the system under the assumption that the magnetization force on the rod acts on the end surface. A graphical representation of the eigenfrequencies is an adequate solution.

9.7. In Fig. 9P.7 two identical thin elastic rods are connected by a thin plate of mass M and area A_M . The plate is positioned between four springs, each having constant K . All

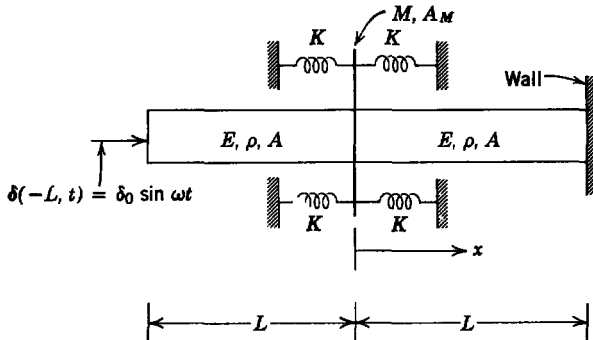


Fig. 9P.7

springs are relaxed when the plate is at $x = 0$. The system is driven on the left with a displacement $\delta(-L, t) = \delta_0 \sin \omega t$. Assume that steady state has been established.

- Write the general solution for the stress $\hat{T}(x)$ and the displacement $\delta(x)$ everywhere in both rods in terms of arbitrary constants.
- What are the boundary conditions that determine the constants in (a)?
- Find the stress $T(x, t)$ everywhere in both rods.

9.8. Example 9.1.5 considers the response of the delay line shown in Fig. 9.1.14 to a transient input signal. In this example design approximations were made concerning the effect of the self-inductance in the output circuit [see approximation following (t)]. You wish to compute the sinusoidal steady-state response of this system without making this approximation. Confine your attention to the sinusoidal steady-state response $v_o = \text{Re}(\hat{v}_o e^{j\omega t})$, to the input $i_i = \text{Re}(\hat{i}_i e^{j\omega t})$, and find the transfer function $H(\omega) = \hat{v}_o / \hat{i}_i$.

9.9. A magnetic transducer is used to excite a thin elastic rod, as shown in Fig. 9P.9. The mass M is attached to the rod, which in turn is fixed to a rigid support at $x = 0$. The driving current $I(t)$ is much smaller than I_0 and is given by $I(t) = \text{Re}[\hat{I} \exp(j\omega t)]$. Under the assumption that the displacements of the mass from its equilibrium position are small, complete the following:

- Find an expression for the position of the mass $y(t)$. You may assume that when there is no applied current the mass is centered between the pole faces and the rod has a length L .
- Under what conditions would you say that it is meaningful to consider the magnetic yoke and plunger as rigid but to recognize that the rod is made of an elastic material?

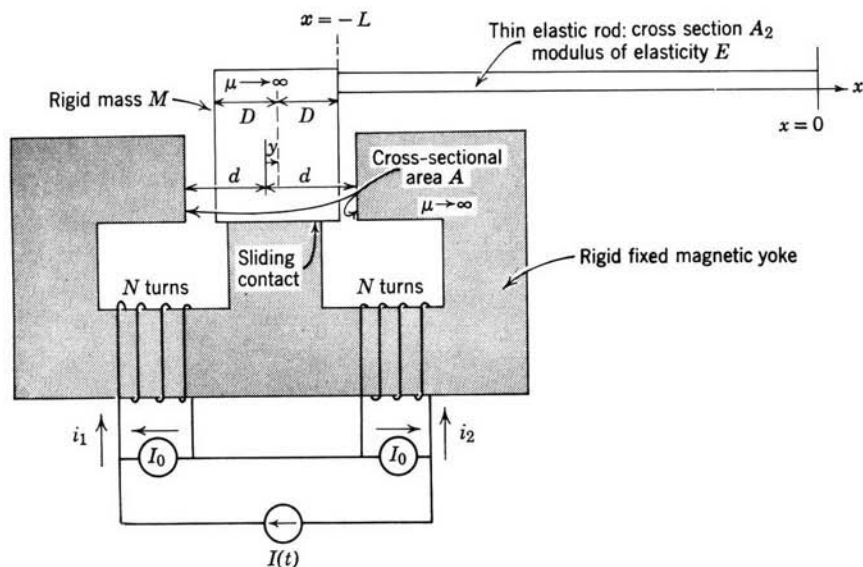


Fig. 9P.9

9.10. A long thin rod is fixed at $x = 0$ and driven at $x = l$, as shown in Fig. 9P.10. The driving transducer consists of a rigid plate with area A attached to the end of the rod, where it undergoes the displacement $\delta(l, t)$ from an equilibrium position exactly between two fixed plates. These fixed plates are biased by potentials V_0 and driven by the voltage $v = \text{Re}(\hat{V}_e^{j\omega t})$, as shown. ($|\hat{V}| \ll V_0$)

- Derive a boundary condition relating $\delta(l, t)$, $(\partial\delta/\partial x)(l, t)$ and $v(t)$.
- Compute the driven deflection of the rod $\delta(x, t)$.

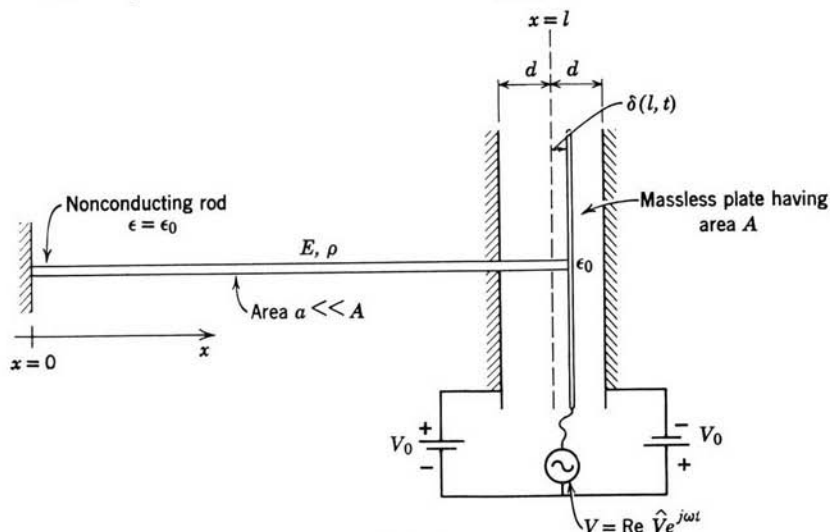


Fig. 9P.10

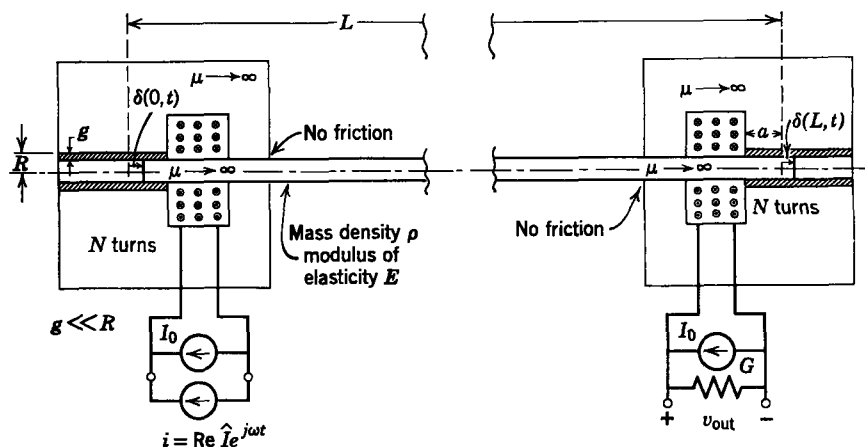


Fig. 9P.13

- Use the energy method to find the magnetic forces acting on the ends of the rod.
- Check the result of part (a) by using the magnetic stress tensor $T_{ij} = \mu H_i H_j - \frac{1}{2} \delta_{ij} \mu H_k H_k$ to find the forces acting on the ends of the rods.
- Assume that the magnetic forces act just on the ends and compute the transfer function $G(\omega) = \hat{v}_{out}/\hat{I}$, where $v_{out} = \text{Re}(\hat{v}_{out} e^{j\omega t})$.
- How would you adjust the system parameters so that $|\hat{v}_{out}|$ is proportional to $|\hat{I}|$, independent of frequency (over some range of frequencies)?

9.14. The electromechanical delay line shown in Fig. 9P.14 consists of a thin elastic rod terminated at both ends by capacitor plates that are massless and very thin. The elastic rod

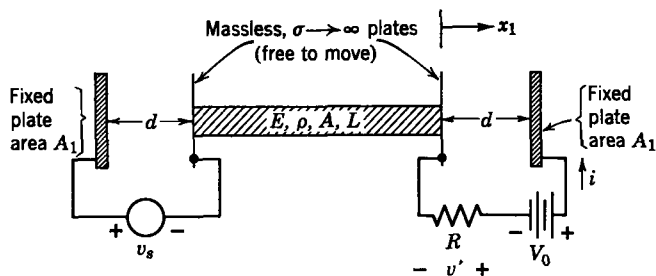


Fig. 9P.14

has length L in the absence of electrical excitations ($v_s = 0$ and $V_0 = 0$). Assume that $\delta(0) \ll d$ and that $\delta(-L) \ll d$.

- If $v_s = V_0$ and the battery at the other capacitor is connected as shown, find the stress everywhere in the rod.
- Suppose that $v_s = V_0 + v_o(t)$, where $v_o(t)$ is a short pulse. How long will it be before the signal is detected as a pulse in the output current i ? Compute a numerical value for the delay, assuming that the rod is steel with length 1 m.

- (c) What must the value of the resistance R be, so that if $v_s = V_0 + v_o(t)$, where v_o is again a short pulse, no pulse will travel back in the $-x_1$ -direction after encountering the capacitor plate at $x_1 = 0$? You may wish to do this by assuming that $v_o(t)$ is a sinusoid and requiring that there be no reflected wave in the rod for all frequencies of excitation. Assume that

$$v' \gg \frac{\epsilon_0 A_1 R}{d} \frac{dv'}{dt} \quad \text{or} \quad \frac{\omega \epsilon_0 A_1 R}{d} \ll 1,$$

where v' is the voltage across R .

9.15. This problem is intended to help you apply the techniques presented in Section 9.1 by considering an analogous situation, namely, the *torsional vibrations* of a thin, cylindrical elastic rod.

- (a) A static experiment is performed on the rod as follows. First, imagine that a line has been painted along one radius at every cross section of the rod when it is unstressed (see Fig. 9P.15). Now suppose that a *constant*, twisting torque τ is applied to the rod. If we single out a small length Δz of the rod (Fig. 9P.15b), we

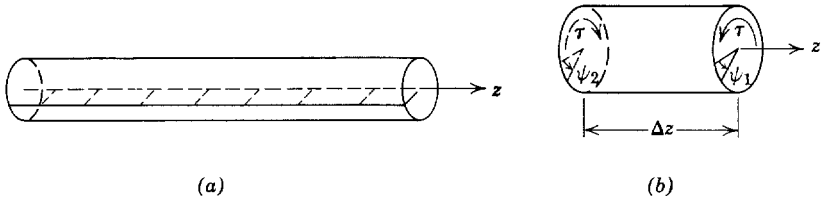


Fig. 9P.15

find that the *difference* between the angular deflections ψ_1 and ψ_2 of the painted line on the two faces is a constant β times the product of the applied torque and the length Δz . Find the relation between the torque τ and the deflection angle ψ in the limit as $\Delta z \rightarrow 0$.

- (b) We wish to find the dynamic equations for the rod. Assume that the twisting torque τ is now a function of z and t , $\tau = \tau(z, t)$. Also, $\psi = \psi(z, t)$. The rod has a moment of inertia J (per unit length) about the z -axis. Write an equation of motion for a length Δz of the rod. Then take the limit of this expression as Δz approaches zero.
- (d) Find a single partial differential equation for $\psi(z, t)$.

9.16. The elastic bar shown in Fig. 9P.16a extends to infinity in the z -direction and has free surfaces at $y = 0$ and $y = a$. Shearing stresses T_{zx} are applied uniformly over the surface at $x = 0$ to set the bar into a mode of vibration, where the material is displaced from equilibrium by the amount $\delta_z(x, t)\mathbf{i}_z$. We wish to find a differential equation of motion for δ_z .

- (a) Write the differential force equation in the z -direction by using the differential slice of material shown in Fig. 9P.16b and the stress function $T_{zx}(x, t)$ N/m².
- (b) Define a shear strain e_{zx} and relate it to the displacement function δ_z . Your strain function should be defined such that we would expect $T_{zx} = 2Ge_{zx}$ (G -N/m²), where G is a constant property of the material. Figure 9P.16c provides a starting point in the derivation.
- (c) Combine the results of (a) and (b) to obtain a single differential equation for δ_z . What is the propagational velocity of disturbances in the x -direction?

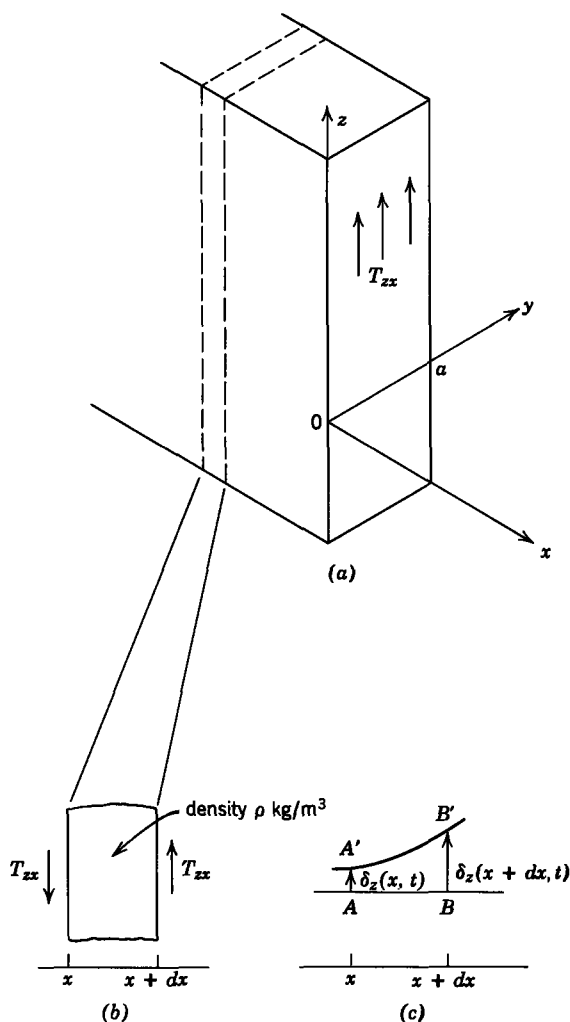


Fig. 9P.16

9.17. In general, an inviscid fluid differs grossly in its dynamics from an elastic solid. If, however, only normal stresses are involved and particle displacements are small, the dynamics of fluids and solids become similar. You are familiar with the derivation of the equation of motion for material in a thin elastic rod. The following derivation is somewhat similar. A tube filled with an initially stationary gas is shown in Fig. 9P.17a. We wish to derive an equation for the particle velocity $v(x, t)$ within the tube. To do this write an equation expressing conservation of mass for a slice of the material, as shown in Fig. 9P.17b. The fluid pressure $p(x, t)$ is a simple form of the stress tensor $T_{ij} = -\delta_{ij}p(x, t)$. We can write a differential equation to express conservation of momentum ($f = ma$) by using the slice of material shown in Fig. 9P.17c.

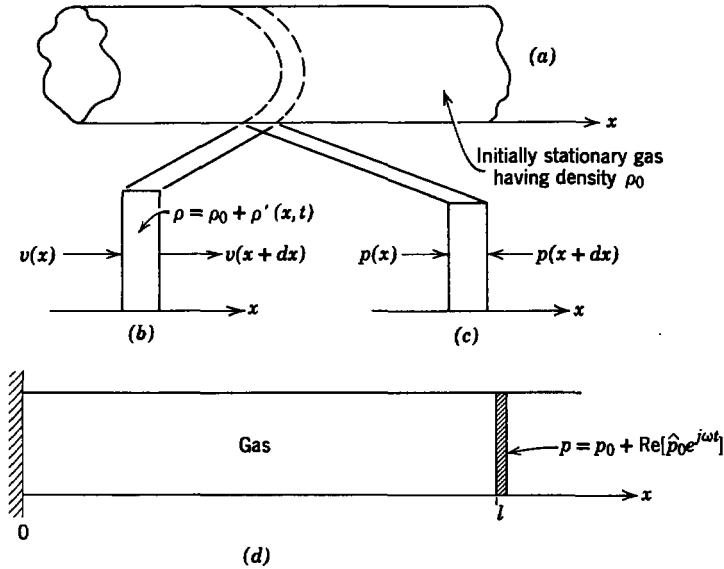


Fig. 9P.17

- (a) Show that conservation of mass is expressed by

$$\rho_0 \frac{\partial v}{\partial x} = -\frac{\partial \rho'}{\partial t},$$

where we ignore products of perturbation quantities and $\rho = \rho_0 + \rho'(x, t)$; ρ_0 is the density of the gas when it is stationary.

- (b) Show that conservation of momentum is expressed by

$$\rho_0 \frac{\partial v}{\partial t} = -\frac{\partial p'}{\partial x},$$

where $p = p_0 + p'(x, t)$ and p_0 is the pressure when the gas is stationary.

- (c) Physical measurements show that the pressure and density are related by a given function $p = p(\rho)$ or

$$p = p_0(\rho_0) + \left(\frac{\partial p}{\partial \rho} \right)_{\rho_0} \rho'.$$

Use the results of (a) and (b) to find one equation for the velocity.

- (d) Figure 9P.17d shows the tube driven at $x = l$ by a piston and terminated at $x = 0$ in a rigid wall. What is $v(x, t)$?

9.18. Figure 9P.18 shows the distribution of velocity $\partial \delta / \partial t$ at $t = 0$ in a thin elastic rod of infinite length. Assume that the rod is characterized by the constants E , ρ , and A , where E is Young's modulus, ρ is the density and A is the cross-sectional area of the rod. Given that $T(x, 0) = 0$, make a plot of $T(x, t)$ in the x - t plane. This plot should be similar to that shown in Fig. 9.1.8.

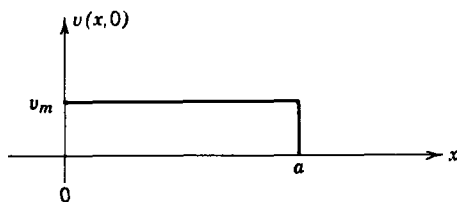


Fig. 9P.18

9.19. Figure 9P.19 shows the distribution of stress $T(x, 0) = E(\partial\delta/\partial x)$ at $t = 0$ in a thin elastic rod of infinite length. Assume that the rod is characterized by the constants E , ρ , and A , where E is Young's modulus, ρ is the density, and A is the cross-sectional area of the rod. Given that $v(x, 0) = 0 = \partial\delta/\partial t$, make a plot of $T(x, t)$ and $v(x, t)$ in the x - t plane. This plot should be similar to that shown in Fig. 9.1.8.

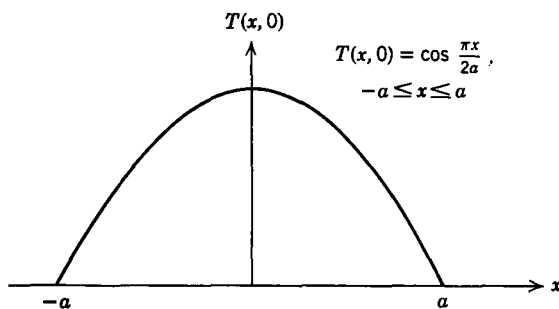


Fig. 9P.19

9.20. A long thin rod supports longitudinal motions $\delta(x, t)$.

- (a) Consider first the case in which the material initially has the velocity distribution $(\partial\delta/\partial t)(x, 0)$ shown in Fig. 9P.20a and initially $(\partial\delta/\partial x)(x, 0) = 0$. The rod has a free end at $x = 0$ but extends to infinity in the positive x -direction. Sketch the resulting velocity $v(x, t)$ in the x - t plane.
- (b) Now, in addition to the initial velocity of Fig. 9P.20a, the end of the rod at $x = 0$ is driven by a force that constrains the stress $T(0, t)$ as shown in Fig. 9P.20b. Sketch the resulting velocity $v(x, t)$ in the x - t plane.

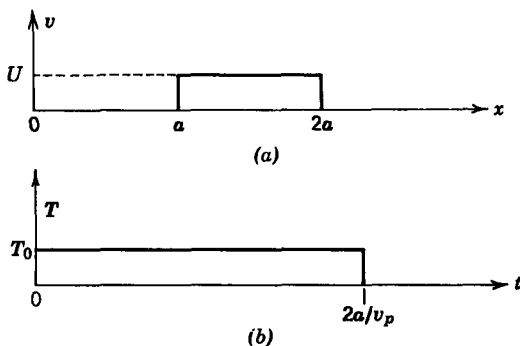


Fig. 9P.20

9.21. An electromechanical device that transduces a pulse of current $I(t)$ into a delayed stress pulse to rupture a diaphragm is shown in Fig. 9P.21. The current $I(t)$ passes through parallel, highly conducting plates shorted by a movable block with conductivity σ . The resulting motion initiates a stress at the left end of the rod, which propagates to the right end to provide the rupturing stress. The current has the form

$$I(t) = \begin{cases} \frac{I_0}{2} \left(1 - \cos \frac{2\pi t}{\tau} \right), & 0 < t < \tau, \\ 0, & t < 0, t > \tau. \end{cases}$$

The time required for the pulse to traverse the length of the rod is long compared with τ .

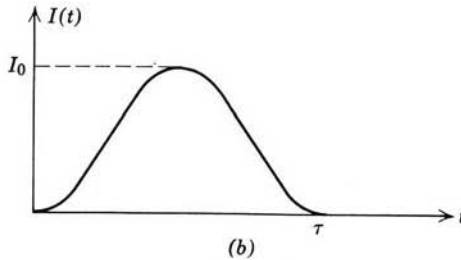
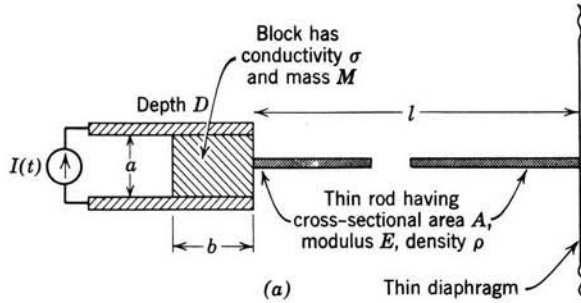


Fig. 9P.21

- Model the electromechanics by assuming that the current is returned on the inside surface of the block and write the boundary condition imposed on the left end of the rod by the magnetic force.
- For what values of τ (in terms of b , σ , etc.) would the surface current model be appropriate?
- Under what conditions can the mass M of the block be ignored in the boundary condition of (a)? Under this condition what is the current I_0 required to produce the peak stress T_r at the right end of the rod? (For this calculation assume that the right end of the rod is fixed in displacement.)

9.22. You are given the following pair of nonlinear differential equations:

$$\frac{\partial W}{\partial t} + W \frac{\partial W}{\partial x} + \frac{\partial U}{\partial x} + \frac{K}{U^3} \frac{\partial U}{\partial x} = 0, \quad (1)$$

$$\frac{\partial U}{\partial t} + \frac{\partial(UW)}{\partial x} = 0, \quad (2)$$

where K is a positive constant. In an equilibrium condition W is zero and U is a positive constant given by $U = C$.

- Linearize (1) and (2), using the given equilibrium conditions.
- By use of the equations of part (a), what is $W(x, t)$ if at $x = 0$, $W = 0$, at $x = -L$, $W = W_0 \cos \omega t$?

9.23. Two identical elastic membranes of mass σ_m per unit area and equilibrium tension S are joined together at $x = 0$ (Fig. 9P.23). Also attached to the membranes at $x = 0$ is a bar of mass M per unit width (kg/m). Both membranes are tied to rigid walls at their other ends ($x = \pm L$).

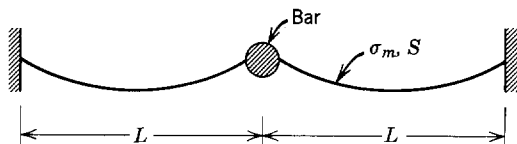


Fig. 9P.23

- Compute the natural frequencies of the system.
- Explain the effect of the mass/unit length M on the natural frequencies. In particular, give a physical reason for what happens when $M = 0$ and $M \rightarrow \infty$.

9.24. Figure 9P.24 shows an elastic membrane fixed at $x = L$ and $x = -L$ and coupled to a pair of capacitor plates at $x = 0$.

- Find the natural frequencies of the system.
- What is the effect on the natural frequencies of raising the voltage V_0 ?

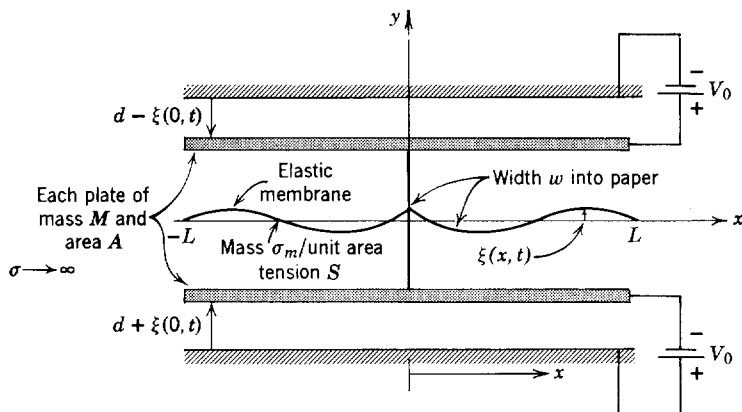


Fig. 9P.24

9.25. A wire is pinned at $x = 0$, where a potentiometer is attached. With the help of an amplifier G , this potentiometer produces a current i_1 proportional to the slope.

$$i_1 = G \frac{\partial \xi}{\partial x}(0, t), \quad G = \text{constant}.$$

The current i_1 is used to drive transducers, which in turn motivate the end of the spring at $x = -l$. The two transducers, which are alike, are connected so that in the absence of i_1

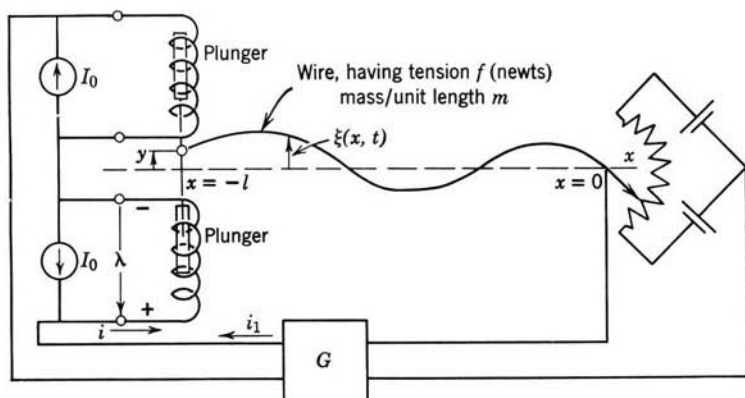


Fig. 9P.25

there is no net force on the wire. The terminal relation for the lower transducer is

$$\lambda = \left(\frac{L_0}{a} y \right) i,$$

where $y = \xi(-l, t)$ as shown in Fig. 9P.25.

- One boundary condition is $\xi(0, t) = 0$. What is the other condition? Ignore the mass of the plungers.
- Find a transcendental expression for the eigenfrequencies (natural frequencies) of the system. Make a plot that shows the graphical solution of this equation.
- What are the natural frequencies when $G = 0$?
- What is the effect of raising the gain G on the first and second nonzero eigenfrequencies?
- Is it possible that this system can be unstable? If so, for what values of G ?

9.26. A magnetic transducer is used to excite a thin elastic membrane of length L , as shown in Fig. 9P.26. The mass M is attached to the membrane, which in turn is fixed to a rigid support at $x = 0$. The driving current $I(t)$ is much smaller than I_0 and is given by

$$I(t) = \text{Re} [\hat{I} \exp(j\omega t)].$$

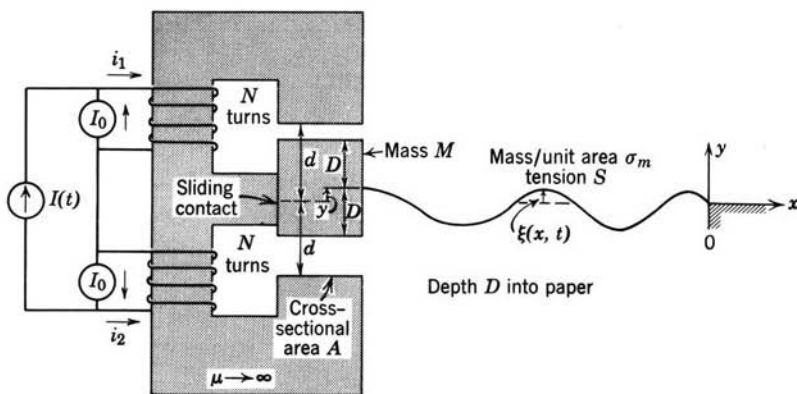


Fig. 9P.26

Work under the assumption that the displacements of the mass from its equilibrium position are small:

- Find an expression for the position of the mass, $y(t)$. You may assume that when there is no applied current the mass is centered between the pole faces.
- Find the resonance frequencies of the system. Your solution may be represented graphically (e.g., see Fig. 9.2.11).

9P.27. A string with tension f and a mass per unit length m is fixed at $x = \pm l$, as shown in Fig. 9P.27. At $x = 0$ it is subject to the force $F(t)$ shown in Fig. 9P.27a.

- Write the boundary condition at $x = 0$ which relates $F(t)$ to ξ . For this purpose divide the function $\xi(x, t)$ into two functions valid to the left and to the right of $x = 0$.
- The displacement $\xi(x, t)$ can be divided into an odd and an even function of x . Show that the odd function $\xi(x, t) = -\xi(-x, t)$ is not excited by the driving force.
- We now confine ourselves to displacements that are even functions of x , $\xi(x, t) = \xi(-x, t)$. For $t < 0$ the string assumes a static shape with the force $F(t) = F_0$, where F_0 is a given constant (see Fig. 9P.27b). Use the equation of motion to find $\xi(x)$, $x > 0$, for $t < 0$.
- When $t = 0$, the force $F(t)$ becomes a cosinusoid, as shown in the figure. Find the displacement $\xi(x, t)$, $x > 0$, $t > 0$. (Reference. Section 9.2.1.)

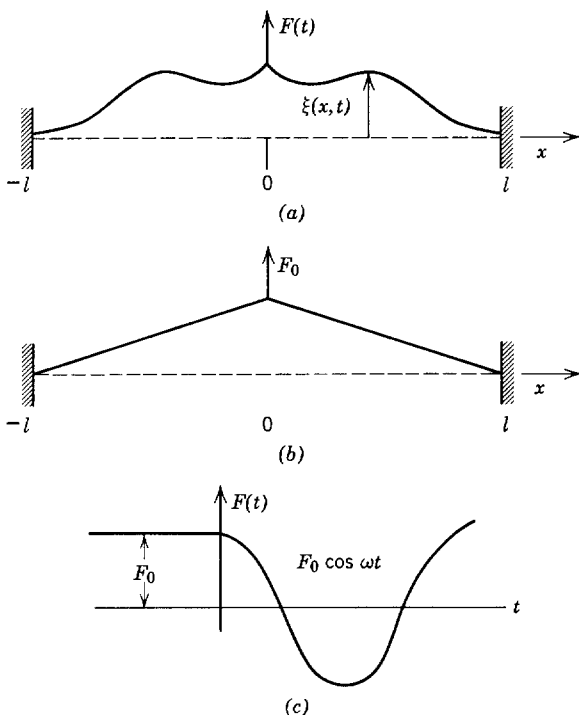


Fig. 9P.27

Chapter 10

DYNAMICS OF ELECTROMECHANICAL CONTINUA

10.0 INTRODUCTION

In Chapter 9 we treated simple examples of mechanical continua to establish the basic techniques of making mathematical models and to illustrate the kinds of dynamic behavior and the mathematical methods needed in analyses. In that chapter simple elastic continua at rest were excited at boundaries so that the resulting continuum dynamics were determined by mechanical characteristics alone.

In this chapter we still restrict our attention to simple elastic continua but we generalize on the treatment of Chapter 9 to include the effects of distributed forces of electric origin and material motion. By the use of simple models we illustrate the basic phenomena that occur in a wide variety of physical systems and the analytical techniques used in their mathematical description. In spite of the diversity of physical situations in which continuum electromechanical interactions are important, a unity results from mathematical techniques that are common to all of the situations. It is our purpose here to illuminate, in the simplest context possible, these mathematical techniques and the physical phenomena they describe.

As stated earlier, the techniques presented are fundamental to a wide variety of physical situations. It is therefore helpful for the purpose of appreciating our objectives to review some of the technical areas concerned with continuum electromechanics.

Magnetohydrodynamics (MHD) is concerned with the interactions of free currents and magnetic fields in fluids (liquids and gases) which have high enough electrical conductivity that a quasi-static magnetic field model is appropriate for describing the electromagnetic part of the system. To reflect more accurately the nature of the mechanical medium this area is sometimes referred to as *magnetogasdynamics (MGD)* or as *magnetofluidynamics (MFD)*. Areas of application include pumping and levitation of liquids (usually

metals), orientation and confinement of extremely hot ionized gases or plasmas, as, for example, in thermonuclear fusion experiments,* electric power generation from ionized gases produced by combustion of fossil fuels or from heat produced in a fission reactor,† and space propulsion achieved by electromagnetic acceleration of ionized gases.‡ Scientific interest in this area includes such geophysical and astrophysical topics as the origin of the earth's magnetic field in its liquid metal core and the dynamics of stellar structures composed of highly ionized gases.

A similar area is *ferrohydrodynamics*, which is concerned with magnetization interactions of magnetic fields with a ferromagnetic fluid.§

Electrohydrodynamics (EHD) is concerned with interaction between electric fields and free or bound (polarization) charges in fluids. The fluids may be extremely good insulators, slightly conducting, or even highly conducting. The distinguishing feature is that the electromagnetic part of the system is described by a quasi-static electric field model. Applications of EHD include pumping and levitation of liquids and gases, extraction of contaminants from gases such as smoke,** mixing of liquids, orientation of liquids in near-zero-gravity environments, augmentation of heat transfer, and property measurements in fluid systems. EHD interactions also occur in meteorology, in which charge distribution in the atmosphere (as in a thunderstorm) is important, and in surface physics, in which the distribution of charges at an interface is significant, as in frictional electrification.††

The engineering and scientific applications of *electron and ion beams* involve continuum electromechanical interactions. Electron beams, confined by magnetic fields and interacting through electric fields with distributed electric circuits, are commonly used to generate power at microwave frequencies.‡‡ In such applications the beam is represented by quasi-static equations, but the distributed electric circuits support electromagnetic waves and are not amenable to quasi-static analysis. Electron beams are also used for heating, welding, forming, and purifying metals. Charged particle beams, electrons and ions, are used for medical treatment, for measuring collision cross sections, and for heating plasmas.§§

* D. J. Rose and M. Clark, Jr., *Plasmas and Controlled Fusion*, M.I.T. Press and Wiley, New York, 1961.

† G. W. Sutton and A. Sherman, *Engineering Magnetohydrodynamics*, McGraw-Hill, New York, 1965.

‡ *Ibid.*, p. 447.

§ R. E. Rosensweig, "Magnetic Fluids," *Intern. Sci. Technol.*, **55**, 48-56, 90 (July 1966).

** H. J. White, *Industrial Electrostatic Precipitation*, Addison-Wesley, Reading, Mass., 1963.

†† L. B. Loeb, *Static Electrification*, Springer, Berlin, 1958.

‡‡ C. C. Johnson, *Field and Wave Electrodynamics*, McGraw-Hill, New York, 1965, p. 275.

§§ T. H. Stix, *The Theory of Plasma Waves*, McGraw-Hill, New York, 1962, p. 107.

*Plasma dynamics** is concerned with the behavior of gases composed at least in part of charged particles. Thus continuum electromechanical interactions will affect the behavior of a plasma. Probably the most common example of a plasma is the ionized gas in a fluorescent lamp. Other examples are gas-filled rectifiers, flames such as rocket exhausts, and the sun. The physical characteristics of ionized gases can assume many forms. In certain cases, such as proposed fusion devices and MHD generators, the plasma behaves as a highly conducting fluid and its dynamic behavior is described by a magnetohydrodynamic approximation. In other cases the plasma is only slightly ionized and the electrohydrodynamic equations are appropriate. In still other cases the plasma may be so tenuous that it is best described as a collection of noninteracting particles in imposed magnetic and electric fields. In all of these regimes the plasma exhibits the basic phenomena of wave propagation and instability, subjects that are treated in this chapter.

Electrons and holes in semiconductors, usefully modeled as plasmas, give rise to the name *solid-state plasmas*.† These charges behave collectively like gaseous plasmas and are thus amenable to analysis with the same types of mathematical model. Electromechanical interactions in solid-state plasmas are used to achieve microwave power generation and to make electronic components with a variety of useful terminal characteristics.

Electromechanical interactions of several types occur with elastic solids and lead to useful devices such as transducers. *Electroelasticity* and *piezoelectricity*‡ result from polarization interactions in elastic dielectrics and are modeled as quasi-static electric field systems. *Magnetoelasticity* and *piezomagnetism*§ result from magnetization interactions in elastic solids and are modeled as quasi-static magnetic field systems. Magnetoelastic phenomena have found applications in *microwave magnetism*** in which electromechanical interactions lead to useful microwave components.

The foregoing examples illustrate the diversity of physical situations, in which continuum electromechanical interactions are important, and the variety of applications resulting therefrom. We now proceed to study the kinds of dynamical phenomena that can result from these interactions.

In the study of lumped parameter systems defined by linear, constant-coefficient equations, the temporal behavior is characterized by e^{st} , as illustrated in Chapter 5. Similarly, the dynamics of a continuum with a single

* L. Spitzer, Jr., *Physics of Fully Ionized Gases*, Interscience, New York, 1956. (The plasma, as defined on p. 17 of this reference, is somewhat more specifically defined than in our discussion.)

† R. Bowers and M. C. Steele, *Proc. I.E.E.E.*, **52**, 1105 (1964).

‡ W. P. Mason, *Physical Acoustics*, Vol. 1, Part A, Academic, New York (1964, pp. 169–270).

§ *Ibid.*

** W. F. Brown, *Micromagnetism*, Interscience, New York, 1963.

space dependence x can be represented in terms of solutions in the form*

$$\xi = \text{Re } \hat{\xi} e^{j(\omega t - kx)}. \quad (10.0.1)$$

The (angular) frequency ω can, in general, be complex

$$\omega = \omega_r + j\omega_i, \quad (10.0.2)$$

just as s can be complex in the lumped-parameter case. As we shall see, the physical significance of a complex ω is not so clear as it is in the lumped-parameter case because the temporal dependence is only part of the story. The wavenumber k , which represents the spatial dependence, can also be complex:

$$k = k_r + jk_i. \quad (10.0.3)$$

The sections that follow have the objective of imparting some physical and mathematical insight as to how wave solutions in the form of (10.0.1) are used to describe the dynamics of one-dimensional continuous media.

10.1 WAVES AND INSTABILITIES IN STATIONARY MEDIA

It is the purpose of this chapter to illustrate the dynamics of electromechanical continua. This is done by making use of the one-dimensional model for the wire and membrane, introduced in Chapter 9. It should be evident from the introduction that the cases considered in the following sections illustrate the dynamics of a variety of purely mechanical and purely electrical as well as electromechanical systems. A salient feature of the dynamics of systems involving continuous media is the effect of material motion. Our development in this section is confined to the dynamics of systems in which the mechanical medium is at rest. The effect of convection, or gross material motion, on these systems is taken up in the next section.

It is helpful to associate fundamental types of dynamical behavior with a particular physical situation. One of the simplest we can imagine for this purpose is the "string" or wire under tension. If the wire has an equilibrium tension f and a mass per unit length m , its transverse deflections are governed by the equation

$$m \frac{\partial^2 \xi}{\partial t^2} = f \frac{\partial^2 \xi}{\partial x^2} + S_z, \quad (10.1.1)$$

where S_z is a transverse force per unit length (see Table 9.2). In the sections that follow we consider electromechanical coupling to the string through

* The use of $j\omega$ rather than s is simply a matter of convention.

S_z . Before embarking on this development it would be helpful to recall the essential features of the string dynamics in the absence of coupling.

10.1.1 Waves without Dispersion

In the absence of external forces the string deflections $\xi(x, t)$ must satisfy the wave equation [(10.1.1) with $S_z = 0$]

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2}, \quad (10.1.2)$$

where $v_s = \sqrt{f/m}$. In Section 9.2.1 we found both the sinusoidal steady-state and the transient response (normal modes) of the string by making appropriate use of solutions in the form of (10.0.1). Substitution of the assumed solution into the equation of motion (10.1.2) shows that ω and k must be related by the dispersion equation

$$\omega^2 = v_s^2 k^2. \quad (10.1.3)$$

Thus the ω - k plot for waves on the simple string appears as shown in Fig. 10.1.1. For the real value of frequency ω shown in the figure there are two solutions ($\pm\beta$) to (10.1.3) for the wavenumber k .

From the form of the wave solution (10.0.1) it follows that points of constant phase move along the x -axis with a phase velocity

$$v_p = \frac{\omega}{k}. \quad (10.1.4)$$

Figure 10.1.1 shows that for a given frequency there are two waves with phase velocities that are negatives. Note that the phase velocities of the waves are

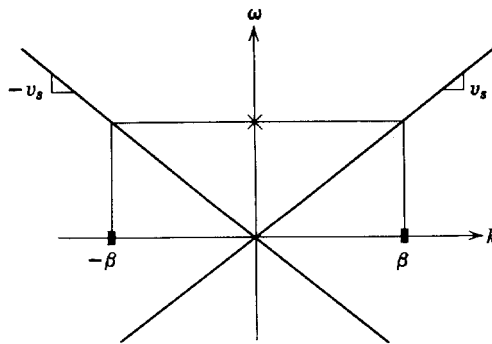


Fig. 10.1.1 Dispersion equation for waves on the simple string.

simply $\pm v_s$ and are independent of the frequency; that is, the dispersion equation for a given wave is a linear relationship between ω and k . For the simple string, waves with differing frequencies (or wavenumbers) propagate with the same phase velocity and are therefore said to propagate without dispersion.

10.1.2 Cutoff or Evanescent Waves

The string, described by (10.1.1), can be a taut conducting wire (or tightly wound helical spring) carrying a current that interacts with a magnetic field (Fig. 10.1.2). Here the string is stretched along the x -axis between two magnet coils arranged to give a null in magnetic field along the x -axis. The wire is free to vibrate in the horizontal plane of the magnet coils. For small excursions from the x -axis the wire experiences a magnetic flux density $\mathbf{B}$, which is essentially a linear function of ξ . Hence, when a current I is passed through the wire in the direction shown, a restoring force per unit length $\mathbf{I} \times \mathbf{B}$ tends to return the wire to the x -axis:

$$S_z \dot{\mathbf{i}}_z = \mathbf{I} \times \mathbf{B} = -Ib \xi \dot{\mathbf{i}}_z; \quad (10.1.5)$$

b is shown in Fig. 10.1.2b. This is the force-displacement relationship of a simple spring. The current $\mathbf{I}$ and flux density $\mathbf{B}$ interact to produce a spring-like force per unit length that tends to return the wire to its equilibrium position.

With the addition of coupling to the magnetic field, the equation of motion

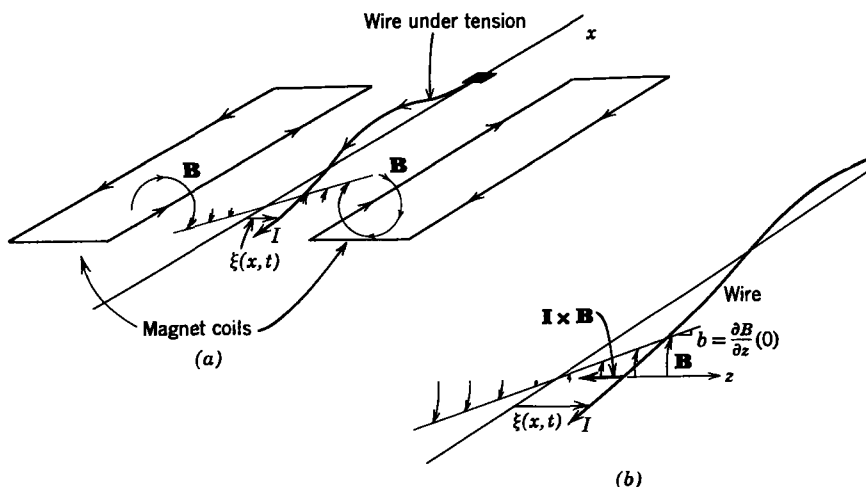


Fig. 10.1.2 (a) A conducting wire is stretched along the x -axis and is free to undergo transverse motions in the horizontal plane. Magnet coils produce a field $\mathbf{B}$ which is zero along the x -axis; (b) the wire carries a current I so that deflections from the x -axis result in a force that tends to restore the wire to its equilibrium position.

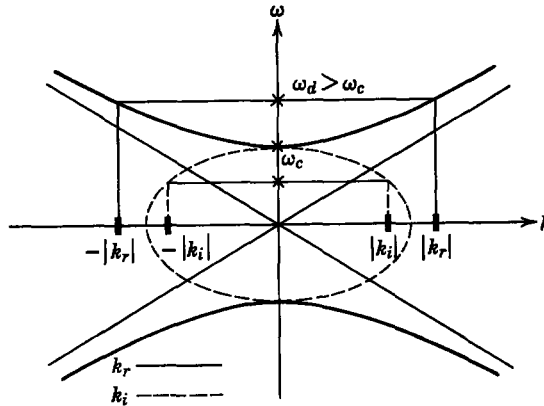


Fig. 10.1.3 Dispersion relation for the wire subject to a restoring force distributed along its length (for the case shown in Fig. 10.1.2). Complex values of k are shown as functions of real values of ω .

(10.1.1) becomes

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2} - \omega_c^2 \xi, \quad \omega_c^2 = \frac{Ib}{m}. \quad (10.1.6)$$

Now, substitution of solutions in the form of (10.0.1) gives a dispersion equation that is not simply the linear relationship between ω and k of (10.1.3). Rather

$$\omega^2 = v_s^2 k^2 + \omega_c^2. \quad (10.1.7)$$

For large values of k , ω does not depend significantly on ω_c , hence the asymptotes of an ω - k plot are simply the two straight lines of the ω - k plot with $\omega_c = 0$ (Fig. 10.1.1). If we note further that $\omega = \pm \omega_c$, where $k = 0$, the ω - k plot of Fig. 10.1.3 can be easily sketched. The effect of the current I is to evolve a pair of hyperbolas from the two straight lines of Fig. 10.1.1. As the current I is raised (with B fixed), the intersections ω_c move out along the ω -axis.

The dispersion equation is quadratic in both ω and k . Hence for each real value of k two values of ω can be seen from the solid curves of Fig. 10.1.3 always to be real. This relationship, however is not true if ω and k are interchanged. Real values of ω give real values of k only if $|\omega| > \omega_c$, as can also be seen in the figure. Solution of (10.1.7) shows that in the range $|\omega| < \omega_c$ the dispersion equation gives imaginary values of k represented by the dashed curve in Fig. 10.1.3. Note that when $k = jk_i$ in (10.1.7) the analytical geometric relation between ω and k_i is an ellipse; for example at the points at which $\omega = 0$ (10.1.7) shows that $k = \pm j\omega_c/v_s$.

The most evident consequences of introducing the additional force, which we can think of as resulting because of the current I , are illustrated by

considering the response to a sinusoidal, steady-state driving function. For purposes of illustration the spring is fixed at $x = 0$ and given a sinusoidal displacement at $x = -l$:

$$\xi(0, t) = 0, \quad (10.1.8)$$

$$\xi(-l, t) = \xi_a \sin \omega_d t. \quad (10.1.9)$$

These conditions are the same as those considered in Section 9.2.1a for the wire with no external force.

To match the driving condition the frequency ω of the dispersion equation is taken as ω_d , and solutions have the form of (10.0.1) for two values of k which can be found by solving (10.1.7).

$$k = \pm \frac{\sqrt{\omega_d^2 - \omega_c^2}}{v_s}. \quad (10.1.10)$$

These wavenumbers are shown graphically in Fig. 10.1.3.

To avoid ambiguity the wavenumbers consistent with the frequency $\omega = \omega_d$, given by (10.1.10) are written as

$$\begin{aligned} k &= \pm |k_r|, & \omega_d &> \omega_c, \\ k &= \pm j |k_i|, & \omega_d &< \omega_c. \end{aligned} \quad (10.1.11)$$

This is the notation used in Fig. 10.1.3. At this point we have established that there are two waves on the wire with the frequency ω_d , and the boundary conditions are satisfied by taking a linear combination of these waves. For $\omega_d > \omega_c$

$$\xi = \text{Re} (C e^{-j|k_r|x} + D e^{j|k_r|x}) e^{j\omega_d t} \quad (10.1.12)$$

and a similar equation with $|k_r| \rightarrow j|k_i|$ for $\omega_d < \omega_c$. The constants C and D are determined by the boundary conditions (10.1.8) and (10.1.9). It follows that for

$$\omega_d > \omega_c, \quad \xi = -\xi_a \frac{\sin |k_r| x}{\sin |k_r| l} \sin \omega_d t, \quad (10.1.13)$$

and for

$$\omega_d < \omega_c, \quad \xi = -\xi_a \frac{\sinh |k_i| x}{\sinh |k_i| l} \sin \omega_d t, \quad (10.1.14)$$

The significance of the imaginary wavenumbers is now apparent. Figure 10.1.4 shows the effect of raising the current I (i.e., ω_c) with the driving frequency ω_d fixed. With no current, waves on the wire have the same familiar appearance as in Section 9.2.1a (see Fig. 9.2.6). As long as $\omega_d > \omega_c$, this same general appearance prevails and the wire tends to bow outward and assume the shape usually found with standing waves. When ω_c has been raised to the point at which it coincides with ω_d , the wire takes on the appearance shown in Fig. 10.1.4b. Here the effect of inertia, which tends to make the wire bow outward,

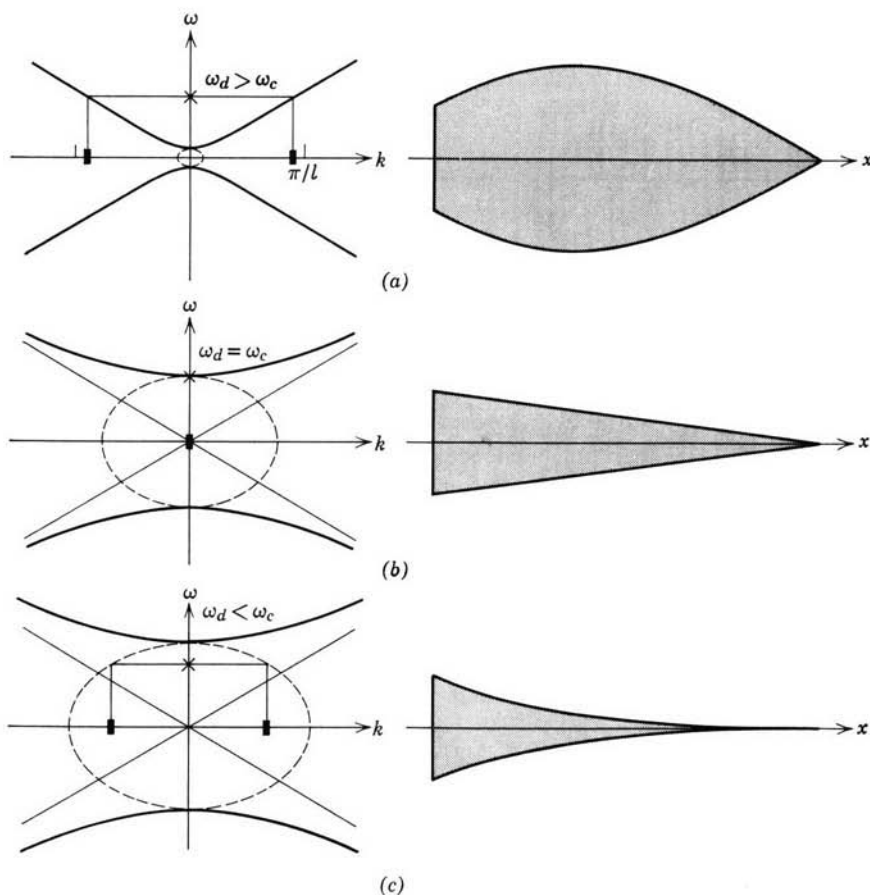


Fig. 10.1.4 Envelope of wire deflection in magnetic field. The wire is fixed at the right end and driven at a fixed sinusoidal frequency at the left end. The ω - k plots show the effect of the current I on the dispersion equation. The current I (or cutoff frequency ω_c) is being raised so that (a), $I \approx 0$, (b) I is sufficient just to cut off the propagation ($\omega_d = \omega_c$), and (c) the waves are evanescent, $\omega_d < \omega_c$. This experiment can be seen in the film "Complex Waves I," produced for the National Committee on Electrical Engineering films by Education Development Center, Newton, Mass.

is just canceled by the restoring force due to I . The wavenumbers are zero and the envelope of the wire deflection is a linear function of x . The limit of either (10.1.13) or (10.1.14) as $k \rightarrow 0$ yields

$$\xi = -\xi_d \left(\frac{x}{l} \right) \sin \omega_d t. \quad (10.1.15)$$

This is called the cutoff condition, hence ω_c is the *cutoff frequency*. As ω_c is raised still further, the wire bows inward, as shown in Fig. 10.1.4c. Here the

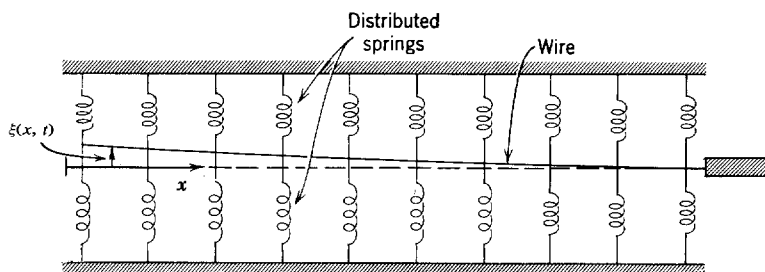


Fig. 10.1.5 With the current as shown, the system in Fig. 10.1.2 gives rise to forces on the wire that are equivalent to springs distributed along the length of the wire.

deflection amplitude simply decays spatially as we move away from the point of excitation. The spatial dependence of (10.1.14) is monotonic in x , so that the wire deflections have the same phase as the excitation. Note that these decaying or *evanescent* waves do not involve dissipation. They are present here because each segment of the wire is subject to a springlike restoring force that tends to push it toward the x -axis. An equivalent physical situation is shown in Fig. 10.1.5. Without recourse to mathematics we expect that when the left end of the wire is slowly displaced upward the wire tends to bow inward toward the x -axis. This is all that (10.1.14) is telling us. The restoring force that we would feel in slowly displacing the end of the wire would be the same as if we were displacing the end of a spring.

As we shall see, continuous media excited in the sinusoidal steady state can also support spatially decaying waves because of damping (dissipation of energy). Even more interesting are situations in which complex wavenumbers mean that waves grow spatially rather than decaying. Both cases, which are considered in later sections, involve dispersion equations that give complex values of k for real values of ω , just as we have here, and it is important to establish a physical picture of their significance.

Evanescent or cutoff waves are often found in studies of propagation through guiding structures. We shall see this in Section 10.4.1 in which waves that propagate in the x -direction on a membrane fixed along its edges at $y = 0$ and $y = b$ have a cutoff frequency below which they are evanescent. This is an example of a wide class of two- and three-dimensional situations in which propagation in a longitudinal (x) direction is restricted by boundary conditions in the transverse directions (y and z). The transverse boundaries restrict (or squash) the dynamical motions, just as they do in Fig. 10.1.5. Before propagation can take place the frequency of the drive must exceed some cutoff value in which the constraints produced by the transverse boundaries are canceled by a dynamic effect such as that resulting here from inertia. This is illustrated for waves in elastic structures in Sections 11.4.2b and 11.4.3.

Cutoff waves are found also in electromagnetic waveguides,* in which they are of considerable significance in establishing the propagation of a single mode (all modes but one are cutoff) and in which they are sometimes used to make microwave attenuators.

When an electromagnetic wave propagates through a medium, such as a plasma, it is possible that the medium will have the same effect on the waves as the springs of Fig. 10.1.5 have on waves propagating on the string. Hence cutoff or evanescent waves are of considerable importance in studies of wave propagation through plasmas.† Example 10.1.1 considers a situation, similar to that found in studies of the hydromagnetic equilibrium of fusion machines, in which evanescent waves account for the stabilizing (stiffening) influence of transverse boundaries.

Before considering what happens when I is reversed, it would be worthwhile to draw attention to the effect of the current I shown in Fig. 10.1.2 on the natural modes of vibration. Remember (from Section 9.2.1b) that it is these modes that characterize the response of the wire to initial conditions.

To find the eigenfrequencies with both ends ($x = 0$ and $x = -l$) of the string fixed, we once again take a linear combination of waves with wave-numbers k found from the dispersion equation (10.1.7) for a given value of ω . At the outset this frequency ω remains to be found.

$$\xi = \text{Re} (Ce^{-i\beta x} + De^{i\beta x})e^{i\omega t}. \quad (10.1.16)$$

For convenience we have written the two values of k as $k = \pm\beta$, with β defined as

$$\beta = \frac{\sqrt{\omega^2 - \omega_e^2}}{v_s}. \quad (10.1.17)$$

It is clear that to satisfy the boundary condition $\xi(0, t) = 0$, the constants C and D are related by $C = -D$, so that (10.1.16) becomes

$$\xi = \text{Re} \xi \sin \beta x e^{i\omega t}, \quad (10.1.18)$$

where ξ includes the remaining arbitrary constant. Then to satisfy the boundary condition $\xi(-l, t) = 0$ it follows from (10.1.18) that

$$\sin \beta l = 0. \quad (10.1.19)$$

* See R. B. Adler, L. J. Chu, and R. M. Fano, *Electromagnetic Energy Transmission and Radiation*, Wiley, New York, 1960, pp. 369–378, or S. Ramo, J. R. Whinnery, and T. Van Duzer, *Fields and Waves in Communications and Electronics*, Wiley, New York, 1965, p. 420.

† See T. H. Stix, *The Theory of Plasma Waves*, McGraw-Hill, New York, 1962, p. 13, or W. P. Allis, S. J. Buchsbaum, and A. Bers, *Waves in Anisotropic Plasmas*, M.I.T. Press, Cambridge, Mass., 1963, p. 13.

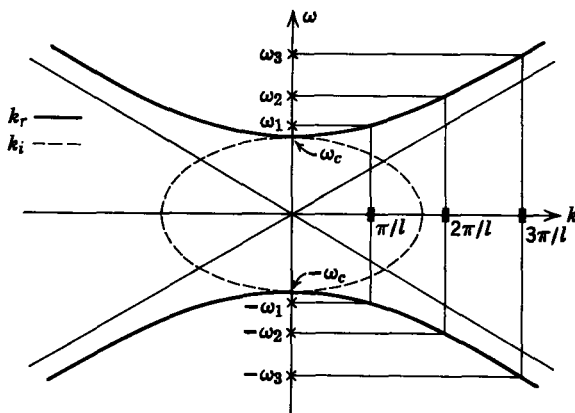


Fig. 10.1.6 A dispersion equation for waves on the wire in Fig. 10.1.2 showing the relationship between the eigenfrequencies ω_n and the eigenvalues $k = n\pi/l$.

This eigenvalue equation is the same as that for a wire without the magnetic force. It has solutions

$$\beta l = n\pi, \quad n = 1, 2, 3, \dots \quad (10.1.20)$$

Now that we have seen a case in which complex wavenumbers have physical significance it is worthwhile to observe that the solutions to (10.1.19), given by (10.1.20), are the only solutions and that they constrain β to be real. We have omitted negative values of n , for the modes represented by these roots are redundant. Note that $n = 0$ is not included because it leads to an eigenvalue $\beta = 0$; hence from (10.1.18) it leads to no deflection. Solutions to the eigenvalue equation of this type, which lead to a vanishing eigenfunction, are referred to as trivial solutions. Now that the eigenvalues have been determined, (10.1.20), we can solve (10.1.17) for the corresponding eigenfrequencies:

$$\omega = \pm \omega_n = \pm \left[\left(\frac{n\pi}{l} \right)^2 + \omega_c^2 \right]^{1/2}. \quad (10.1.21)$$

These are the natural frequencies of the wire in the presence of the magnetic force. The natural modes can be seen from (10.1.18) to have the same form as those for the wire with $I = 0$:

$$\xi = \text{Re} (A_n^+ e^{j\omega_n t} + A_n^- e^{-j\omega_n t}) \sin \frac{n\pi x}{l}. \quad (10.1.22)$$

Note, however, that the frequencies ω_n are affected by the current I . Their relationship to the dispersion equation is shown graphically in Fig. 10.1.6. The eigenvalues are indicated along the k -axis. (Note that taking negative values of k results in redundant frequencies.) The corresponding eigenfrequencies are increased in magnitude by an increase in the current (increase

in ω_e). As should be expected, the magnetic force has a tendency to stiffen the wire, with the largest effect on the longest wavelengths or lowest frequencies (those least affected by the tension of the wire). The most significant effect of the current is on the lowest eigenfrequency ω_1 , which approaches ω_e more rapidly than the other frequencies as the current is increased. Note that because of the current the eigenfrequencies are no longer harmonically related.

Because the natural frequencies are purely real, they appear as resonances of the driven response. This was discussed in detail in Section 9.2.1a for the wire without magnetic coupling. It could have been discerned earlier in this section by observing the possibility of a finite response from (10.1.13), even as $\xi_a \rightarrow 0$, if the denominator also approaches zero. This requires that

$$\sin |k_r| l = 0, \quad (10.1.23)$$

which is the same condition as that in (10.1.19) except that $|k_r|$ is a function of ω_a (the driving frequency) rather than an unspecified ω . Hence, if the frequency ω_a is tuned to one of the eigenfrequencies, there is a resonance in the response. The resonance frequencies of the wire are shifted upward by increasing the current I (or increasing the cutoff frequency ω_e).

Example 10.1.1. Magnetic fields are sometimes used to contain and orient highly conducting media; for example, in devices proposed to achieve thermonuclear fusion the magnetic field is used to make a "bottle" for the plasma. For many purposes this plasma, or highly ionized gas, can be considered as a perfectly conducting fluid. The example discussed here illustrates the continuum dynamics of a medium which, like the plasma, is assumed to have an infinite conductivity.

A perfectly conducting membrane forms the deformable part of the system shown in Fig. 10.1.7. When the membrane is undeformed, it lies in the x - y plane and is immersed in a uniform constant magnetic field $\mathbf{H} = H_0 \mathbf{i}_x$. We consider deformations of the membrane

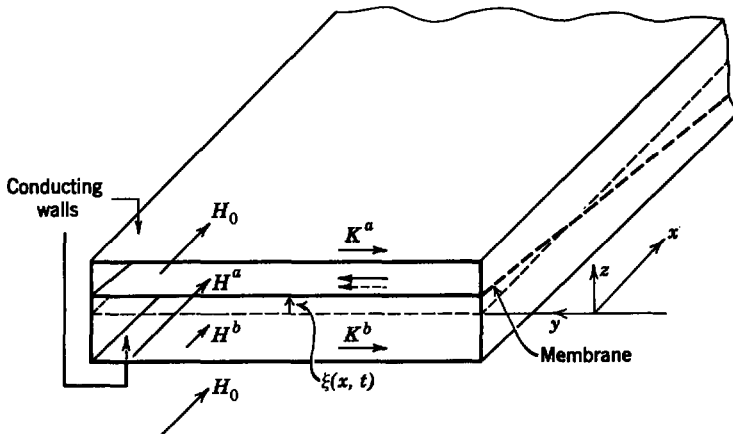


Fig. 10.1.7 A perfectly conducting membrane is bounded from above and below by fixed conducting walls and is immersed in an initially uniform magnetic field H_0 .

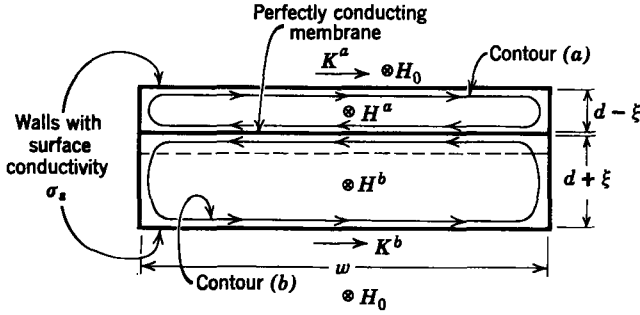


Fig. 10.1.8 Cross-sectional view of the apparatus shown in Fig. 10.1.7. The current loops (a) and (b), with the imposed magnetic field H_0 , couple the membrane to the upper and lower conducting walls.

$\xi(x, t)$ that do not depend on y ; hence the membrane has the general appearance shown in Fig. 10.1.7.

The membrane has a width w with its edges making electrical contact with the walls of a rectangular conducting conduit. Thus the membrane and the upper wall form a conducting loop [contour (a) shown in Fig. 10.1.8]. Similarly, the lower wall and membrane form the conducting loop (b). In the limit in which the conductivity of the walls (σ_s) is infinite the magnetic flux linked by each of these circuits must remain constant. Hence an upward deflection is accompanied by an increase in the field intensity above the membrane and a decrease in the field intensity below it. The conservation of flux is realized because of the flow of surface currents K^a and K^b as shown. This type of flux conservation dynamics is familiar from Section 5.1.3, in which it was encountered in the context of lumped-parameter dynamics.

The induced currents shown in Fig. 10.1.8 are responsible for creating a magnetic force on the membrane that tends to restore it to the plane of zero deflection. Therefore we can expect from the outset to obtain an equation of motion similar to (10.1.6).

The equation of motion for the membrane is

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \frac{\partial^2 \xi}{\partial x^2} + T_z, \quad (a)$$

where T_z is the force per unit area acting in the z -direction (see Table 9.2). The magnetic field intensity above the membrane (a) and below the membrane (b) can be written as

$$H^a = [H_0 + h^a(x, t)]\mathbf{i}_x, \quad (b)$$

$$H^b = [H_0 + h^b(x, t)]\mathbf{i}_x, \quad (c)$$

where h^a and h^b represent perturbations from the equilibrium magnetic field intensity caused by deflections of the membrane. We assume that the transverse displacements $\xi(x, t)$ vary slowly enough with x that at each cross section of the system shown in Fig. 10.1.7 the magnetic field is essentially x -directed and is independent of y and z .

Ampère's law requires that the jump in magnetic field intensity on the upper and lower plates be equal to their respective surface currents. Hence

$$h^a = K^a, \quad (d)$$

$$h^b = -K^b, \quad (e)$$

and we can find the perturbation field intensities by writing circuit equations for the currents K^a and K^b . The integral law of induction used in conjunction with the contour (a) of Fig. 10.1.8 is

$$\oint_{C_a} \mathbf{E}' \cdot d\mathbf{l} = - \frac{\partial}{\partial t} \int_{S_a} \mu_0 \mathbf{H} \cdot \mathbf{n} \, da, \quad (\text{f})$$

where the time derivative is written as a partial because the flux is a function of the longitudinal position x of the cross-sectional surface of integration S_a enclosed by the contour C_a . To evaluate the integral on the left note that $\mathbf{E}'$ is zero both in the perfectly conducting membrane and in the perfectly conducting end walls. In the upper plate the surface current density and electric field are related by Ohm's law, which takes the form

$$\mathbf{K}^a = \sigma_s \mathbf{E} \quad (\text{g})$$

because the upper wall is fixed. Equation f then becomes

$$\frac{wK^a}{\sigma_s} = - \frac{\partial}{\partial t} [w(d - \xi) \mu_0 (H_0 + h^a)]. \quad (\text{h})$$

Similar arguments, used with contour (b) in Fig. 10.1.8, produce the electrical equation

$$\frac{wK^b}{\sigma_s} = \frac{\partial}{\partial t} [w(d + \xi) \mu_0 (H_0 + h^b)]. \quad (\text{i})$$

We confine our attention to small displacements of the membrane so that it is appropriate to linearize these last two equations, which in view of (d) and (e) become

$$\frac{h^a}{\sigma_s} = - \frac{\partial}{\partial t} (\mu_0 d h^a - \mu_0 H_0 \xi), \quad (\text{j})$$

$$- \frac{h^b}{\sigma_s} = \frac{\partial}{\partial t} (\mu_0 d h^b + \mu_0 H_0 \xi). \quad (\text{k})$$

These expressions are the required electrical equations of motion that relate the field intensities above and below the membrane to the displacement ξ . The remaining equation is (a), with T_z written in terms of h^a and h^b .

To find T_z , the surface force density is related to the Maxwell stress T_{mn}^* by (8.4.2)

$$T_z = (T_{zm}^a - T_{zm}^b) n_m, \quad (\text{l})$$

where $\mathbf{n}$ is the vector normal to the membrane surface. Because $\mathbf{n} \approx i_z$, the summation on m has only one contribution, $m = z$, and (l) reduces to

$$T_z = T_{zz}^a - T_{zz}^b = -\frac{1}{2} \mu_0 [(H^a)^2 - (H^b)^2]. \quad (\text{m})$$

Now, if we introduce (b) and (c), this expression can be linearized to obtain

$$T_z = -\mu_0 H_0 (h^a - h^b). \quad (\text{n})$$

* Alternatively, the surface force density can be derived by using the energy method of Chapter 3 and viewing each section of the membrane as forming a pair of parallel plate inductors with the side walls. If this is done, remember that in addition to the two terminal pairs for these inductors there is a third terminal pair, constrained to constant current, for the magnet that makes H_0 .

The effect of finite conductivity in the walls has been included up to this point because it offers the possibility of illustrating at least two important dynamical effects. These are discussed when this example is continued in Sections 10.1.4 and 10.2.4. For now, we consider the limit in which σ_s is large enough that the fluxes linked by contours (a) and (b) remain essentially constant. Because the quantities in parentheses in (j) and (k) are the flux perturbations, they must therefore be zero and it follows that

$$h^b = \frac{H_0 \xi}{d}, \quad (o)$$

$$h^b = \frac{-H_0 \xi}{d}. \quad (p)$$

If we combine these expressions with (n) and (a), an equation of motion results:

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2} - \omega_c^2 \xi, \quad (q)$$

where

$$v_s = \left(\frac{S}{\sigma_m} \right)^{1/2},$$

$$\omega_c^2 = \frac{2\mu_0 H_0^2}{\sigma_m d}.$$

This expression has the same form as (10.1.6), and we can conclude that in the limit in which $\sigma_s \rightarrow \infty$ the membrane of Fig. 10.1.7 will have the same dynamical behavior found for the wire shown in Fig. 10.1.5.

10.1.3 Absolute or Nonconvective Instability

When the current is reversed in the experiment of Fig. 10.1.2, the wire dynamics are found to be altogether different from those described in the preceding section. As the current I (shown in Fig. 10.1.9) is raised, there is a threshold beyond which the wire bows outward. Under this condition, no matter how carefully the wire is placed on the center line between the coils, it bows outward when it is released.

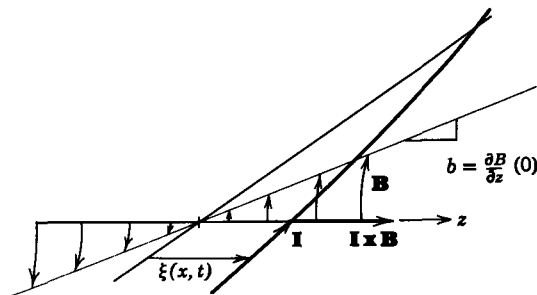


Fig. 10.1.9 Wire carrying current I in a magnetic field that is zero along the axis $\xi = 0$. Current is reversed from the situation shown in Fig. 10.1.2.

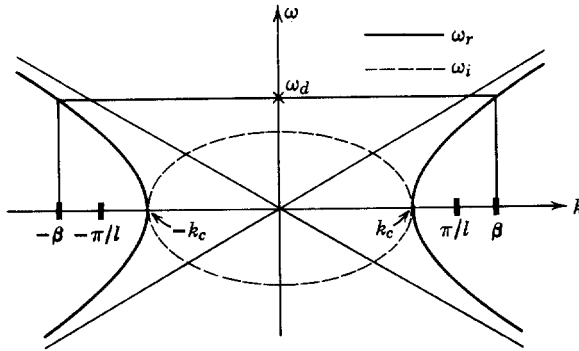


Fig. 10.1.10 Plot of the dispersion equation for physical situation shown in Fig. 10.1.2 with the current reversed, as in Fig. 10.1.9. Complex values of ω are shown for real values of k .

This might be expected. Reversing the current gives rise to a magnetic force, as in (10.1.5) with $I \rightarrow -I$, that tends to carry the wire in the same direction as the displacement. Hence it tends to produce an instability that is the continuum analogue of the static instability of lumped-parameter systems described in Section 5.1.2*a*. A transverse deflection, shown in Fig. 10.1.9, leads to a magnetic force that tends to make the deflection even larger.

The equation of motion is again (10.1.1), but now S_z is

$$S_z = Ib\xi \quad (10.1.24)$$

and the equation governing deflections of the wire is

$$\frac{1}{v_s^2} \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial^2 \xi}{\partial x^2} + k_c^2 \xi, \quad (10.1.25)$$

where

$$k_c^2 = \frac{Ib}{f}.$$

Again, the dispersion equation is found by substituting (10.0.1) into this equation of motion to obtain

$$\omega^2 = v_s^2(k^2 - k_c^2). \quad (10.1.26)$$

The asymptotes of this expression, plotted in the ω - k plane, are again straight lines with slopes $\pm v_s$. The two branches of the plot, however, now pass through the k -axis, as can be seen by setting $\omega = 0$ and solving (10.1.26) to obtain $k = \pm k_c$. Hence the real values of ω , as a function of real values of k , appear as shown by the solid lines in Fig. 10.1.10. For each value of ω there is a corresponding pair of real wavenumbers k . This is evident either from Fig. 10.1.10 or from (10.1.26) solved for k . By contrast only real values of k with a magnitude greater than k_c lead to real values of ω .

Note that the roles of ω and k are the reverse of what they were in the preceding section. Now, it is appropriate to plot complex values of ω for real values of k , and this is the significance of the broken line in Fig. 10.1.10. In the range of wavenumbers $-k_c < k < k_c$ the frequency ω is purely imaginary. Substitution of $\omega = j\omega_i$ in (10.1.26) shows that the relation between ω_i and k is an ellipse (Fig. 10.1.10).

Consider first the sinusoidal steady-state vibrations with the same boundary conditions (10.1.8) and (10.1.9) used in the preceding section. Then the same steps used to compute (10.1.13) give

$$\xi = -\xi_a \frac{\sin \beta x}{\sin \beta l} \sin \omega_a t, \quad (10.1.27)$$

where

$$\beta = \left(\frac{\omega_a^2}{v_s^2} + k_c^2 \right)^{1/2}. \quad (10.1.28)$$

This solution to (10.1.25) is simply a linear combination of waves for which the two wavenumbers $k = \pm\beta$ are found by solving the dispersion equation (10.1.26) with $\omega = \omega_a$. These solutions are shown graphically in Fig. 10.1.10.

The sinusoidal steady-state vibrations have the same general spatial appearance as the wire with no magnetic force. The frequency response, however, has been altered by the current I . As in Section 10.1.2, there are resonances in this response at the natural frequencies. They occur when the denominator of (10.1.27) goes to zero or when $\beta l = n\pi$, $n = 1, 2, 3, \dots$. It follows from (10.1.28) that the resonance frequencies are

$$\omega_a = \pm v_s \left[\left(\frac{n\pi}{l} \right)^2 - k_c^2 \right]^{1/2}, \quad n = 1, 2, 3, \dots \quad (10.1.29)$$

Remember that k_c can be increased by increasing the current I . The equation shows that the response frequencies are reduced by the interaction with the magnetic field. This is just the opposite of the effect found in the preceding section, in which these resonance frequencies were increased by the magnetic interaction.

As the current I is increased, the point is reached at which the lowest resonance frequency is reduced to zero. From (10.1.29) this critical condition occurs when

$$k_c = \frac{\pi}{l} \quad (10.1.30)$$

For larger values of k_c the lowest resonance frequency is no longer real. Because this resonance results when the driving frequency is tuned to the natural frequency, we suspect that the lowest natural frequency is complex after k_c exceeds π/l . Note that the driven response given by (10.1.27) is still perfectly valid even if k_c is larger than the critical value given by (10.1.30).

It is of crucial importance to recognize that once k_c exceeds π/l the sinusoidal steady-state response is of little significance. The lowest natural mode of the system is unstable and the transient solution, initiated by the initial conditions, eventually dominates the driven response. To see this we now consider the natural modes of the wire with both ends fixed.

The steps used in Section 10.1.2 in going from (10.1.18) to (10.1.22) are equally valid here. Hence the eigenmodes are given by (10.1.22). Now, however, the eigenfrequencies are given by the revised dispersion equation (10.1.26) with $k^2 = (n\pi/l)^2$.

$$\omega = \pm \omega_n = \pm v_s \left[\left(\frac{n\pi}{l} \right)^2 - k_c^2 \right]^{1/2}. \quad (10.1.31)$$

These eigenfrequencies are shown in Fig. 10.1.11, in which a graphical representation of the dispersion equation shows how the discrete values of k allowed by the fixed ends of the spring are related to the ω_n 's. From the figure it can readily be visualized how the frequencies of each mode shift inward as $k_c(l)$ is increased. As k_c is increased beyond $n\pi/l$, the frequencies of the n th mode move onto the ellipse and take the form

$$\omega_n = j |\omega_i|,$$

as can be seen from (10.1.31). The negative imaginary eigenfrequency gives rise to an eigenmode (10.1.22) that has an exponentially increasing amplitude.

In Section 9.2.1*b* we saw that the eigenmodes were orthogonal. Physically this meant that if initial conditions were such that only one of the modes was excited that mode would persist indefinitely without involving the others. [See Example (9.2.1) for a particular illustration of this point.] Suppose that

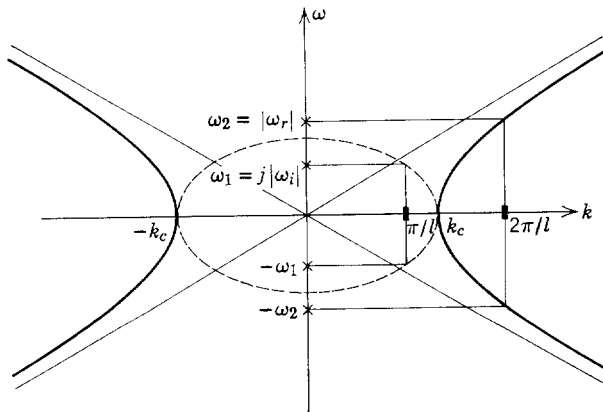


Fig. 10.1.11 The dispersion equation for the system of Fig. 10.1.2 with current as shown in Fig. 10.1.9. Complex values of ω are shown for real values of k . The allowed values of k give rise to the eigenfrequencies as shown.

the A_n 's in (10.1.22) were adjusted so that just the lowest mode was initiated. Then, with $k_e < \pi/l$, the vibrations of the wire would have the oscillatory dependence on x and t shown in Fig. 10.1.12a. With $k_e > \pi/l$, wire displacements would appear as the instability shown in Fig. 10.1.12b. This is an example of an absolute or nonconvective instability because, unlike instabilities that we shall study in Section 10.2, it involves deflections that become unbounded as time increases at a fixed point in space; That is, if the wire extended to $x = \pm \infty$ and a pulse were initiated at $x = 0$, the deflection at $x = 0$ would become unbounded with time.

The "unboundedness" of the unstable deflections is, of course, a prediction of the mathematical model that sooner or later is not accurate because of nonlinear effects that were neglected in both the derivation of the equation of motion for the wire and in writing S_z as in (10.1.24). This same limitation was involved in the lumped-parameter, linear stability theory of Section 5.1.2.

The dynamics that determine when the deflections will become large vary from the mundane to the spectacular and from the disastrous to the useful. In the system of Fig. 10.1.9 the deflections may increase until the wire encounters one of the magnet coils or they may simply reach some saturation

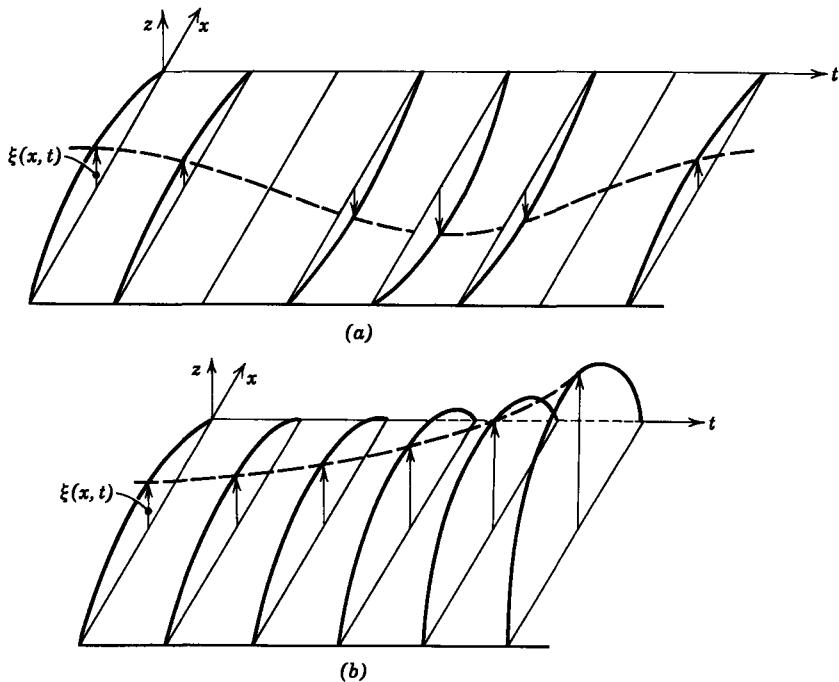


Fig. 10.1.12 Space-time dependence of lowest eigenmode: (a) $k_e < \pi/l$, stable; (b) $k_e > \pi/l$, unstable.

amplitude dictated by the large amplitude variation of the magnetic field. In Section 10.4.3 we shall see a grossly nonlinear consequence of absolute instability—the destruction of the system. On the other hand, as discussed later, instabilities of some types form the basis for making oscillators.

The example of instability presented in this section has the virtue of being extremely simple to understand mathematically, by comparison with instabilities often found in continuous media.* Nevertheless, there are practical situations in which this model has engineering significance. Example 10.1.2 considers how instability, predicted by this simple model, imposes a limitation on the levitation of continuous media with an electric field.

Absolute instability is of great interest to those concerned with fluid dynamics and the dynamics of elastic media. A classic example in the first category, which is closely related to the case considered in this section, is shown in Fig. 10.1.13. In Fig. 10.1.13*a* a heavy fluid (which is dark) is supported on a lighter fluid (which is clear) by hydrostatic pressure. In this initial equilibrium state each element of the fluid is in force equilibrium. With the heavy fluid on top, however, the equilibrium is unstable in the same sense that the wire of Fig. 10.1.9 was unstable. The only reason that the heavy fluid holds the position shown in Fig. 10.1.13*a* is that it is subject to an electric field, which induces polarization forces (discussed in Section 8.5) that stabilize the equilibrium. When this electric field is removed, the equilibrium becomes unstable, as shown in Figs. 10.1.13*b–e*, and the heavier fluid falls to the bottom of the tank. This is called a Rayleigh-Taylor† instability, and although the three-dimensional fluid motions are more complicated mathematically than those of the wire considered here the properties of these instabilities are in many ways similar. Both are absolute instabilities, and both instabilities have small amplitude motions characterized by a purely exponential growth with time.

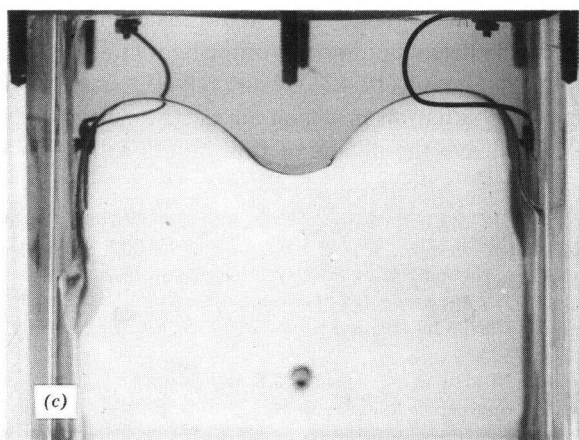
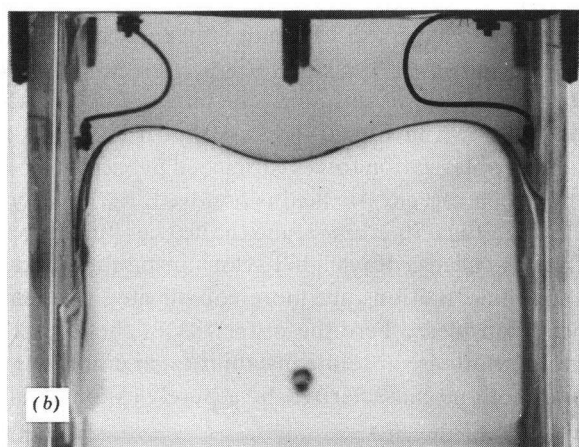
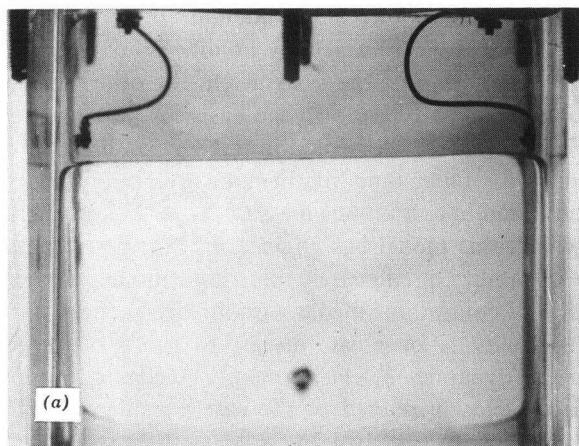
Rayleigh-Taylor types of instability are found in a variety of situations. Many found in the hydromagnetic equilibria of fusion machines have characteristics similar to the heavy fluid on top of the light fluid.‡ This connection is explored further in Section 10.4.3. Figure 10.1.13 actually illustrates an electrohydrodynamic situation in which an electric field prevents instability (although we must leave the details of this situation for further reading§).

* The downward shift of the natural frequencies and instability of the lowest mode are illustrated in the film "Complex Waves I" produced by the Education Development Center, Newton, Mass., for the National Committee on Electrical Engineering Films.

† S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Oxford, 1961, p. 428.

‡ D. J. Rose and M. Clark, Jr., *Plasmas and Controlled Fusion*, M.I.T. Press and Wiley, New York, 1961, p. 258.

§ J. R. Melcher and M. Hurwitz, "Gradient Stabilization of Electrohydrodynamically Oriented Liquids," *J. Spacecraft Rockets*, 4, No. 7, 864–881 (July 1967).



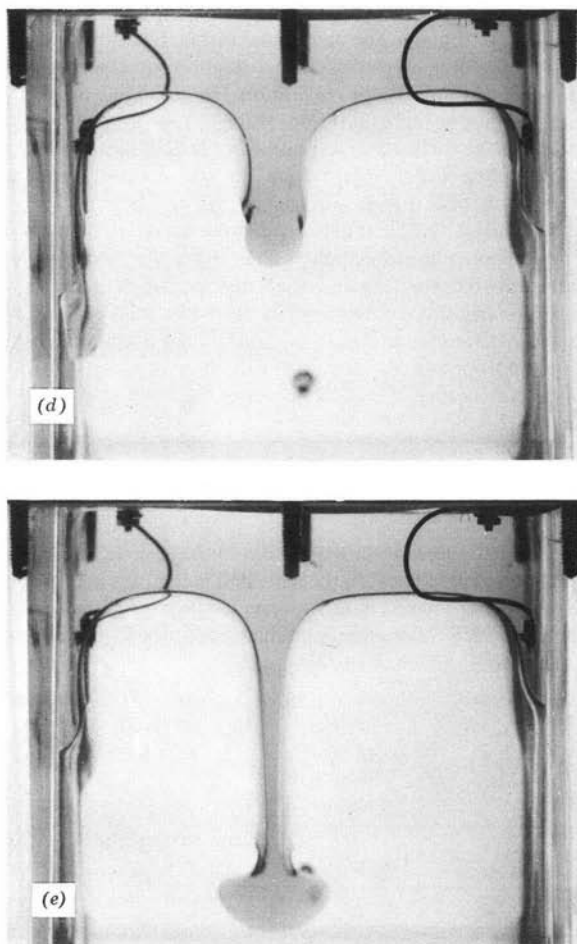


Fig. 10.1.13 (a) A heavy (dark) liquid is supported above a light (clear) liquid by hydrostatic pressure; (b – e) show growth of Rayleigh-Taylor instability resulting after stabilizing electric polarization forces in (a) are removed. (Photograph courtesy of Dynatech Corp., Cambridge, Mass.) This phenomenon can be seen in the film, “Complex Waves II,” available from Education Development Center, Inc., Newton, Mass.

Courtesy of Education Development Center, Inc. Used with permission.

Electric fields most often produce instabilities. The literature of plasma dynamics abounds in illustrations of absolute instabilities.*

Example 10.1.2. Electric and magnetic fields are often used to separate, levitate, or confine continuous media. A simple example of levitation is given in Fig. 10.1.14. An electrically conducting elastic film (membrane), with a mass per unit area σ_m , is stretched horizontally with equilibrium tension S between two rigid supports. With no other forces acting, the gravitational force makes the membrane sag in the middle. We propose to remove the sag without making physical contact with the membrane by placing a parallel conducting electrode, as shown in Fig. 10.1.14, and applying a potential difference between the membrane and the fixed plate. The membrane can be placed in a perfectly horizontal static equilibrium by providing an electrostatic attraction force per unit area between the plate and the membrane just equal to the gravitational force per unit area $\sigma_m g$. An important question, however, is this: Is the equilibrium stable? That is, can we expect that small transverse deflections of the surface due to noise will not become continuously larger? The electromechanical coupling occurs everywhere on the surface; hence to answer this question we must analyze a distributed surface-coupled problem.

It is assumed at the outset that the transverse displacements of the membrane, which are critical, do not depend on the y -coordinate. This is justified if the y -dimension of the membrane is small compared with l . In an actual situation this would mean that the tension in the y -direction of the membrane shown in Fig. 10.1.14 would be considerably less than the tension S in the x -direction, a fact that would make a one-dimensional membrane equation even more appropriate.

Because there is a gravitational force per unit area $\sigma_m g$ acting in the $-z$ -direction, the equation of motion (9.2.3a) becomes

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \frac{\partial^2 \xi}{\partial x^2} - \sigma_m g + T_z^e, \quad (a)$$

where T_z^e is the transverse force per unit area due to the electric field.

The electric field in the region between the plate and membrane is in general a function of both x and z . We restrict our interest here, however, to the "long-wave limit" (see Example 6.2.4) in which the slope $\partial \xi / \partial x$ is small enough that the electric field is essentially z -directed. Hence we write the electric field intensity as

$$\mathbf{E} = \frac{V}{d - \xi} \mathbf{i}_z. \quad (b)$$

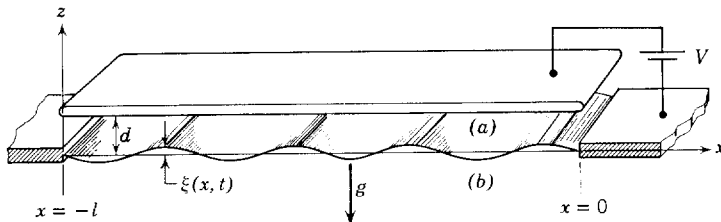


Fig. 10.1.14 Conducting elastic membrane held horizontal in a gravitational field by an electrostatic force.

* J. G. Linhart, *Plasma Physics*, North-Holland, Amsterdam, 1960, p. 101.

We have used the assumption that the membrane and the fixed plate are equipotentials; that is, the conductivities are high enough to allow charges to relax from one position to another on the conductors in a time that is short compared with a characteristic time of the mechanical motions (e.g., period of oscillation of the membrane or, as we shall see, time constant for instability). This is the idealization of the zero electric Reynolds number discussed in Section 7.2.

The force per unit area in the z -direction on a surface with the normal vector $\mathbf{n}$ is in general [from (8.4.2)]

$$T_z^e = (T_{zm}^a - T_{zm}^b)n_m. \quad (c)$$

Because $n_m \cong n_z \cong 1$ and there is no electric field below the membrane, this reduces to

$$T_z^e = T_{zz}^{(a)}. \quad (d)$$

Recall from (8.3.10) that because $E_x \approx 0$ and $E_y \approx 0$ the component of the stress tensor T_{zz} is

$$T_{zz} = \frac{1}{2}\epsilon_0 E_z^2, \quad (e)$$

and it follows from (b), (d), and (e) that

$$T_z^e = \frac{1}{2}\epsilon_0 \left(\frac{V}{d - \xi} \right)^2. \quad (f)$$

Expansion of this surface force density in a Taylor series about the equilibrium position $\xi = 0$ and retention of only linear terms in displacement yield

$$T_z^e = \frac{\epsilon_0}{2d^2} V^2 + \frac{\epsilon_0}{d^3} V^2 \xi. \quad (g)$$

This result is now substituted into (a) to obtain the differential equation of motion

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \frac{\partial^2 \xi}{\partial x^2} - \sigma_m g + \frac{\epsilon_0 V^2}{2d^2} + \frac{\epsilon_0}{d^3} V^2 \xi. \quad (h)$$

It is desired that the voltage V be adjusted so that in static equilibrium the membrane has no sag ($\xi = 0$). In this case, (h) reduces to

$$-\sigma_m g + \frac{\epsilon_0 V^2}{2d^2} = 0, \quad (i)$$

which simply states that the net electrical force upward (the upward attraction of the upper plate) must balance the downward force of gravity.

Subtraction of the equilibrium equation (i) from (h) gives an expression that must be satisfied by the perturbation deflections

$$\frac{1}{v_s^2} \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial^2 \xi}{\partial x^2} + k_c^2 \xi, \quad (j)$$

where

$$v_s = \left(\frac{S}{\sigma_m} \right)^{1/4} \quad (k)$$

$$k_c^2 = \frac{\epsilon_0 V^2}{Sd^3}. \quad (l)$$

This equation has the same form as (10.1.25). We can immediately conclude that because the boundary conditions are also the same as those considered in writing (10.1.30) the equilibrium is stable if

$$\frac{\epsilon_0 V^2}{Sd^3} < \left(\frac{\pi}{l}\right)^2. \quad (m)$$

This institutes an upper bound on the voltage V consistent with stability. If we recognize that a requirement is also imposed on the voltage by the levitation condition (i), it follows that a largest mass-per-unit-area σ_m can be levitated by an apparatus with given dimensions and membrane tension S . This follows by eliminating V^2 from (m) and (i):

$$\sigma_m < \left(\frac{\pi}{l}\right)^2 \frac{Sd}{2g}. \quad (n)$$

It is a limitation of this kind that absolute instability often imposes on practical systems.

In an actual design a further restriction on allowable voltages arises because the material between the plate and membrane in Fig. 10.1.14 will be able to withstand only a finite electric field intensity; for example, if this region is filled with air at atmospheric pressure, the electric field intensity cannot exceed about 3×10^6 V/m.*

10.1.4 Waves with Damping, Diffusion Waves

Any real system is affected to some extent by energy dissipative effects, whether they take the form of mechanical friction or are caused by resistive heating. This fact alone would justify at least brief attention to the dynamics of a continuum influenced by losses. Even more important, however, are the physical situations in which the effect of energy dissipation is essential. This was seen in Chapter 7, in which magnetic diffusion and charge relaxation were governed by loss-dominated continuum equations. The discussion presented in this section adds further physical significance to the diffusion equation, which in Chapter 7 represented the dynamics of the magnetic field in a conducting material and in this section the dynamics of a string in a viscous fluid. One of our reasons for introducing this topic here becomes even more evident in Section 10.2.4, in which the simple ideas introduced are extended to show how losses, in conjunction with material convection, can give rise to a class of instabilities. These “resistive wall” instabilities assume importance in a variety of situations.

For purposes of illustration suppose that the wire is immersed in a viscous liquid. Then transverse deflections ξ produce a viscous drag on each section of the wire which is analogous in its effect to the viscous damper (see Section 2.2.1b) of lumped-parameter systems. A small section of the wire, shown in Fig. 10.1.15, experiences a (retarding) force per unit length to oppose the

* A. E. Knowlton, *Standard Handbook for Electrical Engineers*, McGraw-Hill, 9th ed., 1957, p. 416.

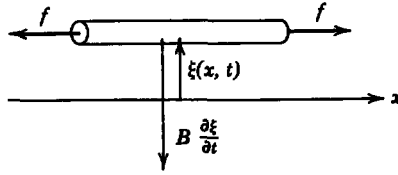


Fig. 10.1.15 A viscous drag per unit length acts to retard the motion of each element of the wire.

motion. If the velocity is low enough (or the viscosity is high enough*), this force is simply proportional to the transverse velocity; that is,

$$S_z = -B \frac{\partial \xi}{\partial t}, \quad (10.1.32)$$

where B is a damping coefficient per unit length.

With this type of force acting on the string, the equation of motion (10.1.1) becomes

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2} - \nu \frac{\partial \xi}{\partial t}, \quad (10.1.33)$$

where, as before,

$$v_s = \left(\frac{f}{m} \right)^{1/2} \quad (10.1.34)$$

and a normalized “damping frequency” has been defined as

$$\nu = \frac{B}{m}. \quad (10.1.35)$$

Now substitution of the standard form of solution (10.0.1) into (10.1.33) gives the dispersion equation

$$\omega^2 = v_s^2 k^2 + j\nu\omega. \quad (10.1.36)$$

This equation, like those considered in the preceding sections, is a simple quadratic either for ω or for k . Note that by contrast with the preceding cases, which did not involve dissipation, the dispersion equation now has a complex coefficient. Although the problem is formally no more complicated than it was before, the complex coefficient is responsible for making the algebra more involved.

First of all, consider the driven response to a sinusoidal steady-state excitation with the frequency $\omega = \omega_d$. Then it is appropriate to solve the dispersion

* The drag on a cylinder in a liquid obeys this law, provided the Reynolds number is small enough. See H. Rouse, *Elementary Mechanics of Fluids*, Wiley, New York, 1946, p. 247.

equation (10.1.36) for k , given that the frequency is ω_d :

$$k = \pm \frac{\sqrt{\omega_d^2 - j\nu\omega_d}}{v_s} \quad (10.1.37)$$

The radical makes it convenient to recognize that this expression has the form

$$k = \pm (|k_r| - j|k_i|). \quad (10.1.38)$$

The graphical construction in the complex plane (Fig. 10.1.16) shows how $|k_r|$ and $|k_i|$ are related to the driving frequency ω_d . The vectors provide a convenient way to picture the result of taking the square root of (10.1.37).

In view of the possible wavenumbers expressed by (10.1.38), the driven response is made up of two waves with amplitudes A^+ and A^- determined by the boundary conditions.

$$\xi = \text{Re} [A^+ e^{-j(|k_r| - j|k_i|)x} + A^- e^{j(|k_r| - j|k_i|)x}] e^{j\omega_d t}. \quad (10.1.39)$$

Remember that the real part of the wavenumber contains information about the periodicity and the phase velocity of the wave. Hence the first term in (10.1.39) represents a wave with the wavelength (measured between points of zero phase)

$$\lambda = \frac{2\pi}{|k_r|} \quad (10.1.40)$$

and a phase velocity (again for points of zero phase) of

$$v_p = \frac{\omega_d}{|k_r|}. \quad (10.1.41)$$

These points of zero phase propagate in the positive x -direction. From the form of (10.1.39) it is also clear that this wave decays in the positive x -direction at a rate determined by $|k_i|$. The A^- wave has a similar physical significance,

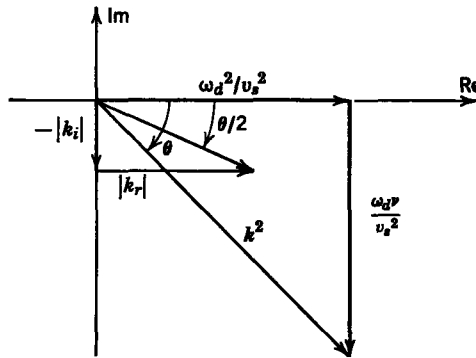


Fig. 10.1.16 Graphical solution of (10.1.37) to show complex nature of k .

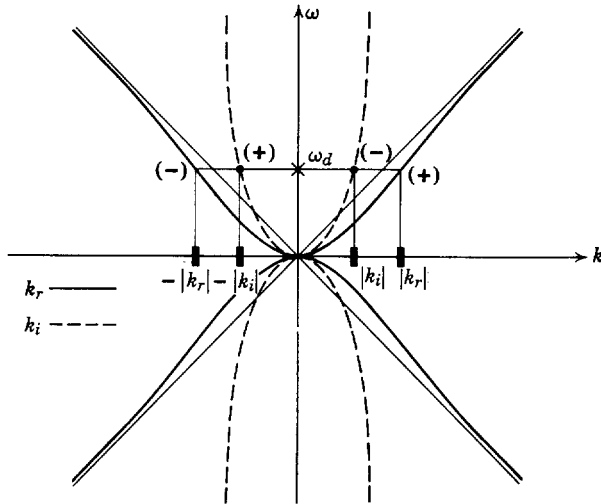


Fig. 10.1.17 Dispersion equation (10.1.37) for a string immersed in a viscous liquid, showing complex values of k for real values of ω .

with points of zero phase propagating in the $-x$ -direction and an amplitude that decays also in the $-x$ -direction.

What we have found is not too surprising. The effect of the damping on the driven waves is to produce a spatial decay in the direction of propagation. Figure 10.1.17 shows the ω - k plot predicted by (10.1.37). Here we are considering the driven response and plot complex values of k for real values of ω . Note that the positive real part of k (corresponding to a given frequency $\omega = \omega_d$) goes with a negative imaginary part of k on this diagram. As the driving frequency is raised, the real parts of k approach the same values they would have in the absence of damping whereas the imaginary parts (which reflect the spatial rate of decay) reach the asymptotic value $|k_i| = \nu/2v_s$.

The terms in (10.1.36) have the same physical significance as the corresponding terms in (10.1.33); that is, the ω^2 on the left represents the effects of inertia, whereas $v_s^2 k^2$ and $j\nu\omega$, respectively, symbolize effects of the tension and viscous drag. If the damping is sufficiently large that

$$\nu \gg \omega_d, \quad (10.1.42)$$

the effect of damping dominates that of the inertia and the ω^2 term in (10.1.36) can be ignored. (This is similar to the class of dynamics discussed in Section 5.2.2 in the context of lumped parameters.) The resulting dispersion equation takes the form

$$k^2 = -\frac{j\omega_d\nu}{v_s^2} \quad (10.1.43)$$

In this simple form the radical found in solving for k [as in (10.1.37)] can be taken to obtain

$$k = \pm \left(\frac{-j\omega_d v}{v_s^2} \right)^{1/2} = \pm \left(\frac{\omega_d v}{2v_s^2} \right)^{1/2} (1 - j). \quad (10.1.44)$$

This simplified dispersion equation is familiar from Section 7.1.3a [see (7.1.70)], where it was found in discussing the sinusoidal steady-state behavior of a magnetic field in a conducting medium. Instead of the magnetic skin depth δ , we now have $\delta \rightarrow \sqrt{2v_s^2/\omega_d v}$. The mathematical analogy between the viscous dominated spring and the magnetic diffusion could have been seen at the outset by comparing the one-dimensional form of (7.1.11) with (10.1.33), with the inertia term omitted. Both systems are represented by the diffusion equation.*

The value in recognizing the analogy between the deflection of the spring as it “oozes” through the viscous liquid and the magnetic diffusion through a conducting material is not confined to the mathematics. We can obtain a good physical “feel” for the diffusion process as it occurs in a diversity of situations by keeping in mind that the dynamics are similar to those found in the experiment shown in Fig. 10.1.18. Here a tightly wound helical spring (which is equivalent to a wire or string with a small tension f) is fixed at one end and driven up and down in a sinusoidal fashion at the other. The spring is immersed in glycerin. If we were to produce this excitation by hand, we would be most aware of a viscous retarding force. This is in contrast to the force required to excite the evanescent waves of Section 10.1.2, which was of the same nature as that required to compress a spring. (The comparison is worthy of note, for in both cases we are concerned with waves that decay spatially away from the point of excitation.)

The dispersion relation for the diffusion wave is characterized by equal real and imaginary parts of k for real values of ω . This is the region near the origin in Fig. 10.1.17, as is evident either from the ω - k plot or from the condition of (10.1.42).

In the preceding sections reference has been made to the effect of damping on the natural frequencies. This effect was clearly necessary for an accurate picture of the transient dynamics over many oscillations of the string. With the ends of the string fixed (as in the last two sections), the natural modes are found by again recognizing that solutions for k have the form $k = \pm \beta$ (where ω is now unknown and (10.1.36) is solved for k) and writing solutions in the form of (10.1.16). The same arguments used then, lead to the conclusion that

$$\beta = \frac{n\pi}{l}, \quad n = 1, 2, 3, \dots, \quad (10.1.45)$$

* Reference to other systems represented by this equation was made in the footnote to (7.1.11).

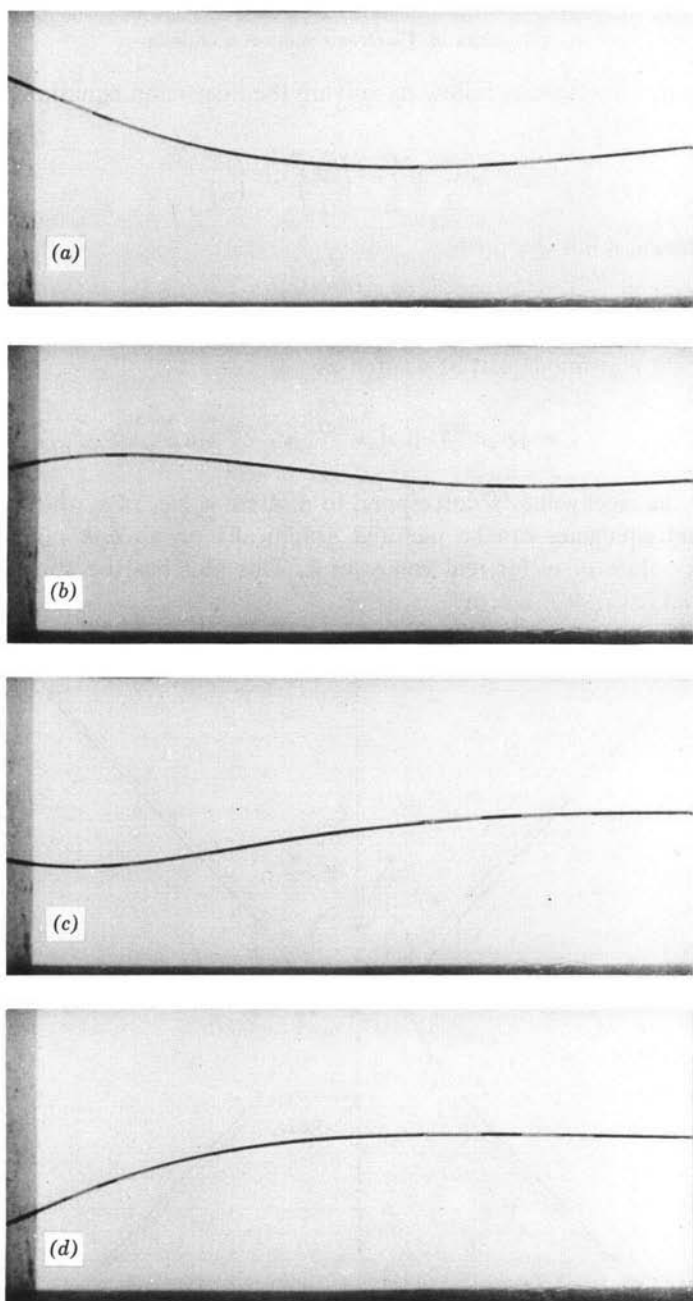


Fig. 10.1.18 A spring under tension is driven sinusoidally at the left end and fixed at the right end. It is immersed in glycerine which is sufficiently viscous that the effect of damping exceeds that of inertia. Hence the resulting motions which are shown in sequence over about one half cycle exemplify diffusion waves (or skin effect waves) in the sinusoidal steady state. (From four-minute film "Diffusion Waves" made for M.I.T. by Education Development Center, Inc., Newton, Mass.)

and the eigenfrequencies follow by solving the dispersion equation (10.1.36) with $k^2 = (n\pi/l)^2$.

$$\omega_n^\pm = \frac{j\nu}{2} \pm \left[v_s^2 \left(\frac{n\pi}{l} \right)^2 - \left(\frac{\nu}{2} \right)^2 \right]^{1/2}. \quad (10.1.46)$$

This expression has the form

$$\omega_n^\pm = \frac{j\nu}{2} \pm \Omega_n, \quad (10.1.47)$$

and the n th eigenmode can be written as

$$\xi = [A_n^+ e^{j\Omega_n t} + A_n^- e^{-j\Omega_n t}] e^{-(\nu/2)t} \sin \frac{n\pi x}{l}. \quad (10.1.48)$$

Because the eigenvalues β correspond to discrete values of k which are real, the eigenfrequencies can be pictured graphically on an ω - k plot to show complex values of ω for real values of k . This plot has the appearance of Fig. 10.1.19.

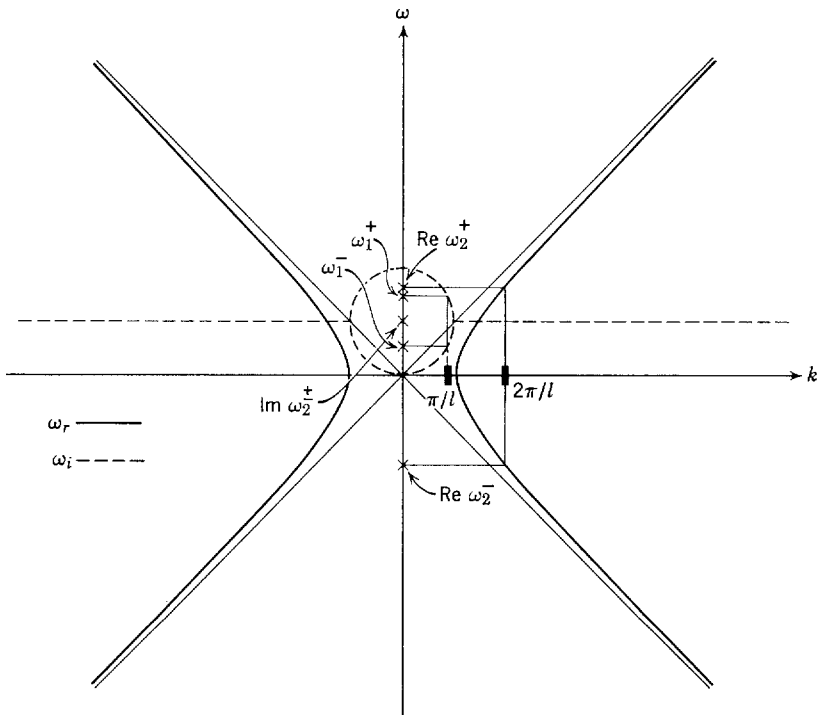


Fig. 10.1.19 Dispersion equation for string in a viscous liquid showing complex values of ω for real values of k . The lowest pairs of eigenfrequencies for a string fixed at each end are shown.

Note that for a given mode n both eigenfrequencies are imaginary (and represent pure damping) if the second (viscous drag) term under the radical in (10.1.46) exceeds the first (tension) term; for example, in Fig. 10.1.19 the $n = 1$ mode is characterized by two exponentially decaying deflections whereas the $n = 2$ mode takes the form of two oscillating deflections with amplitudes with a temporally decaying envelope. In this case we say that the lowest mode is overdamped and the others are underdamped.

Example 10.1.3. Of course, the damping can occur because of electrical as well as mechanical dissipation. The system introduced in Example 10.1.1 serves as an illustration of this fact. Suppose that the conducting top and bottom walls in Fig. 10.1.7 are so highly resistive that the voltage induced by the motion of the membrane is entirely absorbed by the resistance of the plates. This is the opposite extreme to the case considered in Example 10.1.1 in which this voltage was absorbed by the self-inductance of the current loops (a) and (b) shown in Fig. 10.1.8. (These extremes were discussed in the context of lumped parameters in Section 5.1.3.) Then, in writing the electrical loop equations (j) and (k) of Example 10.1.1, we can make the approximation that

$$\left| \frac{h^a}{\sigma_s} \right| \gg \left| \frac{\partial}{\partial t} \mu_0 dh^a \right|, \quad (a)$$

$$\left| \frac{h^b}{\sigma_s} \right| \gg \left| \frac{\partial}{\partial t} \mu_0 dh^b \right|, \quad (b)$$

to obtain

$$h^a = \sigma_s \mu_0 H_0 \frac{\partial \xi}{\partial t}, \quad (c)$$

$$h^b = -\sigma_s \mu_0 H_0 \frac{\partial \xi}{\partial t}. \quad (d)$$

The equation of motion is obtained by combining (a) and (n) in Example 10.1.1 with the last two equations:

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2} - \nu \frac{\partial \xi}{\partial t}, \quad (e)$$

where

$$v_s = \left(\frac{S}{\sigma_m} \right)^{1/2} \quad (f)$$

$$\nu = \frac{2\sigma_s(\mu_0 H_0)^2}{\sigma_m}. \quad (g)$$

Hence in the limit in which the upper and lower walls in Fig. 10.1.7 are very lossy, the effect of the magnetic field is to damp the transverse motions of the perfectly conducting membrane. It is clear from (e) that the discussion of this section applies equally well to the resistively loaded membrane in which the damping is of electrical origin.

This example is continued in Section 10.2.4, in which the effects of electrical damping are exhibited in a considerably less obvious way than found here.

10.2 WAVES AND INSTABILITIES IN THE PRESENCE OF MATERIAL MOTION

We are concerned in this chapter with continuous media and have confined our interest so far to cases in which the continuum is initially at rest or in

static equilibrium. Gross motion of a medium can have a profound effect on the dynamics found, for example, when attempts are made to carry on a conversation in a high wind. In this section we use strings and membranes as models for introducing concepts related to the dynamics of media initially in a state of uniform motion. Again, our remarks are related to particular physical situations chosen for their simplicity, but with implications for a wide range of physical situations; for example, the string might be replaced by a beam of electrons in vacuum, by a stream of holes in a semiconductor, or by a streaming gas or plasma.

The systems to be considered now have the same basic physical nature as those in Section 10.1. The mechanical continuum is a string or a membrane undergoing one-dimensional transverse displacements $\xi(x, t)$. Now we introduce a new ingredient and specify that the continuum is moving in the longitudinal (x) direction with an equilibrium velocity U . Although the remarks that follow use the string as an example, it should be clear that they also apply directly to the one-dimensional motion of a membrane.

Our interest here is in exciting the moving string at a fixed point and in describing the dynamical behavior as viewed from a fixed frame. Consequently, the equation of motion developed in Section 9.2 is not adequate because it does not account for the equilibrium velocity (convection) of the continuum. It does, however, give a proper description of the string viewed from a reference frame moving in the x -direction with a velocity U because in that frame the string has no velocity in the x -direction. We denote variables (both independent and dependent) measured in this moving frame by primes and write (10.1.1) for string deflections ξ' in the moving frame as

$$m \frac{\partial^2 \xi'}{\partial t'^2} = f \frac{\partial^2 \xi'}{\partial x'^2} + S'_z. \quad (10.2.1)$$

We are dealing with velocities much lower than the velocity of light; consequently we use Fig. 10.2.1 to relate the coordinates and time in the two frames by the Galilean transformation*:

$$x = x' + Ut', \quad (10.2.2)$$

$$z = z' \quad (10.2.3)$$

$$t = t'. \quad (10.2.4)$$

It follows from (10.2.3) that $\xi = \xi'$ and because the moving frame is not accelerating, we can also say that $S'_z = S_z$. Hence to write (10.2.1) in the fixed

* This discussion represents a review of ideas developed in Section 6.1, now applied to transforming mechanical rather than electrical equations of motion. Hence once again we encounter the substantial or convective derivative.

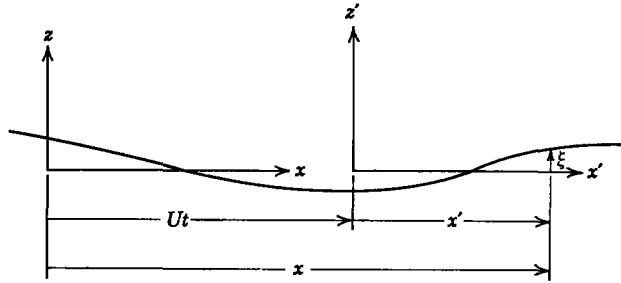


Fig. 10.2.1 The fixed coordinate system x, y, z or the moving coordinate system x', y', z' can be used to define a given position along the equilibrium axis of the string. Because the moving coordinate system has the same velocity U in the x -direction as the string, the string appears to move only in the z' -direction when viewed from the primed frame.

frame we need only compute the derivatives in terms of (x, t) ; that is

$$\frac{\partial \xi}{\partial t'} = \frac{\partial \xi}{\partial t} \frac{\partial t}{\partial t'} + \frac{\partial \xi}{\partial x} \frac{\partial x}{\partial t'} = \frac{\partial \xi}{\partial t} + U \frac{\partial \xi}{\partial x}. \quad (10.2.5)$$

Repeating this procedure, we have

$$\frac{\partial^2 \xi}{\partial t'^2} = \frac{\partial}{\partial t} \left(\frac{\partial \xi}{\partial t} + U \frac{\partial \xi}{\partial x} \right) + U \frac{\partial}{\partial x} \left(\frac{\partial \xi}{\partial t} + U \frac{\partial \xi}{\partial x} \right). \quad (10.2.6)$$

The space derivative is found in the same way to be

$$\frac{\partial^2 \xi}{\partial x'^2} = \frac{\partial^2 \xi}{\partial x^2}, \quad (10.2.7)$$

hence (10.2.1) becomes

$$m \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 \xi = f \frac{\partial^2 \xi}{\partial x^2} + S_z, \quad (10.2.8)$$

where the quantity in brackets on the left is recognized as the one-dimensional form of the substantial or convective derivative operating twice on ξ .^{*} Remember from Section 6.1 (6.1.19) that the substantial or convective derivative is the time rate of change viewed from a frame moving with the velocity U . Our derivation of (10.2.8) makes this interpretation apparent once again. Here the convective derivative provides the means of expressing the equation of motion in the fixed frame.

^{*} From the derivation $\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 \xi \equiv \frac{\partial^2 \xi}{\partial t^2} + 2U \frac{\partial^2 \xi}{\partial x \partial t} + U^2 \frac{\partial^2 \xi}{\partial x^2}$.

10.2.1 Fast and Slow Waves

Before introducing further complications, it is worthwhile to consider the nature of waves that propagate on a convecting string with no electro-mechanical coupling. Thus we set $S_z = 0$ and (10.2.8) becomes

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right)^2 \xi = v_s^2 \frac{\partial^2 \xi}{\partial x^2}, \quad (10.2.9)$$

where once again $v_s = \sqrt{f/m}$.

10.2.1a Traveling Waves

Now, if $U = 0$, the problem is mathematically identical to the one discussed in Section 9.1.1 in which we also found that the wave equation was pertinent and that deflections took the form of waves (9.1.14). We expect that in the moving frame shown in Fig. 10.2.1 solutions will have this same form and that we can now write solutions as

$$\xi = \xi_+(\alpha) + \xi_-(\beta), \quad (10.2.10)$$

where

$$\alpha = x' - v_s t', \quad C^+, \quad (10.2.11)$$

$$\beta = x' + v_s t', \quad C^-. \quad (10.2.12)$$

The parameters α and β can be written in terms of the fixed frame variables (x, t) by using the transformation equations (10.2.2) to (10.2.4):

$$\alpha = x - (v_s + U)t, \quad C^+, \quad (10.2.13)$$

$$\beta = x + (v_s - U)t, \quad C^-, \quad (10.2.14)$$

and direct substitution into (10.2.9) shows that (10.2.10) is indeed a solution.

Equations 10.2.13 and 10.2.14 for α and β are recognized as the characteristics discussed in Section 9.1.1a but altered by the convection. The convection increases the velocity of waves propagating in the positive x -direction along C^+ characteristic lines and decreases the velocity of waves propagating in the negative x -direction along C^- characteristics.

The characteristic lines that originate at $x = 0$ when $t = 0$ (α and β equal to zero in (10.2.13) and (10.2.14)) are shown in Fig. 10.2.2. In (a) of this figure waves propagate relative to the string with a velocity exceeding U ($v_s = 2U$). Note that the slopes of the characteristics are the velocities of propagation for the forward and backward waves ξ_+ and ξ_- . Hence the wave propagating in the $+x$ -direction does so more rapidly than the wave propagating against the convection in the $-x$ -direction.

By contrast Fig. 10.2.2b shows the case in which the convection makes both characteristics point downstream ($+x$ -direction) as time increases. Hence a

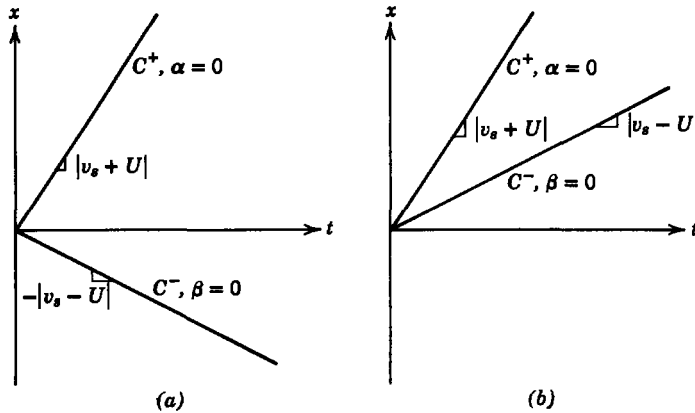


Fig. 10.2.2 Characteristic lines in the space-time plane along which a disturbance at $x = 0$ when $t = 0$ will propagate: (a) when $v_s = 2U$; (b) when $v_s = U/2$.

disturbance at $x = 0$ when $t = 0$ will result in two waves that propagate in the $+x$ -direction. There is no effect from such a disturbance on that part of the string in which $x < 0$. As we shall find in Chapter 13, sound waves in a gas have a behavior analogous to that of the elastic waves considered here. In the language of gas dynamics we say that the two cases of Fig. 10.2.2 are “subsonic” and “supersonic,” respectively.

An example serves to clarify further the effect of material convection on waves.

Example 10.2.1. In Example 9.1.2 of Section 9.1.1a we studied the behavior of an initially static pulse propagating on an elastic rod. A mathematically similar problem is now considered in which the string is given an initial displacement and velocity and we inquire about the resulting motions; that is, we are given

$$\xi(x, 0) = \xi_0(x), \quad (a)$$

$$\frac{\partial \xi}{\partial t}(x, 0) = \dot{\xi}_0(x). \quad (b)$$

We can evaluate ξ_+ and ξ_- by using the initial conditions. This is done by taking the derivative of (10.2.10) with respect to x and using (a). (Here we use the fact that $\partial \alpha / \partial x = \partial \beta / \partial x = 1$.)

$$\frac{d\xi_+}{d\alpha} + \frac{d\xi_-}{d\beta} = \frac{d\xi_0}{dx}. \quad (c)$$

Similarly, the use of the time derivative of (10.2.10) with (b) gives

$$-(v_s + U) \frac{d\xi_+}{d\alpha} + (v_s - U) \frac{d\xi_-}{d\beta} = \dot{\xi}_0(x). \quad (d)$$

Simultaneous solution of (c) and (d) yields

$$\frac{d\xi_+}{d\alpha} = \left(\frac{v_s - U}{2v_s} \right) \frac{d\xi_0}{dx} - \frac{1}{2v_s} \dot{\xi}_0(x), \quad (e)$$

$$\frac{d\xi_-}{d\beta} = \left(\frac{v_s + U}{2v_s} \right) \frac{d\xi_0}{dx} + \frac{1}{2v_s} \dot{\xi}_0(x). \quad (f)$$

Remember that in (e) and (f) x is the position along the string when $t = 0$. To find the solution at a general point C in space and time, as shown in Fig. 10.2.3, we use (e) with x evaluated at A , where $x = \alpha$. In the same way (f) is evaluated at point B , where $x = \beta$. Then (e) provides $d\xi_+/d\alpha$ anywhere along the C^+ characteristic shown in Fig. 10.2.3. The particular values of α and β required to give characteristics passing through the point $C(x, t)$ are given by (10.2.13) and (10.2.14). The space derivative of (10.2.10) can now be evaluated in terms of the initial conditions by using (e) and (f).

$$\frac{\partial \xi}{\partial x}(x, t) = \left(\frac{v_s - U}{2v_s} \right) \frac{d\xi_0}{dx}(\alpha) + \left(\frac{v_s + U}{2v_s} \right) \frac{d\xi_0}{dx}(\beta) - \frac{\dot{\xi}_0(\alpha) - \dot{\xi}_0(\beta)}{2v_s}. \quad (g)$$

Suppose that when $t = 0$ the string is static ($\dot{\xi}_0 = 0$) and the displacement is a uniform pulse of magnitude A distributed between $x = -a$ and $x = a$. Then we have

$$\frac{d\xi_0}{dx}(x) = A[u_0(x + a) - u_0(x - a)], \quad (h)$$

where $u_0(x + a)$ is a unit impulse at the position $x = -a$. The resulting space derivative of the displacement follows from (g) which becomes

$$\frac{\partial \xi}{\partial x}(x, t) = A \left\{ \left(\frac{v_s + U}{2v_s} \right) [u_0(\beta + a) - u_0(\beta - a)] + \left(\frac{v_s - U}{2v_s} \right) [u_0(\alpha + a) - u_0(\alpha - a)] \right\}. \quad (i)$$

Remember that α and β are given by (10.2.13) and (10.2.14). The first two terms in this expression are recognized as impulses propagating along the C^- characteristics originating

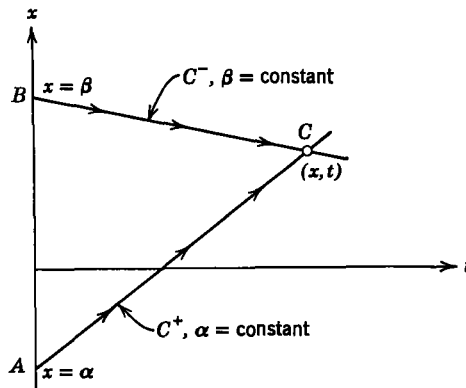


Fig. 10.2.3 Characteristics that intersect at a general point in space and time (x, t) showing the points A and B at which the initial conditions determine the dependent variables at C .

at $x = -a$ and $x = a$, respectively. The last two terms are impulses propagating along the C^+ characteristics. We can obtain the displacement ξ by integrating this equation with respect to x . The resulting values of ξ in two cases are shown in Figs 10.2.4 and 10.2.5. In the first of these figures U is less than v_s and we see that the original pulse divides into a part propagating slowly in the $-x$ -direction (upstream) and a part that is propagating rapidly in the $+x$ -direction. In Fig. 10.2.5 the convection velocity U is greater than the propagation velocity v_s and, although the pulse divides into two parts, both parts are carried downstream by the convection. Note that the convection also makes the ξ_+ and ξ_- waves unequal in magnitude. In fact, when U exceeds v_s , the ξ_+ wave is inverted, as shown in Fig. 10.2.5.

We are now in a position to make an important observation about the appropriate initial conditions and boundary conditions when the differential equation of motion is (10.2.9). We first observe that this expression is a partial differential equation that is second-order in both time (t) and space (x). Consequently, we need two initial conditions and two boundary conditions to evaluate all the constants in the solution. Where these conditions can be applied depends on whether the convection velocity U is greater or less than the propagation velocity v_s and is essentially the question whether a disturbance can propagate upstream (in the negative x -direction).

Consider first the case in which $U < v_s$ and disturbances can propagate upstream. As an example, assume that the string deflection is fixed at $x = l$. Then we can say that $\xi = 0$ along the line $x = l$ in the x - t plane. This is one boundary condition. A second boundary condition is that at $x = 0$ the string position is driven independently

$$\xi(0, t) = \xi_a(t). \quad (10.2.15)$$

It is helpful to picture these conditions as shown in Fig. 10.2.6. The two initial conditions can be specified as the initial displacement $\xi(x, 0)$ and the initial transverse velocity $(\partial \xi / \partial t)(x, 0)$. The solution for $\xi(x, t)$ at an arbitrary point C can be found as follows. First, the value of $\xi_+(\alpha)$ that propagates from A to B along a C^+ characteristic is determined by the initial conditions at point A . Similarly, the $\xi_-(\beta)$ wave propagating along a C^- characteristic from E to D is found from the initial conditions at point E . These incident waves, together with the boundary conditions at B and D , determine the reflected waves that travel from B to C along a C^- characteristic [a $\xi_-(\beta)$ wave] and from D to C along a C^+ characteristic [a $\xi_+(\alpha)$ wave] described in Section 9.1.1*b*. Hence we have found the solution for the displacement at point C . From similar arguments, perhaps involving many reflections of the waves, we can find the solution at any point in the x - t plane in the interval of space $0 < x < l$.

Consider next the case in which $U > v_s$. As indicated by Fig. 10.2.6*b*) both the C^+ and C^- characteristics point downstream (positive x -direction) as t increases and no disturbances can propagate upstream. Consequently, we expect that initial conditions and boundary conditions applied downstream

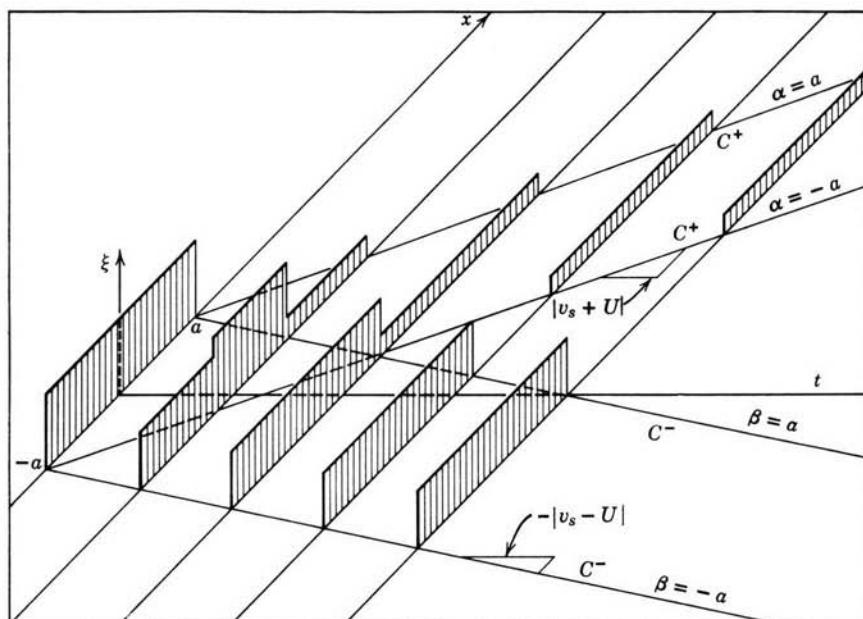


Fig. 10.2.4 Propagation of an initially static displacement pulse on a string moving with the velocity $U = v_s/2$.

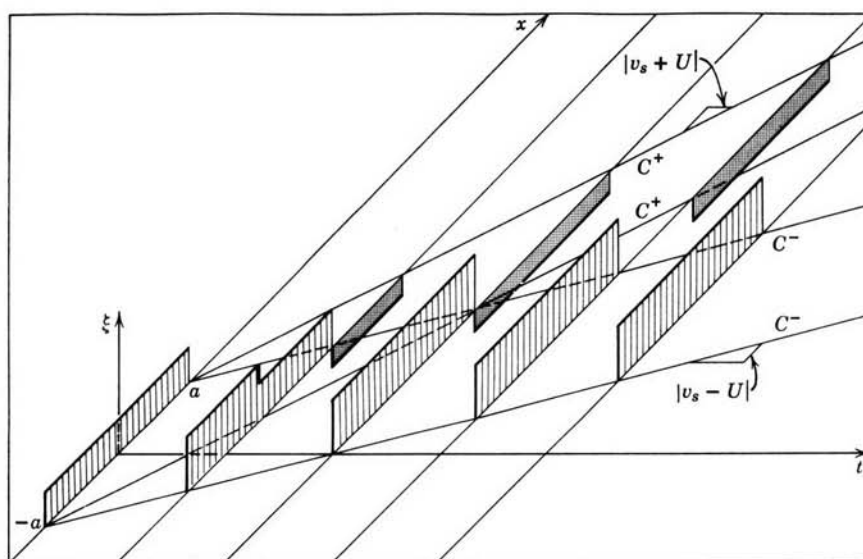


Fig. 10.2.5 Propagation of an initially static displacement pulse on a string moving with the velocity $U = 2v_s$.

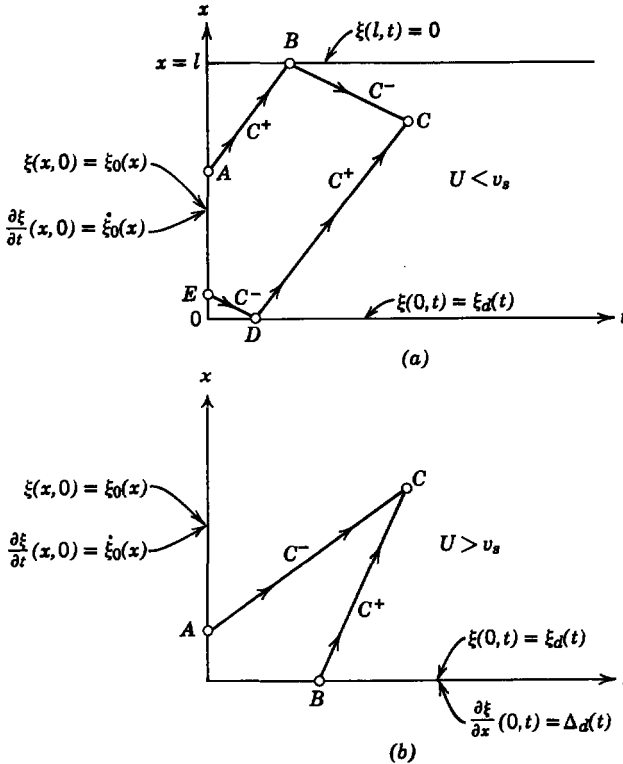


Fig. 10.2.6 Boundary and initial conditions to establish solution $\xi(x, t)$ at a point C : (a) $U < v_s$; (b) $U > v_s$.

from point C will not affect the displacement ξ at point C because the effects of initial and boundary conditions are transmitted in space and time by waves that propagate along the characteristics. To make this discussion explicit we refer to Fig. 10.2.6b and ask for the displacement at an arbitrary point C . Note that the solution at C can depend on only conditions at A and B . Hence causality is introduced as a physical law that must be obeyed by our mathematical solution, and we assume that solutions evolve from left to right (in the direction of increasing time) in Fig. 10.2.6. Hence it is necessary to impose two conditions at A and two conditions at B . Note that this means that boundary conditions along the $(x = 0)$ -axis have the same nature as initial conditions along the $(t = 0)$ -axis. In addition to the two initial conditions when $t = 0$, we can impose

$$\xi(0, t) = \xi_d(t) \quad (10.2.16)$$

and the slope at $x = 0$,

$$\frac{\partial \xi}{\partial x}(0, t) = \Delta_d(t). \quad (10.2.17)$$

With the initial and boundary conditions thus specified, we can calculate the $\xi_+(\alpha)$ wave that propagates along the C^+ characteristic from B to C and the $\xi_-(\beta)$ wave that propagates along the C^- characteristic from A to C in Fig. 10.2.6b. Thus we have found the displacement at point C .

A comparison of Figs. 10.2.6a and b shows that the essential difference between the two cases is that with $U < v_s$ boundary conditions must be specified both upstream and downstream from the point in question because waves propagate in both directions. For the case $U > v_s$, waves propagate downstream only; thus the boundary conditions must be specified upstream from the point in question.

In the supersonic case ($U > v_s$) it would violate causality to impose only one boundary condition at $x = 0$ and the second condition at $x = l$. Waves propagating along the C^+ and C^- characteristics would have to know in advance that they had to satisfy a boundary condition at C in Fig. 10.2.6b. This is an important point because there can be no mathematical objection to a solution to (10.2.9) with $U > v_s$, which satisfies one upstream and one downstream boundary condition. Rather our objection is based on the physical requirement of causality.*

10.2.1b Sinusoidal Steady State

To appreciate fully the drastic effect of the convection it is important to recognize the manner in which a sinusoidal steady-state condition is established on the "supersonic" string. Suppose that with $U > v_s$ a sinusoidal excitation is applied when $t = 0$ at $x = 0$. This is shown in Fig. 10.2.7, in

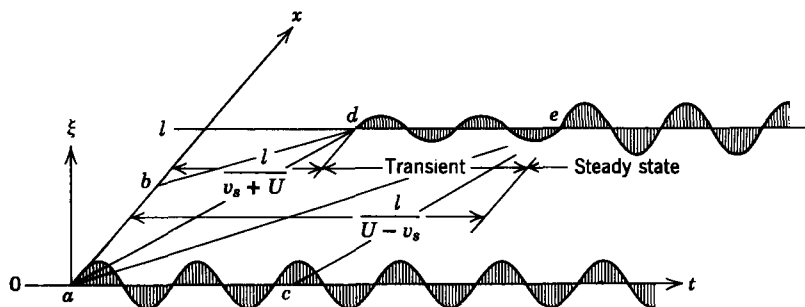


Fig. 10.2.7 A sinusoidal excitation turned on at $x = 0$ when $t = 0$ is applied to a moving string ($U > v_s$). At the downstream position $x = l$ the sinusoidal steady state is established when the slow wave arrives [$t = l/(U - v_s)$].

* Nonlinear disturbances can propagate upstream even when $U > v_s$. In the language of gas dynamics these are "shock waves" that travel faster than the speed v_s of small amplitude disturbances. These nonlinear phenomena are beyond the scope of this work and are not predicted by our simple, linear model. See, for example, R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves*, Interscience, New York, 1948, Part II.

which two sinusoidal driving conditions are imposed along the t -axis when $t > 0$ and when $t = 0$ the string has a dynamical state specified by the initial conditions.

Consider the string deflections that would be observed at the downstream position $x = l$. We have learned in this section that the solution at any given time is determined by a superposition of fast and slow waves propagating downstream on the C^+ and C^- characteristics, respectively. These characteristics originate either on the x -axis, where the initial conditions are arbitrary, or on the t -axis, where the boundary conditions are in the sinusoidal steady state. From this it follows that there is no response to the sinusoidal drive observed at $x = l$ until the fast wave initiated by the excitation at (a) in Fig. 10.2.7 arrives at (d). Hence, if the initial conditions are zero, there is no response until $t = l/(U + v_s)$. In the interval of time between the points d and e at $x = l$ the slow wave is determined by the initial conditions, whereas the fast wave is determined by the sinusoidal driving conditions. After the slow wave arrives when $t = l/(U - v_s)$, however, both characteristic lines, which determine the solution at a given point along the line $x = l$, originate in the sinusoidal steady-state driving condition. Because the conditions that determine the solution are then periodic in time, it follows that so also is the solution; that is, after the time $t = l/(U - v_s)$ when the slow wave arrives at $x = l$ the sinusoidal steady-state condition has been established. After this time there is no evidence of the initial conditions.

Recall the nature of the response to the initial conditions, as described in Section 10.1, in which the string has no longitudinal velocity. There the effect of initial conditions was described in terms of the normal modes, and the consequences of the initial conditions persisted in the form of these modes for an indefinite period of time. As we saw in the examples of Section 10.1, these modes are purely oscillatory unless other effects, such as damping, are introduced.

In the sections that follow we introduce additional forces on the string, with the longitudinal convection included ($U > v_s$). Hence the examples considered involve the same transverse forces that are developed in Section 10.1 but in addition include the effect of convection. With this in mind, remember that in the examples in Section 10.1 the normal modes, with amplitudes specified by the initial conditions, determine whether the string deflections become unbounded in time. In any of these cases the sinusoidal steady-state solution can be found, but in cases in which the system is unstable this steady-state solution is eventually dominated by the unstable normal modes. With the convection, this dominance of the transient resulting from the initial conditions is not possible. The sinusoidal steady-state solution is all that remains after the time $l/(U - v_s)$, which means that in the supersonic case ($U > v_s$) it is not possible for deflections to become unbounded with time at a given position x .

The significance of these remarks becomes more evident in Section 10.2.3. In any case it should be evident that with $U > v_s$ the driven response assumes primary importance

If once again we assume complex waves (10.0.1) as solutions, substitution into (10.2.9) shows that the dispersion relation between the frequency ω and wavenumber k is

$$(\omega - kU)^2 = v_s^2 k^2. \quad (10.2.18)$$

Note that this is the relation obtained in Section 10.1.1, except that now $\omega \rightarrow \omega - kU$, which reflects the fact that the effect of the convection is to replace the time derivative with the convective derivative. (As we emphasize in Section 10.2.4, care must be taken in transforming from one frame of reference to another using this simple substitution by noting the parts of the system that are moving.)

The dispersion equation has the graphical form of two straight lines, as can be seen by solving (10.2.18) for ω :

$$\omega = k(U \pm v_s). \quad (10.2.19)$$

The subsonic and supersonic cases are shown in Fig. 10.2.8. The effect of increasing U is to rotate the two straight lines of Fig. 10.1.1 until finally, when $U > v_s$, as shown in Fig. 10.2.8b, both straight lines are in the first and third quadrants of the ω - k plane.

The dispersion equation can be used to find the response of the string to a drive with the frequency ω_d by solving (10.2.19) for the wavenumbers k , given that $\omega = \omega_d$:

$$k = \eta \pm \gamma, \quad (10.2.20)$$

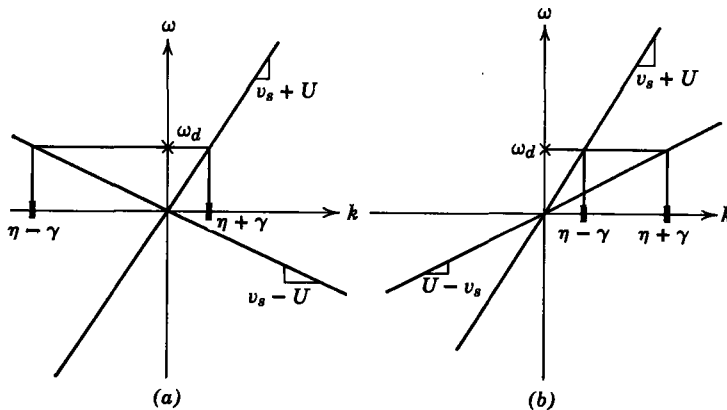


Fig. 10.2.8 Dispersion relations for waves on a string moving with the equilibrium velocity U in the positive x -direction. Although the geometry for the subsonic and supersonic cases is the same as that for the characteristic lines in Fig. 10.2.2, the axes here are ω - k , whereas in Fig. 10.2.2 they are x - t : (a) $U < v_s$; (b) $U > v_s$.

where for the string without additional forces

$$\begin{aligned}\eta &= \frac{U}{v_s} \gamma, \\ \gamma &= \frac{\omega_d v_s}{U^2 - v_s^2}\end{aligned}\tag{10.2.21}$$

A graphic representation of these two wavenumbers corresponding to the frequency ω_d is given in Fig. 10.2.8. Remember that the phase velocity of a wave is ω/k . With $U < v_s$, phases of one wave propagate rapidly downstream with the velocity $v_p = U + v_s$, whereas those of the other propagate upstream at the lesser velocity $v_p = v_s - U$. With $U > v_s$, as shown in Fig. 10.2.8b, phases of both waves propagate downstream. These results are not surprising in view of the wave dynamics found in Section 10.2.1a.

Consider the sinusoidal steady-state dynamics resulting when $U > v_s$. Then, upstream boundary conditions in the form of (10.2.16) and (10.2.17) are appropriate. In particular, suppose that

$$\xi(0, t) = 0 \tag{10.2.22}$$

$$\frac{\partial \xi}{\partial x}(0, t) = \Delta_0 \cos \omega_d t. \tag{10.2.23}$$

Then we can take a linear combination of waves with wavenumbers given by (10.2.20) to satisfy the first of these conditions:

$$\xi(x, t) = \text{Re } A e^{j(\omega_d t - \eta x)} \sin \gamma x. \tag{10.2.24}$$

The remaining arbitrary constant A is determined by the second condition to be $A = \Delta_0/\gamma$, and so the required sinusoidal steady-state driven response is

$$\xi(x, t) = \frac{\Delta_0}{\gamma} \cos(\omega_d t - \eta x) \sin \gamma x. \tag{10.2.25}$$

We have found that when sinusoidal steady-state waves are excited on the supersonic string they combine to form an envelope with nulls spaced by the distance $\pi/\gamma = \pi(U^2 - v_s^2)/\omega_d v_s$. Within this envelope points of zero phase on the string move in the x -direction with the velocity $\omega_d/\eta = (U^2 - v_s^2)/U$. The deflections are shown at an instant in time in Fig. 10.2.9.

The periodic envelope of the waves is stationary in space and therefore has the same character as the standing waves found for the stationary string. These peaks and nulls in the deflection are sometimes referred to as beats*

* The beating of two sinusoidal signals is commonly encountered in demodulation processes, such as those that occur in the ear when tuning two musical instruments. Tones that differ slightly in frequency beat together at the difference frequency. This illustrates how phenomena observed as functions of time are found as functions of space when the supersonic motion of a medium is involved.

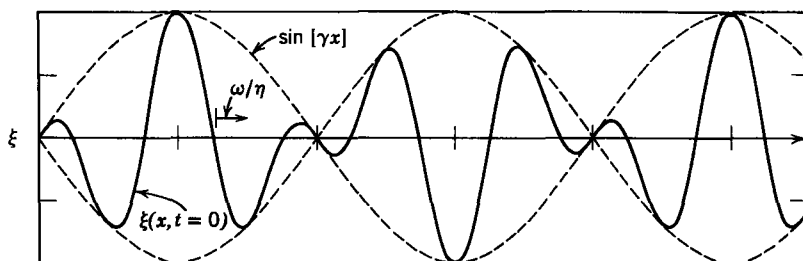


Fig. 10.2.9 Fast and slow waves interfere to form beats in space when $U > v_s$. The deflection is as the string appears if illuminated once each period of excitation $2\pi/\omega_d$.

because they result from interference between the two waves propagating downstream. At some positions the deflections from the two waves tend to add, whereas at others the deflections cancel.

In the sections that follow we undertake to show the effect of convection on each of the classes of interactions developed in Section 10.1. In cases in which the convection is equivalent to the entire system being in a state of uniform translation this is a simple matter, since the effect of the convection is represented by replacing the time derivatives in the differential equation with convective derivatives.

10.2.2 Evanescence and Oscillation with Convection

The developments of Section 10.2.1 pertained to the effect of convection on the dynamics of an ordinary string. In the absence of convection this string supports waves that propagate without dispersion. In this section we begin a discussion of the effects of convection on dispersive waves. In particular, we reconsider the systems that support the cutoff or evanescent waves described in Section 10.1.2. Now the system is as shown in Fig. 10.1.2, except that the wire has an equilibrium longitudinal velocity U . Because the magnetic restoring force is the same, whether evaluated in the frame moving with the string or in a fixed frame, the equation of motion found in Section 10.1.2 (10.1.6) is valid in a frame moving with the velocity U in the x -direction. As illustrated in Section 10.2, the equation of motion in the fixed frame is obtained by replacing the time derivative with the convective derivative. From this it follows that the dispersion equation with convection is obtained by simply replacing ω in the dispersion equation for the fixed system (10.1.7) with $\omega - kU$. Hence the required dispersion equation with convection is

$$(\omega - kU)^2 = v_s^2 k^2 + \omega_c^2. \quad (10.2.26)$$

The graphical representation of this equation is shown in Fig. 10.2.10, in which $U < v_s$ and $U > v_s$ are shown. These plots give complex values of k

for real values of ω . It should be clear from the plots and the fact that the dispersion equation is quadratic in ω that only real values of ω are given by (10.2.26) for real values of k .

The asymptotes of the dispersion equation are the two straight lines of Fig. 10.2.8*a, b*. Hence the result of increasing the velocity U is to rotate the hyperbolas in a counterclockwise direction. To make further deductions about the dispersion equation, it is helpful to solve (10.2.26) for k .

$$k = \eta \pm \gamma, \quad (10.2.27)$$

where

$$\begin{aligned} \eta &= \frac{\omega_d U}{U^2 - v_s^2}, \\ \gamma &= \frac{\sqrt{v_s^2 \omega_d^2 + (U^2 - v_s^2) \omega_c^2}}{U^2 - v_s^2}. \end{aligned} \quad (10.2.28)$$

This makes it evident that in the case in which $U < v_s$, there is a range of driving frequencies over which the wavenumbers are complex. From the last two equations, this is the range of ω_d over which the quantity under the radical is negative.

$$|\omega_d| < \left| \frac{(v_s^2 - U^2) \omega_c^2}{v_s^2} \right|^{1/2}. \quad (10.2.29)$$

The complex values of k are shown in Fig. 10.2.10*a*, in which the imaginary part of k takes the form of an ellipse and the real part of k (the straight line) is the same for both waves.

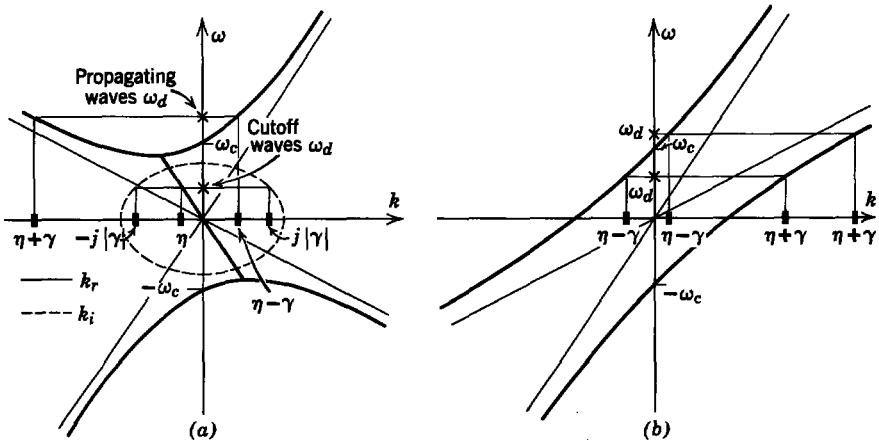


Fig. 10.2.10 Dispersion relations for system of Section 1.2, with the addition of a longitudinal velocity U . When U exceeds v_s , both waves propagate without decay, regardless of the frequency, and the effect of the convection is to eliminate the evanescence. (a) $U < v_s$; (b) $U > v_s$.

Two cases are shown in Fig. 10.2.10a. At the lower of the two driving frequencies waves with wavenumbers whose real part is η and imaginary parts, $\pm j|\gamma|$ are cutoff. At the larger driving frequency both wavenumbers are real. Note that the effect of the convection has been to make the cutoff wavenumbers complex rather than purely imaginary, as they were when the string was stationary.

As U is raised, the ellipse of Fig. 10.2.10a becomes smaller, until under the critical condition that $U = v_s$ it disappears and only real values of k are possible. Hence in Fig. 10.2.10b there are two real wavenumbers corresponding to a given driving frequency ω_d .

It is evident from what we have found that the effect of the convection has been to eliminate the evanescence. Two cases are illustrated in Fig. 10.2.10b. At the larger of the two driving frequencies both wavenumbers are positive and points of constant phase on the waves propagate downstream. At the lower driving frequency, however, the phase velocity of the slow wave is in the negative x -direction (upstream). This is perhaps our first disenchantment with the physical significance of the phase velocity. In fact, a pulse initiated at some point along the moving string would not propagate upstream in the face of the convection. The phase velocity does *not* indicate the manner in which a wavefront would propagate.

We introduced a discussion of the propagation of pulses on the moving string in Section 10.2 because it clearly indicated the direction in which a disturbance would propagate on the string. The string alone, however, is a rather special case in that such sinusoidal steady-state quantities as the phase velocity are identical with the actual propagational velocity of wavefronts. This is why the string merits the designation "dispersionless medium." In general, the phase velocity has no more significance than that associated with the dynamics of the sinusoidal steady state. We return to a discussion of propagational velocities in Section 10.3. For now, suffice it to say that regardless of the types of dispersion introduced with the examples of this and the preceding section wavefronts propagate with the velocities $U + v_s$ and $U - v_s$. Hence boundary conditions are imposed in accordance with the relative values of U and v_s , just as they are with the string alone.

If U is less than v_s , a pulse on the string can propagate upstream. Hence in Fig. 10.2.10a it is appropriate to use the wavenumbers to provide solutions that satisfy one upstream and one downstream boundary condition. Just as in the case considered in Section 10.1.2 the complex wavenumbers indicate that the deflections excited at one end decay spatially from the point of excitation. In the supersonic case shown in Fig. 10.2.10b it is appropriate to impose two boundary conditions at an upstream position. The sinusoidal steady state is then established in essentially the same way as for the moving string alone (Fig. 10.2.7). In fact, the wavenumbers now have the same form

considered in Section 10.2.1 (10.2.20), except that η and γ are defined by (10.2.27). Without any further mathematical developments, we can see that driving conditions in the sinusoidal steady-state form of (10.2.22) and (10.2.23) lead to the deflections given by (10.2.24) and shown in Fig. 10.2.9. The waves combine to form spatial beats as an envelope. The most salient consequences of the dispersion introduced by the addition of the magnetic restoring force can be seen if we consider the limit in which the tension on the string is of negligible importance. Then $v_s \rightarrow 0$ and η and γ , as given by (10.2.28), reduce to

$$\begin{aligned}\eta &= \frac{\omega_d}{U}, \\ \gamma &= \frac{\omega_c}{U}.\end{aligned}\tag{10.2.30}$$

It follows that the string deflections for the sinusoidal steady-state solution of (10.2.25) become

$$\xi(x, t) = \frac{\Delta_0 U}{\omega_c} \cos \left[\omega_d \left(t - \frac{x}{U} \right) \right] \sin \frac{\omega_c x}{U}.\tag{10.2.31}$$

The envelope of the waveform is determined by the cutoff frequency ω_c , whereas the phase velocity of the waves within this envelope is simply the convection velocity U .

In the absence of a tension f each section of the string is subject only to the spring like restoring force of the magnetic field. Because each section has a mass, the resulting dynamics can be pictured in terms of the mechanical model shown in Fig. 10.2.11. Here the mass has been lumped at discrete positions so that it is clear that each section of the string behaves as a simple mass-spring oscillator. As in Section 10.1.2, the spring represents the effect of the field. The convection carries these oscillators in the x -direction with the velocity U .

The resonance frequency of the mass-spring oscillators is ω_c . We can see this by recognizing that in the moving frame the equation of motion takes the form found with a simple harmonic oscillator.

$$\frac{\partial^2 \xi}{\partial t'^2} = -\omega_c^2 \xi.\tag{10.2.32}$$

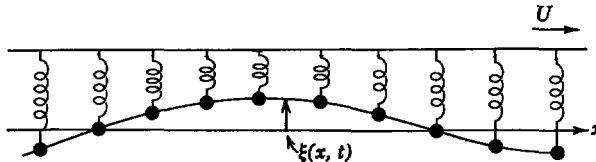


Fig. 10.2.11 When $f \rightarrow 0$ ($v_s \rightarrow 0$), the moving string subject to the restoring force of the magnetic field is equivalent to a distribution of mass-spring oscillators with the resonance frequency ω_c and moving with the velocity U in the x -direction.

The example that follows illustrates how this type of dynamics develops in dealing with electron beams.

Example 10.2.2. The dynamics of moving oscillators is of importance in dealing with interactions between streaming electrons and various media, such as distributed circuits, plasmas, and solid-state lattice structures. The similarity between the electron and string dynamics treated in this section can be illustrated by considering a simple one-dimensional example.

Suppose that a given region of space is filled with electrons with the number density n_e and ions with the number density n_i . We assume that these particles do not interact with one another, except through their electric fields. In the equilibrium situation both the electrons and the ions are static. Moreover, the ion charge density is equal to the electron charge density, so that there is no net charge so long as the system of particles is in equilibrium; for example, if both the ions and electrons have a single electronic charge e , the ion charge density en_i just balances the electron charge density $-en_e$ when the system is in equilibrium.

We are concerned here about the dynamics of the electrons, which have a much smaller mass than the ions. For this reason, it is a good approximation to consider the more massive ions as fixed, hence as having a constant number density. Then the perturbation from equilibrium charge density is

$$\rho_e = (n_i - n_e)e = -ne, \quad (a)$$

where $n(x, t)$ is the amount by which the number density of electrons exceeds the equilibrium value n_e at any given (x, t) .

To describe the electric part of this interaction it is recognized that the charge density is related to the electric field intensity by Gauss's law. In one dimension this is

$$\frac{\partial E}{\partial x} = -\frac{ne}{\epsilon_0}. \quad (b)$$

In addition, conservation of charge in one dimension becomes

$$\frac{\partial J_f}{\partial x} + \frac{\partial \rho_e}{\partial t} = 0, \quad (c)$$

where J_f is the current density. In the present case currents arise only because of the motion of the electrons in the x -direction; hence

$$J_f = -(n_e + n)ev \simeq -n_e ev, \quad (d)$$

where v is the longitudinal (x) electron velocity and the last approximate equality results from a linearization.

The mechanical equation of motion is Newton's law expressed for each of the electrons (having the mass m).

$$m \frac{\partial v}{\partial t} = -eE. \quad (e)$$

Now, a combination of (a) through (e) gives

$$\frac{\partial^2 n}{\partial t^2} + \omega_p^2 n = 0, \quad (f)$$

where

$$\omega_p = \left(\frac{n_e e^2}{\epsilon_0 m} \right)^{1/2}. \quad (g)$$

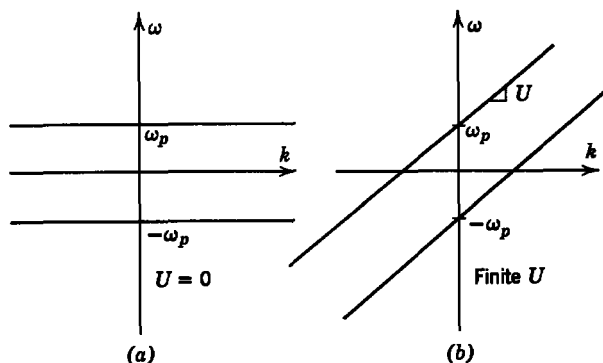


Fig. 10.2.12 Dispersion plot for electron oscillations having the plasma frequency ω_p : (a) the electrons with a fixed equilibrium state; (b) electrons with an equilibrium velocity U in the x -direction.

On the basis of the developments in this and the preceding section, it is a simple matter to see that if the electrons have an equilibrium state of motion in the x -direction with the velocity U (f) is altered to

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 n + \omega_p^2 n = 0. \quad (\text{h})$$

The dispersion equations for these two situations are shown in Fig. 10.2.12. The plasma frequency ω_p plays the same role here as the cutoff frequency ω_c in the discussion of this section. The string dynamics in the magnetic field are described by the same dispersion equation found here in the limit in which $v_s \rightarrow 0$. Note, however, that the electron motions considered here are longitudinal, whereas the deflections of the oscillators (Fig. 10.2.11) are transverse.

In electron beam devices similar oscillations are obtained on a beam with finite cross-sectional dimensions by imposing a large longitudinally directed magnetic field. The Lorentz force $-ev \times \mathbf{B}$ then tends to confine these space-charge oscillations to the longitudinal direction.*

10.2.3 Convective Instability or Wave Amplification

Continuum instability is the subject of Section 10.1.3, in which the simple string, under the influence of a destabilizing magnetic force (Fig. 10.1.9), sustained deflections that grew exponentially with time.

In this section we shall find that if the unstable medium moves fast enough the instability will be carried downstream and a perturbation can grow in amplitude, but not at the same point in space at which it originated. This

* For a discussion of electron beam dynamics, see C. C. Johnson, *Field and Wave Electrodynamics*, McGraw-Hill, 1965, p. 277. The analogy between the moving string and the electron beam should make it clear to those who are familiar with the two-cavity klystron that the moving string with $U > v_s$ can be used to make a "stringtron" or amplifier using the longitudinal kinetic energy of the string as a source of energy.

type of instability can be excited in the sinusoidal steady state, in which case it appears as a spatially growing wave. Hence this *convective instability* is also called an *amplifying wave*, since it can be used to amplify a driving signal.

Some electron-beam devices, such as traveling wave tubes (TWT), make practical use of amplifying waves. In addition, convective instabilities play important roles in a variety of physical situations, ranging from fluid dynamics (boundary layer dynamics) to physical acoustics (ultrasonic amplification). In some cases the convective instability can be put to work, and in others the convective instability occurs inherently in a system and must be understood.

We can illustrate the basic nature of the amplifying wave by considering the effect of convection on the instability studied in Section 10.1.3. We now include the fact that the string is moving in the x -direction with the velocity U .

The effect of the convection is accounted for by replacing the time derivative in the equation of motion by the convective derivative. Hence (10.1.25) becomes

$$\frac{1}{v_s^2} \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 \xi = \frac{\partial^2 \xi}{\partial x^2} + k_c^2 \xi. \quad (10.2.33)$$

Remember, k_c^2 is proportional to the current I in the wire and the gradient of the imposed magnetic field.

Substitution of the traveling wave solution (10.0.1) into this equation of motion yields

$$(\omega - kU)^2 = v_s^2(k^2 - k_c^2). \quad (10.2.34)$$

We could just as well have obtained this dispersion equation by replacing ω in the dispersion equation of Section 10.1.3 with $\omega - kU$. The effect of the convection is simply to translate the entire system in the x -direction with the velocity U . As in Section 10.2.2, however, the excitations and frame of observation remain fixed, and it is this fact that makes the situation inherently different from the case with no convection.

Equation 10.2.34 is quadratic in either ω or k . Solving it for ω shows that there are “unstable” values of ω for real values of k ($|k| < |k_c|$)

$$\omega = kU \pm \sqrt{v_s^2(k^2 - k_c^2)}. \quad (10.2.35)$$

We have seen situations in which a negative imaginary value of ω resulting from a real value of k indicates that the deflections become unbounded with time (Section 10.1.3). In fact, if $U = 0$ and the string is fixed at each end, the allowed values of $k = n\pi/l$, and the associated eigenfrequencies are given by (10.2.35). With the string moving with a velocity U that exceeds v_s , it is no longer consistent with causality to impose a downstream boundary condition. This means that the response to initial conditions is no longer defined in terms of eigenmodes, but rather that it has the character described

in Section 10.2.1. With $U > v_s$, we impose two boundary conditions at $x = 0$. At any downstream position the transient response to the initial conditions is over when a slow wavefront arrives from $x = 0$. Hence the deflections of the string can no longer become unbounded with time at a fixed position x . If the boundary conditions at $x = 0$ are both zero, the response will be zero once the transient is completed.

Consider the consequences of a sinusoidal steady-state boundary condition at $x = 0$. Solution of the dispersion equation shows that for $\omega = \omega_d$, the wavenumbers k are

$$k = \eta \pm j\gamma, \quad (10.2.36)$$

where

$$\eta = \frac{\omega_d U}{U^2 - v_s^2},$$

$$\gamma = \frac{v_s \sqrt{(U^2 - v_s^2)k_c^2 - \omega_d^2}}{U^2 - v_s^2}.$$

The dispersion equation is shown graphically in Fig. 10.2.13. In this ω - k plot complex values of k are shown as functions of real values of ω . As can be seen from (10.2.36), over the range of frequencies

$$\omega_d^2 < (U^2 - v_s^2)k_c^2, \quad (10.2.37)$$

the wavenumbers are complex. Each wavenumber has the same real part over this range, which depends linearly on ω , as shown by the straight line in Fig. 10.2.13. As a function of ω , the imaginary part of k forms an ellipse. Of

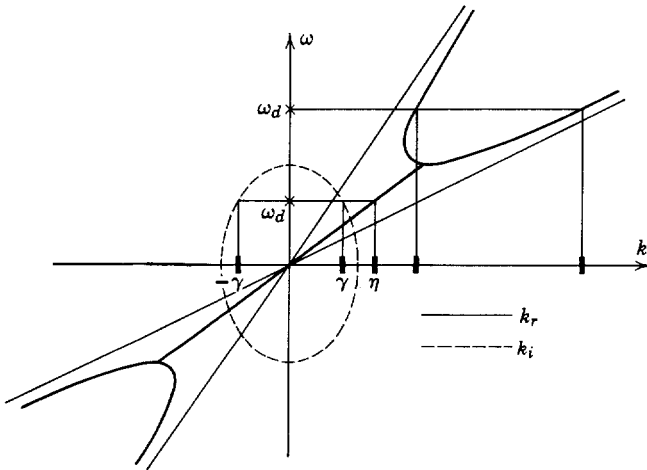


Fig. 10.2.13 Dispersion plot for supersonic string with a destabilizing magnetic force that shows complex values of k for real values of ω .

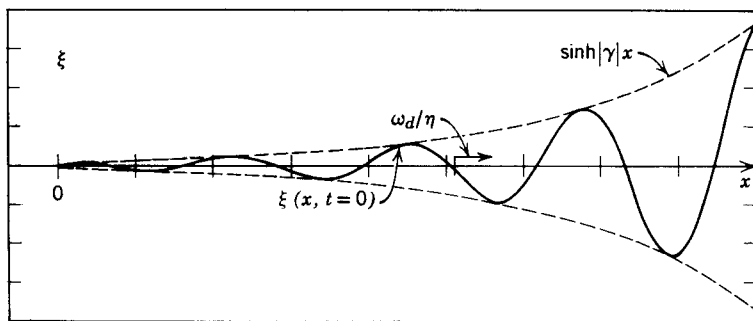


Fig. 10.2.14 Instantaneous view of string deflections when they are convectively unstable.

course, if the driving frequency exceeds a maximum value (10.2.37), both wavenumbers become real and the driven response takes on the form discussed in Section 10.2.1b.

To appreciate the significance of what we have found, consider once again the sinusoidal steady-state response to the boundary conditions of (10.2.22) and (10.2.23). Following the procedures outlined in Section 10.2.1b, we take a linear combination of the two waves which satisfies the boundary conditions at $x = 0$.

$$\xi(x, t) = \frac{\Delta_0}{|\gamma|} \sinh |\gamma| x \cos (\omega_d t - \eta x). \quad (10.2.38)$$

Here we have assumed that the driving frequency ω_d is the lower of the two shown in Fig. 10.2.13 so that the wavenumbers are complex.

The displacement given by (10.2.38) is shown as a function of x at a given instant of time in Fig. 10.2.14.*

Points of zero deflection move downstream with the phase velocity ω_d/η . Most important, the displacements have an envelope that grows exponentially with increasing x . The effect of the motion on the instability is now apparent. Rather than having an amplitude that is a monotonically increasing function of time at a given point in space, the displacements are now bounded in time but exhibit an exponential growth in space. The convection is responsible for washing the instability downstream.

The convective and absolute instabilities impose very different limitations on the engineering of systems. The largest amplitude obtained by the string within a given length is determined by the input of signals, perhaps in the form of noise, to the system. The more nearly the excitations are eliminated from the system, the more nearly will the string maintain its equilibrium

* The reason for the term “amplifying wave” is that an output transducer downstream from the excitation can extract a signal of the same frequency as the excitation but at a higher power level. The source of power is the moving medium.

position. This is by contrast to the case of absolute instability, in which, inadvertently, no matter how carefully the conditions of the string are set, initial noise will result in displacements that become unbounded in time.

The following example affords an opportunity to gain further physical insight into the nature of the convective instability.

Example 10.2.3. In Section 10.1.3 (on absolute instability) an example was considered in which a highly conducting membrane stressed by an electric field was found to be absolutely unstable (Example 10.1.2). Figure 10.2.15 shows a somewhat similar situation in which a jet of water passes between plane-parallel electrodes. The jet moves in the x -direction with the equilibrium velocity U . Transverse or kinking motions of the stream can be modeled by the string equation with the effect of convection included. In the jet the tension f is due to surface tension.

The jet is grounded, but the plates are at the same constant potential V shown in the figure. This is an electric field system, and because the time required for free charges to

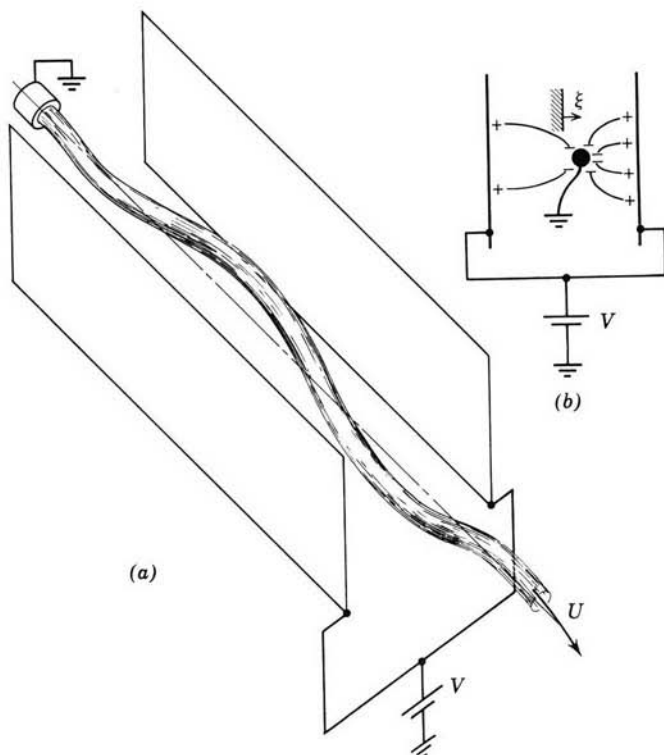


Fig. 10.2.15 (a) A jet of water has a streaming velocity U midway between plane-parallel electrodes. A potential difference V is applied between the plates and the jet. (b) As the jet is deflected toward one of the plates, an unbalance in the electric force of attraction tends to carry it even farther in the direction of deflection. The instability resulting from this force is washed downstream to form the convective instability sketched in Fig. 10.2.16.

relax from one point to another on the water jet is extremely short compared with dynamical times of interest, the jet can be regarded as perfectly conducting; that is, the potential difference between any point on the jet and either of the electrodes is the constant V . This is the same electrical situation as in Example 10.1.2, in which the membrane and electrodes retained a constant potential difference even as the membrane deflected. The dependence of the electric force on the jet displacement is much the same as in the membrane.

A cross section of the jet and plates is shown in Fig. 10.2.15*b*. When the jet is centered, charges induced on its surface are attracted equally toward their images on each of the plates. Hence there is no net transverse force on the jet so long as it remains centered. If, however, it moves off center by the amount ξ , there is a net force. More charges are induced on the side of the jet that is nearer one of the plates (remember that the potential difference is constant and so the jet and plates form a constant potential capacitor). This surplus of

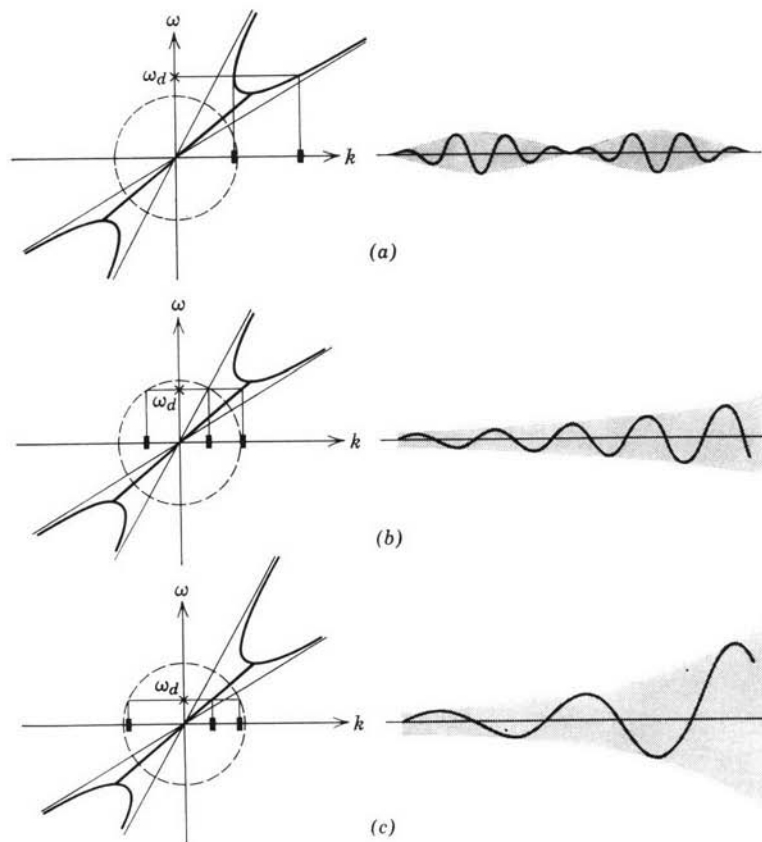


Fig. 10.2.16 Sinusoidal excitation of jet shown in Fig. 10.2.15. Driving frequency decreases from (a) to (c). This experiment together with animated ω - k plots can be seen in the film "Complex Waves II," produced for the National Committee on Electrical Engineering Films by the Education Development Center, Newton, Mass.

charge is attracted toward its image on the plate. Hence the force is (to linear terms) proportional to ξ and in the same direction as ξ . Both the amount of surface charge and the electric field are proportional to V , hence the force, which is proportional to the product of these quantities, is proportional to V^2 . We conclude that the force per unit length acting on the jet transverse to its direction of streaming has the form

$$S_z = bV^2\xi, \quad (a)$$

where b is a geometric constant.

If we combine the electric force of (a) with the equation of motion for the jet alone (10.2.9), it follows that the equation for the jet in the electric field will take the form of (10.2.33) with

$$k_c^2 = \frac{bV^2}{f}. \quad (b)$$

A jet of tap water with the diameter of a pencil and moving with a longitudinal velocity of 2 m/sec has a convection velocity U that exceeds v_s by a factor of 10 or more. Hence in the presence of the electric field the jet is subject to convective instability. It can be excited in the sinusoidal steady state at the upstream end to obtain the response illustrated by (10.2.38) and sketched in Fig. 10.2.16.

In the sequence of sketches the driving frequency ω_d is reduced, as indicated on the corresponding ω - k plots. Hence in Fig. 10.2.16*a* the frequency is high enough that wavenumbers are real and beats form on the jet, as described in Section 10.2.2. In Fig. 10.2.16*b,c* the frequency is reduced to the point at which the wavenumbers are complex and the waves exhibit spatial growth. Note that as the frequency is reduced the wavelength is increased and the rate of spatial growth is increased. This is consistent with the prediction of the ω - k plot, since the rate of growth is proportional to the imaginary part of k , whereas the wavelength is inversely proportional to the real part of k .

The physical significance of the complex wavenumbers found in this section is altogether different from that for the evanescent waves. In Section 10.1.2 complex wavenumbers were used to represent waves that decayed spatially away from the point of excitation. Here the deflection amplitude grows as a function of distance from the excitation.

The contrast between evanescent and amplifying waves must be emphasized, for even though these waves are physically very different they can be confused mathematically; for example, we could arbitrarily impose two upstream sinusoidal steady-state boundary conditions on the evanescent waves found in Section 10.1.2. The resulting mathematical solution would exhibit spatial growth and would appear to be an amplifying wave. Similarly, we could use the waves represented by (10.2.34) with $U > v_s$ to satisfy boundary conditions at two points in space and to conclude that waves decay away from the point of excitation. What we have called a convective instability in this section would then appear to be an evanescent wave. Both sinusoidal steady-state solutions, however, would be inconsistent with causality. The boundary conditions would make it impossible to establish the sinusoidal steady-state solution with the past always affecting the future and not vice versa.

The dispersion equation, as we have used it here, simply guarantees that solutions with the form $\exp j(\omega t - kx)$ satisfy the equation of motion. To make appropriate use of these solutions requires that proper regard be taken of the physical realizability of boundary conditions imposed on the solutions. The examples considered here are so simple that we could use physical intuition or a knowledge of how the system behaves without dispersion effects to establish the correct boundary conditions. Our arguments hinge essentially on observing how the system responds from the moment it is "turned on" until the sinusoidal steady state has been established. If the sinusoidal steady state can be established without violating conditions of causality, it is physically significant. We return to this topic in Section 10.3, in which propagation in the presence of dispersion is considered.*

10.2.4 "Resistive Wall" Wave Amplification

As shown in Section 10.2.3, the instability of continuous media in motion can give rise to amplifying waves. In the frame of reference attached to the moving medium the convective instability appears to grow in time, but in the fixed frame it is bounded in time. Hence it can be excited in the sinusoidal steady state with an attendant spatial growth.

Although the combination of material convection and absolute instability is one way of obtaining a convective instability, it is by no means true that this is the only way in which convective instabilities arise. Convective instabilities or amplifying waves often result when a stream interacts with a fixed structure. This is how amplifying waves are obtained in a traveling wave (electron beam) tube. A beam of electrons is coupled to a fixed electromagnetic transmission line. Similarly, beams of electrons or holes can couple to solid-state crystal structures to produce amplifying waves. These systems are more complicated than those introduced in this section because they involve more than two waves. Not only do the beams support waves, but so also do the fixed structures. Here we can illustrate this class of phenomenon by coupling the moving continuum to a fixed structure that does not support waves—simply a resistive wall. In fact, resistive wall instabilities are also found in electron beam devices and particle accelerators.†

With the developments of Section 10.1.4 in view, it is a simple matter to illustrate how convection, in conjunction with the damping, can produce wave amplification. Consider once again the situation in which the string is

* For a general discussion of this topic see A. Bers and R. J. Briggs, "Criteria for determining absolute instabilities and distinguishing between amplifying and evanescent waves," *Bull. Amer. Phys. Soc.*, Ser 2, 9, 304 (1964) or R. J. Briggs, *Electron-Stream Interaction With Plasmas*, M.I.T. Press, Cambridge, Mass., 1964, pp. 8–46.

† F. H. Clauser, *Symposium on Plasma Dynamics*, Addison-Wesley, Reading, Mass., 1960, pp. 78–118.

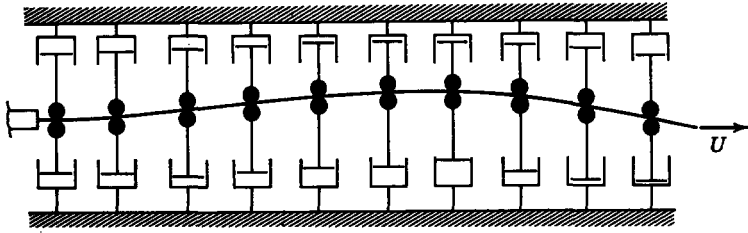


Fig. 10.2.17 The string moves unimpeded in the x -direction with the velocity U , and transverse motions are retarded by dashpots that interact with the string through rollers. The dashpots are fixed to the laboratory frame of reference.

immersed in a viscous liquid. Now, however, the string has a longitudinal velocity U . We assume that the convection of the string in the x -direction is unimpeded by the viscous fluid, even though the transverse motions lead to transverse viscous retarding forces. For conceptual purposes it is helpful to think of the system as being equivalent to that shown in Fig. 10.2.17.

The damping remains in the fixed frame. This means that the effect of the string motion is not simply accounted for by replacing time derivatives in (10.1.33) with convective derivatives, as would be appropriate with the total system in motion. We do, however, know the equation of motion for the string alone (10.2.8). Because the damping force does not depend on the motions, it remains as given by (10.1.32). Hence the equation of motion is (10.1.33) with the second time derivative (acceleration term) replaced by the second convective derivative but with the damping term left unaltered.

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right)^2 \xi = v_s^2 \frac{\partial^2 \xi}{\partial x^2} - \nu \frac{\partial \xi}{\partial t}. \quad (10.2.39)$$

As in Section 10.1.4, the wavefront velocity v_s and damping frequency ν are

$$v_s = \left(\frac{f}{m}\right)^{1/2},$$

$$\nu = \frac{B}{m}.$$

It follows from (10.2.39) that traveling wave solutions must satisfy the dispersion equation

$$(\omega - kU)^2 = v_s^2 k^2 + j\omega\nu, \quad (10.2.40)$$

which has the same form as (10.1.36), except that the ω on the left has been replaced by $\omega - kU$. Now, this dispersion equation, like others introduced in this and the preceding section, is simply quadratic in ω and k . Hence we

can solve for either ω or k . Observe first that $\omega(k)$ is

$$\omega = kU + \frac{j\nu}{2} \pm \left[\left(kU + \frac{j\nu}{2} \right)^2 - (U^2 - v_s^2)k^2 \right]^{1/2}. \quad (10.2.41)$$

If U exceeds v_s , this result shows that for real values of k there are negative imaginary values of ω . The ω - k plot, which shows complex values of ω for real values of k , is sketched in Fig. 10.2.18. This plot should be compared with Fig. 10.1.19 to appreciate the effect of convection. The convection is responsible for replacing the damping found (in Section 10.1.4) with $U < v_s$ by an instability. It would be erroneous, however, to assume on the basis of (10.2.41) that deflections of the string would become unbounded with time at a fixed position x . As in Section 10.2.3, the fact that U exceeds v_s requires that two upstream boundary conditions be imposed. In a manner similar to that discussed in Section 10.2.1*b*, the response to initial conditions vanishes in a finite time at a given position. Hence a sinusoidal steady state can be established, and the corresponding mathematical solution is found by solving (10.2.40) for k as a function of ω .

$$k = \frac{\omega U \pm \sqrt{v_s^2 \omega^2 + (U^2 - v_s^2)(j\omega\nu)}}{(U^2 - v_s^2)}. \quad (10.2.42)$$

This dispersion equation is shown in Fig. 10.2.19, in which complex values of k are shown for real values of ω . Note that the phase velocity of both waves is now downstream. More significant is the fact that because $U > v_s$ the wave with a positive imaginary wavenumber grows spatially. With the

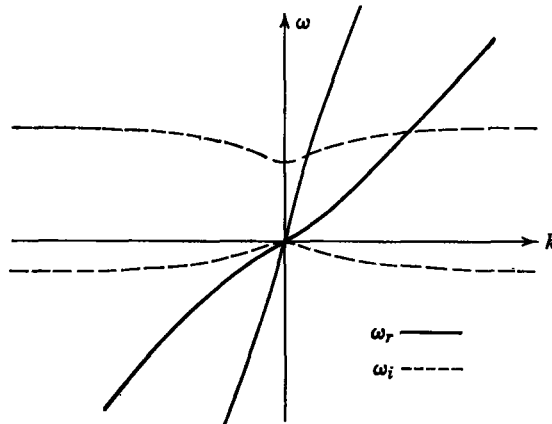


Fig. 10.2.18 Dispersion equation for resistive wall instability. Complex values of ω are shown for real values of k . Although this plot indicates that the string deflections are unstable, the instability is exhibited as a growth in space rather than in time.

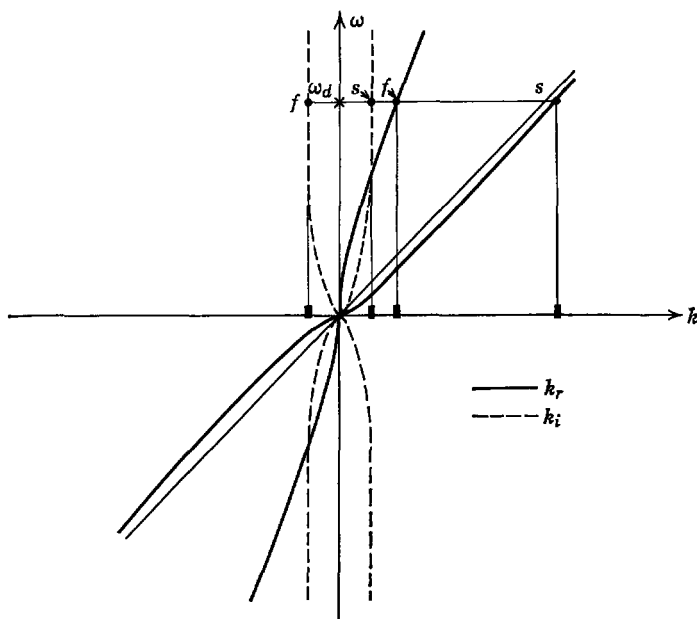


Fig. 10.2.19 Dispersion relation for resistive wall instability. Complex values of k are shown as functions of real values of ω . The effect of convection is apparent if this plot is compared with that shown in Fig. 10.1.17 (where $U = 0$).

damping removed, this is the wave that tries to buck the convection and, because $U > v_s$, propagates slowly downstream. The amplifying slow wave is denoted by (s) in Fig. 10.2.19; the spatially decaying fast wave is labeled (f). This amplifying wave has been termed a resistive wall instability because it is often found in electron beam devices when the beam is coupled to a resistive wall. The mechanical dampers might also be viewed as a resistive wall. The same effect as the dampers, however, is produced by coupling the continuum through a magnetic field to a dissipative structure, as shown in Example 10.1.3. With motion, that example quite literally illustrates a resistive wall instability, as discussed in the following example.

Example 10.2.4. Consider once again the perfectly conducting membrane shown in Fig. 10.1.7. Recall that the membrane is immersed in a longitudinal magnetic field that is uniform when the membrane is undeflected. Because the edges of the membrane make contact with side walls, which in turn form electric circuits above and below the membrane, deformations are accompanied by currents that tend to conserve the total flux above and below the membrane. We consider here the effect of convection when the resistance of the walls is so high that the self-inductance of the external current loops is unimportant. This limit was discussed in Example 10.1.3, in which it was found that the wall had the effect of damping the transverse motions.

We now introduce the additional complication that the membrane is moving in the x -direction with a velocity $U > v_s$. The resistive walls remain fixed. Consequently, the equation of motion, which for the membrane without convection takes the form of (e) in Example 10.1.3, becomes

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 \xi = v_s^2 \frac{\partial^2 \xi}{\partial x^2} - \nu \frac{\partial \xi}{\partial t}, \quad (\text{a})$$

where $v_s = \sqrt{S/\sigma_m}$ and $\nu = 2\sigma_s(\mu_0 H_0)^2/\sigma_m$. The left-hand side of this equation reflects the effect of convection on the moving membrane, whereas the damping force on the right remains unaffected. This can be shown by using a contour of integration for Faraday's law [contour (a) or (b) in Fig. 10.1.8] just as in Example 10.1.1 but with the segment through the membrane moving in the x -direction with velocity U . Because the membrane is perfectly conducting $\mathbf{E}' = 0$ over that segment and, moreover, there is no addition to the rate of change of the flux linked by the contour.

Because (a) has the same form as (10.2.39), we conclude that the moving membrane coupled to the resistive wall is subject to convective instability.

A physical picture of the variety of convective instability studied in Section 10.2.3 is not difficult to obtain. In a frame of reference moving with the string the amplitude becomes unbounded with time for the same reason that it does when the string is stationary. Without the motion, however, the system we consider in this section is stable. In fact, we normally think of damping as a stabilizing influence. Yet the combination of the convection and damping leads to instability!

An understanding of this instability can be obtained by considering string deflections in a frame of reference moving with the string, represented by Fig. 10.2.20. In this frame the string appears to have no longitudinal velocity but the resistive wall (the dampers) moves in the $-x$ -direction with the velocity U . The viscous force, expressed in terms of the coordinates (x', t') , is

$$S_z = -B \left(\frac{\partial}{\partial t'} - U \frac{\partial}{\partial x'} \right) \xi. \quad (10.2.43)$$

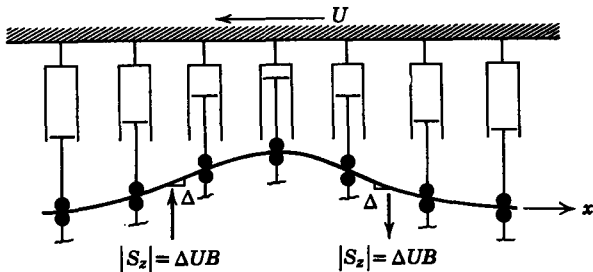


Fig. 10.2.20 A pulse is initially stationary when viewed from a frame moving with the string. Then, as the dashpots move to the left, they produce a force with a sign determined by the slope of the string.

The first term, familiar as the retarding force due to the transverse velocity, will always act in a direction opposite to that in which ξ is temporally increasing. The second term enters because, with the dashpots moving to the left, the plungers must change their positions at a rate determined by U and the slope of the string; for example, suppose that the pulse is instantaneously stationary in the x' -frame. The moving dampers will tend to increase the deflection when the slope is positive and decrease it when the slope is negative. Hence the back of the pulse tends to grow, whereas the front of the pulse is flattened by the moving dampers. As the regions of positive slope grow, they move to the right when viewed in the fixed frame, and we have the ingredients of a convective instability.

10.3 PROPAGATION

The situations developed in the last two sections provide an ample background for embarking on a discussion of wave propagation in distributed systems exhibiting dispersive waves. In retrospect, it is clear that we have either confined attention to sinusoidal steady-state behavior or, when transient conditions were considered, to nondispersive waves (with and without convection); for example, in Section 10.2.1 we used the moving string without external forces to illustrate how conditions of causality restrict the boundary conditions that can be imposed. Throughout our discussion we have assumed that even with dispersive waves (an extreme example is the convective instability) we are justified in imposing two upstream conditions if $U > v_s$ and one upstream and one downstream condition if $U < v_s$. In fact, our assumption is correct but this is not obvious. One way to ask the question is this: if a pulse is initiated on the string, does any part of it propagate upstream? If so, we must impose a downstream boundary condition. We have illustrated that the phase velocity of a periodic wave is not a reliable basis for answering this question. We begin by reviewing this point.

10.3.1 Phase Velocity

Suppose that we restrict attention to waves that are purely sinusoidal in space and time so that both ω and k are real. Then points of constant phase propagate with the velocity,

$$v_p = \frac{\omega}{k}. \quad (10.3.1)$$

On an ω - k plot this phase velocity has the geometrical significance shown in Fig. 10.3.1. As long as we are dealing with nondispersive waves, the phase velocity is identical with the velocity of propagation of wavefronts, which is

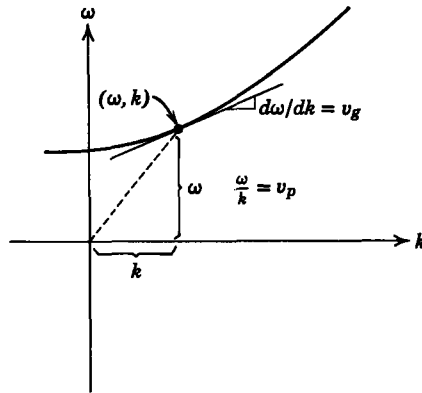


Fig. 10.3.1 The slope of the line joining the point (ω, k) and the origin is the phase velocity. The tangent of the ω - k curve at this same point is the group velocity.

why the ω - k plots of Fig. 10.2.8 appear so similar to the characteristic lines in the x - t plane, shown in Fig. 10.2.2. For the moving string without external forces wavefronts propagate with the velocities $U \pm v_s$ (the slopes of the characteristics), and these are the phase velocities of the fast and slow waves in the sinusoidal steady state.

It would be an assumption to say that the phase velocity is always the velocity of propagation of a wavefront, and we have examples from which to choose to show that this assumption is not well founded. In the case in which we have the possibility of evanescent waves (Fig. 10.1.3) the phase velocity approaches infinity as $k \rightarrow 0$, and it is certainly unreasonable to assume that a pulse will propagate with this velocity. Even more obvious, since v_p is a function of ω or k , which value do we select as the velocity of a wavefront? Considerations of this type lead us to consider other velocities such as the group velocity.

10.3.2 Group Velocity

The group velocity of waves with frequencies in the neighborhood of ω is defined as

$$v_g = \frac{d\omega}{dk} \quad (10.3.2)$$

or as the slope shown in Fig. 10.3.1. The velocity v_g has the physical significance of being the velocity of propagation of a group of waves with essentially the same wavenumber and frequency.

A straightforward way to show the dynamical meaning of the group velocity is to consider an excitation in the form of

$$\xi(0, t) = \xi_0[1 + \cos \omega_m t] \cos \omega_s t, \quad (10.3.3)$$

which is recognized as amplitude modulation of a sinusoidal excitation with a frequency ω_s . The amplitude of the excitation varies between $2\xi_0$ and 0 with a frequency ω_m . The nature of the resulting waves is most easily demonstrated if it is assumed that the string has an infinite extent in the x -direction and excitations are applied in such a way that only one of the two possible waves is excited.

A trigonometric identity* converts (10.3.3) to

$$\xi(0, t) = \xi_0 \cos \omega_s t + \frac{1}{2} \xi_0 [\cos (\omega_s + \omega_m)t + \cos (\omega_s - \omega_m)t], \quad (10.3.4)$$

which is the familiar statement that a sinusoidal signal with a frequency ω_s , amplitude modulated at the frequency ω_m , is equivalent to three signals with frequencies ω_s , $\omega_s + \omega_m$, and $\omega_s - \omega_m$. Because the wavenumber depends on the frequency of excitation, each of the terms in (10.3.4) excites a wave with a different wavelength.

Attention is confined to situations in which $\omega_m \ll \omega_s$. Then the wavenumbers corresponding to $\omega_s \pm \omega_m$ are given approximately by $k = k_s \pm \omega_m/v_g$. Figure 10.3.2 shows graphically the relationship between frequencies and wavenumbers. The wavenumbers adjacent to k_s are obtained by assuming that in the vicinity of the point (ω_s, k_s) the dispersion equation can be approximated by a straight line whose slope v_g is the same as that in the ω - k plot.

If only one wave is excited, the first term in (10.3.4) gives rise to the string deflections

$$\xi(x, t) = \xi_0 \cos (\omega_s t - k_s x). \quad (10.3.5)$$

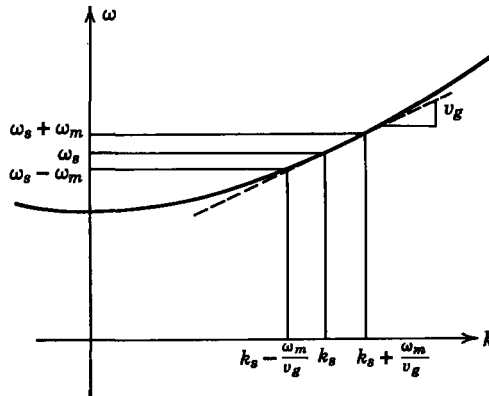


Fig. 10.3.2 For waves with very nearly the same frequency the group velocity v_g can be used to represent the dispersion.

* $\cos A \cos B = \frac{1}{2} [\cos (A + B) + \cos (A - B)]$.

The excitations at the sum and difference frequencies $\omega_s \pm \omega_m$ lead to similar waves with wavenumbers $k_s \pm \omega_m/v_g$ which are now superimposed to find waves resulting from the total excitation.

$$\begin{aligned}\xi(x, t) = & \xi_0 \cos(\omega_s t - k_s x) \\ & + \frac{1}{2} \xi_0 \cos \left[(\omega_s t - k_s x) + \omega_m \left(t - \frac{x}{v_g} \right) \right] \\ & + \frac{1}{2} \xi_0 \cos \left[(\omega_s t - k_s x) - \omega_m \left(t - \frac{x}{v_g} \right) \right].\end{aligned}\quad (10.3.6)$$

We again use a trigonometric identity* to write this expression as

$$\xi(x, t) = \xi_0 \left[1 + \cos \omega_m \left(t - \frac{x}{v_g} \right) \right] \cos(\omega_s t - k_s x), \quad (10.3.7)$$

so that it is clear that displacements take the form of a wave, with a frequency ω_s , which is modulated by another wave with a frequency ω_m . At an instant of time the string appears as shown in Fig. 10.3.3. A point of constant phase (a zero crossing) moves to the right with the phase velocity ω_s/k_s . By contrast, a point of constant phase on the *envelope* moves to the right with group velocity v_g . The group velocity is therefore interpreted as the velocity at which the group of waves, defined by the envelope, propagates in the x -direction. Note that because the phase and group velocities are different the waves will appear to move with respect to the envelope. In the ω - k plot of Fig. 10.1.3 (which was the subject of Section 10.1.2) the group velocity is less than the phase velocity; hence the phases move downstream more rapidly than the envelope.

Our analysis presumes that the wavetrain of Fig. 10.3.3 extends to infinity in the x -direction and that the sinusoidal steady state has been established.

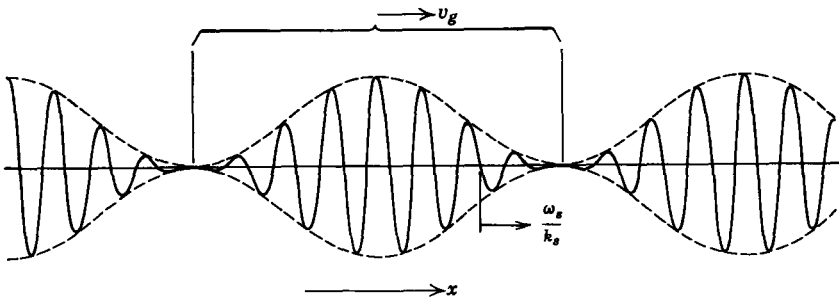


Fig. 10.3.3 Wavetrain amplitude modulated at a frequency that is much lower than the wave frequency. The string appears as shown by the solid line at an instant in time.

* $\cos(A + B) = \cos A \cos B - \sin A \sin B$.

Hence it does not directly indicate the manner in which a pulse propagates. Suppose, however, that we were concerned with the propagation of the pulse obtained by eliminating all the wavetrain except the one full pulse in Fig. 10.3.3 bounded on either end by the nulls in the envelope. It is not possible to describe this signal as the superposition of three sinusoidal signals, but, if we take the Fourier transform of this signal*

$$\hat{\xi}(\omega, k) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \xi(x, t) e^{-j(\omega t + kx)} dx dt, \quad (10.3.8)$$

we find that its major frequency and wavenumber components are in the neighborhood of (ω_s, k_s) . Provided that the spectrum of frequency (hence, because of the dispersion equation, the wavenumbers) of a pulse is confined essentially to a small region, the group velocity gives a measure of the velocity of the pulse. In terms of (x, t) this pulse is almost a pure sinusoid, except for a slowly varying envelope.

From this discussion it should be clear that the group velocity can be used to ascertain the "over-all" velocity of a high-frequency pulse, but because only those portions of the pulse that have a slowly varying envelope can be so described the group velocity is not the velocity of a wavefront. As we have seen, in the region of a wavefront deflections vary rapidly and, in fact, can be discontinuous. Hence the group velocity is not appropriate for determining the boundary conditions consistent with causality. As for the phase velocity, we might have suspected this limitation from the outset, since the group velocity is a function of ω or k . We hope that the question whether two upstream boundary conditions or one upstream and one downstream boundary condition should be used is independent of the dynamical quantities ω and k .

We must make one further reservation concerning the physical significance of the group velocity. It does not give a reliable indication of pulse propagation, even subject to the limitations outlined, if the system is unstable.† This is true whether the instability is convective or nonconvective. We have not attempted to prove this here but have used the examples of the two previous sections to make this further limitation plausible. The ω - k plot for the unstable string is shown in Fig. 10.1.11. Note that as $k \rightarrow k_e$ the group velocity approaches infinity. It would be unreasonable to conclude that a pulse could be made to propagate with an arbitrarily large velocity.

With convection (Fig. 10.2.13) the insignificance of v_g becomes even more apparent. In this case we can select ω or k to give any value of v_g , positive,

* R. G. Brown and J. W. Nilsson, *Introduction to Linear Systems Analysis*, Wiley, New York, 1962, p. 209.

† R. J. Briggs, *Electron-Stream Interaction With Plasmas*, M.I.T. Press, Cambridge, Mass., 1964, p. 33.

negative, zero, and infinite! From this it is clear that the notion of a group velocity must be used with care. Cases in which instability can develop make this particularly clear. A pulse of finite extent contains all wavenumbers. The group velocity is meaningful if the dynamics are dominated by signal components confined to a small range of wavenumbers. The wavenumber spectrum that is unstable tends to amplify and cannot be ignored.

10.3.3 Characteristics and the Velocity of Wavefronts

Wavefronts propagate along the characteristic lines in the x - t plane. This is illustrated by considering a convecting string with a destabilizing force, discussed in Section 10.2.3. The equation of motion (10.2.33) is

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right)^2 = v_s^2 \left[\frac{\partial^2 \xi}{\partial x^2} + k_c^2 \xi \right]. \quad (10.3.9)$$

A general discussion of how characteristics can be found for a given differential equation would lead us afield.* Thus for our purposes we view the characteristics as a convenient transformation of independent variables from the coordinate system (x, t) to the new coordinate system (α, β) according to the relations

$$\alpha = x - (U + v_s)t, \quad (10.3.10)$$

$$\beta = x - (U - v_s)t. \quad (10.3.11)$$

The resulting simplification in the differential equation and its interpretation and solution justify the transformation.

Note that the characteristic lines are the same as those used in Sections 9.1.1a and 10.2.1, in which nondispersive waves were studied.

To write (10.3.9) in terms of the new independent variables α and β , we use (10.3.10) and (10.3.11) to evaluate the partial time derivative as

$$\frac{\partial \xi}{\partial t} = \frac{\partial \xi}{\partial \alpha} \frac{\partial \alpha}{\partial t} + \frac{\partial \xi}{\partial \beta} \frac{\partial \beta}{\partial t} = -(U + v_s) \frac{\partial \xi}{\partial \alpha} - (U - v_s) \frac{\partial \xi}{\partial \beta}. \quad (10.3.12)$$

In a similar manner

$$\frac{\partial \xi}{\partial x} = \frac{\partial \xi}{\partial \alpha} \frac{\partial \alpha}{\partial x} + \frac{\partial \xi}{\partial \beta} \frac{\partial \beta}{\partial x} = \frac{\partial \xi}{\partial \alpha} + \frac{\partial \xi}{\partial \beta}. \quad (10.3.13)$$

* For a general discussion of the method of characteristics see, for example, R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves*, Interscience, New York, 1948, Chapters, II, III, and IV.

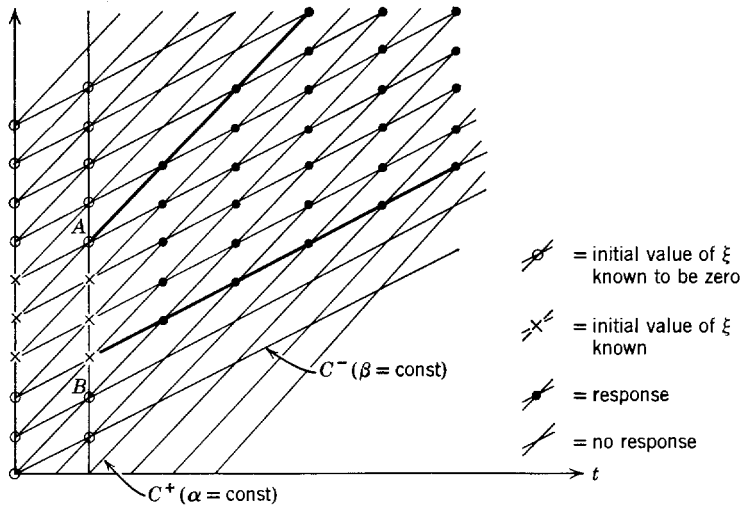


Fig. 10.3.4 Grid of characteristics in the x - t plane for $U > v_s$.

A repetition of the process and transformation of the derivatives in (10.3.9) yields

$$\frac{\partial^2 \xi}{\partial \alpha \partial \beta} + \frac{k_c^2}{4} \xi = 0. \quad (10.3.14)$$

This form of the equation of motion makes it particularly clear why $\xi(\alpha)$ and $\xi(\beta)$ are both solutions when $k_c = 0$.

In the x - t plane the C^+ characteristics ($\alpha = \text{constant}$) are parallel straight lines with the slope $U + v_s$ and the C^- characteristics ($\beta = \text{constant}$) are parallel straight lines with slope $U - v_s$. These characteristics fill the x - t plane as illustrated in Fig. 10.3.4. In Fig. 10.3.5 the four intersections of two pairs of closely spaced characteristics are shown. This configuration can be used to obtain an approximate integral of (10.3.14). The solution thus obtained yields both numerical answers and insight into how disturbances propagate.

We assume that the displacements at points A , B , and D in Fig. 10.3.5 are

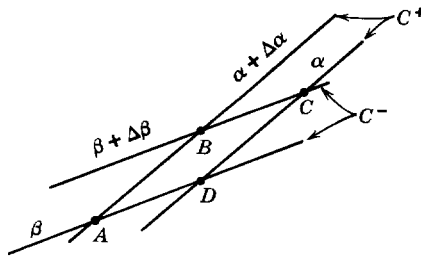


Fig. 10.3.5 Intersections A , B , C , and D of two pairs of characteristics with $U > v_s$.

known and we wish to evaluate the displacement at point C (at some future time at a downstream point) When $\Delta\alpha$ and $\Delta\beta$ are small, we can write

$$\frac{\partial \xi}{\partial \alpha}(\alpha, \beta + \Delta\beta) \approx \frac{\xi(\alpha + \Delta\alpha, \beta + \Delta\beta) - \xi(\alpha, \beta + \Delta\beta)}{\Delta\alpha} = \frac{\xi_B - \xi_C}{\Delta\alpha}, \quad (10.3.15)$$

$$\frac{\partial \xi}{\partial \alpha}(\alpha, \beta) \approx \frac{\xi(\alpha + \Delta\alpha, \beta) - \xi(\alpha, \beta)}{\Delta\alpha} = \frac{\xi_A - \xi_D}{\Delta\alpha} \quad (10.3.16)$$

Continuation of this process one step further yields

$$\frac{\partial^2 \xi}{\partial \alpha \partial \beta}(\alpha, \beta) \approx \frac{(\partial \xi / \partial \alpha)(\alpha, \beta + \Delta\beta) - (\partial \xi / \partial \alpha)(\alpha, \beta)}{\Delta\beta} \quad (10.3.17)$$

or

$$\frac{\partial^2 \xi}{\partial \alpha \partial \beta}(\alpha, \beta) \approx \frac{(\xi_B - \xi_C) - (\xi_A - \xi_D)}{\Delta\alpha \Delta\beta} \quad (10.3.18)$$

The use of this approximate derivative in (10.3.14) and solution for ξ_C yield the desired result*:

$$\xi_C = \xi_B + \xi_D - \xi_A \left(1 - \frac{k_c^2 \Delta\alpha \Delta\beta}{4} \right). \quad (10.3.19)$$

The solution to our problem $\xi(x, t)$ or $\xi(\alpha, \beta)$ can be viewed in a three-dimensional plot as the height of a surface above the x - t or α - β plane. Hence the initial conditions $\xi(x, 0)$ and $(\partial \xi / \partial t)(x, 0)$ give the height of the surface and its slope in the t -direction near the x -axis. In approximate terms this is equivalent to giving the value of ξ at the intersections of characteristics on and adjacent to the x -axis, as shown in Fig. 10.3.4. From this initial data (10.3.19) can be used to find ξ at the next (third) vertical column of intersections in the x - t plane of Fig. 10.3.4. These values, in turn, can be used to calculate ξ at the next vertical column of intersections.

In Fig. 10.3.4 we have indicated an initial pulse by an $\times$ at the characteristic intersections; that is, ξ has some finite value at these points. The initial displacements are zero at intersections marked $\circ$. The approximate solution (10.3.19) of the equation of motion then shows that there is no deflection ξ except possibly at the intersections marked. Hence the initial deflection in the interval AB results in wavefronts propagating within the region in the x - t plane bounded by the C^+ characteristic originating at A and the C^- characteristic originating at B . Because $U > v_s$, the initial disturbance propagates downstream, as we assumed in Section 10.2.3.

* Note that the term $k_c^2 \xi / 4$ has been approximated by $k_c^2 \xi_A / 4$. The error implicit in this approximation becomes more and more significant as iterative use is made of (10.3.19).

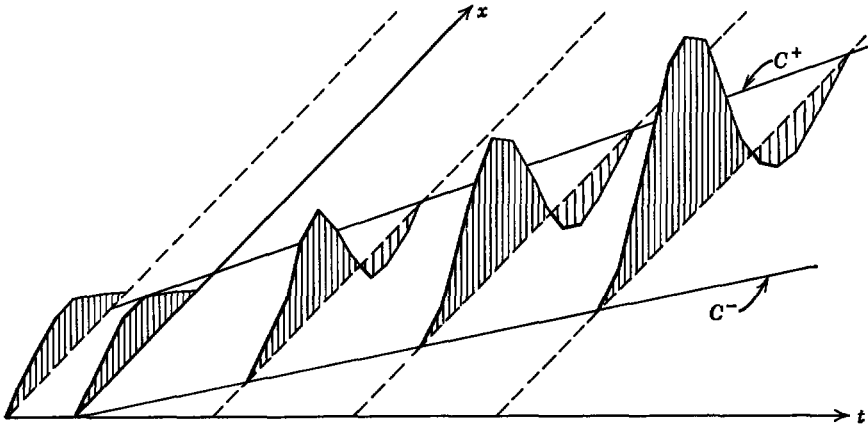


Fig. 10.3.6 Transient deflection following the initiation of a static pulse on a moving unstable string ($U = 3v_s$). The deflection is convectively unstable.

It can be seen that as long as both families of characteristics go downstream as t increases, a driving condition (along the t -axis) that involves two conditions on ξ is no different mathematically from the two initial conditions along the x -axis considered here.

Our approximate solution for the equation of motion shows also that waves are likely to grow. Consider the case in which points A , B , and D all have the deflection $\xi = \xi_0$ (the initial condition of a static, spatially uniform pulse of height ξ_0). Then

$$\xi_C = \xi_0 \left(1 + \frac{k_c^2 \Delta \alpha \Delta \beta}{4} \right), \quad (10.3.20)$$

and the amplitude has increased by the time the pulse reaches the point C .

A numerical example is shown in Fig. 10.3.6. For all values of $t < 0$ the initial pulse has the same x -dependence, as shown; that is, the pulse is initially static. For this transient solution seven grid points have been used to describe the initial pulse and $k_c^2 \Delta \alpha \Delta \beta / 4 = 0.1$. The pulse grows in amplitude but propagates downstream. At any given position x , the deflection remains bounded. These are the salient features of a convective instability.

The leading edge of the pulse becomes inverted, as is expected from the results found in Example 10.2.1 for an initially static pulse in a stable situation but with $U > v_s$ (see Fig. 10.2.5).

10.4 DYNAMICS IN TWO DIMENSIONS

The techniques described for dealing with continuum electromechanical interactions are not limited to a single space dimension, as implied by the

examples given. In this section we consider several cases that show how the dispersion equation also accounts for motions that depend on two or three dimensions.

10.4.1 Membrane Dynamics: Two-Dimensional Modes

A classic demonstration of wave propagation, as it depends on two dimensions, is given by considering a membrane. In the absence of external surface forces the equation of motion (as derived in Section 9.2) is

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right), \quad (10.4.1)$$

where

$$v_s = \left(\frac{S}{\sigma_m} \right)^{1/2}.$$

Solutions to this equation have the general form

$$\xi = \text{Re } \hat{\xi} e^{j(\omega t - k_x x - k_y y)}, \quad (10.4.2)$$

where we must now distinguish between the wavenumber k_y , which indicates the dependence on y , and k_x , which indicates the x -dependence. Substitution of (10.4.2) into (10.4.1) shows that ω is related to k_y and k_x by the dispersion equation

$$\omega^2 = v_s^2 (k_x^2 + k_y^2). \quad (10.4.3)$$

Now suppose that the edges of the membrane at $y = 0$ and $y = b$ are fixed, as shown in Fig. 10.4.1; that is, deflections satisfy the boundary conditions

$$\xi(x, 0, t) = 0, \quad (10.4.4)$$

$$\xi(x, b, t) = 0. \quad (10.4.5)$$

Any linear combination of solutions with the traveling waveform of (10.4.2) and (ω, k_x, k_y) satisfying the dispersion equation (10.4.3) will satisfy the equation of motion. In particular, we take the linear combination

$$\xi = \text{Re } [\hat{\xi}_1 \sin k_y y + \hat{\xi}_2 \cos k_y y] e^{j(\omega t - k_x x)}. \quad (10.4.6)$$

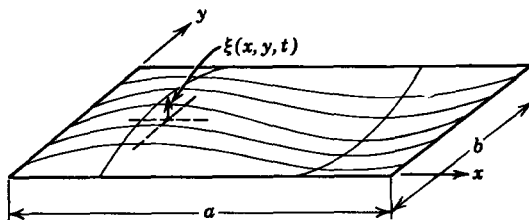


Fig. 10.4.1 Membrane having static equilibrium in x - y plane.

Here it is more convenient to write the complex exponential dependence on y in terms of trigonometric functions because it is clear from (10.4.4) that $\xi_2 = 0$. Moreover, if boundary condition (10.4.5) is to be satisfied,

$$k_y = \frac{n\pi}{b}. \quad (10.4.7)$$

We have found that modes satisfying the transverse boundary conditions take the form

$$\xi(x, y, t) = \text{Re } \xi_1 \sin\left(\frac{n\pi y}{b}\right) e^{j(\omega t - k_x x)}, \quad (10.4.8)$$

where in view of (10.4.7) the dispersion equation relating (ω, k_x) for any given mode (n) is

$$\omega^2 = v_s^2 \left[k_x^2 + \left(\frac{n\pi}{b}\right)^2 \right]. \quad (10.4.9)$$

This expression takes the same form as the dispersion equation (10.1.7) developed in Section 10.1.2, in which evanescent waves were studied. Hence for each mode the dispersion equation takes the geometrical form shown in Fig. 10.1.3. The cutoff frequency is [compare (10.1.7) and (10.4.9)]

$$\omega_c^2 = v_s^2 \left(\frac{n\pi}{b}\right)^2. \quad (10.4.10)$$

Suppose that the membrane is driven at one end by a sinusoidal steady-state excitation with a frequency $\omega = \omega_d$. Then we have found that the response is composed of an infinite number of waves, each having n half-wavelengths in the y -direction. If $\omega_d < \omega_c$, the waves in the x -direction are evanescent. Hence for any given driving frequency a finite number (or, if $\omega_d < v_s\pi/b$, none) of the waves will propagate (have real values of k_x), whereas all others will be evanescent.

The evanescent or cutoff waves are a consequence of the stiffening effect of the transverse boundaries. Curvature of the membrane leads to an elastic restoring force (remember the membrane is under the longitudinal and transverse tension S). Because of the transverse boundaries, the membrane cannot deform without incurring a curvature. Waves are cutoff if the elastic restoring force resulting from this curvature (proportional to k_y^2) outweighs the effect of inertia.

If, in addition to the boundary conditions already imposed, the membrane is fixed at $x = 0$ and $x = a$, it is appropriate to look for the natural modes or eigenmodes of the membrane. This time we take a linear combination of solutions in the form of (10.4.8) to determine that

$$\xi(x, y, t) = \text{Re } \xi_3 \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{m\pi x}{a}\right) e^{j\omega t}, \quad (10.4.11)$$

where k_x has been made $m\pi/a$; hence the dispersion equation gives the eigenfrequencies as

$$\omega_{mn} = \pm v_s \left[\left(\frac{n\pi}{b} \right)^2 + \left(\frac{m\pi}{a} \right)^2 \right]^{1/2}. \quad (10.4.12)$$

The indices (n, m) can have any integer values. Note, however, that if both n and m are zero the solution is trivial, since (10.4.11) then shows that the deflections are zero.

The eigenfrequencies of the membrane constrained on all of its edges have the same physical significance as those of the simple string; for example, if the membrane is driven in some manner by a sinusoidal steady-state forcing function, there will be a resonance in the response at the eigenfrequencies, with the membrane assuming the corresponding deflections given by (10.4.11). Figure 10.4.2 is a graphic example of membrane modes $n = 1$. A soap film is attached to a rectangular wire frame by surface tension. Each of the modes is then excited by vibrating the frame by hand.

10.4.2 Moving Membrane: Mach Lines

We have seen that convection can have a marked effect on the dynamics of one-dimensional continua. This is especially true if the motion is supersonic, in the sense that the convection velocity (U) is greater than the propagation velocity (v_s) of small disturbances. The moving membrane provides an opportunity to determine how these effects are displayed in two dimensions.

With the membrane moving in the x -direction with the velocity U , the equation of motion is

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 \xi = v_s^2 \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right). \quad (10.4.13)$$

Here we have replaced the time derivative with the convective derivative, as justified in Section 10.2.

Now suppose we consider the case in which the *Mach number* M is greater than unity,

$$M = \frac{U}{v_s} > 1, \quad (10.4.14)$$

and in which steady-state conditions prevail ($\partial/\partial t = 0$). Then (10.4.13) becomes

$$(M^2 - 1) \frac{\partial^2 \xi}{\partial x^2} = \frac{\partial^2 \xi}{\partial y^2}. \quad (10.4.15)$$

This expression has the same familiar form as the wave equation. The time dependence, however, has been replaced by a dependence on the second dimension y . With this in mind, we expect that deformations of the membrane,

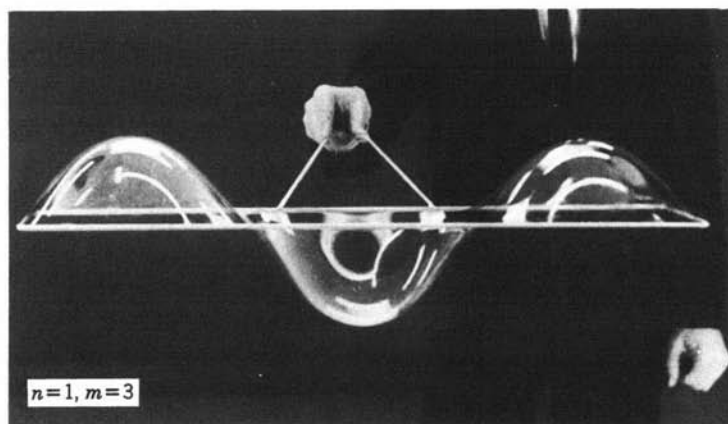
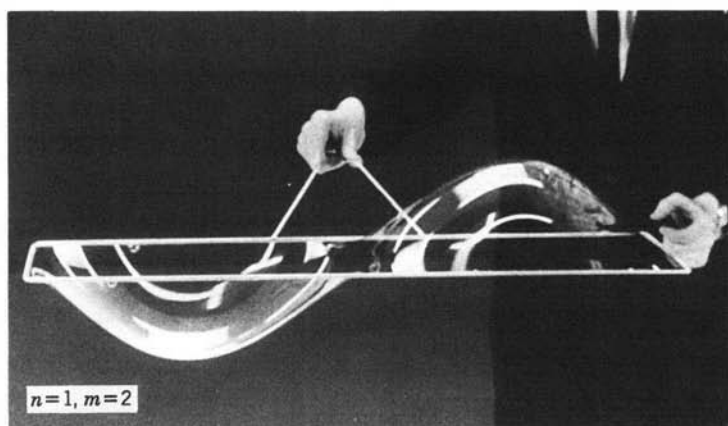
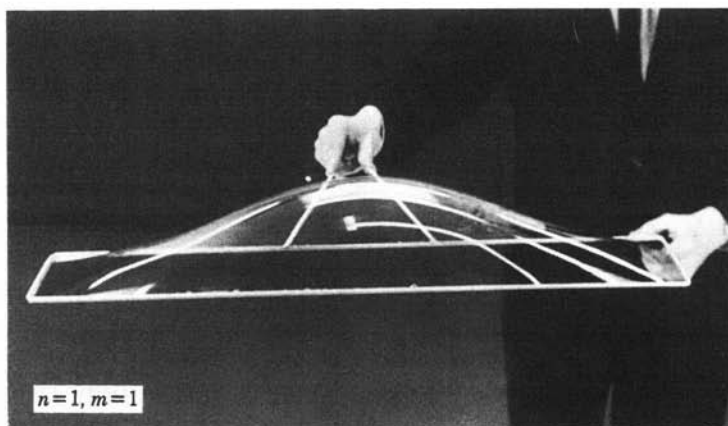


Fig. 10.4.2 Membrane motions illustrated by soap film on a vibrating wire frame. Pictures from film "Soap Film Oscillations." Courtesy of A. M. Hudson, Occidental College and Film Studio, Education Development Center, Inc., Newton, Mass.

as they depend on (x, y) , will assume the same wavelike character found in Section 9.1.1a.

Solution of (10.4.15) is completely analogous to the solution of the wave equation in (x, t) coordinates. Reference to Section 9.1.1a shows that we have solutions

$$\xi = \xi_+(\alpha) + \xi_-(\beta), \quad (10.4.16)$$

where

$$\alpha = x - \sqrt{M^2 - 1} y, \quad (10.4.17)$$

$$\beta = x + \sqrt{M^2 - 1} y. \quad (10.4.18)$$

The lines $\alpha = \text{constant}$ and $\beta = \text{constant}$ are again the characteristic lines along which we expect that the membrane can undergo abrupt deformations. These lines along which the equivalent of wavefronts in the x - y space can exist are referred to as *Mach lines*.^{*} Their physical significance is more apparent in the context of a simple example.

Example 10.4.1. A sheet of molten plastic is to be given the cross-sectional shape shown in Fig. 10.4.3a. It is proposed that this be done by ejecting the molten plastic as a sheet

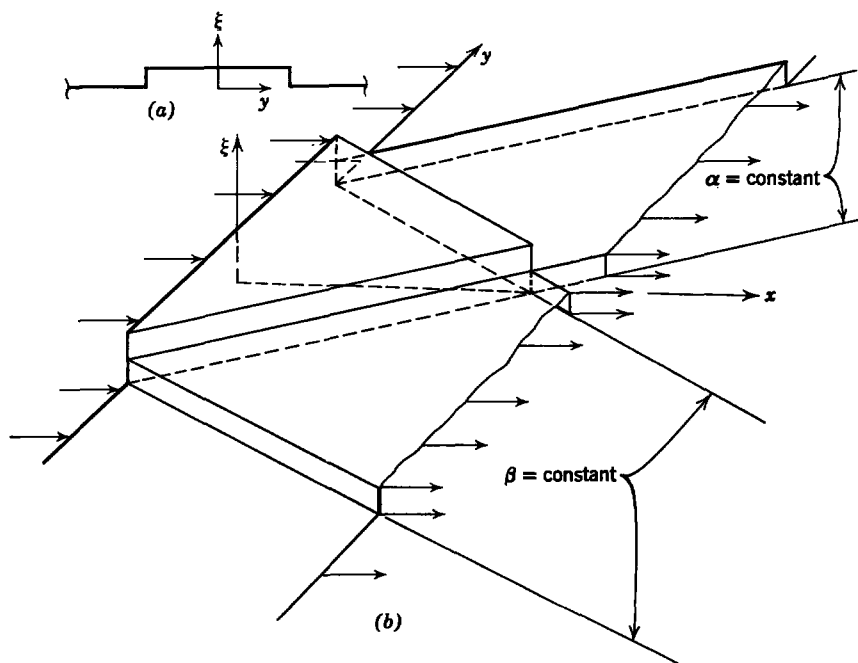


Fig. 10.4.3 (a) A membrane is constrained to have the shape shown at $x = 0$; (b) the resulting deflections for $M > 1$.

^{*} A. H. Shapiro, *The Dynamics and Thermodynamics of Compressible Fluid Flow*, Ronald, New York, 1953, Vol. I, p. 462.

through an orifice in the shape of Fig. 10.4.3a. As long as the sheet is molten and thin, it is reasonable to obtain an idea of the likely success of our method by modeling the sheet as a membrane with $M > 1$ because the effect of surface tension, hence v_s , is small. The membrane is constrained at $x = 0$ so that $\xi(0, y)$ has the form shown in Fig. 10.4.3a and

$$\frac{\partial \xi}{\partial x}(0, y) = 0. \quad (a)$$

This problem then reduces to one that is essentially the same as that discussed in Example 9.1.2. The deformations of the membrane, as they depend on (x, y) , are shown in Fig. 10.4.3b. The pulse that originates at $x = 0$ divides into two pulses that tend toward infinity in the $\pm y$ -directions as x increases. On the basis of this finding, we expect that our technique of forming the plastic would not be a good one unless the plastic could be made to solidify within a distance short compared with that required for the pulses to separate from one another.

10.4.3 A Kink Instability

Two-dimensional motions of a wire are very different from those of a membrane and provide an excellent model for demonstrating electro-mechanical coupling between propagating modes. A wire is shown in Fig. 10.4.4. It is now free to deflect in the x - or y -direction; hence we can, in general, find the wire at the transverse position

$$\xi = u(z, t)\mathbf{i}_x + v(z, t)\mathbf{i}_y. \quad (10.4.19)$$

Note that the motions still involve only two independent variables (z, t) by contrast with the two-dimensional motions of the membrane. Now,

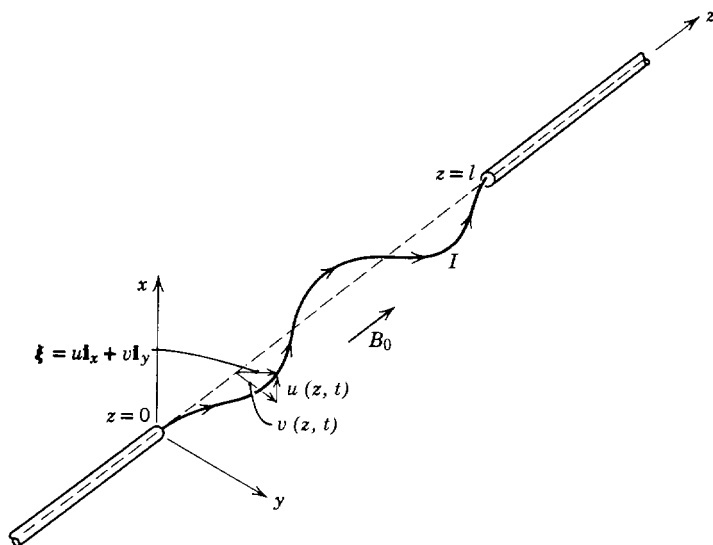


Fig. 10.4.4 A wire stretched along the z -axis.

however, there are two dependent variables $u(z, t)$ and $v(z, t)$, the displacements along the x - and y -axes, respectively.

We have already found that motions in the x - or y -direction are defined by the string equation (Table 9.2). To linear terms deflections v in the y -direction do not produce an x -directed force on the wire, and vice versa; hence the string equation remains valid for two-dimensional motions. A force equation for deflections in the x - and y -directions is

$$m \frac{\partial^2 \xi}{\partial t^2} = f \frac{\partial^2 \xi}{\partial z^2} + S. \quad (10.4.20)$$

The external force per unit length S has components S_x and S_y . With $S = 0$ motions in the x - and y -directions are uncoupled. In either case they are governed by a simple wave equation and disturbances propagate with the velocity $v_s = \sqrt{f/m}$.

We now use a magnetic field interaction to couple the motions in the x - and y -directions. In the process the continuum model of a constant-current constraint is illustrated.

The elastic wire is assumed to be conducting and is stretched between the pole pieces of a magnet (along the z -axis). As shown in Fig. 10.4.4, a current I is carried by the wire. In static equilibrium this current is in the same direction as an imposed magnetic flux density B_0 ; hence there is no magnetic force per unit length $\mathbf{I} \times \mathbf{B}_0$. Any slight perturbation of the current path (the wire position), however, will result in a radial component of current, and this component produces an $\mathbf{I} \times \mathbf{B}$ force that tends to rotate the wire about the z -axis. We see that motions of the wire cannot be purely in the x - z or y - z plane but must involve all three space dimensions.

Our constant current approximation is implicit in the statement that the current in the wire has the same direction as the wire. To linear terms we write

$$\mathbf{I} = \mathbf{i}_x \frac{\partial u}{\partial z} I + \mathbf{i}_y \frac{\partial v}{\partial z} I + \mathbf{i}_z I. \quad (10.4.21)$$

Here we have assumed that the cross section of the wire is small enough to make currents induced by the motion negligible. In general, the motion would produce currents that close on themselves in the x - y plane (eddy currents), and they could be crossed with the imposed magnetic field to give a transverse force on the wire. In Section 7.1 we obtained the condition that the currents induced by the motion of a conductor (conductivity σ) in a magnetic field are small if

$$R_m = \frac{\mu_0 \sigma \omega d^2}{2} \ll 1, \quad (10.4.22)$$

where in our case d is the diameter of the wire and ω is the frequency of vibration. Our constant current approximation is the limit of small R_m .

We make the additional assumption that the magnetic flux density B_0 imposed by the external magnet is much larger than the equilibrium flux density generated by the current I . This allows us to compute the force on the wire as

$$\mathbf{S} = \mathbf{I} \times B_0 \mathbf{i}_z \quad (10.4.23)$$

or from Eq. (10.4.21)

$$\mathbf{S} = B_0 I \frac{\partial v}{\partial z} \mathbf{i}_x - B_0 I \frac{\partial u}{\partial z} \mathbf{i}_y. \quad (10.4.24)$$

We now use these forces, and the components of (10.4.20) become

$$\frac{\partial^2 u}{\partial t^2} = v_s^2 \frac{\partial^2 u}{\partial z^2} + \frac{B_0 I}{m} \frac{\partial v}{\partial z}, \quad (10.4.25)$$

$$\frac{\partial^2 v}{\partial t^2} = v_s^2 \frac{\partial^2 v}{\partial z^2} - \frac{B_0 I}{m} \frac{\partial u}{\partial z}. \quad (10.4.26)$$

This pair of equations is sufficient to describe the dynamics of the current-carrying wire. As our intuition suggested, the magnetic field produces a force in the x -direction [the last term in (10.4.25)] in proportion to the tilt of the wire in the y -direction. The equations are now coupled. A motion in one transverse direction cannot occur without involving a motion in the other transverse direction. Put another way, waves polarized in either the x - or y -directions are now coupled as they propagate along the z -axis, and we expect that the coupling will affect the dynamics of both waves.

There are still two waves that propagate in each direction on the wire. We can find them if we assume solutions with the variable separable form

$$u = \text{Re} [\hat{u} \exp j(\omega t - kz)] \quad \text{and} \quad v = \text{Re} [\hat{v} \exp j(\omega t - kz)].$$

The equations of motion are satisfied by these solutions if

$$\hat{u}(\omega^2 - v_s^2 k^2) - \hat{v} \left(\frac{jkB_0 I}{m} \right) = 0, \quad (10.4.27)$$

$$\hat{u} \left(\frac{jkB_0 I}{m} \right) + \hat{v}(\omega^2 - v_s^2 k^2) = 0. \quad (10.4.28)$$

These equations are homogeneous in the amplitudes $\hat{u}$ and $\hat{v}$. Unless both amplitudes are zero (not an interesting situation), the determinant of the coefficients must be zero. It is concluded that ω and k are related by

$$(\omega^2 - v_s^2 k^2) = \pm \left(\frac{kB_0 I}{m} \right). \quad (10.4.29)$$

We have found the dispersion equation for waves on the wire, and the nature of these waves is made evident by solving (10.4.29) for ω^2 :

$$\omega^2 = k^2 \left[v_s^2 \pm \left(\frac{B_0 I}{k m} \right) \right]. \quad (10.4.30)$$

Evidently four waves can propagate on the wire, which is consistent with what we find if there is no magnetic field.

Equation 10.4.30 shows that one pair of waves (+ sign) has a phase velocity that increases with the magnetic field, whereas the other pair has a phase velocity that decreases with the magnetic field. The slower waves can, in fact, be unstable if

$$\left| \frac{B_0 I}{v_s^2 m} \right| > k, \quad (10.4.31)$$

for then $\omega = \pm j\alpha$, where α is a real number. If all real values of k were allowed by the boundary conditions, the wire would be unstable, no matter how small $B_0 I$, since there would always be a small enough wavenumber (long enough wave) to satisfy the condition for instability. The boundary conditions, however, are satisfied only by certain discrete values of k , which we enumerate shortly.

For the purpose of establishing the physical significance of the waves we have found, consider the case in which the wire is excited in the sinusoidal steady state at the (real) frequency ω . Then (10.4.29) is

$$k^2 \pm k \left(\frac{B_0 I}{v_s^2 m} \right) - \frac{\omega^2}{v_s^2} = 0, \quad (10.4.32)$$

and we can solve this relation to find four possible wavenumbers. Using the + sign in (10.4.32), we obtain

$$k = k_2, -k_1, \quad (10.4.33)$$

and using the minus sign we obtain

$$k = k_1, -k_2, \quad (10.4.34)$$

where k_1 and k_2 are the positive real numbers

$$k_1 = \frac{B_0 I}{2v_s^2 m} + \left[\left(\frac{B_0 I}{2v_s^2 m} \right)^2 + \frac{\omega^2}{v_s^2} \right]^{1/2}, \quad (10.4.35)$$

$$k_2 = -\frac{B_0 I}{2v_s^2 m} + \left[\left(\frac{B_0 I}{2v_s^2 m} \right)^2 + \frac{\omega^2}{v_s^2} \right]^{1/2}. \quad (10.4.36)$$

(Just to keep things straight, we assume for now that $B_0 I > 0$.) Note that $k_1 > k_2$. The dispersion equation (10.4.29), together with (10.4.27) or (10.4.28), shows that

$$\hat{v} = \pm j\hat{u} \quad (10.4.37)$$

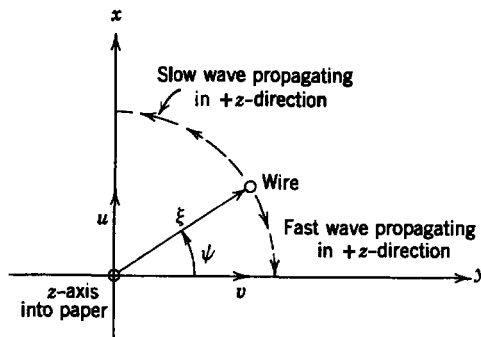


Fig. 10.4.5 Polar coordinates of wire deflection. At a given position along the z -axis the wire appears to rotate about it. The fast and slow waves have opposite directions of rotation.

We must remember to use the upper sign with those waves with wavenumbers given by (10.4.33) and the lower sign with those given by (10.4.34). Hence, if we have a wave given by

$$u = \text{Re} [\hat{u}_+^a e^{j(\omega t - k_1 z)} + \hat{u}_+^b e^{j(\omega t - k_2 z)} + \hat{u}_-^a e^{j(\omega t + k_1 z)} + \hat{u}_-^b e^{j(\omega t + k_2 z)}], \quad (10.4.38)$$

it follows from (10.4.37) that

$$v = \text{Re} j[\hat{u}_+^a e^{j(\omega t - k_1 z)} - \hat{u}_+^b e^{j(\omega t - k_2 z)} - \hat{u}_-^a e^{j(\omega t + k_1 z)} + \hat{u}_-^b e^{j(\omega t + k_2 z)}]. \quad (10.4.39)$$

The first two terms in these equations are waves with points of constant phase propagating in the $+z$ -direction. Because $k_1 > k_2$, the (a) wave propagates more slowly than the (b) wave. The third and fourth terms are similar waves propagating in the $-z$ -direction.

To obtain some insight into the nature of these waves, consider the first wave alone with $\hat{u}_+^a = -ju_0$. Then

$$\xi = u\mathbf{i}_x + v\mathbf{i}_y = u_0[\sin(\omega t - k_1 z)\mathbf{i}_x + \cos(\omega t - k_1 z)\mathbf{i}_y]. \quad (10.4.40)$$

This deflection is easier to interpret if we consider the magnitude ξ and angle ψ defined in Fig. 10.4.5. Then

$$\begin{aligned} \xi &= \sqrt{u^2 + v^2}, \\ \psi &= \tan^{-1} \left(\frac{u}{v} \right) = \omega t - k_1 z. \end{aligned} \quad (10.4.41)$$

At an instant in time the deflection given by this wave is a spiral that circles the z -axis in the clockwise direction, as shown in Fig. 10.4.6b. Points of constant phase advance in the z -direction with the velocity ω/k_1 . The phases of the (b) wave, which propagates in the $+z$ -direction, move more rapidly with the velocity ω/k_2 . If we repeat the arguments that led to (10.4.41),

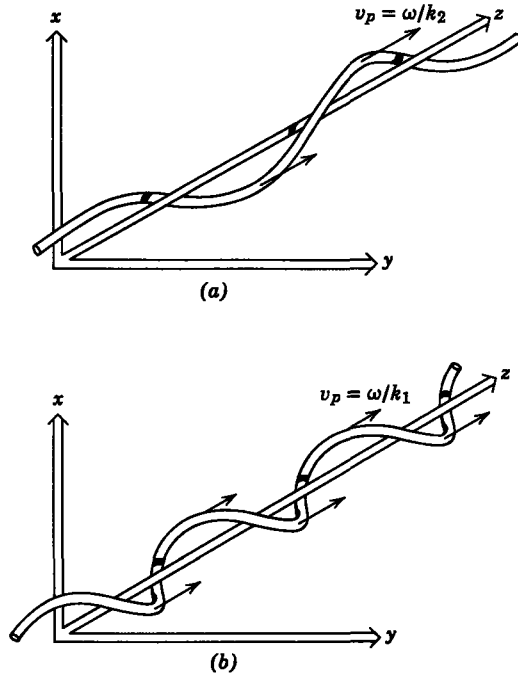


Fig. 10.4.6 Waves propagating in the $+z$ -direction: (a) fast wave with a wavelength $2\pi/k_2$; and (b) slow wave with a wavelength $2\pi/k_1$.

using the u_+^b wave, we obtain a spiral that at an instant in time circles the z -axis in the counterclockwise direction (Fig. 10.4.6a). At any position z the wire will circle the z -axis in the counterclockwise direction for the slow wave and in the clockwise direction for the fast wave. Except that the propagation is in the opposite direction, the u_- waves have the same physical appearance.

We address ourselves now to superimposing these four wave solutions to find the free modes that satisfy the boundary conditions of Fig. 10.4.4 in which the wire is fixed at $z = 0$ and at $z = l$. This is done by finding the allowed eigenvalues k and then using (10.4.30) to find the associated eigenfrequencies.

In a boundary value problem of this type it is easier to work with trigonometric functions than with the complex exponentials. Hence we use the fact that $e^{j\theta} = \cos \theta + j \sin \theta$ and write the solution given by (10.4.38) as

$$u = \text{Re} \{ e^{j\omega t} [A \sin k_1 z + B \cos k_1 z + C \sin k_2 z + D \cos k_2 z] \}, \quad (10.4.42)$$

where we have defined a new set of four constants as

$$\begin{aligned} A &= j(\hat{u}_-^a - \hat{u}_+^a), & C &= j(\hat{u}_-^b - \hat{u}_+^b), \\ B &= (\hat{u}_+^a + \hat{u}_-^a), & D &= (\hat{u}_+^b + \hat{u}_-^b). \end{aligned} \quad (10.4.43)$$

In terms of these same constants the solution for the y -deflection becomes (10.4.39)

$$v = -\operatorname{Re} \{e^{j\omega t}[A \cos k_1 z - B \sin k_1 z - C \cos k_2 z + D \sin k_2 z]\}. \quad (10.4.44)$$

The four constants are, of course, determined by the boundary conditions

$$\begin{aligned} u(0, t) &= 0, & v(0, t) &= 0, \\ u(l, t) &= 0, & v(l, t) &= 0. \end{aligned} \quad (10.4.45)$$

The advantage of the trigonometric form that we have used for our solution is the simplicity of the relation among the constants that results from the boundary conditions at $z = 0$. Because the sine functions are zero at the origin,

$$\begin{aligned} B &= -D, \\ A &= C. \end{aligned} \quad (10.4.46)$$

The remaining two conditions at $z = l$ require that

$$\begin{aligned} A(\sin k_1 l + \sin k_2 l) + B(\cos k_1 l - \cos k_2 l) &= 0, \\ A(\cos k_1 l - \cos k_2 l) - B(\sin k_1 l + \sin k_2 l) &= 0. \end{aligned} \quad (10.4.47)$$

Because A and B are not zero, the determinant of the coefficients must be zero; that is,

$$(\sin k_1 l + \sin k_2 l)^2 + (\cos k_1 l - \cos k_2 l)^2 = 0. \quad (10.4.48)$$

Recall that $\sin^2 \theta + \cos^2 \theta = 1$ and $\cos \theta \cos \gamma - \sin \theta \sin \gamma = \cos(\theta + \gamma)$, and (10.4.48) reduces to the simple condition

$$\cos l(k_1 + k_2) = 1. \quad (10.4.49)$$

It follows that the argument $l(k_1 + k_2)$ must have values $0, 2\pi, 4\pi, \dots, 2n\pi, \dots$. If $k_1 + k_2$ is replaced by the sum of (10.4.35) and (10.4.36), we obtain an expression that determines the eigenfrequencies ω_n :

$$\omega_n^2 = \frac{v_s^2}{4} \left[\left(\frac{2n\pi}{l} \right)^2 - \left(\frac{B_0 I}{v_s^2 m} \right)^2 \right], \quad n = 1, 2, \dots \quad (10.4.50)$$

We have left out the case $n = 0$ because this solution is trivial; that is, there is no deflection when $n = 0$, as can be seen by taking the limiting case in which there is no magnetic field.

We have found that if we set the wire into vibration in one of its natural modes and then turn on the magnetic field and current, the frequency will decrease in magnitude. As we raise the magnetic field, the equilibrium is first unstable in the $n = 1$ mode. The system is unstable if

$$\left| \frac{B_0 I}{v_s^2 m} \right| > \frac{2\pi}{l}. \quad (10.4.51)$$

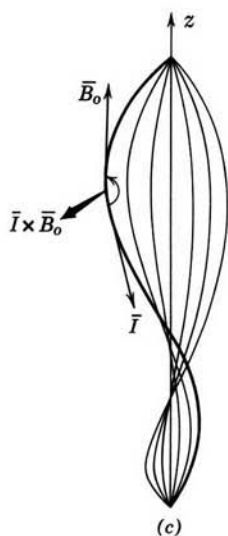
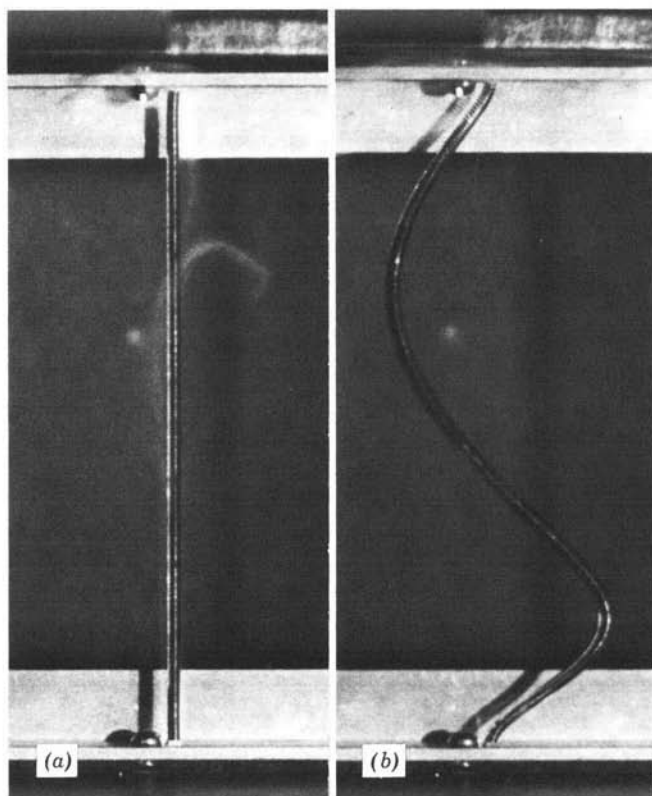


Fig. 10.4.7 (a) Experiment in which a conducting spring is stretched between pole faces of a magnet. There is a vertical field B_0 and a current has just been applied to the spring. (b) A moment later instability occurs. (c) Predicted $n = 1$ mode at point of impending instability. (From Film "Complex Waves I, Propagation, Evanescence and Instability," produced by Education Development Center, Inc., Newton, Mass., for the National Committee on Electrical Engineering Films.)

We have established that the value of k that should be used in (10.4.31) is $2\pi/l$.

Finally, it is worthwhile to see what the physical form of a natural mode is. Given one of the constants, the other three can be determined from (10.4.46) and (10.4.47). For this particular problem, however, the compatibility condition (10.4.49) ensures that the coefficients of both A 's and B 's in (10.4.47) are zero, as can be determined by using trigonometric identities. This means that all conditions are satisfied if A and B are specified independently. Consider the case in which A , hence [from (10.4.46)] C is zero. Then, since $B = -D$, if we determine the time phase by taking B as a real number, (10.4.42) and (10.4.44) become [here, use is made of (10.4.35) and (10.4.36) and the fact that $k_1 + k_2 = 2n\pi/l$]

$$\begin{aligned} u &= -2B \cos \omega t \sin \frac{n\pi z}{l} \sin \frac{B_0 I}{2v_s^2 m} z, \\ v &= 2B \cos \omega t \sin \frac{n\pi z}{l} \cos \frac{B_0 I}{2v_s^2 m} z. \end{aligned} \quad (10.4.52)$$

In terms of the polar variables of (10.4.41) and Fig. 10.4.5,

$$\begin{aligned} \psi &= -\frac{B_0 I}{2v_s^2 m} z, \\ \xi &= 2B \cos \omega t \sin \frac{n\pi z}{l}. \end{aligned} \quad (10.4.53)$$

If we had used the constant A , rather than B , it would have shifted this deflection by 90° about the z -axis.

A plot of (10.4.53) is shown in Fig. 10.4.7c for the first eigenmode with $B_0 I < 0$. Note that because of the condition for instability (10.4.51) the maximum twist ψ that the standing wave can undergo is 180° . Hence the mode shown in Fig. 10.4.7c is at the point at which instability occurs for the $n = 1$ mode.

The wave dynamics that we have found in this section are related to several types of electromechanical interaction. The "pinch" confinement of a plasma, proposed as a scheme for containing a controlled thermonuclear reaction, suffers from an instability with the same kinked property as the one found here.* The plasma is likely to behave more nearly as a perfectly conducting medium or as one with a high rather than a low magnetic Reynolds number. If, however, a current is passed through a liquid column of mercury (where R_m is small) in the presence of an axial magnetic field, an instability with

* D. J. Rose and M. Clarke, Jr., *Plasmas and Controlled Fusion*, M. I. T. Press and Wiley, New York, 1961, p. 336.

essentially the same features as that described here will result.* Often the twisting motions that characterize the dynamics of the wire are found in other electromechanical systems that involve an imposed magnetic field. An example is the cyclotron wave of electron beam theory.†

10.5 DISCUSSION

In this chapter we have explored the consequences of continuum electro-mechanical coupling with simple elastic continua. This has produced mathematical analyses and physical interpretations of evanescent waves, absolute instabilities, and waves and instabilities in convecting systems. The unifying mathematical concept is the dispersion relation presented graphically in the ω - k plots.

Although our examples have been framed in terms of simple physical situations, the phenomena we have discussed occur in the wide variety of practical situations indicated in Section 10.0 and throughout the chapter.

PROBLEMS

10.1. The current-carrying wire described in Section 10.1.2 is attached to a pair of dashpots with damping coefficients B and driven at $x = -l$, as shown in Fig. 10P.1.

- What is the boundary condition at $x = 0$?
- Compute the power absorbed in the dashpots for $\omega < \omega_e$, given the amplitude ξ_0 and other system parameters.

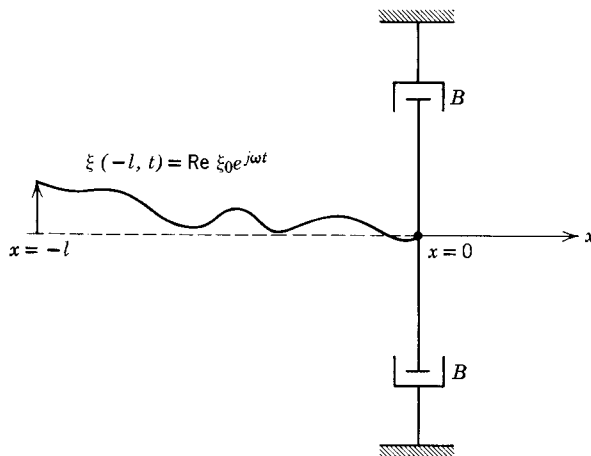


Fig. 10P.1

* Film Cartridge, produced by the National Committee for Fluid Mechanics Films, *Current-Induced Instability of a Mercury Jet*, may be obtained from Education Development Center, Inc., Newton, Mass. The instability seen in this film is convective, as would be the case here if the string were in motion with $U > v_s$.

† W. H. Louisell, *Coupled Modes and Parametric Electronics*, Wiley, 1960, p. 51.

10.2. Consider the same physical situation as that described in Section 10.1.2, except with the current-carrying wire constrained at $x = 0$, so that $(\partial \xi / \partial x)(0, t) = 0$, and driven at $x = -l$ such that $\xi(-l, t) = \xi_a \cos \omega_a t$.

- Find analytical expressions for $\xi(x, t)$ with $\omega_a > \omega_c$ and $\omega_a < \omega_c$.
- Sketch the results of (a) at an instant in time for cases in which $\omega_a = 0$, $\omega_a < \omega_c$, $\omega_a > \omega_c$.
- How could the boundary condition at $x = 0$ be realized physically?

10.3. The ends of the spring shown in Fig. 10.1.2 and discussed in Sections 10.1.2 and 10.1.3 are constrained such that

$$\frac{\partial \xi}{\partial x}(0, t) = 0,$$

$$\frac{\partial \xi}{\partial x}(-l, t) = 0.$$

- What are the eigenfrequencies of the spring with the current as shown in Fig. 10.1.2?
- What are these frequencies with I as shown in Fig. 10.1.9?
- What current I is required to make the equilibrium with $\xi = 0$ unstable? Give a physical argument in support of your answer.

10.4. In Section 10.1.2 a current-carrying wire in a magnetic field was described by the equation of motion

$$m \frac{\partial^2 \xi}{\partial t^2} = f \frac{\partial^2 \xi}{\partial x^2} - Ib\xi + F(x, t), \quad (a)$$

where F is an externally applied force/unit length. We wish to consider the flow of power on the string. Because $F \partial \xi / \partial t$ is the power input/unit length to the string, we can find a conservation of power equation by multiplying (a) by $\partial \xi / \partial t$. Show that

$$P_{\text{in}} = \frac{\partial W}{\partial t} + \frac{\partial P}{\partial x}, \quad (b)$$

where $P_{\text{in}} = F \partial \xi / \partial t$,

W = energy stored/unit length

$$\frac{1}{2} m \left(\frac{\partial \xi}{\partial t} \right)^2 + \frac{1}{2} f \left(\frac{\partial \xi}{\partial x} \right)^2 + \frac{1}{2} Ib \xi^2,$$

P = power flux

$$= -f \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial t}.$$

10.5. Waves on the string in Problem 10.4 have the form

$$\xi(x, t) = \text{Re}[\xi_+ e^{j(\omega t - kx)} + \xi_- e^{j(\omega t + kx)}].$$

This problem makes a fundamental point of the way in which power is carried by ordinary waves in contrast to evanescent waves. The instantaneous power P carried by the string is given in Problem 10.4. Sinusoidal steady-state conditions prevail.

- Compute the time average power carried by the waves under the assumption that k is real. Your answer should show that the powers carried by the forward and

backward waves are independent; that is,

$$\langle P \rangle = \frac{\omega k f}{2} (\xi_+^* \xi_+ - \xi_-^* \xi_-),$$

where ξ^* is the complex conjugate of ξ .

(b) Show that if $k = j\beta$, β real we obtain by contrast

$$\langle P \rangle = \frac{-j\omega\beta f}{2} (\xi_+^* \xi_- - \xi_-^* \xi_+).$$

A single evanescent wave cannot carry power.

(c) Physically, how could it be argued that (b) must be the case rather than (a) for an evanescent wave?

10.6. Use the results of Problem 10.4 to show that the group velocity $v_g = d\omega/dk$ is given by the ratio of the time average power to the time average energy/unit length: $v_g = \langle P \rangle / \langle W \rangle$. Attention should be confined to the particular case of Problem 10.4 with $F = 0$.

10.7. A pair of perfectly conducting membranes has equilibrium spacing d from each other and from parallel rigid walls (Fig. 10P.7). The membranes and walls support currents such

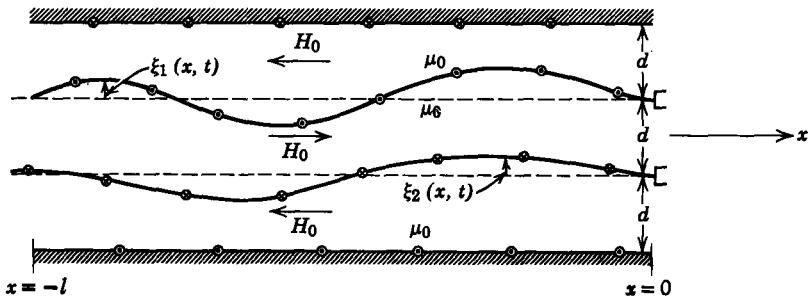


Fig. 10P.7

that with $\xi_1 = 0$ and $\xi_2 = 0$ the static uniform magnetic field intensities H_0 are as shown. As the membranes deform, the flux through each of the three regions between conductors is conserved.

- Assume that both membranes have tension S and mass/unit area σ_m . Write two equations of motion for ξ_1 and ξ_2 .
- Assume that $\xi_1 = \text{Re } \xi_1 \exp j(\omega t - kx)$ and $\xi_2 = \text{Re } \xi_2 \exp j(\omega t - kx)$ and find the dispersion equation.
- Make an ω - k plot to show complex values of k for real values of ω . Show which branch of this plot goes with $\xi_1 = \xi_2$ and which with $\xi_1 = -\xi_2$. What are the respective cutoff frequencies for these odd and even modes?
- The membranes are fixed at $x = 0$ and given the displacements $\xi_1(-l, t) = -\xi_2(-l, t) = \text{Re } \xi_0 \exp j\omega t$. Find ξ_1 and ξ_2 and sketch for $\omega = 0$.

10.8. An electromagnetic wave can be transmitted through or reflected by a plasma, depending on the frequency of the wave relative to the plasma frequency ω_p . This phenomenon, which is fundamental to the propagation of radio signals in the ionosphere, is illustrated by the following simple example of a cutoff wave. In dealing with electromagnetic waves, we require that both the electric displacement current in Ampère's law and the

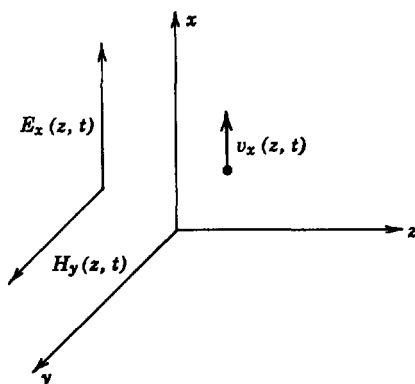


Fig. 10P.8

magnetic induction in Faraday's law (see Section B.2.1) be accounted for. We consider one-dimensional plane waves in which $\mathbf{E} = \mathbf{i}_x E_x(z, t)$ and $\mathbf{H} = \mathbf{i}_y H_y(z, t)$.

(a) Show that Maxwell's equations require that

$$\frac{\partial E_x}{\partial z} = -\frac{\partial \mu_0 H_y}{\partial t}, \quad -\frac{\partial H_y}{\partial z} = \frac{\partial \epsilon_0 E_x}{\partial t} + J_x.$$

(b) The space is filled with plasma composed of equal numbers of ions and electrons. Assume that the more massive ions remain fixed and that n_e is the electron number density, whereas e and m are the electronic charge and mass. Use a linearized force equation to relate E_x and v_x , where v_x is the average electron velocity in the x -direction.

(c) Relate v_x and J_x to linear terms.

(d) Use the equations of (a)–(c) to find the dispersion equation for waves in the form of $\exp j(\omega t - kz)$.

(e) Define the plasma frequency as $\omega_p = \sqrt{n_e e^2 / \epsilon_0 m}$ and describe the dynamics of a wave with $\omega < \omega_p$.

(f) Suppose that a wave in free space were to be normally incident on a layer of plasma (such as the ionosphere). What would you expect to happen? (See Problem 10.9 for a similar situation.)

10.9. A current-carrying string extends from $x = -\infty$ to $x = +\infty$. The section $-l < x < 0$ is subjected to a magnetic field with the distribution shown in Fig. 10.1.2. Hence the sections of string to the left of $x = -l$ and to the right of $x = 0$ support ordinary waves, whereas the section in between can support cutoff waves. Sinusoidal steady-state conditions

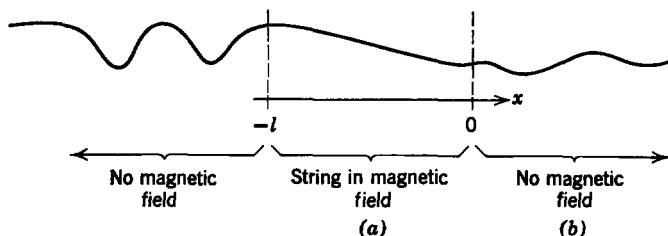


Fig. 10P.9

prevail, with waves incident from the left producing a deflection $\xi(-l, t) = \text{Re } \xi_0 e^{j\omega t}$, $\omega < \omega_c$. Assume that waves propagating to the right are completely absorbed at $x \rightarrow \infty$ so that in interval (b) $\xi_b = \text{Re } \xi_0 e^{j(\omega t - k_b x)}$, $k_b = \omega/v_s$.

- Find the attenuation factor ξ_b/ξ_0 for a wave passing through the cutoff section.
- What is ξ_b/ξ_0 as $l \rightarrow 0$? As $l \rightarrow \infty$?

10P.10. A rigid straight rod supports a charge Q coulombs per unit length and is fixed. Just below this rod an insulating string is stretched between fixed supports, as shown in Fig.

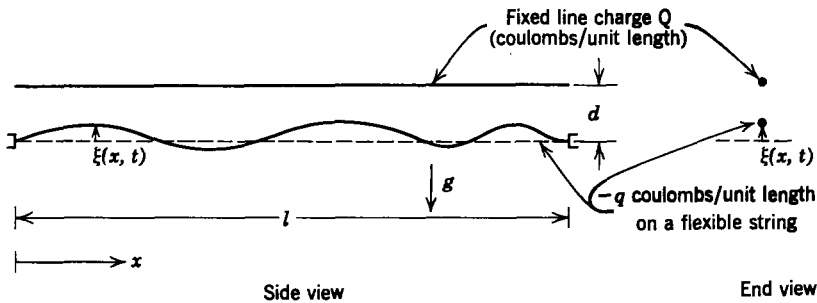


Fig. 10P.10

10P.10. This string, which has a tension f and mass per unit length m , supports a charge per unit length $-q$, where $q \ll Q$ and both Q and q are positive.

- What should qQ be in order that the string have the static equilibrium $\xi = 0$ in spite of the gravitational acceleration g ?
- What is the largest value of m that is consistent with the equilibrium of part (a) being stable?
- How would you alter this physical situation to make the static equilibrium stable even with m larger than given by (b)?

10P.11. A wire with the mass/unit length m and tension f is stretched between fixed supports, as shown in Fig. 10P.11. The wire carries a current I and is subject to the gravitational

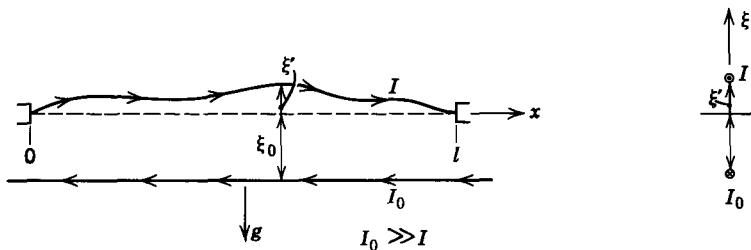


Fig. 10P.11

acceleration g . An adjacent wire carries the much larger current I_0 . Because $I_0 \gg I$, the magnetic field produced by I can be ignored.

- Given all other system parameters, what value of I is required to hold the wire in static equilibrium with $\xi' = 0$?
- Write the differential equation of motion for vertical displacements $\xi'(x, t)$ of the wire from a horizontal equilibrium at $\xi = \xi_0$.
- Under what conditions will the equilibrium be stable?

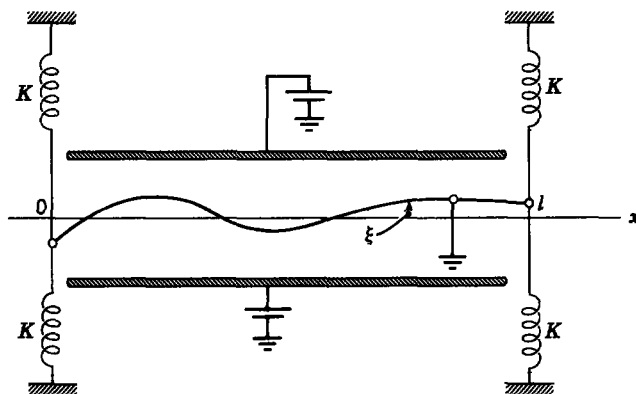


Fig. 10P.12

10.12. The conducting wire shown in Fig. 10P.12 is stressed by a transverse electric field and hence has transverse displacements that satisfy the force equation:

$$m \frac{\partial^2 \xi}{\partial t^2} = f \frac{\partial^2 \xi}{\partial x^2} + P \xi,$$

where m , f , and P are known constants. (P arises from the electric field.) The ends of the string are constrained by springs, but are otherwise free to move in the transverse direction.

- Write the boundary conditions in terms of $\xi(x, t)$ at $x = 0$ and $x = l$.
- Find an expression for the natural frequencies and illustrate its solution graphically. What effect does raising P have on the lowest eigenfrequency?
- What is the largest value of P consistent with stability in the limit where $K \rightarrow 0$?

10.13. A pair of perfectly conducting membranes are stretched between rigid supports at $x = 0$ and $x = L$, as shown in Fig. 10P.13. The membranes have the applied voltage V_0 with respect to each other and with respect to plane-parallel electrodes.

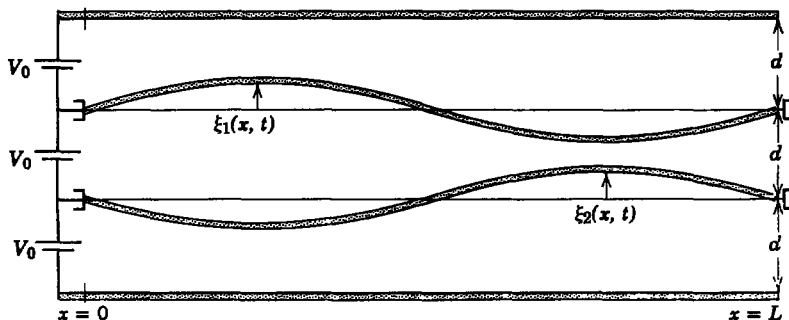


Fig. 10P.13

- Write a pair of differential equations in $\xi_1(x, t)$ and $\xi_2(x, t)$ which describe the membrane motions. Assume that ξ_1 and ξ_2 are small enough to warrant linearization and that the wavelengths are long enough that the membranes appear flat to the electric field at any one value of x .

(b) Assume that

$$\xi_1 = \operatorname{Re} \xi_1 e^{j(\omega t - kx)}$$

$$\xi_2 = \operatorname{Re} \xi_2 e^{j(\omega t - kx)}$$

and find a dispersion equation relating ω and k .

(c) Make an ω - k plot showing the results of part (b), including imaginary values of ω for real values of k . (This equation should be biquadratic in ω .)

(d) At what potential V_0 will the static equilibrium $\xi_1 = 0$ and $\xi_2 = 0$ first become unstable? Describe the mode of instability.

10.14. A spring is immersed in a viscous fluid so that damping forces of the type discussed in Section 10.1.4 are important. The spring is fixed at $x = 0$ and $x = -l$. When $t = 0$, it is static and released from the initial deflection shown.

(a) Find $\xi(x, t)$ in terms of the system normal modes and ξ_0 .

(b) Compare this physical situation with that developed in Section 7.1.1.

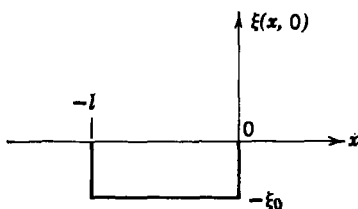


Fig. 10P.14

10.15. A string with tension f and mass/unit length m moves in the x -direction with the velocity $U > \sqrt{f/m}$. The string may be regarded as infinitely long. When $t = 0$, the string has no deflection: $[\xi(x, 0) = 0]$. It has, however, the transverse velocity $(\partial \xi / \partial t)(x, 0)$ given in Fig. 10P.15. Find an analytical expression for $\xi(x, t)$ and sketch it as a function of (x, t) . (Your sketch should have an appearance similar to that of Figs. 10.2.4 and 10.2.5.)

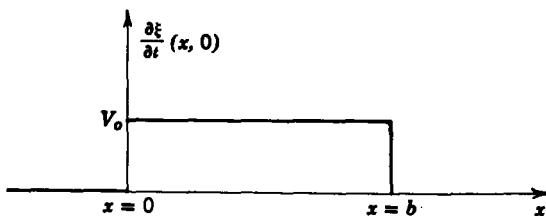


Fig. 10P.15

10.16. A string with the tension f and mass/unit length m has an equilibrium velocity U in the x -direction, where $U > \sqrt{f/m}$. At $x = 0$ it is constrained such that

$$\xi(0, t) = \xi_0 \cos \omega t,$$

$$\frac{\partial \xi}{\partial x}(0, t) = 0.$$

(a) Find the sinusoidal steady-state response $\xi(x, t)$.

(b) Sketch the results of (a) at an instant in time.

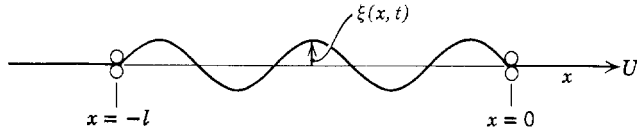


Fig. 10P.17

10.17. A string with the longitudinal velocity U is excited sinusoidally at $x = -l$, $\xi(-l, t) = \text{Re } \xi_0 e^{j\omega t}$, and constrained to zero deflection at $x = 0$ by pairs of rollers.

- Find the driven response $\xi(x, t)$ in the interval $-l < x < 0$.
- What are the natural frequencies of the system? How do they depend on U ?
- For what values of U are the results of (a) and (b) physically meaningful?

10.18. A wire under the tension f is closed on itself as shown. The resulting loop rotates with the constant angular velocity Ω . We consider deflections ξ from a circular equilibrium which have short wavelengths compared with the radius R . Hence each section of the wire is essentially straight and effects of the curvature on the dynamics can be ignored.

- Show that the partial differential equation of motion is

$$m \left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \theta} \right)^2 \xi = \frac{f}{R^2} \frac{\partial^2 \xi}{\partial \theta^2},$$

where m , f , and R are given constants.

- For $t < 0$ the pulse of deflection (Fig. 10P.18b) is imposed externally and is stationary when $t = 0$. At $t = 0$ this pulse is released. You are given that $\Omega R = 2\sqrt{f/m}$. Plot the deflection $\xi(0, t)$ for $0 \leq t \leq 2\pi R/\sqrt{f/m}$.

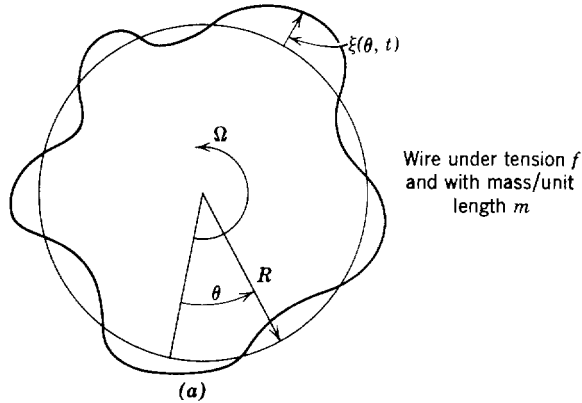


Fig. 10P.18

10.19. A string has the velocity U in the x -direction and is subject to arbitrary inputs of energy from a distributed force $F(x, t)$. Use the equation of motion to find a conservation of power equation in the form $P_{\text{in}} = \frac{\partial W}{\partial t} + \frac{\partial P}{\partial x}$. (See Problem 10.4.)

10.20. Find the sinusoidal steady-state response for the conditions outlined in Problem 10.16 with the additional effect of a destabilizing force included (see Section 10.2.3). Sketch the deflections at an instant in time under conditions in which the response takes the form of an amplifying wave.

10.21. A perfectly conducting membrane with the tension S and mass per unit area σ_m is ejected from a nozzle along the x -axis with a velocity U . Gravity acts as shown in Fig.

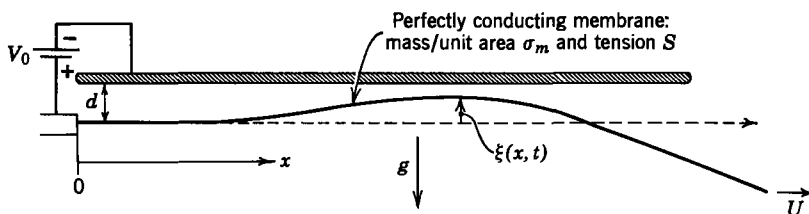


Fig. 10P.21

10P.21. A planar electrode above the membrane has the constant potential V_0 relative to the membrane.

- What value of V_0 is required to make the membrane assume an equilibrium parallel to the electrode?
- Now, under the conditions in (a), the membrane is excited at the frequency ω_d ; what is the lowest frequency of excitation that will not lead to spatially growing deflections? Assume that $U > \sqrt{S/\sigma_m}$.

10.22. An elastic membrane with tension S and mass/unit area σ_m is closed on itself, as shown in Fig. 10P.22. When it is in a steady-state equilibrium, the membrane has radius R

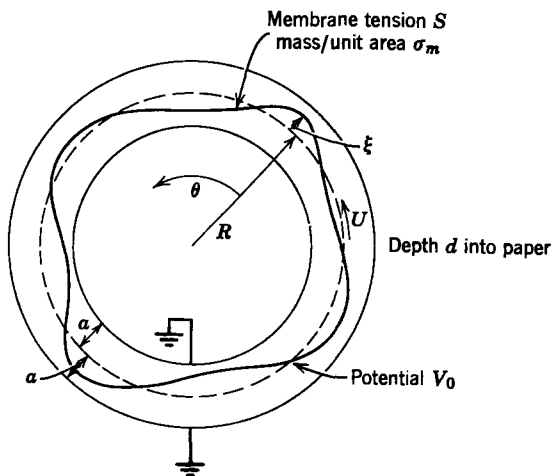


Fig. 10P.22

and rotates with angular velocity Ω . Any point on its surface has an azimuthal velocity $U = \Omega R$. At a distance a to either side of the membrane are coaxial electrodes which, like the membrane, are perfectly conducting. There is a constant potential difference V_0 between the membrane and each of the electrodes. The radius R is very large, so that effects of the membrane and electrode curvatures can be ignored. In addition, wavelengths on the membrane are much greater than a .

- (a) Show that the equation of motion for membrane deflections takes the form

$$\left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \theta}\right)^2 \xi = \Omega_s^2 \left(\frac{\partial^2 \xi}{\partial \theta^2} + m_c^2 \xi\right); \quad \Omega_s \text{ and } m_c = ?$$

- (b) Assume that solutions have the form $\xi = \text{Re } \tilde{\xi} \exp j(\omega t - m\theta)$ and find the ω - m relation. Plot complex ω for real m and complex m for real ω .
 (c) Under what conditions is this system absolutely stable?

10.23. A pair of perfectly conducting membranes move in the x -direction with the velocity U . The membranes have the applied voltage V_0 with respect to one another and to plane-parallel electrodes. They enter the region between the plates from rollers at $x = 0$.

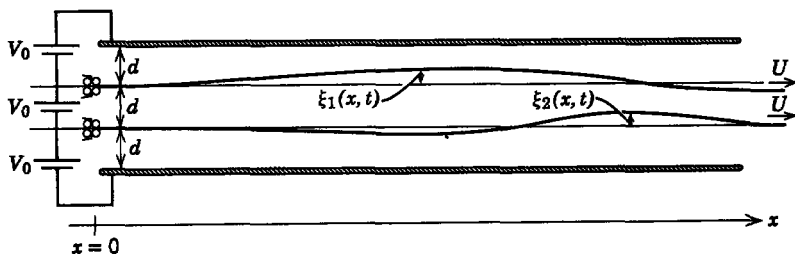


Fig. 10P.23

- (a) Write a pair of differential equations in $\xi_1(x, t)$ and $\xi_2(x, t)$ to describe the membrane motions. Assume that ξ_1 and ξ_2 are small enough to warrant linearization and that the wavelengths are long enough for the membrane to appear flat to the electric field at any one value of x .
 (b) Assume that

$$\begin{aligned} \xi_1 &= \text{Re } \tilde{\xi}_1 e^{j(\omega t - kx)}, \\ \xi_2 &= \text{Re } \tilde{\xi}_2 e^{j(\omega t - kx)} \end{aligned}$$

and find a dispersion equation relating ω and k .

- (c) Make an ω - k plot to show the results of part (b), including complex values of k for real values of ω . This equation can be factored into two quadratic equations for k . Assume that $U > \sqrt{S/\sigma_m}$.
 (d) One of the quadratic factors in part (c) describes motions in which $\xi_1(x, t) = \xi_2(x, t)$, whereas the other describes motions in which $\xi_1(x, t) = -\xi_2(x, t)$. Show that this is true by assuming first that $\xi_1 = \xi_2$ and then that $\xi_1 = -\xi_2$ in parts (a) and (b).
 (e) Now suppose that the rollers at $x = 0$ are given the sinusoidal excitation $\xi_1(0, t) = \text{Re } \tilde{\xi} e^{j\omega t} = -\xi_2(0, t)$, where $\tilde{\xi}$ is the same real constant for each excitation. Also, $0 = \partial \xi_1 / \partial x = \partial \xi_2 / \partial x$ at $x = 0$. Find $\xi_1(x, t)$ and $\xi_2(x, t)$.
 (f) What voltage V_0 is required to make the waves excited in part (e) amplify?
 (g) Sketch the spatial dependence of ξ_1 and ξ_2 at an instant in time with $V_0 = 0$ and with V_0 large enough to produce amplifying waves.

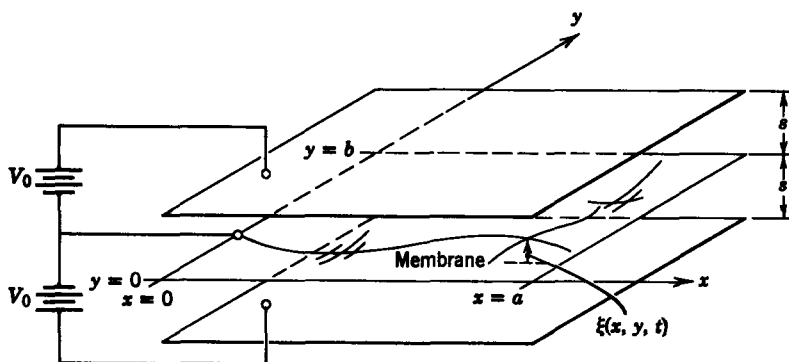


Fig. 10P.24

10.24. A perfectly conducting membrane with tension S and mass/unit area σ_m is fixed at $x=0$ and $x=a$ and at $y=0$ and $y=b$. Perfectly conducting plane-parallel electrodes have an equilibrium distance s from the membrane and a potential V_0 relative to the membrane.

- What is the largest value of V_0 that will still allow the membrane to be in a state of stable static equilibrium? You may assume that $a \gg s$ and $b \gg s$.
- What are the natural frequencies of the membrane?
- Given that the membrane is stationary when $t=0$ and that

$$\xi(x, y, 0) = J_0 u_0 \left(x - \frac{a}{2} \right) u_0 \left(y - \frac{b}{2} \right),$$

where u_0 is the unit impulse and J_0 is an arbitrary constant, find the response $\xi(x, y, t)$.

10.25. A membrane with tension S and mass/unit area σ_m is fixed along its edges at $y=0$ and $y=a$. It is also fixed along the edge $x=b$. At $x=0$ it is driven and has the displacement shown in Fig. 10P.25b. Find the sinusoidal steady-state driven response $\xi(x, y, t)$.

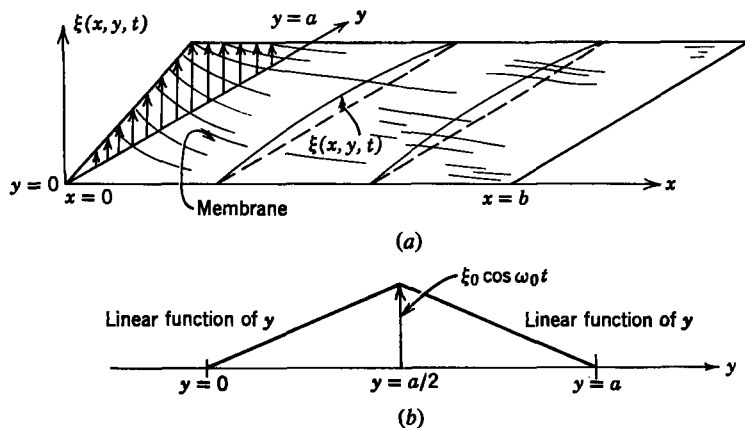


Fig. 10P.25

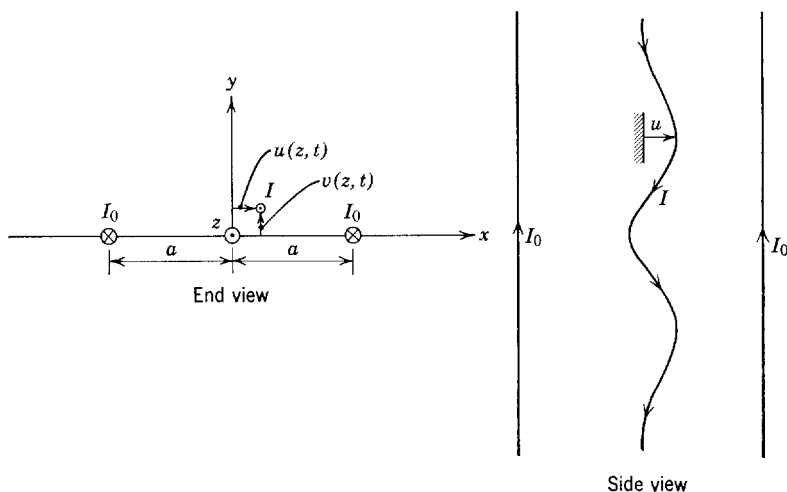


Fig. 10P.26

10.26. A pair of conductors, separated by a distance $2a$, carries currents I_0 (amperes) in the $-z$ -direction, as shown in Fig. 10P.26. A conducting wire of mass density/unit length m is stretched along the z -axis and carries a current I ($I \ll I_0$). Deflections of the wire from the z -axis are given by $u(z, t)\mathbf{i}_x + v(z, t)\mathbf{i}_y$.

- (a) Show that the equations of motion for the wire in the magnetic field have the form

$$m \frac{\partial^2 u}{\partial t^2} = f \frac{\partial^2 u}{\partial z^2} - I b u,$$

$$m \frac{\partial^2 v}{\partial t^2} = f \frac{\partial^2 v}{\partial z^2} + I b v,$$

under the assumption that deflections u and v are small. What is the constant b in terms of I_0 and a ? What fundamental law requires that if $I b u$ appears with a positive sign in the second equation it must appear with a negative sign in the first equation?

- (b) Consider solutions that have the form $u = \text{Re } \hat{u} e^{j(\omega t - kz)}$ and $v = \text{Re } \hat{v} e^{j(\omega t - kz)}$ and find the relationship between ω - k for x and for y displacements. Make dimensioned plots in each case of real and imaginary values of k for real values of ω . Make dimensioned plots in each case of real and imaginary values of ω for real values of k . (Throughout this problem consider $I_0 > 0$, $I > 0$.)
- (c) The wire is now fixed at $z = 0$ and, given the deflection

$$u(-l, t)\mathbf{i}_x + v(-l, t)\mathbf{i}_y = u_0 \cos \omega_0 t \mathbf{i}_x + v_0 \sin \omega_0 t \mathbf{i}_y \quad (\omega_0 \text{ is real}).$$

Find $u(z, t)$ and $v(z, t)$.

- (d) For what values of the currents (I , I_0) will it be possible to establish the sinusoidal steady-state solution of part (c)? For what values of ω_0 , in terms of f , m , and l , will the wire support evanescent waves as x -deflections and remain stable?
- (e) The frequency ω_0 is set $\omega_0 = (\pi/2l)\sqrt{f/m}$. Sketch the peak deflections u and v as functions of z for several values of I (starting with $I = 0$). Summarize in words how the deflections will change as the current I is raised.

10.27. This is a continuation of Problem 10.26. Now the wire has an equilibrium velocity U along the z -axis with $U > \sqrt{f/m}$.

- Write the differential equations for the deflections $u(z, t)$ and $v(z, t)$, including the effect of U .
- Consider solutions $u = \text{Re } \hat{u}e^{j(\omega t - kz)}$ and $v = \text{Re } \hat{v}e^{j(\omega t - kz)}$ and find the relationship between ω - k for x and y displacements. Make dimensioned plots in each case of real and imaginary values of k for real values of ω . Make dimensioned plots in each case of real and imaginary values of ω for real values of k ($I_0 > 0, I > 0$).
- Why would it not be possible to impose the boundary conditions of part (c) in Problem 10.26 to solve this problem? The wire is driven at $z = 0$ by the deflection $u(0, t)\mathbf{i}_x + v(0, t)\mathbf{i}_y = u_0 \cos \omega_0 t \mathbf{i}_x + v_0 \sin \omega_0 t \mathbf{i}_y$ with the slopes

$$\frac{\partial u}{\partial z}(0, t) = 0, \quad \frac{\partial v}{\partial z}(0, t) = 0.$$

Find $u(z, t)$ and $v(z, t)$.

- For a given driving frequency ω_0 sketch the peak deflections u and v as functions of z for several values of I (starting with $I = 0$). Summarize in words how the deflections would change as the current I is raised.
- How would you devise an experiment to demonstrate the results of the preceding parts (i.e., what would you use as the moving "wire" and how would you excite it?).

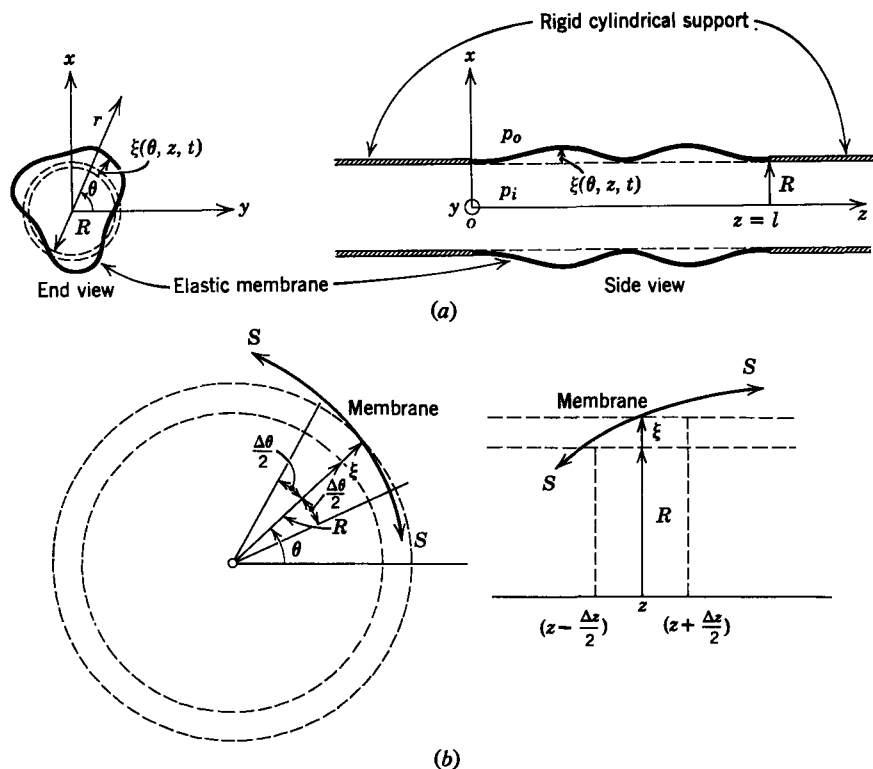


Fig. 10P.28

10.28. An elastic membrane with constant tension S has a circular cylindrical equilibrium geometry, as shown in Fig. 10P.28a. It is supported at $z = 0$ and at $z = l$ by circular rigid tubes. Intuitively, we expect that the membrane will collapse inward if the pressure inside the membrane (p_i) is not larger than that outside (p_o). We could imagine stopping up one of the supporting tubes and pushing a cork into the end of the other tube just far enough to maintain the required pressure difference, as might be done in extruding a hollow section of molten glass or plastic or a soap film.

- Show that for a static equilibrium to exist with $\xi = 0$, $p_i - p_o = S/R$.
- There is, of course, no guarantee that if we establish this pressure difference the equilibrium will be stable. To examine this question write a linearized equation of radial force equilibrium for a small section of the membrane. The sketches of surface deformation shown in Fig. 10P.28b should be helpful in writing the force per unit area due to the tension S . Your equation of motion should be

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \left(\frac{\xi}{R^2} + \frac{1}{R^2} \frac{\partial^2 \xi}{\partial \theta^2} + \frac{\partial^2 \xi}{\partial z^2} \right)$$

- Under what conditions is the equilibrium stable?
- Describe the lowest modes of oscillation for the membrane.
- Reconsider Problem 10.22, taking into account the effect of the curvature.

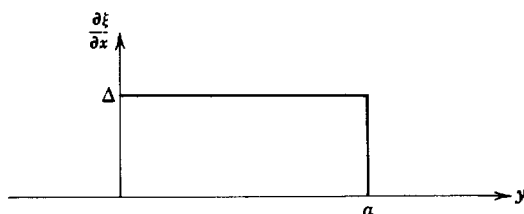


Fig. 10P.29

10.29. A membrane has the velocity $U > v_s$ in the x -direction, as described in Section 10.4.2. At $x = 0$, $\xi = 0$, and $\partial \xi / \partial x$ has the distribution shown. Assume that the membrane is infinitely wide in the y -direction.

- Find and sketch $\xi(x, y)$ for $x > 0$. Assume that $M^2 = 2$.
- Describe how you would physically produce the postulated excitation at $x = 0$.

10.30. A membrane moves with the velocity $U > v_s$ in the x -direction (see Section 10.4.2). Its edges are prevented from undergoing transverse motions along boundaries at $y = 0$ and

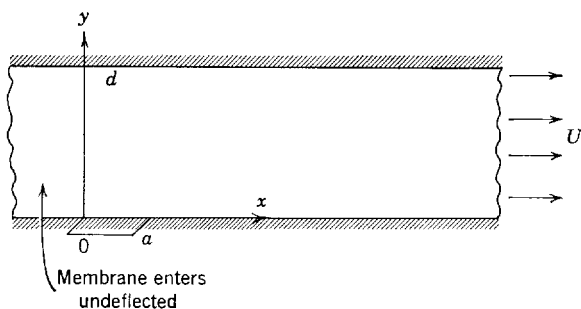


Fig. 10P.30

$y = d$, except for the segment $0 < x < a$, where the membrane is constrained to have the constant amplitude ξ_0 . Assume that $M^2 = 2$ and find the resulting deflection $\xi(x, t)$. Your answer should be presented as a sketch similar to Fig. 10.4.3.

10.31. Plot the ω - k relation (10.4.30) for the example described in Section 10.4.3 to show complex ω for real k and complex k for real ω . Indicate the modes (fast wave or slow wave) represented by each branch of the curves.

10.32. Consider the example of Section 10.4.3, but with the wire having a longitudinal velocity $U > v_s$.

- (a) Find the revised dispersion equation.
- (b) Sketch the ω - k relation and show complex values of k for real values of ω .
- (c) Describe the response of the wire to a sinusoidal excitation.

Appendix D

GLOSSARY OF COMMONLY USED SYMBOLS

Section references indicate where symbols of a given significance are introduced; grouped symbols are accompanied by their respective references. The absence of a section reference indicates that a symbol has been applied for a variety of purposes. Nomenclature used in examples is not included.

Symbol	Meaning	Section
A	cross-sectional area	
A_i	coefficient in differential equation	5.1.1
(A_n^+, A_n^-)	complex amplitudes of components of n th mode	9.2.1
A_w	cross-sectional area of armature conductor	6.4.1
a	spacing of pole faces in magnetic circuit	8.5.1
$a, (a_c, a_s)$	phase velocity of acoustic related waves	13.2.1, 11.4.1
a_b	Alfvén velocity	12.2.3
(a, b, c)	Lagrangian coordinates	11.1
a_i	constant coefficient in differential equation	5.1.1
$\mathbf{a}_p$	instantaneous acceleration of point p fixed in material	2.2.1c
B, B_r, B_s	damping constant for linear, angular and square law dampers	2.2.1b, 4.1.1, 5.2.2
$\mathbf{B}, \mathbf{B}_i, B_0$	magnetic flux density	1.1.1a, 8.1, 6.4.2
B_i	induced flux density	7.0
$(B_r, B_{ra}, B_{rb}, B_{rm})$	radial components of air-gap flux densities	4.1.4
$[B_{rf}, (B_{rf})_{av}]$	radial flux density due to field current	6.4.1
b	width of pole faces in magnetic circuit	8.5
b	half thickness of thin beam	11.4.2b
C	contour of integration	1.1.2a
$C, (C_a, C_b), C_o$	capacitance	2.1.2, 7.2.1a, 5.2.1
C	coefficient in boundary condition	9.1.1
$\mathbf{C}$	the curl of the displacement	11.4
(C^+, C^-)	designation of characteristic lines	9.1.1

Symbol	Meaning	Section
c_p	specific heat capacity at constant pressure	13.1.2
c_v	specific heat capacity at constant volume	13.1.2
D	electric displacement	1.1.1a
d	length	
da	elemental area	1.1.2a
$d\mathbf{f}_n$	total elemental force on material in rigid body	2.2.1c
$d\mathbf{l}$	elemental line segment	1.1.2a
$d\mathbf{T}_n$	torque on elemental volume of material	2.2.1c
dV	elemental volume	1.1.2b
E	constant of motion	5.2.1
E	Young's modulus or the modulus of elasticity	9.1
$\mathbf{E}, E_o$	electric field intensity	1.1.1a, 5.1.2d
E_f	magnitude of armature voltage generated by field current in a synchronous machine	4.1.6a
E_i	induced electric field intensity	7.0
e_{11}, e_{ij}	strain tensor	9.1, 11.2
$\dot{e}_{ij}$	strain-rate tensor	14.1.1a
F	magnetomotive force (mmf)	13.2.2
F	force density	1.1.1a
$\hat{F}$	complex amplitude of $f(t)$	5.1.1
F_0	amplitude of sinusoidal driving force	9.1.3
f	equilibrium tension of string	9.2
f	driving function	5.1.1
$f, \mathbf{f}, f^e, f^s, f_j, f_i, f_1$	force	2.2.1, 2.2.1c, 3.1, 5.1.2a, 3.1.2b, 8.1, 9.1
f	arbitrary scalar function	6.1
f'	scalar function in moving coordinate system	6.1
f	three-dimensional surface	6.2
f	integration constant	11.4.2a
G	a constant	5.1.2c
G	shear modulus of elasticity	11.2.2
G	speed coefficient	6.4.1
G	conductance	3.1
g	air-gap length	5.2.1
$g, \mathbf{g}$	acceleration of gravity	5.1.2c, 12.1.3
$(\mathbf{H}, H_x, H_y, H_z)$	magnetic field intensity	1.1.1a
h	specific enthalpy	13.1.2
I, I, $(I_r, I_s), I_f$	electrical current	10.4.3, 12.2.1a, 4.1.2, 6.4.1
$(i, i_1, i_2, \dots, i_k), (i_{ar}, i_{as}, i_{br}, i_{bs}), i_a, (i_a, i_b, i_c), (i_f, i_t), (i_r, i_s)$	electrical current	2.1, 4.1.3, 6.4.1, 4.1.7, 6.4.1, 4.1

Symbol	Meaning	Section
i_n	unit vector perpendicular to area of integration	6.2.1
i_s	unit vector normal to surface of integration	6.2.1
$(i_x, i_y, i_z), (i_1, i_2, i_3)$	unit vectors in coordinate directions	2.2.1c
J, J_f	current density	7.0, 1.1.1a
$J, J_r, (J_x, J_y, J_z)$	moment of inertia	5.1.2b, 4.1.1, 2.2.1c
J_{xx}, J_{yz}	products of inertia	2.2.1c
j	$\sqrt{-1}$	4.1.6a
K	loading factor	13.2.2
K, K_f	surface current density	7.0, 1.1.1a
K	linear or torsional spring constant	2.2.1a
K_t	induced surface current density	7.0
$k, k_c, (k_r, k_t)$	wavenumber	7.1.3, 10.1.3, 10.0
k	summation index	2.1.1
k	maximum coefficient of coupling	4.1.6b
k_n	n th eigenvalue	9.2
$(L, L_1, L_2), (L_a, L_f),$ $L_m, (L_0, L_2),$ $(L_r, L_s, L_{st}), L_{ss}$	inductance	2.1.1, 6.4.1, 2.1.1, 4.2.1, 4.1.1, 4.2.4
L	length of incremental line segment	6.2.1
l	value of relative displacement for which spring force is zero	2.2.1a
l, l_w, l_y	length	
M	Hartmann number	14.2.2
M	mass of one mole of gas in kilograms	13.1.2
M	Mach number	13.2.1
M	mass	2.2.1c
M	number of mechanical terminal pairs	2.1.1
M, M_s	mutual inductance	4.1.1, 4.2.4
M	magnetization density	1.1.1a
m	mass/unit length of string	9.2
N	number of electrical terminal pairs	2.1.1
N	number of turns	5.2.2
n	number density of ions	12.3.1
n	integer	7.1.1
n	unit normal vector	1.1.2
P	polarization density	1.1.1a
P	power	12.2.1a
P	number of pole pairs in a machine	4.1.8
P	power per unit area	14.2.1
P	pressure	5.1.2d and 12.1.4
P_e, P_g, P_m, P_r	power	4.1.6a, 4.1.6b, 4.1.2, 4.1.6b
Q	electric charge	7.2.1a
q, q_i, q_k	electric charge	1.1.3 and 2.1.2, 8.1, 2.1.2
R, R_i, R_o	radius	

Symbol	Meaning	Section
$R, R_a, R_b, R_f, R_r, R_s$	resistance	
(R, R_g)	gas constant	13.1.2
R_e	electric Reynolds number	7.0
R_m	magnetic Reynolds number	7.0
r	radial coordinate	
$\mathbf{r}$	position vector of material	2.2.1c
$\mathbf{r}'$	position vector in moving reference frame	6.1
$\mathbf{r}_m$	center of mass of rigid body	2.2.1c
S	reciprocal modulus of elasticity	11.5.2c
S	surface of integration	1.1.2a
S	normalized frequency	7.2.4
S	membrane tension	9.2
S_z	transverse force/unit length acting on string	9.2
s	complex frequency	5.1.1
(s, s_{mT})	slip	4.1.6b
s_i	i th root of characteristic equation, a natural frequency	5.1.1
T	period of oscillation	5.2.1
T	temperature	13.1.2
$\mathbf{T}, T, T^e, T_{em}, T_m,$ T_0, T_1	torque	2.2.1c, 5.1.2b, 3.1.1, 4.1.6b, 4.1.1, 6.4.1, 6.4.1
$\mathbf{T}$	surface force	8.4
T_{ij}^m	mechanical stress tensor	13.1.2
T_{mn}	the component of the stress-tensor in the m th-direction on a cartesian surface with a normal vector in the n th-direction	8.1
T_{or}	constant of coulomb damping	4.1.1
T_0	initial stress distribution on thin rod	9.1.1
T	longitudinal stress on a thin rod	9.1.1
T_z	transverse force per unit area on membrane	9.2
T_2	transverse force per unit area acting on thin beam	11.4.2b
t	time	1.1.1
t'	time measured in moving reference frame	6.1
U	gravitational potential	12.1.3
U	longitudinal steady velocity of string or membrane	10.2
u	internal energy per unit mass	13.1.1
u	surface coordinate	11.3
$u_0(x - x_0)$	unit impulse at $x = x_0$	9.2.1
u	transverse deflection of wire in x -direction	10.4.3
$u_{-1}(t)$	unit step occurring at $t = 0$	5.1.2b
V, V_m	velocity	7.0, 13.2.3
V	volume	1.1.2
V, V_a, V_f, V_o, V_s	voltage	
V	potential energy	5.2.1

Symbol	Meaning	Section
$v, \mathbf{v}$	velocity	
$(v, v_1, \dots, v_k)$	voltage	2.1.1
$v', (v_a, v_b, v_c),$ v_f, v_{oc}, v_t	voltage	
v_n	velocity of surface in normal direction	6.2.1
v_o	initial velocity distribution on thin rod	9.1.1
v_p	phase velocity	9.1.1 and 10.2
$\mathbf{v}^r$	relative velocity of inertial reference frames	6.1
v_s	$\sqrt{f/m}$ for a string under tension f and having mass/unit length m	10.1.1
v	longitudinal material velocity on thin rod	9.1.1
v	transverse deflection of wire in y -direction	10.4.3
(W_e, W_m)	energy stored in electromechanical coupling	3.1.1
(W'_e, W'_m, W')	coenergy stored in electromechanical coupling	3.1.2b
W''	hybrid energy function	5.2.1
w	width	5.2.2
w	energy density	11.5.2c
w'	coenergy density	8.5
X	equilibrium position	5.1.2a
$(x, x_1, x_2, \dots, x_k)$	displacement of mechanical node	2.1.1
x	dependent variable	5.1.1
x_p	particular solution of differential equation	5.1.1
$(x_1, x_2, x_3), (x, y, z)$	cartesian coordinates	8.1, 6.1
(x', y', z')	cartesian coordinates of moving frame	6.1
(α, β)	constants along C^+ and C^- characteristics, respectively	9.1.1
(α, β)	see (10.2.20) or (10.2.27)	
α	transverse wavenumber	11.4.3
(α, β)	angles used to define shear strain	11.2
(α, β)	constant angles	4.1.6b
α	space decay parameter	7.1.4
α	damping constant	5.1.2b
α	equilibrium angle of torsional spring	2.2.1a
γ	ratio of specific heats	13.1.2
γ	piezoelectric constant	11.5.2c
$\gamma, \gamma_0, \gamma'$	angular position	
$\Delta_d(t)$	slope excitation of string	10.2.1b
Δ_0	amplitude of sinusoidal slope excitation	10.2.1b
$\Delta \mathbf{r}$	distance between unstressed material points	11.2.1a
$\Delta \mathbf{s}$	distance between stressed positions of material points	11.2.1a
$\delta()$	incremental change in ()	8.5
$\delta, \delta_1, \delta_0$	displacement of elastic material	11.1, 9.1, 11.4.2a
δ	thickness of incremental volume element	6.2.1
δ	torque angle	4.1.6a

Symbol	Meaning	Section
δ_{ij}	Kronecker delta	8.1
(δ_+, δ_-)	wave components traveling in the $\pm x$ -directions	9.1.1
ϵ	linear permittivity	1.1.1b
ϵ_0	permittivity of free space	1.1.1a
η	efficiency of an induction motor	4.1.6b
η	second coefficient of viscosity	14.1.1c
$\theta, \theta_i, \theta_m$	angular displacement	2.1.1, 3.1.1, 5.2.1
θ	power factor angle; phase angle between current and voltage	4.1.6a
θ	equilibrium angle	5.2.1
$\dot{\theta}$	angular velocity of armature	6.4.1
θ_m	maximum angular deflection	5.2.1
$(\lambda, \lambda_1, \lambda_2, \dots, \lambda_k)$	magnetic flux linkage	2.1.1, 6.4.1, 4.1.7,
λ_a		4.1.3, 4.1
$(\lambda_a, \lambda_b, \lambda_c)$		
$(\lambda_{ar}, \lambda_{as}, \lambda_{br}, \lambda_{bs})$		
(λ_r, λ_s)		
λ	Lamé constant for elastic material	11.2.3
λ	wavelength	7.1.4
μ	linear permeability	1.1.1a
$\mu, (\mu_+, \mu_-)$	mobility	12.3.1, 1.1.1b
μ	coefficient of viscosity	14.1.1
μ_d	coefficient of dynamic friction	2.2.1b
μ_0	permeability of free space	1.1.1a
μ_s	coefficient of static friction	2.2.1b
ν	Poisson's ratio for elastic material	11.2.2
ν	damping frequency	10.1.4
(ξ, ξ)	continuum displacement	8.5
ξ_0	initial deflection of string	9.2
ξ_d	amplitude of sinusoidal driving deflection	9.2
$(\xi_n(x), \xi_n(x))$	n th eigenfunctions	9.2.1b
(ξ_+, ξ_-)	amplitudes of forward and backward traveling waves	9.2
$\dot{\xi}_0(x)$	initial velocity of string	9.2
ρ	mass density	2.2.1c
ρ_f	free charge density	1.1.1a
ρ_s	surface mass density	11.3
Σ	surface of discontinuity	6.2
σ	conductivity	1.1.1a
σ_f	free surface charge density	1.1.1a
σ_m	surface mass density of membrane	9.2
σ_o	surface charge density	7.2.3
σ_s	surface conductivity	1.1.1a
σ_u	surface charge density	7.2.3
τ	surface traction	8.2.1
τ, τ_d	diffusion time constant	7.1.1, 7.1.2a
τ	relaxation time	7.2.1a

Symbol	Meaning	Section
τ_e	electrical time constant	5.2.2
τ_m	time for air gap to close	5.2.2
τ_o	time constant	5.1.3
τ_t	traversal time	7.1.2a
ϕ	electric potential	7.2
ϕ	magnetic flux	2.1.1
ϕ	cylindrical coordinate	2.1.1
ϕ	potential for H when $J_f = 0$	8.5.2
ϕ	flow potential	12.2
χ_e	electric susceptibility	1.1.1b
χ_m	magnetic susceptibility	1.1.1a
ψ	the divergence of the material displacement	11.4
ψ	angle defined in Fig. 6.4.2	6.4.1
ψ	angular position in the air gap measured from stator winding (a) magnetic axis	4.1.4
ψ	electromagnetic force potential	12.2
ψ	angular deflection of wire	10.4.3
Ω	equilibrium rotational speed	5.1.2b
Ω	rotation vector in elastic material	11.2.1a
Ω_n	real part of eigenfrequency (10.1.47)	10.1.4
$\omega, (\omega_r, \omega_i)$	radian frequency of electrical excitation	4.1.6a, 4.1.2
ω	natural angular frequency (Im s)	5.1.2b
ω, ω_m	angular velocity	2.2.1c, 4.1.2
ω_c	cutoff frequency for evanescent waves	10.1.2
ω_d	driving frequency	9.2
ω_n	n th eigenfrequency	9.2
ω_o	natural angular frequency	5.1.3
(ω_r, ω_i)	real and imaginary parts of ω	10.0
∇	nabla	6.1
∇_Σ	surface divergence	6.2.1

Appendix E

SUMMARY OF PARTS I AND II AND USEFUL THEOREMS

IDENTITIES

$$\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{B} \times \mathbf{C},$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi,$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B},$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B},$$

$$\nabla(\phi\psi) = \phi \nabla\psi + \psi \nabla\phi,$$

$$\nabla \cdot (\psi\mathbf{A}) = \mathbf{A} \cdot \nabla\psi + \psi \nabla \cdot \mathbf{A},$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B},$$

$$\nabla \cdot \nabla\phi = \nabla^2\phi,$$

$$\nabla \cdot \nabla \times \mathbf{A} = 0,$$

$$\nabla \times \nabla\phi = 0,$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A},$$

$$(\nabla \times \mathbf{A}) \times \mathbf{A} = (\mathbf{A} \cdot \nabla)\mathbf{A} - \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A}),$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

$$\nabla \times (\phi\mathbf{A}) = \nabla\phi \times \mathbf{A} + \phi \nabla \times \mathbf{A},$$

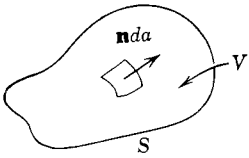
$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}.$$

THEOREMS

$$\int_a^b \nabla \phi \cdot d\mathbf{l} = \phi_b - \phi_a.$$



Divergence theorem $\oint_S \mathbf{A} \cdot \mathbf{n} \, da = \int_V \nabla \cdot \mathbf{A} \, dV$



Stokes's theorem $\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} \, da$

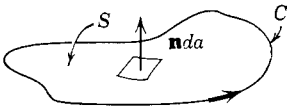
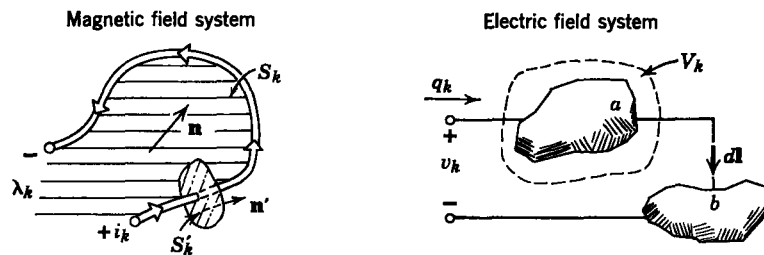


Table 1.2 Summary of Quasi-Static Electromagnetic Equations

	Differential Equations		Integral Equations	
Magnetic field system	$\nabla \times \mathbf{H} = \mathbf{J}_f$	(1.1.1)	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot \mathbf{n} \, da$	(1.1.20)
	$\nabla \cdot \mathbf{B} = 0$	(1.1.2)	$\oint_S \mathbf{B} \cdot \mathbf{n} \, da = 0$	(1.1.21)
	$\nabla \cdot \mathbf{J}_f = 0$	(1.1.3)	$\oint_S \mathbf{J}_f \cdot \mathbf{n} \, da = 0$	(1.1.22)
	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	(1.1.5)	$\oint_C \mathbf{E}' \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da$	(1.1.23)
			where $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$	
Electric field system	$\nabla \times \mathbf{E} = 0$	(1.1.11)	$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	(1.1.24)
	$\nabla \cdot \mathbf{D} = \rho_f$	(1.1.12)	$\oint_S \mathbf{D} \cdot \mathbf{n} \, da = \int_V \rho_f \, dV$	(1.1.25)
	$\nabla \cdot \mathbf{J}_f = -\frac{\partial \rho_f}{\partial t}$	(1.1.14)	$\oint_S \mathbf{J}_f' \cdot \mathbf{n} \, da = -\frac{d}{dt} \int_V \rho_f \, dV$	(1.1.26)
	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$	(1.1.15)	$\oint_C \mathbf{H}' \cdot d\mathbf{l} = \int_S \mathbf{J}_f' \cdot \mathbf{n} \, da + \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{n} \, da$	(1.1.27)
			where $\mathbf{J}_f' = \mathbf{J}_f - \rho_f \mathbf{v}$	
			$\mathbf{H}' = \mathbf{H} - \mathbf{v} \times \mathbf{D}$	

Table 2.1 Summary of Terminal Variables and Terminal Relations



Definition of Terminal Variables

Flux

$$\lambda_k = \int_{S_k} \mathbf{B} \cdot \mathbf{n} \, da$$

Current

$$i_k = \int_{S'_k} \mathbf{J}_f \cdot \mathbf{n}' \, da$$

Charge

$$q_k = \int_{V_k} \rho_f \, dV$$

Voltage

$$v_k = \int_a^b \mathbf{E} \cdot d\mathbf{l}$$

Terminal Conditions

$$v_k = \frac{d\lambda_k}{dt}$$

$$i_k = \frac{dq_k}{dt}$$

$$\lambda_k = \lambda_k(i_1 \cdots i_N; \text{geometry})$$

$$q_k = q_k(v_1 \cdots v_N; \text{geometry})$$

$$i_k = i_k(\lambda_1 \cdots \lambda_N; \text{geometry})$$

$$v_k = v_k(q_1 \cdots q_N; \text{geometry})$$

and M Mechanical Terminal Pairs*

Magnetic Field Systems	Electric Field Systems
Conservation of Energy	
$dW_m = \sum_{j=1}^N i_j d\lambda_j - \sum_{j=1}^M f_j^e dx_j$	(a) $dW_e = \sum_{j=1}^N v_j dq_j - \sum_{j=1}^M f_j^e dx_j$ (b)
$dW'_m = \sum_{j=1}^N \lambda_j di_j + \sum_{j=1}^M f_j^e dx_j$	(c) $dW'_e = \sum_{j=1}^N q_j dv_j + \sum_{j=1}^M f_j^e dx_j$ (d)
Forces of Electric Origin, $j = 1, \dots, M$	
$f_j^e = - \frac{\partial W_m(\lambda_1, \dots, \lambda_N; x_1, \dots, x_M)}{\partial x_j}$	(e) $f_j^e = - \frac{\partial W_e(q_1, \dots, q_N; x_1, \dots, x_M)}{\partial x_j}$ (f)
$f_j^e = \frac{\partial W'_m(i_1, \dots, i_N; x_1, \dots, x_M)}{\partial x_j}$	(g) $f_j^e = \frac{\partial W'_e(v_1, \dots, v_N; x_1, \dots, x_M)}{\partial x_j}$ (h)
Relation of Energy to Coenergy	
$W_m + W'_m = \sum_{j=1}^N \lambda_j i_j$	(i) $W_e + W'_e = \sum_{j=1}^N v_j q_j$ (j)
Energy and Coenergy from Electrical Terminal Relations	
$W_m = \sum_{j=1}^N \int_0^{\lambda_j} i_j(\lambda_1, \dots, \lambda_{j-1}, \lambda'_j, 0, \dots, 0; x_1, \dots, x_M) d\lambda'_j$ (k)	$W_e = \sum_{j=1}^N \int_0^{q_j} v_j(q_1, \dots, q_{j-1}, q'_j, 0, \dots, 0; x_1, \dots, x_M) dq'_j$ (l)
$W'_m = \sum_{j=1}^N \int_0^{i_j} \lambda_j(i_1, \dots, i_{j-1}, i'_j, 0, \dots, 0; x_1, \dots, x_M) di'_j$ (m)	$W'_e = \sum_{j=1}^N \int_0^{v_j} q_j(v_1, \dots, v_{j-1}, v'_j, 0, \dots, 0; x_1, \dots, x_M) dv'_j$ (n)

* The mechanical variables f_j and x_j can be regarded as the j th force and displacement or the j th torque T_j and angular displacement θ_j .

Table 6.1 Differential Equations, Transformations, and Boundary Conditions for Quasi-static Electromagnetic Systems with Moving Media

	Differential Equations		Transformations		Boundary Conditions	
Magnetic field systems	$\nabla \times \mathbf{H} = \mathbf{J}_f$	(1.1.1)	$\mathbf{H}' = \mathbf{H}$	(6.1.35)	$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K}_f$	(6.2.14)
	$\nabla \cdot \mathbf{B} = 0$	(1.1.2)	$\mathbf{B}' = \mathbf{B}$	(6.1.37)	$\mathbf{n} \cdot (\mathbf{B}^a - \mathbf{B}^b) = 0$	(6.2.7)
	$\nabla \cdot \mathbf{J}_f = 0$	(1.1.3)	$\mathbf{J}'_f = \mathbf{J}_f$	(6.1.36)	$\mathbf{n} \cdot (\mathbf{J}_f^a - \mathbf{J}_f^b) + \nabla_\Sigma \cdot \mathbf{K}_f = 0$	(6.2.9)
	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	(1.1.5)	$\mathbf{E}' = \mathbf{E} + \mathbf{v}^r \times \mathbf{B}$	(6.1.38)	$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = v_n(\mathbf{B}^a - \mathbf{B}^b)$	(6.2.22)
	$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$	(1.1.4)	$\mathbf{M}' = \mathbf{M}$	(6.1.39)		
Electric field systems	$\nabla \times \mathbf{E} = 0$	(1.1.11)	$\mathbf{E}' = \mathbf{E}$	(6.1.54)	$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = 0$	(6.2.31)
	$\nabla \cdot \mathbf{D} = \rho_f$	(1.1.12)	$\mathbf{D}' = \mathbf{D}$	(6.1.55)	$\mathbf{n} \cdot (\mathbf{D}^a - \mathbf{D}^b) = \sigma_f$	(6.2.33)
			$\rho'_f = \rho_f$	(6.1.56)		
	$\nabla \cdot \mathbf{J}_f = -\frac{\partial \rho_f}{\partial t}$	(1.1.14)	$\mathbf{J}'_f = \mathbf{J}_f - \rho_f \mathbf{v}^r$	(6.1.58)	$\mathbf{n} \cdot (\mathbf{J}_f^a - \mathbf{J}_f^b) + \nabla_\Sigma \cdot \mathbf{K}_f = v_n(\rho_f^a - \rho_f^b) - \frac{\partial \sigma_f}{\partial t}$	(6.2.36)
	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$	(1.1.15)	$\mathbf{H}' = \mathbf{H} - \mathbf{v}^r \times \mathbf{D}$	(6.1.57)	$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K}_f + v_n \mathbf{n} \times [\mathbf{n} \times (\mathbf{D}^a - \mathbf{D}^b)]$	(6.2.38)
	$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$	(1.1.13)	$\mathbf{P}' = \mathbf{P}$	(6.1.59)		

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Numbers preceded by letters are Appendix references. Appendices A, B, and C are in Part One; Appendices D and E, Part Two; and Appendices F and G, Part Three.

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ELECTROMECHANICAL DYNAMICS

Part III: Elastic and Fluid Media

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ELECTROMECHANICAL DYNAMICS

Part III: Elastic and Fluid Media

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To our parents

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PREFACE

Part III: Elastic and Fluid Media

In the early 1950's the option structure was abandoned and a common core curriculum was instituted for all electrical engineering students at M.I.T. The objective of the core curriculum was then, and is now, to provide a foundation in mathematics and science on which a student can build in his professional growth, regardless of the many opportunities in electrical engineering from which he may choose. In meeting this objective, core curriculum subjects cannot serve the needs of any professional area with respect to nomenclature, techniques, and problems unique to that area. Specialization comes in elective subjects, graduate study, and professional activities.

To be effective a core curriculum subject must be broad enough to be germane to the many directions an electrical engineer may go professionally, yet it must have adequate depth to be of lasting value. At the same time, the subject must be related to the real world by examples of application. This is true because students learn by seeing material in a familiar context, and engineering students are motivated largely by the relevance of the material to the realities of the world around them.

In the organization of the core curriculum in electrical engineering at M.I.T. electromechanics is one major component. As our core curriculum has evolved, there have been changes in emphasis and a broadening of the topic. The basic text in electromechanics until 1954, when a new departure was made, was *Electric Machinery* by Fitzgerald and Kingsley. This change produced *Electromechanical Energy Conversion* by White and Woodson, which was used until 1961. At that time we started the revision that resulted in the present book. During this period we went through many versions of notes while teaching the material three semesters a year.

Our objective has always been to teach a subject that combines classical mechanics with the fundamentals of electricity and magnetism. Thus the subject offers the opportunity to teach both mechanics and electromagnetic theory in a context vital to much of the electrical engineering community.

Our choice of material was to some extent determined by a desire to give the student a breadth of background sufficient for further study of almost any type of electromechanical interaction, whether in rotating machinery,

plasma dynamics, the electromechanics of biological systems, or magnetoelasticity. It was also chosen to achieve adequate depth while maintaining suitable unity, but, most important, examples were chosen that could be enlivened for the engineering student interested in the interplay of physical reality and the analytical model. There were many examples from which to choose, but only a few satisfied the requirement of being both mathematically lucid *and* physically demonstrable, so that the student could “push it or see it” and directly associate his observations with symbolic models. Among the areas of electrical engineering, electromechanics excels in offering the opportunity to establish that all-important “feel” for a physical phenomenon. Properly selected electromechanical examples can be the basis for discerning phenomena that are remote from human abilities to observe.

Before discussing how the material can be used to achieve these ends, a review of the contents is in order. The student who uses this book is assumed to have a background in electrostatics and magnetostatics. Consequently, Chapter 1 and Appendix B are essentially a review to define our starting point.

Chapter 2 is a generalization of the concepts of inductance and capacitance that are necessary to the treatment of electromechanical systems; it also provides a brief introduction to rigid-body mechanics. This treatment is included because many curricula no longer cover mechanics, other than particle mechanics in freshman physics. The basic ideas of Chapter 2 are repeated in Chapter 3 to establish some properties of electromechanical coupling in lumped-parameter systems and to obtain differential equations that describe the dynamics of lumped-parameter systems.

Next, the techniques of Chapters 2 and 3 are used to study rotating machines in Chapter 4. Physical models are defined, differential equations are written, machine types are classified, and steady-state characteristics are obtained and discussed. A separate chapter on rotating machines has been included not only because of the technological importance of machines but also because rotating machines are rich in examples of the kinds of phenomena that can be found in lumped-parameter electromechanical systems.

Chapter 5 is devoted to the study, with examples, of the dynamic behavior of lumped-parameter systems. Virtually all electromechanical systems are mathematically nonlinear; nonetheless, linear incremental models are useful for studying the stability of equilibria and the nature of the dynamical behavior in the vicinity of an equilibrium. The second half of this chapter develops the classic potential-well motions and loss-dominated dynamics in the context of electromechanics. These studies of nonlinear dynamics afford an opportunity to place linear models in perspective while forming further insights on the physical significance of, for example, flux conservation and state functions.

Chapter 6 represents our first departure from lumped-parameter systems into continuum systems with a discussion of how observers in relative motion will define and measure field quantities and the related effects of material motion on electromagnetic fields. It is our belief that dc rotating machines are most easily understood in this context. Certainly they are a good demonstration of field transformations at work.

As part of any continuum electromechanics problem, one must know how the electric and magnetic fields are influenced by excitations and motion. In quasi-static systems the distribution of charge and current are controlled by magnetic diffusion and charge relaxation, the subjects of Chapter 7. In Chapter 7 simple examples isolate significant cases of magnetic diffusion or charge relaxation, so that the physical processes involved can be better understood.

Chapters 6 and 7 describe the electrical side of a continuum electromechanical system with the material motion predetermined. The mechanical side of the subject is undertaken in Chapter 8 in a study of force densities of electric and magnetic origin. Because it is a useful concept in the analysis of many systems, we introduce the Maxwell stress tensor. The study of useful properties of tensors sets the stage for later use of mechanical stress tensors in elastic and fluid media.

At this point the additional ingredient necessary to the study of continuum electromechanics is the mechanical medium. In Chapter 9 we introduce simple elastic continua—longitudinal motion of a thin rod and transverse motion of wires and membranes. These models are used to study simple continuum mechanical motions (nondispersive waves) as excited electromechanically at boundaries.

Next, in Chapter 10 a string or membrane is coupled on a continuum basis to electric and magnetic fields and the variety of resulting dynamic behavior is studied. The unifying thread of this treatment is the dispersion equation that relates complex frequency ω with complex wavenumber k . Without material convection there can be simple nondispersive waves, cut off or evanescent waves, absolute instabilities, and diffusion waves. The effect of material convection on evanescent waves and oscillations and on wave amplification are topics that make a strong connection with electron beam and plasma dynamics. The method of characteristics is introduced as a convenient tool in the study of wave propagation.

In Chapter 11 the concepts and techniques of Chapters 9 and 10 are extended to three-dimensional systems. Strain displacement and stress-strain relations are introduced, with tensor concepts, and simple electromechanical examples of three-dimensional elasticity are given.

In Chapter 12 we turn to a different mechanical medium, a fluid. We first study electromechanical interactions with inviscid, incompressible

fluids to establish essential phenomena in the simplest context. It is here that we introduce the basic notions of MHD energy conversion that can result when a conducting fluid flows through a transverse magnetic field. We also bring in electric-field interactions with fluids, in which ion drag phenomena are used as an example. In addition to these basically conducting processes, we treat the electromechanical consequences of polarization and magnetization in fluids. We demonstrate how highly conducting fluids immersed in magnetic fields can propagate Alfvén waves.

In Chapter 13 we introduce compressibility to the fluid model. This can have a marked effect on electromechanical behavior, as demonstrated with the MHD conduction machine. With compressibility, a fluid will propagate longitudinal disturbances (acoustic waves). A transverse magnetic field and high electrical conductivity modify these disturbances to magnetoacoustic waves.

Finally, in Chapter 14 we add viscosity to the fluid model and study the consequences in electromechanical interactions with steady flow. Hartmann flow demonstrates the effect of viscosity on the dc magnetohydrodynamic machine.

To be successful a text must have a theme; the material must be inter-related. Our philosophy has been to get into the subject where the student is most comfortable, with lumped-parameter (circuit) concepts. Thus many of the subtle approximations associated with quasi-statics are made naturally, and the student is faced with the implications of what he has assumed only after having become thoroughly familiar with the physical significance and usefulness of his approximations. By the time he reaches Chapter 4 he will have drawn a circle around at least a class of problems in which electromagnetic fields interact usefully with media in motion.

In dealing with physical and mathematical subjects, as we are here, in which the job is incomplete unless the student sees the physical laws put to work in some kind of physical embodiment, it is necessary for the thread of continuity to be woven into the material in diverse and subtle ways. A number of attempts have been made, to which we can add our early versions of notes, to write texts with one obvious, pedagogically logical basis for evolving the material; for example, it can be recognized that classes of physical phenomena could be grouped according to the differential equation that describes the pertinent dynamics. Thus we could treat magnetic diffusion, diffusion waves on elastic continua, and viscous diffusion waves in one chapter, even though the physical embodiments are entirely different. Alternatively, we could devise a subject limited to certain technological applications or cover superficially a wide range of basically unrelated topics such as "energy conversion" under one heading. This was the prevalent approach in engineering education a decade or so ago, even at the

undergraduate level. It seems clear to us that organizing material in a teachable and meaningful fashion is far more demanding than this. To confess our own mistakes, our material went originally from the general to the specific; it began with the relativistic form of Maxwell's equations, including the effects of motion, and ended with lumped-parameter devices as special cases. Even if this were a pedagogically tenable approach, which we found it was not, what a bad example to set for students who should be learning to distinguish between the essential and the superfluous! Ideas connected with the propagation of electromagnetic waves (relativistic ideas) must be included in the curriculum, but their connection with media in motion should be made after the student is aware of the first-order issues.

A meaningful presentation to *engineers* must interweave and interrelate mathematical concepts, physical characteristics, the modeling process, and the establishment of a physical "feel" for the world of reality. Our approach is to come to grips with each of these goals as quickly as possible (let the student "get wet" within the first two weeks) and then, while reinforcing what he has learned, continually add something new. Thus, if one looks, he will see the same ideas coming into the flow of material over and over again.

For the organization of this book one should look for many threads of different types. We can list here only a few, in the hope that the subtle reinforcing interplay of mathematical and physical threads will be made evident. Probably the essential theme is Maxwell's equations and the ideas of quasi-statics. The material introduced in Chapter 1 is completely abstract, but it is reinforced in the first few chapters with material that is close to home for the student. By the time he reaches Chapter 10 he will have learned that waves exist which intimately involve electric and magnetic fields that are altogether quasistatic. (This is something that comes as a surprise to many late in life.) Lumped-parameter ideas are based on the integral forms of Maxwell's equations, so that the dynamical effects found with lumped-parameter time constants L/R and RC in Chapter 5 are easily associated with the subjects of magnetic diffusion and charge relaxation. A close tie is made between the "speed voltage" of Chapter 5 and the effects of motion on magnetic fields, as described by field transformations in Chapters 6 to 14. Constant flux dynamics of a lumped coil in Chapter 5 are strongly associated with the dynamics of perfectly conducting continuous media; for example, Alfvén waves in Chapter 12.

Consider another thread of continuity. The book begins with the mathematics of circuit theory. The machines of Chapter 4 are essentially circuits in the sinusoidal steady state. In Chapter 5 we linearize to pursue lumped-parameter ideas of stability and other transient responses and then proceed to nonlinear dynamics, potential-well theory, and other approaches that should form a part of any engineer's mathematical background. By the time

the end of Chapter 10 is reached these ideas will have been carried into the continuum with the addition of tensor concepts, simple cases of the method of characteristics, and eigenvalue theory. The ω - k plot and its implication for all sorts of subjects in modern electrical engineering can be considered as a mathematical or a physical objective. The ideas of stability introduced with ordinary differential equations ($\exp st$) in Chapter 5 evolve into the continuum stability studies of Chapter 10 [$\exp j(\omega t - kx)$] and can be regarded as a mathematical or a physical thread in our treatment. We could list many other threads: witness the evolution of energy and thermodynamic notions from Chapters 3 to 5, 5 to 8, and 8 to 13.

We hope that this book is not just one more in the mathematics of electrical engineering or the technical aspects of rotating machines, transducers, delay lines, MHD converters, and so on, but rather that it is the mathematics, the physics, and, most of all, the engineering combined into one.

The material brought together here can be used in a variety of ways. It has been used by Professors C. N. Weygandt and F. D. Ketterer at the University of Pennsylvania for two subjects. The first restricts attention to Chapters 1 to 6 and Appendix B for a course in lumped-parameter electromechanics that both supplants the traditional one on rotating machines in the electrical engineering curriculum and gives the background required for further study in a second term (elective) covering Chapter 7 and beyond. Professors C. D. Hendricks and J. M. Crowley at the University of Illinois have used the material to follow a format that covers up through Chapter 10 in one term but omits much of the material in Chapter 7. Professor W. D. Getty at the University of Michigan has used the material to follow a one-term subject in lumped-parameter electromechanics taught from a different set of notes. Thus he has been able to use the early chapters as a review and to get well into the later chapters in a one-term subject.

At M.I.T. our curriculum seems always to be in a state of change. It is clear that much of the material, Chapters 1 to 10, will be part of our required (core) curriculum for the foreseeable future, but the manner in which it is packaged is continually changing. During the fall term, 1967, we covered Chapters 1 to 10 in a one-semester subject taught to juniors and seniors. The material from Chapters 4 and 6 on rotating machines was used selectively, so that students had "a foot solidly in the door" on this important subject but also that the coverage could retain an orientation toward the needs of all the diverse areas found in electrical engineering today. We have found the material useful as the basis for early graduate work and as a starting point in several courses related to electromechanics.

Finally, to those who open this book and then close it with the benediction, "good material but unteachable," we apologize because to them we have not made our point. Perhaps not as presented here, but certainly as it is

represented here, this material is rich in teaching possibilities. The demands on the teacher to see the subject in its total context, especially the related problems that lie between the lines, are significant. We have taught this subject many times to undergraduates, yet each term has been more enjoyable than the last. There are so many ways in which drama can be added to the material, and we do not need to ask the students (bless them) when we have been successful in doing so.

In developing this material we have found lecture demonstrations and demonstration films to be most helpful, both for motivation and for developing understanding. We have learned that when we want a student to see a particular phenomenon it is far better for us to do the experiment and let the student focus his attention on what he should see rather than on the wrong connections and blown fuses that result when he tries to do the experiment himself. The most successful experiments are often the simplest—those that give the student an opportunity to handle the apparatus himself. Every student should “chop up some magnetic field lines” with a copper “axe” or he will never really appreciate the subject. We have also found that some of the more complex demonstrations that are difficult and expensive to store and resurrect each semester come through very well in films. In addition to our own short films, three films have been produced professionally in connection with this material for the National Committee on Electrical Engineering Films, under a grant from the National Science Foundation, by the Education Development Center, Newton, Mass.

Synchronous Machines: Electromechanical Dynamics by H. H. Woodson

Complex Waves I: Propagation, Evanescence and Instability by J. R. Melcher

Complex Waves II: Instability, Convection and Amplification by J. R. Melcher

An additional film is in the early stages of production. Other films that are useful have been produced by the Education Development Center for the National Committee on Fluid Mechanics Films and for the College Physics Film Program. Of particular interest, from the former series, is *Magnetohydrodynamics* by Arthur Shercliff.

A book like this can be produced only with plenty of assistance. We gratefully acknowledge the help we received from many directions and hope we have forgotten no one after seven years of work. First of all we want to acknowledge our students with whom we worked as the material developed. They are the one most essential ingredient in an effort of this sort. Next we want to thank Dr. S. I. Freedman, Professor H. H. Richardson, and Dr. C. V. Smith, Jr., for their assistance in framing worthwhile approaches to several of our key topics. In seven years we have had the help of many able

teachers in presenting this material to students. Their discussions and advice have been most useful. In this category we want particularly to mention Professors H. A. Haus, P. L. Penfield, D. C. White, G. L. Wilson, R. Gallager, and E. Pierson and Doctors J. Reynolds, W. H. Heiser, and A. Kusko. Professor Ketterer, who has taught this material at M.I.T. and the University of Pennsylvania, Professors C. D. Hendricks and J. M. Crowley, who have taught it at M.I.T. and the University of Illinois, and Professor W. D. Getty, who has taught it at M.I.T. and the University of Michigan, have been most generous with their comments. Messrs. Edmund Devitt, John Dressler, and Dr. Kent Edwards have checked the correctness of many of the mathematical treatments. Such a task as typing a manuscript repeatedly is enough to try the patience of anyone. Our young ladies of the keyboard, Miss M. A. Daly, Mrs. D. S. Figgins, Mrs. B. S. Morton, Mrs. E. M. Holmes, and Mrs. M. Mazroff, have been gentle and kind with us.

A lengthy undertaking of this sort can be successful only when it has the backing of a sympathetic administration. This work was started with the helpful support of Professor P. Elias, who was then head of the Department of Electrical Engineering at M.I.T. It was finished with the active encouragement of Professor L. D. Smullin, who is presently head of the Department.

Finally, and most sincerely, we want to acknowledge the perseverance of our families during this effort. Our wives, Blanche S. Woodson and Janet D. Melcher, have been particularly tolerant of the demands of this work.

This book appears in three separately bound, consecutively paged parts that can be used individually or in any combination. Flexibility is ensured by including with each part a complete Table of Contents and Index. In addition, for convenient reference, Parts II and III are supplemented by brief appendices which summarize the relevant material from the preceding chapters. Part III, Chapters 11 to 14, introduces three-dimensional elasticity and fluid dynamics while emphasizing important electromechanical phenomena involving these mechanical models.

H. H. Woodson
J. R. Melcher

Cambridge, Massachusetts
January 1968

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Chapter 11

INTRODUCTION TO THE ELECTROMECHANICS OF ELASTIC MEDIA

11.0 INTRODUCTION

Most electromechanical effects and devices involving deformable solid media are best understood in terms of specific models. Elastic membranes and wires are examples of simple one-dimensional and two-dimensional models that can be developed without recourse to a more general theory of elasticity*. Any particular analysis can be developed in this way, and if our objective were to understand specific examples this would be our approach. A more general description of elastic media serves the purpose of giving a larger picture, with the specific examples placed in perspective. Our objective in this chapter is this larger picture. At the same time, examples which emphasize that special models still play an essential role are developed from the general theory.

The material of this chapter is, of course, relevant to the dynamics of a variety of electromechanical interactions with elastic solids. In particular, it relates to areas of electromechanics such as physical acoustics, the microwave electromechanics of crystals, and the development of electromechanical distributed circuits.

The one-dimensional example of a rod subject to longitudinal motions introduced in Section 9.1* illustrates the essential steps required to find the equations of motion for an elastic continuum. First, an equation expressing force equilibrium for a small volume of material is written in terms of the material displacement and the mechanical stresses. Then the stresses are related to the strain, hence to the displacement, by means of the stress-strain relations. The first step, which can be completed without regard to the

* See Chapter 9 or Table 9.2, Appendix G.

elastic properties of the material, is given in the next section. The second step, for isotropic media, is found in Section 11.2. In Section 11.5 we introduce enough of the properties of anisotropic materials to consider certain illustrative examples.

11.1 FORCE EQUILIBRIUM

Our first objective is to write an equation that expresses force equilibrium for an element of material. A small cube of the material, centered at (x_1, x_2, x_3) , is shown in Fig. 11.1.1. To write Newton's law we must know the instantaneous acceleration of the material at this point. The particles of matter found at (x_1, x_2, x_3) have a displacement from their unstressed position given by

$$\delta(x_1 - \delta_1, x_2 - \delta_2, x_3 - \delta_3, t). \quad (11.1.1)$$

This is true because $\delta(a, b, c, t)$ is defined as the displacement of the particle with an unstressed position (a, b, c) .^{*} As pointed out in Section 9.1, displacements are small in many important situations. If we limit ourselves to small displacements, (11.1.1) can be expanded about the position (x_1, x_2, x_3) to give

$$\begin{aligned} \delta(x_1 - \delta_1, x_2 - \delta_2, x_3 - \delta_3, t) \\ = \delta(x_1, x_2, x_3, t) - \frac{\partial \delta}{\partial x_1} \delta_1 - \frac{\partial \delta}{\partial x_2} \delta_2 - \frac{\partial \delta}{\partial x_3} \delta_3 + \cdots \end{aligned} \quad (11.1.2)$$

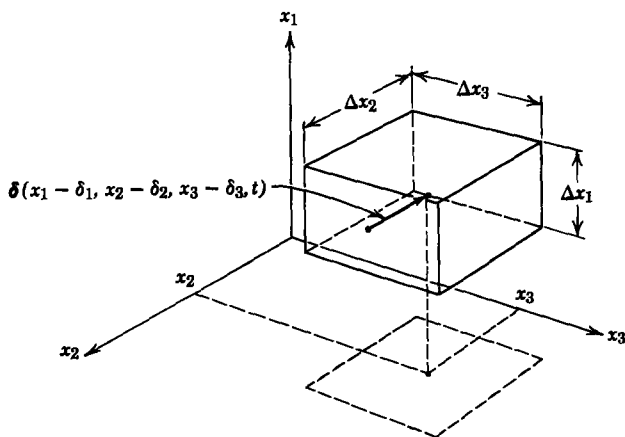


Fig. 11.1.1 Element of material with volume V centered at the position (x_1, x_2, x_3) . The grain of material at the center of the cube has the unstressed position $(x_1 - \delta_1, x_2 - \delta_2, x_3 - \delta_3)$.

^{*} A function such as δ , which follows the motion of a particular particle, is said to be written in Lagrangian coordinates. The coordinates indicate the particle under consideration.

If the displacement and its derivatives are small, we can approximate the displacement of the material at the center of the element (Fig. 11.1.1) by the first term on the right, which is the displacement evaluated at the center of the element. Newton's law for the small cube of Fig. 11.1.1 is then

$$\rho \frac{\partial^2 \delta}{\partial t^2} = \mathbf{F}. \quad (11.1.3)$$

Because we ignore products of perturbation quantities, the mass density ρ is a constant in this expression. In an elastic solid there is always a force density $\mathbf{F}$ due to mechanical stresses imposed on the cube by the surrounding material. In the presence of electric or magnetic fields an additional contribution to $\mathbf{F}$ is made by forces of electrical origin. In the next section we develop the relation between elastic stresses and the displacement δ (for homogeneous media) as a stress tensor, and in Chapter 8 it was found that forces due to free charges or free currents could be written as the divergence of a stress tensor*. Hence we can write (11.1.3) as

$$\rho \frac{\partial^2 \delta_m}{\partial t^2} = \frac{\partial T_{mn}}{\partial x_n}, \quad (11.1.4)$$

where it is understood that the stress tensor T_{mn} is due to mechanical (elastic) interactions and (if they are present in the problem) electrical interactions.

11.2 EQUATIONS OF MOTION FOR ISOTROPIC MEDIA

From (11.1.4) it is apparent that to formulate the equations of motion for an elastic medium it is necessary to make a connection between the applied stresses and the resulting deformations. It was shown in Section 9.1 that for a simple, one-dimensional problem this could be done by introducing the strain, which has a simple relationship both with the deformation of the material and the applied stresses. In the section that follows we consider how the strain gives a description of material deformation, hence the relation between the strain and the displacement of the material. Then in Section 11.2.2 a description is given of the relation between the stress and strain.

11.2.1 Strain-Displacement Relations

Two types of material deformation corresponding to the action of normal and shear stresses on the material, are possible. These are illustrated two-dimensionally in Fig. 11.2.1, in which the points A , B , and C represent tagged grains of material that, when strained, move from A to A' , B to B' , and C to C' . Note that in the unstrained condition the lines joining the tagged points are parallel to the coordinate axes. Because the effects considered are linear,

* See Sec. 8.2 or Appendix G.

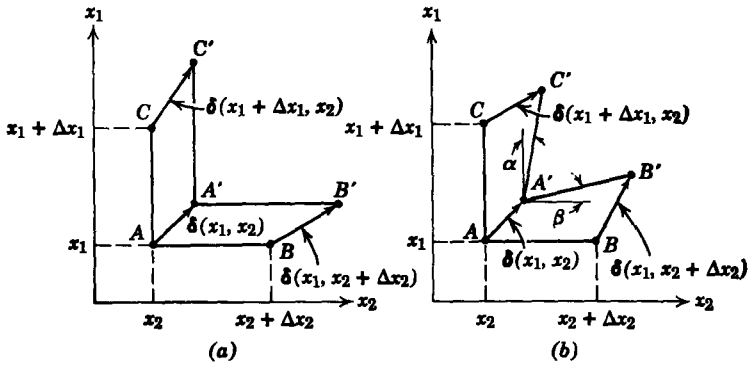


Fig. 11.2.1 (a) Strain produced by normal stresses; (b) strain produced by shear stresses.

the deformations can be considered separately and then superimposed. We shall shortly develop the strain from a formal point of view, and this will make the superposition principle more apparent. At first our development is more intuitive.

In Fig. 11.2.1a the deformation has lengthened the distances AB and AC to the distances $A'B'$ and $A'C'$, whereas in Fig. 11.2.1b these lengths have remained constant but the angles between the sides of the cube of material have changed. The first of the two types of deformation is a generalization of the kind of strain considered in Section 9.1 (the thin elastic rod); that is, we can *define* a normal strain in each of the axis directions as an elongation per unit length. In the x_1 -direction this is

$$e_{11} = \lim_{\Delta x_1 \rightarrow 0} \frac{\delta_1(x_1 + \Delta x_1, x_2) - \delta_1(x_1, x_2)}{\Delta x_1} = \frac{\partial \delta_1}{\partial x_1}, \quad (11.2.1)$$

where $\delta(x_1, x_2, x_3, t)$ is the material displacement at the point x_1, x_2, x_3 , as discussed in Section 11.1. Similarly,

$$e_{22} = \frac{\partial \delta_2}{\partial x_2}, \quad (11.2.2)$$

$$e_{33} = \frac{\partial \delta_3}{\partial x_3}. \quad (11.2.3)$$

In the second kind of deformation (Fig. 11.2.1b) the sides of the cube keep their original length but are deflected with respect to each other to an angle different from the original 90° . This strain is caused by the shear stresses (T_{12} , T_{31} , etc.) and can be visualized by placing the covers of a book under shear, as shown in Fig. 11.2.2. (The significance of the stress components is discussed in Section 8.2 and Appendix G.)

The shear-strains, like the normal strains, are *defined* functions. They are defined as one half the tangent of the change in angle between the originally perpendicular sides of the cube (Fig. 11.2.1b) in the limit in which the cube becomes very small. Hence in the diagram of Fig. 11.2.1b the strain resulting from a change in angle with respect to the x_1 and x_2 axes is designated e_{12} .

In terms of the angles defined in Fig. 11.2.1b the strain e_{12} is

$$e_{12} = \lim_{\substack{\Delta x_1 \rightarrow 0 \\ \Delta x_2 \rightarrow 0}} \frac{1}{2} \tan [\alpha + \beta] \quad (11.2.4)$$

Note that a positive shear strain signifies that the angle between the originally perpendicular lines is less than 90° .

In Chapter 9, Example 9.1.1 was used to illustrate that the deformations commonly encountered in elastic solids are very small. For this reason the angles of deflection due to shear stresses are also commonly small and the tangent function in (11.2.4) can be approximated by the argument $(\alpha + \beta)$. For the same reason the angles α and β can in turn be approximated by their tangents to write

$$\begin{aligned} \alpha &\approx \frac{[\delta_2(x_1 + \Delta x_1, x_2) - \delta_2(x_1, x_2)]}{\Delta x_1}, \\ \beta &\approx \frac{[\delta_1(x_1, x_2 + \Delta x_2) - \delta_1(x_1, x_2)]}{\Delta x_2}, \end{aligned} \quad (11.2.5)$$

(11.2.4) then becomes

$$e_{12} = \lim_{\substack{\Delta x_1 \rightarrow 0 \\ \Delta x_2 \rightarrow 0}} \frac{1}{2} \left\{ \frac{[\delta_2(x_1 + \Delta x_1, x_2) - \delta_2(x_1, x_2)]}{\Delta x_1} + \frac{[\delta_1(x_1, x_2 + \Delta x_2) - \delta_1(x_1, x_2)]}{\Delta x_2} \right\}, \quad (11.2.6)$$

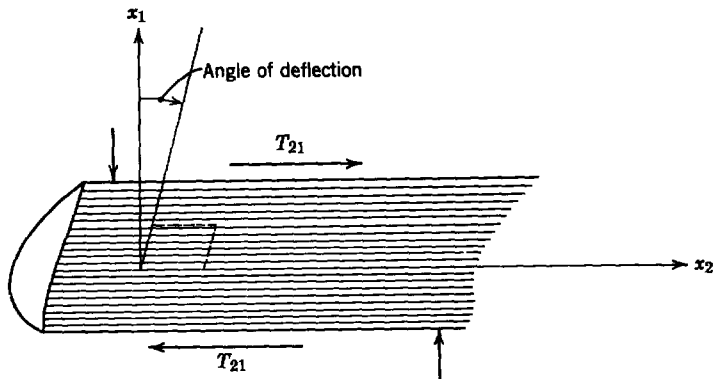


Fig. 11.2.2 Simple situation in which shear stresses result in a shear strain (the angle of deflection).

and in the limit we obtain a point relation between the component of strain e_{12} and the displacements:

$$e_{12} = \frac{1}{2} \left(\frac{\partial \delta_1}{\partial x_2} + \frac{\partial \delta_2}{\partial x_1} \right); \quad (11.2.7)$$

that is, e_{12} is evaluated at the unstressed position A . In the process of taking the limit in which $\Delta x_1 \rightarrow 0$ and $\Delta x_2 \rightarrow 0$ all of the approximations in going from (11.2.4) to (11.2.7) become exact except the one requiring small angular deflections, which remains the basic limitation of the theory.

A two-dimensional picture of shear strain has been used so far because it is easily visualized. A three-dimensional description of the strain follows by considering the deflections between the other two pairs of axes, with the results

$$e_{13} = \frac{1}{2} \left(\frac{\partial \delta_1}{\partial x_3} + \frac{\partial \delta_3}{\partial x_1} \right), \quad (11.2.8)$$

$$e_{23} = \frac{1}{2} \left(\frac{\partial \delta_2}{\partial x_3} + \frac{\partial \delta_3}{\partial x_2} \right). \quad (11.2.9)$$

The last three expressions make it evident that $e_{ij} = e_{ji}$, as would be expected from the definition of the shear strain. Altogether, we have defined nine components of the strain, which are presumably sufficient to describe the distortions of the material in the vicinity of a point x_1, x_2, x_3 related to the stress. We can summarize all the strain components with the expression

$$e_{ij} = \frac{1}{2} \left(\frac{\partial \delta_i}{\partial x_j} + \frac{\partial \delta_j}{\partial x_i} \right). \quad (11.2.10)$$

Note that e_{ij} is a Lagrangian variable in that it is evaluated at the unstressed position of a grain of material (A in Fig. 11.2.1). As discussed in Section 11.1, because the deflections are small it will not be necessary to distinguish between the strain evaluated at the unstressed position of the material and the strain evaluated at the stressed position of the material.

11.2.1a A Formal Derivation

Our remarks about the strain have so far been aimed at establishing the physical significance of each of the nine components. We have not shown in a formal way that the components e_{ij} are sufficient to describe the relative distortions of the material caused by the stresses (although e_{ij} does include all possible space derivatives of δ). To do so we must consider a general material deformation rather than the two particular forms shown in Fig. 11.2.1. The reasoning used now parallels that of the preceding discussion in that we again consider the relative positions of grains of matter.

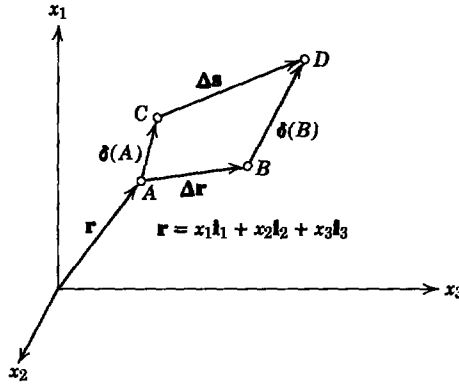


Fig. 11.2.3 Displacement of grains of matter in the unstressed positions A and B and final separation Δs .

Figure 11.2.3 shows two grains of material in the unstrained positions A and B . When the material is subjected to stress, these grains move to the new positions C and D . We are interested in that part of the material distortion that is produced by the stress. Hence we are not interested (when we define the strain) in a uniform translation of the material, or, as we shall see momentarily, we are not interested in uniform rotations of the material. The strain is defined to describe the stretching of the material between the points A and B ; hence attention is given to the distance between the points originally at A and B , as they are distorted to points C and D . At first the relative displacement is $\Delta \mathbf{r}$, whereas after the stress is applied the relative displacement is $\Delta \mathbf{s}$, as shown in Fig. 11.2.3.

The coordinate of the material at A is $\mathbf{r}$. By taking $\Delta \mathbf{r}$ ($\Delta \mathbf{r} = \Delta x_1 \mathbf{i}_1 + \Delta x_2 \mathbf{i}_2 + \Delta x_3 \mathbf{i}_3$) to be small it is possible to find the separation Δs between the points in the strained positions C and D . First, vector addition of the displacement components shown in Fig. 11.2.3 gives

$$\Delta \mathbf{s} = \Delta \mathbf{r} + \delta(B) - \delta(A), \quad (11.2.11)$$

which becomes approximately (writing the i th component)*

$$\begin{aligned} \Delta s_i &= \Delta x_i + \delta_i(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3, t) - \delta_i(x_1, x_2, x_3, t) \\ &\cong \Delta x_i + \frac{\partial \delta_i}{\partial x_1} \Delta x_1 + \frac{\partial \delta_i}{\partial x_2} \Delta x_2 + \frac{\partial \delta_i}{\partial x_3} \Delta x_3 \\ &\equiv \Delta x_i + \frac{\partial \delta_i}{\partial x_k} \Delta x_k. \end{aligned} \quad (11.2.12)$$

* See Appendix G for index notation.

Now, if the quantity $\frac{1}{2}(\partial\delta_k/\partial x_i) \Delta x_k$ is both added to and subtracted from this equation, we obtain

$$\Delta s_i = \Delta x_i + \frac{1}{2}\left(\frac{\partial\delta_i}{\partial x_k} - \frac{\partial\delta_k}{\partial x_i}\right) \Delta x_k + \frac{1}{2}\left(\frac{\partial\delta_i}{\partial x_k} + \frac{\partial\delta_k}{\partial x_i}\right) \Delta x_k. \quad (11.2.13)$$

Here we have an expression for the i th component of the directed distance $\Delta \mathbf{s}$ between the points A and B after the material has been subjected to stress. This distance is written as a function of the initial distance $\Delta \mathbf{r}$ and the displacement $\boldsymbol{\delta}$ of the material in the vicinity of A . Note that in (11.2.13) the derivatives of $\boldsymbol{\delta}$ are evaluated at the unstressed position A .

In writing (11.2.13), we have divided the expression for the relative positions of A and B into a part that is due to a pure (rigid body) rotation of the material in the vicinity of A and a part resulting because of the material distortions produced by the applied stress. We have already agreed that a pure translation, and similarly a pure rotation, involve no strain deformation in the material. The bracketed part of the second term on the right in (11.2.13) is one component of the vector $\nabla \times \boldsymbol{\delta}$. Hence it describes a rotation of the material about the unstressed position A . This may be verified by defining a rotation vector $\boldsymbol{\Omega}$ in terms of the components of $\boldsymbol{\delta}$.

$$\boldsymbol{\Omega} = \frac{1}{2}\left(\frac{\partial\delta_3}{\partial x_2} - \frac{\partial\delta_2}{\partial x_3}\right)\mathbf{i}_1 + \frac{1}{2}\left(\frac{\partial\delta_1}{\partial x_3} - \frac{\partial\delta_3}{\partial x_1}\right)\mathbf{i}_2 + \frac{1}{2}\left(\frac{\partial\delta_2}{\partial x_1} - \frac{\partial\delta_1}{\partial x_2}\right)\mathbf{i}_3. \quad (11.2.14)$$

Then the second term in (11.2.13) can be written in terms of $\boldsymbol{\Omega}$ as

$$\frac{1}{2}\left(\frac{\partial\delta_i}{\partial x_k} - \frac{\partial\delta_k}{\partial x_i}\right) \Delta x_k = (\boldsymbol{\Omega} \times \Delta \mathbf{r})_i. \quad (11.2.15)$$

This is not obvious unless one substitutes (11.2.14) into (11.2.15).

Without specifying the direction of $\boldsymbol{\Omega}$, we can conclude from (11.2.15) that since $\boldsymbol{\Omega} \times \Delta \mathbf{r}$ is perpendicular to $\Delta \mathbf{r}$ the relative displacement represented by this term is also perpendicular to $\Delta \mathbf{r}$. In Fig. 11.2.4 the stressed and unstressed positions of the material are shown in the case in which the contribution of the last term in (11.2.13) is zero. The material initially at points A and B undergoes a uniform translation and then, because $\boldsymbol{\Omega}$ is perpendicular to $\Delta \mathbf{r}$, a rigid body rotation.

It is apparently the last term in (11.2.13) that represents a distortion of the material and therefore should be defined as the strain. This is, of course, consistent with our definition of the strain e_{ij} in the preceding section (11.2.10). In view of this definition, (11.2.13) provides an expression for the relative positions of two material particles with the initial relative positions $\Delta \mathbf{r}$.

$$\Delta s_i = \Delta x_i + (\boldsymbol{\Omega} \times \Delta \mathbf{r})_i + e_{ij} \Delta x_j. \quad (11.2.16)$$

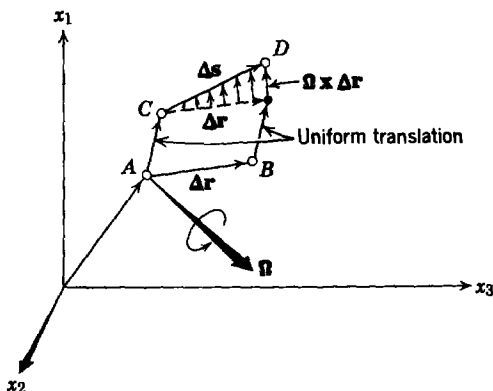


Fig. 11.2.4 Displacements of grains of matter in the undeformed positions A and B and deformed positions C and D . This material has suffered a uniform translation and a rigid body rotation.

The strain e_{ij} is evaluated at the unstressed position of the material. As mentioned in Section 11.2.1, however, we commit a negligible error by evaluating it at the stressed position of the material.

The strain has been defined in such a way that it has the transformation properties of a tensor. This purely mathematical fact is shown in the next section.

11.2.1b The Strain as a Tensor

We can confirm that the components e_{ij} form a tensor by using the fact that the displacement δ is a vector. Our discussion here parallels that given in Section 8.2.2 in which we used the transformation properties of the vector traction τ to show that the stress was a tensor. In a similar way we begin here with the transformation of δ to a primed coordinate system:

$$\delta'_i = a_{il}\delta_l. \quad (11.2.17)$$

A discussion of this vector transformation and the direction cosines a_{il} was given in Section 8.2.2 and is summarized in Appendix G.

It follows from (11.2.17) that since components of a_{il} are not functions of x_i

$$\frac{\partial \delta'_j}{\partial x'_j} = a_{il} \frac{\partial \delta_l}{\partial x'_j} = a_{il} \frac{\partial x_k}{\partial x'_j} \frac{\partial \delta_l}{\partial x_k}. \quad (11.2.18)$$

The position vector $\mathbf{r} = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3$ is also transformed from a primed coordinate system by an equation in the form of [see (8.2.18) and (8.2.19)]

$$x_k = a_{jk}x'_j; \quad (11.2.19)$$

from this expression it follows that

$$\frac{\partial x_k}{\partial x'_j} = a_{jk} \quad (11.2.20)$$

and (11.2.18) becomes

$$\frac{\partial \delta'_i}{\partial x'_j} = a_{il} a_{jk} \frac{\partial \delta_l}{\partial x_k}. \quad (11.2.21)$$

The steps leading to this equation can be repeated with the indices i and j reversed.

The strain e'_{ij} in the primed coordinate system is by definition

$$e'_{ij} = \frac{1}{2} \left(\frac{\partial \delta'_i}{\partial x'_j} + \frac{\partial \delta'_j}{\partial x'_i} \right). \quad (11.2.22)$$

The first derivative on the right in this expression is replaced by (11.2.21), whereas the second derivative is replaced by (11.2.21), with i and j reversed:

$$e'_{ij} = \frac{1}{2} \left(a_{il} a_{jk} \frac{\partial \delta_l}{\partial x_k} + a_{jl} a_{ik} \frac{\partial \delta_l}{\partial x_k} \right). \quad (11.2.23)$$

We are required to sum over the indices l and k , and so these indices can be reversed in the second term on the right-hand side of this expression, which then becomes the desired transformation equation for e_{ij} :

$$e'_{ij} = a_{ik} a_{jl} e_{kl}. \quad (11.2.24)$$

This expression for the transformation of the strain is the same as that found in Section 8.2.2 for the transformation of the stress T_{ij} ; for example, the expressions for the components of stress in a cylindrical coordinate system, as derived in Example 8.2.5, could also be used here by replacing $T_{ij} \rightarrow e_{ij}$.

11.2.2 Stress-Strain Relations

Our objective in defining the strain was to provide a function of the material displacements δ that could be directly related to the stress. The relations between the components of strain and stress depend on material properties. A particular stress-strain relation was discussed in Section 9.1, in which the modulus of elasticity was introduced as an experimentally determined constant of proportionality between a one-dimensional normal stress and strain*. In this section we generalize this simple stress-strain relation to three-dimensional isotropic solids but again confine ourselves to those solids that can be modeled by an algebraically linear dependence of strain on the stress. This is not unduly restrictive because virtually all elastic media of interest to us are represented adequately by this model. In any case the stress-strain relations are ultimately empirical. Therefore it is reasonable to propose several simple experiments that lead to an understanding of them.

* See Table 9.2, Appendix G.

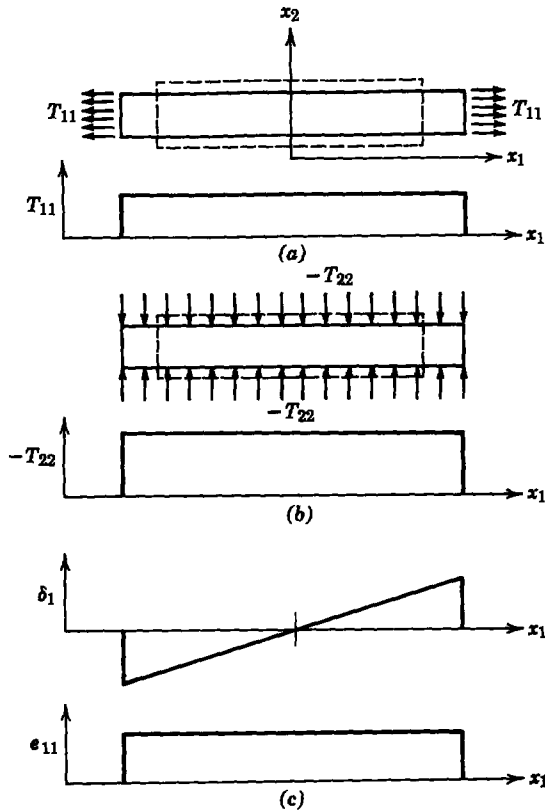


Fig. 11.2.5 (a) Deformations of a block due to a uniform normal stress T_{11} ; (b) deformations identical to (a) produced by a uniform normal stress $-T_{22}$; (c) displacement and normal strain in the x_1 -direction for both (a) and (b).

First, the effect of applying a normal stress to a block of material is considered. In Fig. 11.2.5 two ways of applying a normal stress are shown. Both can result in the same deformation of the material. A normal stress T_{11} is applied to the x_1 -surfaces of the block in Fig. 11.2.5a; the result is elongation of the block in the x_1 -direction. If no stresses are applied to the x_2 (or x_3)-surfaces, there is, in addition, a contraction of the material in the x_2 (and x_3)-directions. This “necking down” of the material is familiar to anyone who has observed what happens when a rubber band is stretched.

Figure 11.2.5b shows the same deformation of the bar as in Fig. 11.2.5a, except that the stress is now compressional and normal to the x_2 -surfaces. Here the material is “squeezed out” in the x_1 -directions by the stress T_{22} , which also reduces the thickness of the block. The x_1 -displacement $\delta_1(x_1)$ of the block in each situation is shown in Fig. 11.2.5c. We see that e_{11} is

uniform over the length of the bar. In Fig. 11.2.5 the block is assumed to be constrained so that there are no displacements in the x_3 -direction. Of course, in three-dimensional displacements a stress T_{11} could also produce displacements in the x_3 -direction and a stress T_{33} could produce displacements in the x_1 -direction.

To account for this experiment two constants E and ν are defined such that

$$e_{11} = \frac{1}{E} [T_{11} - \nu(T_{22} + T_{33})]. \quad (11.2.25)$$

This equation provides that a negative T_{22} can produce the same strain e_{11} as a positive T_{11} . The x_3 -direction is equivalent to the x_2 -direction in our experiment, hence T_{33} enters in (11.2.25) in the same way as T_{22} .

Because the material is isotropic, we can make the same arguments for the other components of the strain and write

$$e_{22} = \frac{1}{E} [T_{22} - \nu(T_{33} + T_{11})], \quad (11.2.26)$$

$$e_{33} = \frac{1}{E} [T_{33} - \nu(T_{11} + T_{22})]. \quad (11.2.27)$$

As pointed out in Chapter 9, E is called the modulus of elasticity, or Young's modulus, and (11.2.25) reduces to the stress-strain relation for a thin rod by setting $T_{22} = T_{33} = 0$. We comment further on the significance of this approximation in Section 11.4. The constant ν , which accounts for the necking down of the material in Fig. 11.2.5a, is called *Poisson's ratio*. Materials that are isotropic, hence could possibly be modeled by (11.2.25) to (11.2.27), are usually a conglomeration of minute crystals. Although each individual crystal is not isotropic, the conglomeration is isotropic on a macroscale. The physical properties of such materials are extremely difficult to predict. For this reason E and ν may be regarded as experimentally determined constants.

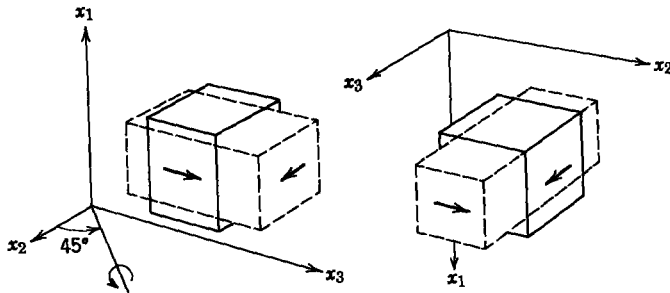


Fig. 11.2.6 Hypothetical situation in which a normal strain results from a shear stress.

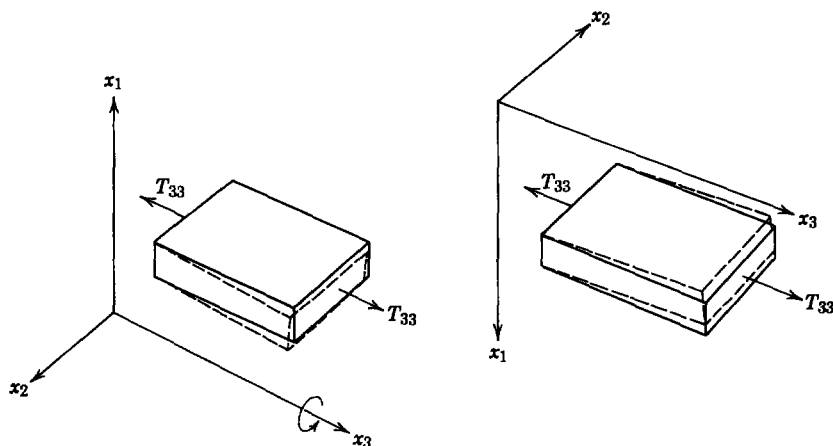


Fig. 11.2.7 Hypothetical situation in which a shear strain results from a normal stress.

In writing (11.2.25) to (11.2.27), we have not only assumed that the material is isotropic but that the normal strains do not depend on the shear stresses. A simple mental experiment shows that these assumptions are the same. Suppose a situation occurs in which normal strain results from a shear stress, as shown in Fig. 11.2.6. A rotation of the coordinates makes it evident that the same stress would give a very different strain, a result that contradicts our assumption of an isotropic material (a material with properties that do not depend on the orientation of the coordinates relative to the material).

This same kind of isotropy argument can be used to show that shear strains cannot depend on normal stresses. Now the conjecture is that we have shear strains that result from normal stresses, as shown in Fig. 11.2.7. Again, a rotation of the coordinate system as shown requires that the same normal stress produce the opposite shear strain.

Physical intuition tells us that each shear strain should be proportional to the corresponding shear stress. As an example, Fig. 11.2.8 shows a block of material subject to the shear stresses T_{13} and T_{31} . The change in angle

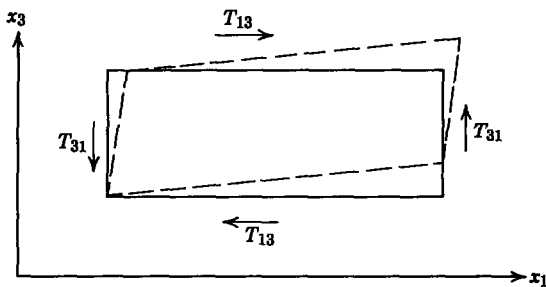


Fig. 11.2.8 The shear strain e_{13} results from the shear stresses T_{13} and T_{31} .

between the originally perpendicular sides of the block is in direct proportion to the applied stress. Hence

$$e_{ij} = \frac{T_{ij}}{2G}, \quad i \neq j, \quad (11.2.28)$$

where G is an experimentally determined constant called the *shear modulus*. In drawing Fig. 11.2.8 we have assumed that $T_{13} = T_{31}$, for otherwise there would be a net torque on the material. This assumption is implicit in (11.2.8), for we have already shown that $e_{ij} = e_{ji}$ (11.2.10).

Given the stress, we can use (11.2.25) to (11.2.28) to find any component of the strain. If, however, we made independent measurements of ν , E , and G , we would be expending more effort than necessary, since, in fact, these constants are related. An example illustrates this point.

Example 11.2.1. Figure 11.2.9 shows a cube of material that is subject to the shear stresses $T_{12} = T_{21} = T_0$ in the x_1, x_2, x_3 -coordinate system. It is clear from this diagram that the components of the stress, viewed from the x'_1, x'_2, x'_3 -coordinate system, are not in

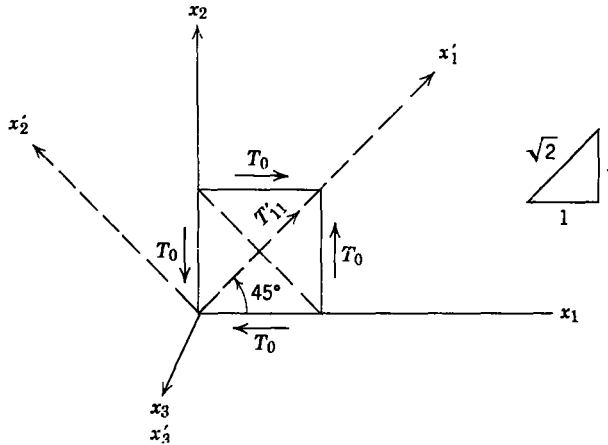


Fig. 11.2.9 A simple example of a pure shear in the x_j frame which transforms into a pure tension and compression in the x'_i frame.

shear but in tension and compression. It is because of this fact that E , ν , and G are not independent constants. A calculation of the strain e'_{ij} , viewed in the x'_i -frame serves to illustrate this point.

The x'_i -coordinates of a point in space can be found from the x_j -coordinates by the transformation $x'_i = a_{ij}x_j$ (Section 8.2.2 or Appendix G), in which

$$a_{ij} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (a)$$

We are given that the material supports the stress

$$T_{ij} = \begin{bmatrix} 0 & T_0 & 0 \\ T_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (b)$$

hence from (11.2.28) that the material undergoes the strain

$$e_{ij} = \begin{bmatrix} 0 & \frac{T_0}{2G} & 0 \\ \frac{T_0}{2G} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (c)$$

The advantage of writing the stress and strain as tensors is that their components can be found in the x'_i -frame by means of the transformations $T'_{ij} = a_{ik}a_{jl}T_{kl}$ and $e'_{ij} = a_{ik}a_{jl}e_{kl}$. Hence from the last three equations it follows that

$$e'_{ij} = \begin{bmatrix} \frac{T_0}{2G} & 0 & 0 \\ 0 & -\frac{T_0}{2G} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (d)$$

and

$$T'_{ij} = \begin{bmatrix} T_0 & 0 & 0 \\ 0 & -T_0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (e)$$

or, as we suspected, the components of the stress and strain are purely diagonal. Because the material is isotropic, the stress-strain relations must hold, regardless of the coordinate system; that is, (11.2.25) to (11.2.28) must also hold for e'_{ij} and T'_{ij} , and it follows from the above equations that

$$\frac{T_0}{2G} = \frac{1}{E} (T_0 + \nu T_0) \quad (f)$$

or

$$G = \frac{E}{2(1 + \nu)}. \quad (g)$$

This result is important, for it indicates that there are only two independent constants necessary to define the stress-strain relations for an isotropic material; for example, given the modulus of elasticity E and Poisson's ratio ν , the shear modulus G can be found from (g).

Characteristic values of ν and G are given in Table 11.2.1.

Table 11.2.1 Shear Modulus and Poisson's Ratio for Various Materials*

Material	G -units of 10^{10} N/m^2	ν
Aluminum (pure and alloy)	2.55–3.65	0.32–0.34
Brass (60–70% Cu, 40–30% Zn)	3.6–4.1	0.33–0.36
Copper	4.0–4.6	0.33–0.36
Iron, cast (2.7–3.6% C)	3.6–5.6	0.21–0.30
Steel (carbon and low alloy)	7.6–8.2	0.26–0.29
Stainless steel (18% Cr, 8% Ni)	7.3	0.30
Titanium (pure and alloy)	4.1	0.34
Glass	2.6–3.2	0.21–0.27

* See Table 9.1 Appendix G for references and values of E , ρ , and $\sqrt{E/\rho}$.

11.2.3 Summary of Equations

We shall be occupied with electromechanical problems in which the stress T_{ij} and the displacement δ_i are the important variables. Therefore it is desirable to eliminate the strain as a variable from (11.2.25) through (11.2.28).

For the off-diagonal terms this leads directly to the stress as a function of the displacement, but for the diagonal terms three simultaneous conditions on the components T_{11} , T_{22} , and T_{33} result:

$$\begin{aligned}
 e_{11} &= \frac{1}{E} [T_{11} - \nu(T_{22} + T_{33})] = \frac{\partial \delta_1}{\partial x_1}, \\
 e_{22} &= \frac{1}{E} [T_{22} - \nu(T_{33} + T_{11})] = \frac{\partial \delta_2}{\partial x_2}, \\
 e_{33} &= \frac{1}{E} [T_{33} - \nu(T_{11} + T_{22})] = \frac{\partial \delta_3}{\partial x_3}.
 \end{aligned} \tag{11.2.29}$$

These equations can be solved for T_{11} , T_{22} , and T_{33} in terms of the derivatives of δ to provide the diagonal terms in the expression (remember, we sum on a subscript that appears twice),

$$T_{ij} = \begin{bmatrix} 2G \frac{\partial \delta_1}{\partial x_1} + \lambda \frac{\partial \delta_k}{\partial x_k} & G \left(\frac{\partial \delta_1}{\partial x_2} + \frac{\partial \delta_2}{\partial x_1} \right) & G \left(\frac{\partial \delta_1}{\partial x_3} + \frac{\partial \delta_3}{\partial x_1} \right) \\ G \left(\frac{\partial \delta_2}{\partial x_1} + \frac{\partial \delta_1}{\partial x_2} \right) & 2G \frac{\partial \delta_2}{\partial x_2} + \lambda \frac{\partial \delta_k}{\partial x_k} & G \left(\frac{\partial \delta_2}{\partial x_3} + \frac{\partial \delta_3}{\partial x_2} \right) \\ G \left(\frac{\partial \delta_3}{\partial x_1} + \frac{\partial \delta_1}{\partial x_3} \right) & G \left(\frac{\partial \delta_3}{\partial x_2} + \frac{\partial \delta_2}{\partial x_3} \right) & 2G \frac{\partial \delta_3}{\partial x_3} + \lambda \frac{\partial \delta_k}{\partial x_k} \end{bmatrix}, \tag{11.2.30}$$

where (g) of Example 11.2.1 has been used to reduce the number of constants to two and λ is an elastic constant given by

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}. \quad (11.2.31)$$

The off-diagonal terms in 11.2.30 are given by (11.2.28). The parameter λ has been introduced purely for convenience and is sometimes called the Lamé constant.

The expression for the stress in terms of the displacements, given by (11.2.30), can be summarized in a compact form by using the Kronecker delta function introduced in Section 8.1.

$$T_{ij} = G \left(\frac{\partial \delta_i}{\partial x_j} + \frac{\partial \delta_j}{\partial x_i} \right) + \lambda \delta_{ij} \frac{\partial \delta_k}{\partial x_k}. \quad (11.2.32)$$

Remember that $\delta_{ij} = 0$ for $i \neq j$ [the off-diagonal terms in (11.2.30)], whereas $\delta_{ij} = 1$ if $i = j$ [the diagonal terms in (11.2.30)].

If there are no other forces acting on the elastic material besides the force arising from the elastic stresses given by (11.2.32), we shall have completed the task of finding the equations of motion for the material; that is, the stress given by (11.2.32) can be substituted into the force equation (11.1.4) to provide one vector equation for δ . In this equation the force density of elastic origin is $F_i = \partial T_{ij} / \partial x_j$. It is often more convenient to write the force density in vector notation. The following manipulations illustrate the use of tensor notation.

First, we simply write out the tensor divergence of (11.2.32):

$$F_i = \frac{\partial T_{ij}}{\partial x_j} = G \left(\frac{\partial^2 \delta_i}{\partial x_j \partial x_j} + \frac{\partial^2 \delta_j}{\partial x_i \partial x_j} \right) + \lambda \delta_{ij} \frac{\partial^2 \delta_k}{\partial x_j \partial x_k}. \quad (11.2.33)$$

The first term on the right will be recognized as $G \nabla^2 \delta$, the second is the i th component of $G \nabla (\nabla \cdot \delta)$ and the last has value only when $i = j$ so that it is the i th component of $\lambda \nabla (\nabla \cdot \delta)$. Hence we can write (in vector notation)

$$\mathbf{F} = G \nabla^2 \delta + (G + \lambda) \nabla (\nabla \cdot \delta). \quad (11.2.34)$$

It must be remembered that $\nabla^2 \delta$ is a vector Laplacian defined by $\nabla^2 \delta = \nabla (\nabla \cdot \delta) - \nabla \times (\nabla \times \delta)$, so that (11.2.34) can also be written

$$\mathbf{F} = (2G + \lambda) \nabla (\nabla \cdot \delta) - G \nabla \times (\nabla \times \delta). \quad (11.2.35)$$

This is a useful form of the force density because the material displacements leading to $\nabla \cdot \delta$ and $\nabla \times \delta$ are easily visualized. We defer this point until Section 11.4.

The elastic forces, represented by (11.2.35), are held in dynamical equilibrium by other forces that act on the material. As pointed out in Section 11.1,

Table 11.2.2 Equations Which Describe the Motions of Isotropic Perfectly Elastic Media*Force equation*

$$\rho \frac{\partial^2 \delta_i}{\partial t^2} = \frac{\partial T_{ij}}{\partial x_j} + (F_{\text{ex}})_i \quad (\text{tensor form}) \quad (11.1.4)$$

$$\rho \frac{\partial^2 \delta}{\partial t^2} = (2G + \lambda) \nabla(\nabla \cdot \delta) - G \nabla \times (\nabla \times \delta) + F_{\text{ex}} \quad (\text{vector form}) \quad (11.2.35)$$

Stress equation

$$T_{ij} = 2Ge_{ij} + \lambda \delta_{ij} e_{kk} \quad (\text{Hooke's law}) \quad (11.2.32)$$

$$T_{ij} = G \left(\frac{\partial \delta_i}{\partial x_j} + \frac{\partial \delta_j}{\partial x_i} \right) + \lambda \delta_{ij} \frac{\partial \delta_k}{\partial x_k} \quad (\text{stress-displacement}) \quad (11.2.32)$$

Strain equation

$$e_{ij} = \frac{1}{2} \left(\frac{\partial \delta_i}{\partial x_j} + \frac{\partial \delta_j}{\partial x_i} \right) \quad (\text{strain-displacement}) \quad (11.2.10)$$

$$e_{ij} = \frac{1}{2G} T_{ij} - \frac{\nu}{E} \delta_{ij} T_{kk} \quad (\text{Hooke's law}) \quad \begin{matrix} (11.2.28) \\ (11.2.29) \end{matrix}$$

Relations among constants

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \quad (11.2.31)$$

$$G = \frac{E}{2(1 + \nu)} \quad (\text{g) of Example 11.2.1}$$

one of these forces (per unit volume) is an inertial force. In addition, there may be force densities produced by gravity or electromagnetic fields. The last two externally produced forces are called F_{ex} in the summary of equations given in Table 11.2.2. Other basic equations and relations of elasticity are shown in Table 11.2.2; equation numbers indicate their places in the text.

11.3 ELECTROMECHANICAL BOUNDARY CONDITIONS

Electromechanical coupling with elastic media often occurs through boundary conditions. One-dimensional illustrations of this type of problem were given in Sections 9.1.2. and 9.2.2, in which the boundary condition entered as the requirement of equilibrium for a mechanical terminal pair. In these examples the boundary condition related the stress and displacement at a given point in space. In this section we consider the more general three-dimensional situation.

Boundary conditions are required to describe solutions for the stress and displacement in a region in which material properties undergo abrupt changes. We have made general comments about boundary conditions in connection with the magnetic and electric field equations (Section 6.2)*. We have assumed that the field equations hold in the region of the discontinuity and performed integrations of these equations over the appropriate volumes or surfaces to provide the required "jump" conditions on the fields. Although the displacement vector and stress, like the electric and magnetic fields, are defined by differential equations that can be integrated through an abrupt change in material properties, the analogy is not complete. We were able to assume that Maxwell's equations applied throughout all the volume of interest. The equations of elasticity, however, apply only to a region occupied by an elastic solid and not, for example, to an adjoining region filled with fluid. Hence the boundary conditions resulting from an integration of (11.2.35) over a volume enclosing a section of the interface between two elastic materials are restricted to problems involving just elastic materials. Actually, the situation is not so complicated because a variety of physical problems is modeled by equations in the *form* of (11.1.4), if we are willing to recognize the stress T_{mn} as the total stress acting on the material. Because in writing this equation there are no implications regarding the relationship between T_{mn} and the material motions, we can use (11.1.4) to write a boundary condition of some generality.

In Section 11.1 it was pointed out that because the displacements δ are small no distinction need be made between the Lagrangian and Eulerian representations. We find it convenient here to view the equations of motion as though they were written in Lagrangian coordinates, that is, as though (x_1, x_2, x_3) denoted the unstrained position of the particle that is instantaneously displaced from (x_1, x_2, x_3) by the amount $\delta(x_1, x_2, x_3)$. We can define a surface in three dimensions by the equations

$$\begin{aligned}x_1 &= a(u, v), \\x_2 &= b(u, v), \\x_3 &= c(u, v),\end{aligned}\tag{11.3.1}$$

where (u, v) are parameters, each pair of which defines a particular point on the boundary. When the boundary deforms, due to a material strain, particles on the boundary are then found at the position

$$\begin{aligned}x_1 &= a + \delta_1(a, b, c, t), \\x_2 &= b + \delta_2(a, b, c, t), \\x_3 &= c + \delta_3(a, b, c, t).\end{aligned}\tag{11.3.2}$$

Hence the motion of a particular particle on the boundary in the unstrained position (a, b, c) is defined by (11.3.2). We now consider the situation in

* See Table 6.1, Appendix G.

which elastic media [regions (1) and (2) in Fig. 11.3.1] are joined along the boundary defined by (11.3.2). It is clear that if the boundary is to be well defined one of our boundary conditions is

$$\delta^{(2)}(a, b, c, t) = \delta^{(1)}(a, b, c, t). \quad (11.3.3)$$

This condition can also be considered a necessary consequence of our equations of motion, for if the displacement is not a continuous function the strain, hence the stress (which depends on rates of change of the displacement with respect to position), becomes singular at the boundary.

We are now in a position to integrate (11.1.4) over a small volume that includes the boundary.

$$\int_V \rho \frac{\partial^2 \delta_i}{\partial t^2} dV = \int_V \frac{\partial T_{ij}}{\partial x_j} dV. \quad (11.3.4)$$

The volume V is fixed with its center at the position (a, b, c) , as shown in Fig. 11.3.1. The integration is carried out over the Lagrangian variables (x_1, x_2, x_3) . Hence the time derivative and space integration on the left side of (11.3.4) can be reversed in order. The integral of the divergence of a stress tensor over a volume (see Section 8.1 or Appendix G) can be converted to a surface integral, and (11.3.4) becomes [variations in ρ with time are of the same order as δ , hence are second order in (11.3.4)]

$$\frac{\partial^2}{\partial t^2} \int_V \rho \delta_i dV = \oint_S T_{ij} n_j da. \quad (11.3.5)$$

We consider the situation in which the dimensions of the surfaces A (shown in Fig. 11.3.1) are small compared with the radius of curvature of the boundary but large compared with the thickness Δ of the volume element

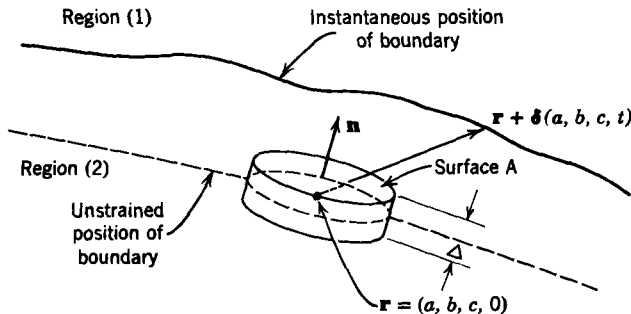


Fig. 11.3.1 Unstrained and strained (instantaneous) interface between regions (1) and (2). A small volume V , with normal $\mathbf{n}$, encloses a section of the interface. Note that an integration over the volume centered at $(a, b, c, 0)$ is an integration over a volume that remains centered on the moving interface.

Then, if ρ and δ are finite, the volume integral of (11.3.5) vanishes as $\Delta \rightarrow 0$. Sometimes physical situations can be described by an elastic medium, bounded by a heavy layer of material. In these cases the effect of the layer of material is approximated by including a surface mass density ρ_s . Mathematically, the surface mass density is a singularity in the mass density ρ in the same way that the surface charge density is a singularity in the charge density (Section 6.2.2). Then the integral over the volume retains a finite value, and as $\Delta \rightarrow 0$ (11.3.5) can be approximated as

$$\rho_s \frac{\partial^2 \delta_i}{\partial t^2} = [T_{ij}^{(1)} - T_{ij}^{(2)}] n_j, \quad \text{on the boundary,} \quad (11.3.6)$$

where we have divided through by the area A and assumed that the stresses are always finite.

The boundary condition used in Section 9.1.2 was a special case of (11.3.6), in which the stresses T_{ij} were in part due to the elastic strains and in part to a force of electric origin. The condition of (11.3.5) is the continuum-mechanical extension of the requirement used in Chapter 2 that the sum of all forces applied to a mechanical node must be equal to the inertial force associated with that node. The right-hand side of (11.3.6) is the net traction (force per unit area), whereas the left side is an inertial force per unit area.

11.4 WAVES IN ISOTROPIC ELASTIC MEDIA

This section is devoted to establishing a picture of the kinds of dynamical behavior that can be expected in dealing with elastic materials. To this end, we extend the notions introduced in Chapter 9 and recognize that the vibrations of continuous media in three dimensions can also be understood in terms of waves and normal modes. We have already used simple elastic models in Chapters 9 and 10 to illustrate transverse and longitudinal motions in one and two dimensions (the thin rod and membrane). We now consider these motions in three dimensions.

11.4.1 Waves in Infinite Media

In the absence of externally applied forces $\mathbf{F}_{\text{ex}}$ the motions of an elastic material are described by (11.2.35), written as

$$\rho \frac{\partial^2 \delta}{\partial t^2} = (2G + \lambda) \nabla(\nabla \cdot \delta) - G \nabla \times (\nabla \times \delta) \quad (11.4.1)$$

This equation is in a particularly convenient form because it makes it possible to distinguish between two essentially different kinds of material displacement. If we take the divergence of (11.4.1), the time and space derivatives can

be permuted to obtain

$$\rho \frac{\partial^2 \psi}{\partial t^2} = (2G + \lambda) \nabla^2 \psi, \quad (11.4.2)$$

where

$$\psi = \nabla \cdot \delta$$

and where use has been made of the identity $\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0$. In the same way the curl of (11.4.1) gives*

$$\rho \frac{\partial^2 \mathbf{C}}{\partial t^2} = G \nabla^2 \mathbf{C}, \quad (11.4.3)$$

where $\mathbf{C} = \nabla \times \delta$ and use has been made of the identity $\nabla \times (\nabla f) \equiv 0$. The scalar function ψ and vector function $\mathbf{C}$ represent kinds of displacement that are analogous to the field intensities $\mathbf{E}$ and $\mathbf{H}$ used to formulate Maxwell's equations. The function ψ can be thought of as a source of the displacement δ in the same sense as the charge density ρ_f is a source of the electric displacement $\mathbf{D}$ [see (1.1.12)]†. Hence the displacements represented by ψ have the same character as the electric displacement that originates or terminates on the charge ρ_f . An intuitive example is shown in Fig. 11.4.1. In a region in which ψ is found to be positive the material displacements tend to diverge. Similarly, the material converges toward regions in which ψ is negative (just as electric lines of force end on negative charge).

Deformations that can be represented by the function ψ are referred to as dilatational, for they represent outward or inward displacements of the material that lead to a change in the volume occupied by the material.

In a similar way $\mathbf{C}$ can be thought of as a "current" that gives rise to a displacement δ in the same way as an electrical current gives rise to a magnetic field $\mathbf{H}$ [see (Eq. 1.1.1)]†. The material displacements tend to follow circular paths about the vector $\mathbf{C}$, as shown in Fig. 11.4.2.

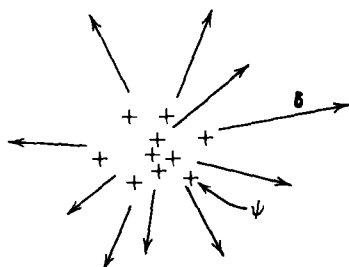


Fig. 11.4.1 Dilatational displacements δ represented by the source function ψ .

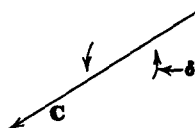


Fig. 11.4.2 Curl displacements represented by $\mathbf{C}$. The material tends to circulate about the vector field $\mathbf{C}$.

* $\nabla \times (\nabla \times \mathbf{C}) = \nabla(\nabla \cdot \mathbf{C}) - \nabla^2 \mathbf{C}$

† Table 1.2, Appendix G.

The components of C and the function ψ satisfy equations of the same form. Equations 11.4.2 and 11.4.3 are three-dimensional forms of the wave equation. The essential feature emphasized by these equations [illustrated one-dimensionally by (9.1.13) to (9.1.15)] is the propagating nature of the solutions. Dilatational motions apparently propagate more rapidly than the rotational motions. The wave dynamics are most easily seen by considering two one-dimensional special cases.

Example 11.4.1. Consider a one-dimensional dilatational motion that depends on $x_1 = x$. Then $\partial/\partial x_2 = \partial/\partial x_3 = 0$ and (11.4.2) becomes

$$\frac{\partial^2 \psi}{\partial t^2} = a_c^2 \frac{\partial^2 \psi}{\partial x^2}, \quad (a)$$

where

$$a_c = \left(\frac{2G + \lambda}{\rho} \right)^{1/2}.$$

A discussion of solutions to this wave equation was given in Section 9.1.1. To obtain a physical picture of the mechanics we consider a solution that is sinusoidal in space and time.

$$\psi = \psi_0 \sin \left[\omega \left(t - \frac{x}{a_c} \right) \right]. \quad (b)$$

This solution can be justified by direct substitution into (a) and can be thought of as a wave propagating with the phase velocity a_c in the x -direction.

Within an arbitrary constant that would be determined by the boundary conditions, the actual displacements follow from $\psi = \nabla \cdot \delta$.

$$\delta = \frac{\psi_0 a_c}{\omega} \cos \left[\omega \left(t - \frac{x}{a_c} \right) \right]. \quad (c)$$

At a given instant these displacements appear as shown in Fig. 11.4.3. Note that the material is displaced out of the regions of positive ψ and into regions of negative ψ . We can imagine painting equidistant parallel lines in the unstressed material. Then a wave propagating perpendicular to these lines would distort their relative positions as shown in Fig. 11.4.3b. The material density is increased where the lines are closest together and where ψ is negative. Points of constant phase in the density distribution propagate to the right with the phase velocity a_c . Longitudinal waves of this kind are referred to as *compressional*, *acoustic*, or *dilatational*. Actually, they are a close relative of the compressional waves on a thin rod, encountered in Section 9.1.1. If the expressions for λ and G given in Tables 11.2.2 and 11.4.1 are used to write a_c as a function of E and ν (Poisson's ratio), we obtain

$$a_c = \left(\frac{E}{\rho} \right)^{1/2} N(\nu), \quad (d)$$

where

$$N(\nu) = \left[\frac{1 - \nu}{(1 + \nu)(1 - 2\nu)} \right]^{1/2}.$$

Measured values of ν are given in Table 11.2.1 and can be seen to fall between about 0.2 and 0.5. The function $N(\nu)$ in this range is shown in Fig. 11.4.4 and is greater than 1.

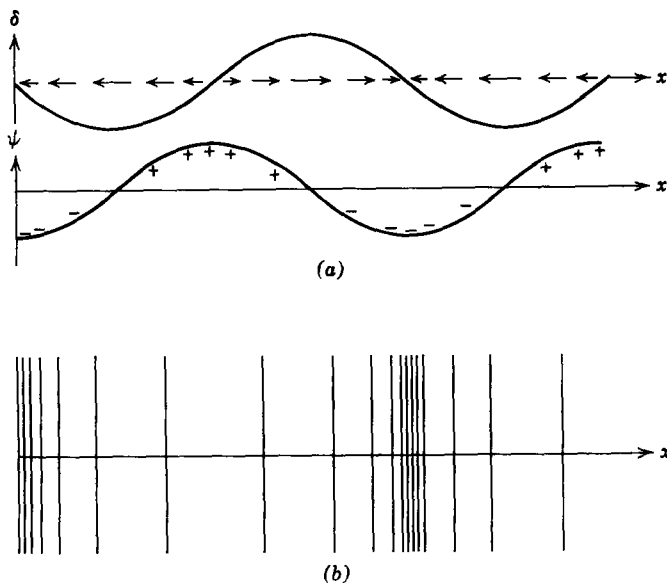


Fig. 11.4.3 Instantaneous view of the displacements δ and source function ψ for a one-dimensional dilatational wave: (a) relative distributions of δ and ψ ; (b) exaggerated appearance of originally equidistant lines painted on the material. Lines compressed together indicate a compression of the material.

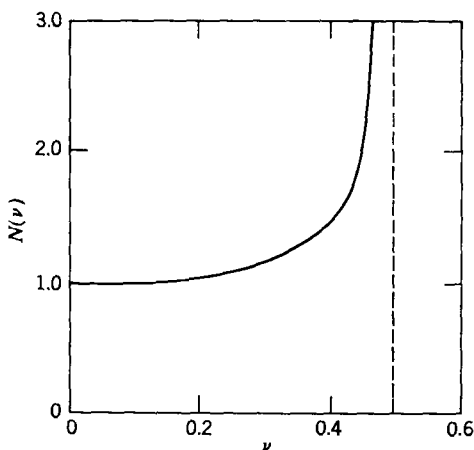


Fig. 11.4.4 The function $N(\nu)$, where ν is Poisson's ratio. Plane dilatational waves propagate with the velocity $a_c = (\sqrt{E/\rho})N$, whereas waves on a thin rod have the velocity $\sqrt{E/\rho}$. Hence N is the ratio of a_c to the acoustic velocity on a thin rod.

Remember that Poisson's ratio entered in the stress-strain relations because a longitudinal stress could lead to transverse displacements (Fig. 11.2.5). By assuming that the material motions were one-dimensional in nature, we have required that there be no transverse displacements. This means that there is a transverse stress (T_{22} or T_{33}) that can be computed from (11.2.26) and (11.2.27) with $e_{22} = e_{33} = 0$; for example,

$$T_{22} = \frac{\nu(1 + \nu)}{1 - \nu^2} T_{11}. \quad (e)$$

This stress tends to constrain the material from the sides and, through Poisson's ratio, to stiffen the material to longitudinal deformations. For this reason we have found a phase velocity a_c that always exceeds the velocity of waves on a thin rod $\sqrt{E/\rho}$. In the thin rod the transverse stresses are zero because of the free surfaces on the rod and longitudinal motions are not affected by Poisson's ratio. We see now that there are actually transverse material displacements on a thin rod. This point is discussed further in Section 11.4.2a, where we define the conditions under which a thin rod model can be used.

Dilatational waves involve normal stresses and normal strains. By contrast the rotational motions constitute a shearing of the medium. The next example illustrates these shear deformations in a one-dimensional case.

Example 11.4.2. In one dimension ($x_1 = x$) the rotational equations (11.4.3) become

$$\frac{\partial^2 C_2}{\partial t^2} = a_s^2 \frac{\partial^2 C_2}{\partial x^2}, \quad (a)$$

$$\frac{\partial^2 C_3}{\partial t^2} = a_s^2 \frac{\partial^2 C_3}{\partial x^2}, \quad (b)$$

where $a_s = \sqrt{G/\rho}$ and because $\partial/\partial x_2 = \partial/\partial x_3 = 0$, $C_1 = 0$. By definition, the components C_2 and C_3 are related to the displacement by

$$C_2 = -\frac{\partial \delta_3}{\partial x}, \quad C_3 = \frac{\partial \delta_2}{\partial x}. \quad (c)$$

Once again, the equations of motion (a) and (b) are wave equations. Now, however, the phase velocity a_s of the waves is less than the compressional wave velocity a_c in Example 11.4.1 and the corresponding material deformations are altogether different. The component C_2 represents displacements in the x_3 -direction. Similarly, C_3 represents transverse motions of the elastic material in the x_2 -direction. Because the stresses and strains are in shear rather than compression, transverse waves of this kind are referred to as *shear waves* or *waves of distortion*.

If we assume that the boundary conditions are such that only C_3 is excited, a traveling wave solution to (b) appears as shown in Fig. 11.4.5. In this figure the material displaces in the x_2 -direction or in a direction that is perpendicular to both C_3 and the direction of propagation. Note that material tends to rotate about the vector C (Fig. 11.4.2) and that the local material density does not change as it did in the dilatational waves.

In this section we have seen that in an infinite medium we can separate rotational or shearing deformations from dilatational motions. Except in a few simple cases, elastic materials deform in such a way that both shearing

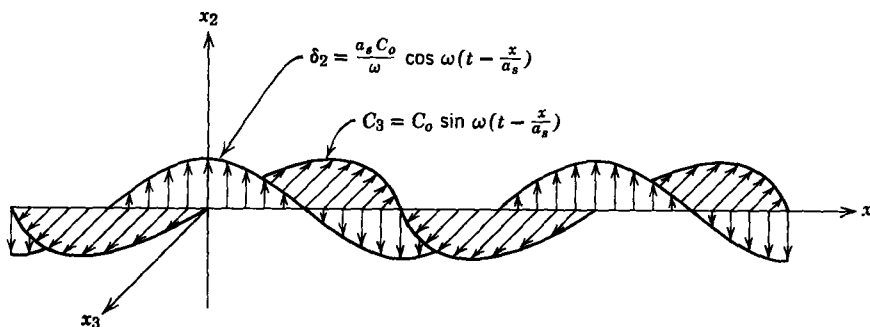


Fig. 11.4.5 Traveling shear wave, showing the spacial relation between material displacements δ_2 and the rotational vector component C_3 . Note that material tends to “rotate” about the vector $C_3 \mathbf{i}_3$, as shown in Fig. 11.4.2.

and dilatation are present. This is true because both types of motion must be present to satisfy boundary conditions.

A simple case in which the one-dimensional dilatational motions predicted by (11.4.2) are an exact solution even in the presence of boundaries is shown in Fig. 11.4.6. Here the transverse boundaries of a bar are constrained by rigid walls that prevent transverse motions but do not inhibit longitudinal motions. Given a driving condition at one end and a boundary condition at the other, the problem can be solved in a manner identical to that used for the thin rod in Section 9.1. If, however, the transverse walls constrain the bar in the x -direction or fail to constrain the transverse displacements, the motions are no longer purely dilatational. Shear strains are required to satisfy the boundary conditions.

The block of material shown in Fig. 11.4.7 is subject to boundary conditions that are satisfied by purely shearing motions. Here one edge is rigidly attached to a wall that prevents both perpendicular motion and slip. The opposite end is driven by a time-varying stress $T_{21} = T_o(t)$. The resulting motions are predicted by (11.4.3) if the boundaries transverse to the x -axis are driven by the same time-varying shear stress $T_{21}(x, t)$ (see Example 11.4.3 for a solution), but if these transverse boundaries are constrained by a rigid wall, or are free

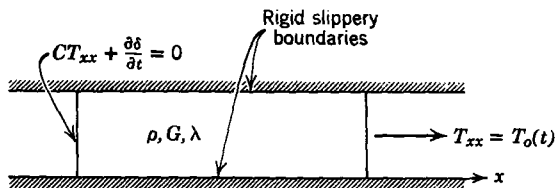


Fig. 11.4.6 An elastic bar with boundaries that permit purely dilatational motions in the x -direction.

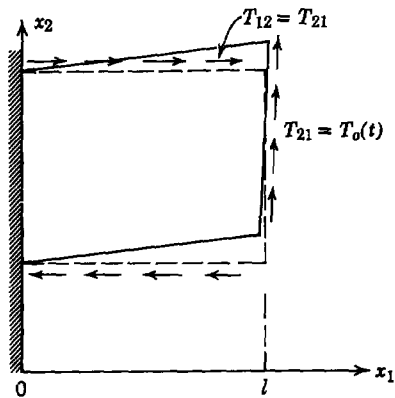


Fig. 11.4.7 An elastic bar with boundaries and driving stresses that permit purely shearing motions in the x_2 -direction.

of stress, the motions must include a dilatational part; that is, any other boundary condition than that shown in Fig. 11.4.7 will couple the rotational and dilatational motions.

Because boundary conditions usually couple the compressional and shearing motions, any dynamical problem will involve a combination of the characteristic velocities a_s and a_c . These velocities are tabulated, along with characteristic values of λ , in Table 11.4.1.

Table 11.4.1 Phase Velocities for Shear and Compressional Waves in an Infinite Medium*

Material	λ (units of 10^{11} N/m ²)	a_s (units of 10^3 m/sec)	a_c (units of 10^3 m/sec)
Aluminum	0.626	3.0	6.35
Brass	1.04	2.2	4.7
Copper	1.17	2.3	4.8
Iron, cast	0.836	2.8	5.2
Steel	1.18	3.2	6.0
Stainless steel	1.19	3.0	5.8
Titanium	0.904	3.0	6.2
Glass	0.366	2.9	5.1

* When ranges of E , ν , and G are given in Tables 9.1 (Appendix G) and 11.2.1, the largest values have been used.

Example 11.4.3. In this example we seek to establish a further familiarity with shearing deformations. Figure 11.4.8 shows a slab of material rigidly attached to a wall at $x = 0$ and driven with a shear stress $T_{21} = \text{Re} [\hat{T}e^{j\omega t}]$ at $x = l$. The slab has infinite extent in the x_2 - and x_3 -directions; hence it is reasonable to assume that the motions are one-dimensional

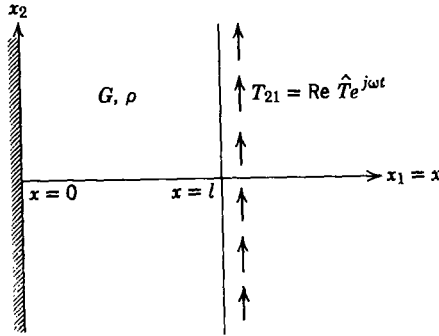


Fig. 11.4.8 An infinite slab of elastic material subjected to the uniformly distributed shear stress $T_{21} = \text{Re} (\hat{T} e^{j\omega t})$ at $x = l$ and fixed at $x = 0$.

($\partial/\partial x_2 = \partial/\partial x_3 = 0$). The following questions are to be answered: (1) What is the displacement of the material as a function of (x, t) ? (2) If boundaries are introduced at $x_2 = 0$ and $x_2 = L$, what boundary conditions are required to make the one-dimensional assumption correct? (3) if the peak shear stress applied at $x = l$ is equal to 1 atm, what is the largest displacement of the material (as an example, consider a slab made of brass, $l = 1$ m, and the low frequency limit at which $\omega \rightarrow 0$). (4) At a frequency of 1 kc what is the least value of l required to produce a resonance?

1. The excitation tends to produce displacements δ_2 , hence we guess that $C_2 = 0$. Our guess is justified if we can satisfy both differential equations and boundary conditions. Equations (b) and (c) of Example 11.4.2 then give

$$\frac{\partial^2 \delta}{\partial t^2} = a_s^2 \frac{\partial^2 \delta}{\partial x^2}, \quad (\text{a})$$

where $\delta_2 = \delta$ and $x_1 = x$. Two boundary conditions are necessary to determine fully the sinusoidal steady-state solution to this equation. These conditions are evident from the statement of the problem

$$T_{21}(l, t) = \text{Re} (\hat{T} e^{j\omega t}) = G \frac{\partial \delta}{\partial x}(l, t) \quad (\text{b})$$

and

$$\delta(0, t) = 0. \quad (\text{c})$$

Now, if we assume solutions with the same frequency as the excitation

$$\delta = \text{Re} (\hat{\delta} e^{j\omega t}), \quad (\text{d})$$

the unknown function $\hat{\delta}(x)$ can be found by substituting (d) into (a) and solving the resulting ordinary differential equation. Hence

$$\hat{\delta} = A \sin kx + B \cos kx, \quad (\text{e})$$

where

$$k = \frac{\omega}{a_s}$$

and A and B are arbitrary constants determined by the boundary conditions. Condition (c) shows that $B = 0$, whereas condition (b) determines A as

$$A = \frac{\hat{T}}{kG \cos kl}. \quad (f)$$

It follows from (d) and (e) that the required solution for the displacement is

$$\delta = \text{Re} \left[\frac{\hat{T} \sin kx e^{j\omega t}}{kG \cos kl} \right]. \quad (g)$$

2. In our solution all displacements are zero except $\delta_a = \delta$, as given by (g). There are two components of stress (11.2.32). One was used to match the boundary condition at $x = l$:

$$T_{21} = G \frac{\partial \delta}{\partial x} = \text{Re} \left[\frac{\hat{T} \cos kx e^{j\omega t}}{\cos kl} \right]. \quad (h)$$

The other is present because $T_{ij} = T_{ji}$ or, in particular,

$$T_{12} = T_{21}. \quad (i)$$

Hence, if the slab has boundaries at $x_2 = 0$ and $x_2 = L$, our solution will be correct only if there is a shearing stress on these boundaries given by (h) and (i). Note that this stress is a function of both x and t . In the limit at which $L \gg l$, we expect that the stress excitation on the transverse boundaries can be ignored and our one-dimensional solution will be approximately correct, regardless of the boundary conditions at $x_2 = 0$ and $x_2 = L$.

3. In the limit at which $\omega \rightarrow 0$ (quasi-static motions) $k \rightarrow 0$ and (g) shows that the peak δ occurs at $x = l$, where (since $1 \text{ atm} = 1.013 \times 10^5 \text{ N/m}^2$ and G can be found from Table 11.2.1) (g) gives

$$\begin{aligned} |\delta|_{\text{peak}} &= \frac{|\hat{T}|l}{G} = (1.01 \times 10^5)(1)/4.1 \times 10^{10} \\ &= 2.5 \times 10^{-6} \text{ m} \quad \left(\text{about } \frac{1}{10,000} \text{ in.} \right) \end{aligned} \quad (j)$$

We see that static deflections are likely to be very small.

4. The slab is in a resonant state when the denominator of (g) becomes zero or when

$$kl = \frac{n\pi}{2}, \quad n = 1, 3, 5, \dots \quad (k)$$

Hence the smallest value of l that will produce a resonance at 1 kc (a_s in Table 11.4.1) is

$$l = \frac{\pi a_s}{2\omega} = \pi 2.2 \times \frac{10^3}{2(2\pi \times 10^3)} = 0.55 \text{ m}. \quad (l)$$

Under these conditions the 0.55-m thickness of the brass slab represents one quarter of a wavelength.

11.4.2 Principal Modes of Simple Structures

In most dynamical situations involving elastic media boundaries play an important role. Our development makes it natural to think of these boundaries

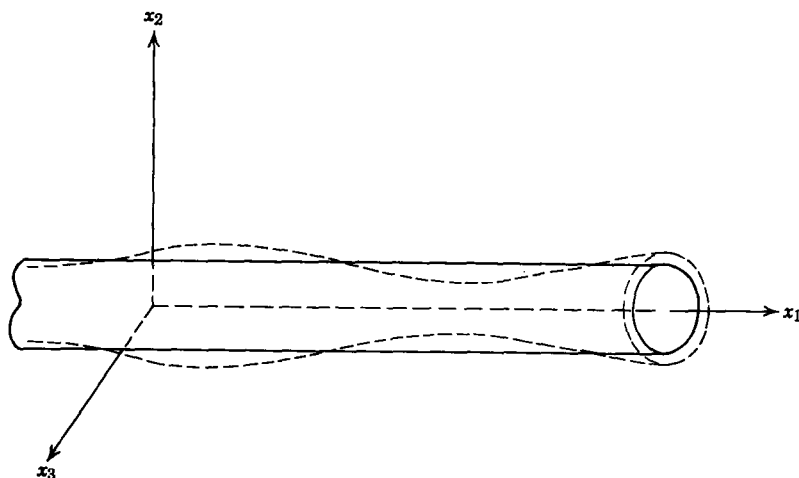


Fig. 11.4.9 A thin elastic rod with a longitudinal axis x_1 and transverse coordinates x_2 and x_3 . An exaggerated transverse distortion of the material is shown as it accompanies longitudinal compression and extension of the rod.

as two types: longitudinal and transverse; that is, the elastic structure usually has one dimension that is greater than the others, such as the x -direction for the thin rod of Fig. 11.4.9, and it is natural to analyze the dynamics in terms of wave propagation in that direction. Conditions applied at discrete positions along the x_1 -axis, referred to as longitudinal boundary conditions, were discussed in Section 9.1.1*b*. The extremities of the material in the x_2 - and x_3 -directions are referred to as transverse boundaries. It is the purpose of this section to introduce the effect that the transverse boundaries have on wave propagation in the longitudinal direction.*

Even in the absence of electromechanical interactions, wave propagation in the presence of material boundaries is an involved subject. It serves our purposes here to consider two cases, both of which make the essential point in a simple way and have practical value. First, we reconsider in the light of three-dimensional elasticity the thin rod. Then in Section 11.4.2*b* the transverse motions of a thin beam are analyzed. In each of these cases it is assumed that the longitudinal wavelengths of interest are large compared with the transverse dimensions. In the general case in which no approximations are made concerning the wavelength, thin elastic structures support an infinite set of modes, each having a different dependence on the transverse coordinates. Most of these modes do not propagate at low frequencies. The waves on a thin rod and on a thin beam, as considered in Sections 11.4.2*a* and 11.4.2*b*,

* For those familiar with the theory of guided electromagnetic waves the waves of Section 11.4.1 are waves in "free space," whereas those discussed here are guided waves analogous to those found in a waveguide.

propagate even as the frequency approaches zero. Among all modes that are generally possible, they assume considerable practical importance and are called the *principal modes* of their respective structures. In Section 11.4.3 we illustrate the nature of the higher order modes by considering the dynamics of a plate subject to a shearing excitation. Here we find that at low frequencies the higher order modes appear as evanescent waves; hence we again encounter the topic of spatially growing (decaying) waves discussed in Section 10.1.2. A detailed presentation of wave propagation in elastic plates and cylinders is of interest* in the design of delay lines and electromechanical filters to be used at high frequencies (e.g., megahertz). At high frequencies the higher order modes are inadvertently excited because longitudinal wavelengths are on the order of the transverse dimensions.

11.4.2a The Thin Rod

A thin elastic rod is shown in Fig. 11.4.9. In static equilibrium it has the geometry of a right circular cylinder, with its axis in the x_1 -direction. An approximate description of the longitudinal motions was given in Section 9.1†. We are interested in having a second look at the dynamics to see what transverse motions of the material are implied by the three-dimensional equations of elasticity. We can argue that the equation of motion is the same as that found in Section 9.1 for longitudinal deformations by observing that the transverse surfaces of the rod are free of externally applied stresses. Hence, if the rod is very thin, the stresses T_{22} , T_{33} (essentially normal to the transverse surface) and T_{12} and T_{13} (essentially the shear stress on the surface) cannot be very different from zero inside the rod. This is the starting point in writing an approximate equation of motion.

Because we take $T_{12} = T_{13} = 0$, the force equation in the x_1 -direction (11.1.4) becomes

$$\rho \frac{\partial^2 \delta_1}{\partial t^2} = \frac{\partial T_{11}}{\partial x_1}. \quad (11.4.4)$$

In addition, because $T_{22} = T_{33} = 0$, (11.2.29) of Table 11.2.2 shows that

$$e_{11} = \frac{\partial \delta_1}{\partial x_1} = \left(\frac{1}{2G} - \frac{\nu}{E} \right) T_{11}, \quad (11.4.5)$$

and because $G = E/2(1 + \nu)$ we obtain

$$T_{11} = E \frac{\partial \delta_1}{\partial x_1}. \quad (11.4.6)$$

* See W. P. Mason, *Physical Acoustics*, Vol. I, Part A, Academic, New York, 1964, p. 111.

† Table 9.2, Appendix G.

This will be recognized as the relation used in Section 9.1. It follows from (11.4.4) and (11.4.6) that the longitudinal displacement is predicted by the equation

$$\rho \frac{\partial^2 \delta_1}{\partial t^2} = E \frac{\partial^2 \delta_1}{\partial x_1^2}. \quad (11.4.7)$$

Although they do not enter in the equation of motion, transverse displacements do accompany $\delta_1(x_1, t)$. They can be computed under the assumption that δ_1 is a known function. From (11.2.32) and the condition that the normal stress on the transverse boundaries be zero we have

$$T_{22} = 0 = (2G + \lambda) \frac{\partial \delta_2}{\partial x_2} + \lambda \left(\frac{\partial \delta_1}{\partial x_1} + \frac{\partial \delta_3}{\partial x_3} \right), \quad (11.4.8)$$

$$T_{33} = 0 = (2G + \lambda) \frac{\partial \delta_3}{\partial x_3} + \lambda \left(\frac{\partial \delta_1}{\partial x_1} + \frac{\partial \delta_2}{\partial x_2} \right). \quad (11.4.9)$$

Presumably, we have solved (11.4.7). Hence these last two equations can be simultaneously solved for $\partial \delta_2 / \partial x_2$ or $\partial \delta_3 / \partial x_3$; for example,

$$\frac{\partial \delta_2}{\partial x_2} = \frac{-\lambda}{2(G + \lambda)} \frac{\partial \delta_1}{\partial x_1}(x_1, t). \quad (11.4.10)$$

The right-hand side of this equation is dependent only on (x_1, t) . Hence it can be integrated to obtain

$$\delta_2 = \frac{-\lambda}{2(G + \lambda)} \frac{\partial \delta_1}{\partial x_1} x_2 + f(x_1, x_3, t), \quad (11.4.11)$$

where f is an arbitrary function determined by the cross-sectional geometry; for example, if the rod is a right-circular cylinder, coaxial with the x_1 -axis, symmetry requires that $\delta_2(x_1, 0, x_3) = 0$ or that $f = 0$. Similarly,

$$\delta_3 = \frac{-\lambda}{2(G + \lambda)} \frac{\partial \delta_1}{\partial x_1} x_3 + g(x_1, x_2, t), \quad (11.4.12)$$

where $g = 0$ for a right-circular cylinder. The last two equations show that the transverse displacements are largest for the material with the greatest distance from the x_1 -axis. In regions in which the rate of change with respect to x_1 is large the transverse displacements are also large.

As an example, consider the traveling wave

$$\delta_1 = \delta_0 \sin(\omega t - kx_1), \quad (11.4.13)$$

where (11.4.7) shows that $\omega = k\sqrt{E/\rho}$. Then from (11.4.11)

$$\delta_2 = \frac{\delta_0 \lambda k x_2}{2(G + \lambda)} \cos(\omega t - kx_1). \quad (11.4.14)$$

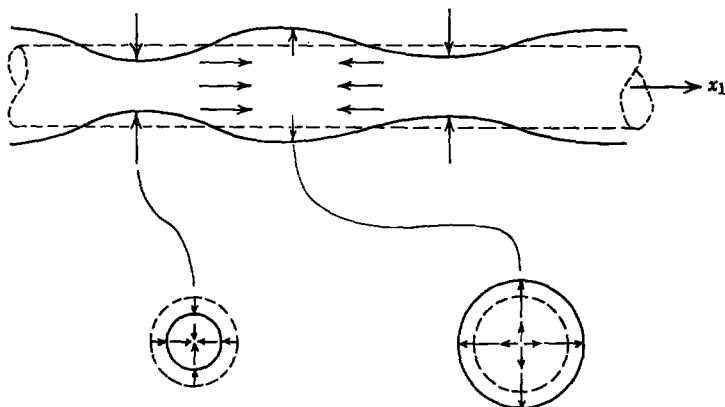


Fig. 11.4.10 An instantaneous view of displacements that accompany a compressional wave traveling in the x_1 -direction on a thin rod. The transverse displacements are exaggerated in this figure.

The displacements represented by the last two equations can be pictured as shown in Fig. 11.4.10.

In retrospect, we see that δ_2 and δ_3 were ignored in the longitudinal force equation and were then found from the predicted displacements δ_1 . This procedure is analogous to a quasi-static analysis (Sections B.2.2 and 9.1.3) in which variations with respect to time are at first ignored and then computed as second-order effects. In the rod, two-dimensional effects are second order and the analysis may be referred to as quasi-one-dimensional. Other quasi-one-dimensional models are introduced in the next section and in Chapter 13. Such models, which reduce the significant effects to a dependence on a single coordinate, are of considerable importance not only in continuum electromechanics but in many other areas as well. Very often they are referred to as “long-wave” limits, because the quasi-one-dimensional model is correct, provided wavelengths in the longitudinal direction are long enough. We can illustrate this point by recognizing that δ_2 is small compared with δ_1 if [from (11.4.13) and (11.4.14)]

$$\frac{\lambda k R}{2(G + \lambda)} \ll 1, \quad (11.4.15)$$

where we have used the rod radius R to evaluate δ_2 . Remember that one wave-length is $2\pi/k$, and we see that (11.4.15) is fulfilled if wavelengths are large compared with the radius R .

11.4.2b The Thin Beam

The principal longitudinal or dilatational mode in the presence of boundaries is the subject of Section 11.4.2a. In this section we consider principal

shearing modes on a thin beam of elastic material. Vibrating reeds or bars (tuning forks), commonly used in electromechanical transducers, provide a familiar context for broadening our understanding of distributed dynamic systems.

By now it is a well-established notion that the dynamics of continuous media are closely related to the propagation of waves. The examples of Chapter 9, which describe thin rods and membranes, illustrate this point. Transverse motions of a beam are similar but involve several complications that prevent a misleading generalization from the simple systems considered so far. We find that beam deflections involve four boundary conditions, compared with the two conditions required for the rods and membranes. As a result, the eigenmodes are not simple sinusoids in space but rather have both propagating and evanescent components and the eigenfrequencies of the beam are not usually harmonics. We have encountered this effect of dispersion before (Chapter 10) but not in so familiar a context. If we clamp the end of a beam (steel ruler) at one end with the other end free, as shown in Fig. 11.4.11, the lowest eigenmode can be excited by releasing the beam from a deflected position. In a rod or membrane halving the length l will double the frequency (which can be measured with a strob tachometer). As we shall see in Example 11.4.4, the thin beam lacks this property.

Our object is to use the fact that the bar is thin (in the direction of the deflection) to write an equation of motion that contains only the longitudinal coordinate x_1 and the time t . As is usually the case in developing quasi-one-dimensional models, the starting point is motivated physically. A cross-sectional view of the bar, subject to a hypothetical deformation, is shown in Fig. 11.4.12. If there is no equilibrium (static) longitudinal tension on the bar (it is not being stretched in the x_1 -direction as the membrane was*), the displacement of a line painted on the side of the bar will be as shown.

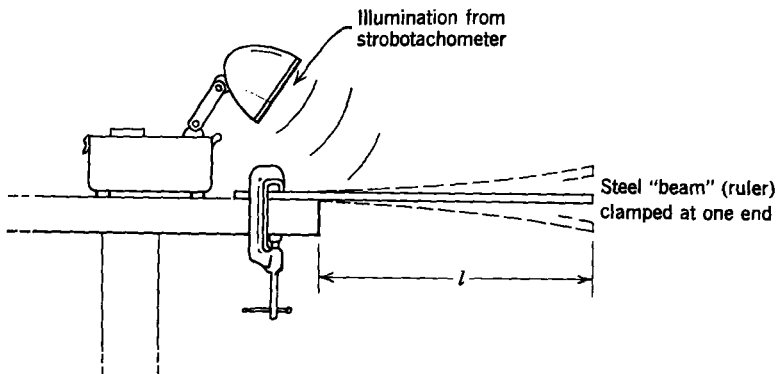


Fig. 11.4.11 The lowest eigenfrequency of a thin beam clamped at one end and free at the other can be measured by using a strob tachometer.

* See Table 9.2, Appendix G.

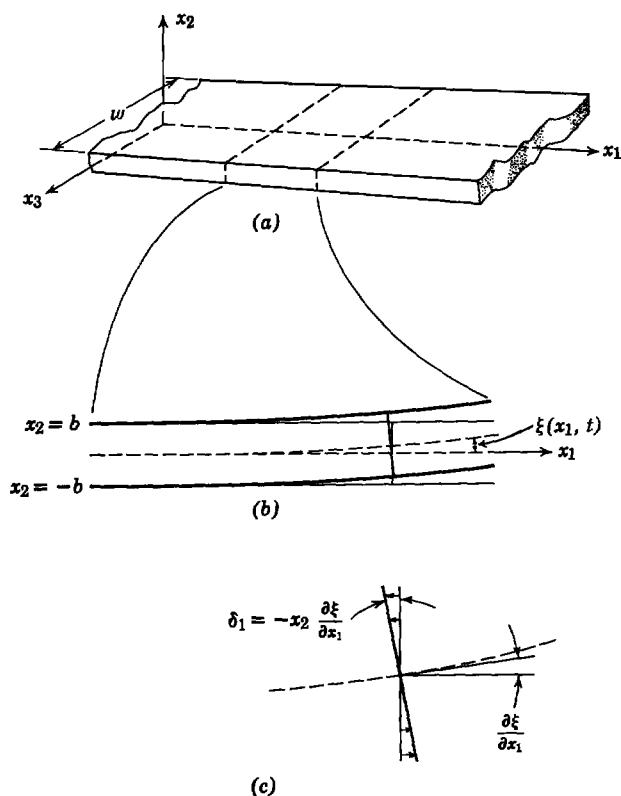


Fig. 11.4.12 Transverse vibrations of a thin bar: (a) in static equilibrium the axis of the bar is in the x_1 -direction with material motions essentially in the x_2 -direction; (b) a small section of the bar shows the deformation of a line perpendicular to the neutral plane; (c) a detailed view of a perpendicular line shows the relation to the transverse displacement ξ of the neutral plane.

Because the bar is not undergoing a net tension, there is an x_2 - x_3 plane (called the neutral plane) in which the material has no x_1 displacement. Then for small deflections of the bar the angle of deflection of a cross-sectional line is given by $\partial \xi / \partial x_1$ (Fig. 11.4.12). The assumption is made that the bar is thin enough that the longitudinal material displacement δ_1 at any cross section can be approximated as having a linear dependence on the transverse dimension x_2 . It follows that this linear dependence is about

$$\delta_1 = -x_2 \frac{\partial \xi}{\partial x_1}(x_1, t). \quad (11.4.16)$$

Then from (11.2.10) the normal strain is

$$e_{11} = \frac{\partial \delta_1}{\partial x_1} = -x_2 \frac{\partial^2 \xi}{\partial x_1^2}(x_1, t). \quad (11.4.17)$$

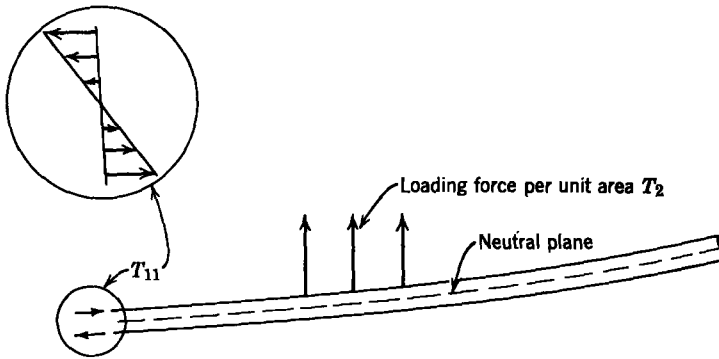


Fig. 11.4.13 Hypothetical beam deflection due to a surface force density T_2 . Because the thickness is small compared with the length, the stresses T_{11} are much greater than either T_{22} or T_{33} .

A second approximation is now made. It is assumed that the strain e_{11} is largely created by T_{11} or that $T_{11} \gg T_{22}$ or T_{33} . A section of the bar is shown in Fig. 11.4.13. Because the bar is thin, the stress T_{22} is on the order of any loading force per unit area T_2 . The stresses T_{11} must hold the vertical forces in force equilibrium, and because the beam is thin compared with its length it is apparent that we must have $T_{11} \gg T_{22}$. Because there are no loading forces in the x_3 -direction, it is even more reasonable that the stresses T_{33} can be ignored, compared with T_{11} . It follows from (11.2.29) that

$$T_{11} = Ee_{11} = -x_2 E \frac{\partial^2 \xi}{\partial x_1^2}. \quad (11.4.18)$$

Altogether, we shall make four approximations based on the thinness of the beam and the transverse nature of the deflections under consideration. The third of these assumptions is now introduced—that the longitudinal inertial force makes no essential contribution to the dynamics. This is reasonable because the deflection considered is mainly in the x_2 -direction. Then the x_1 -component of the momentum equation for the elastic material [see (11.1.4)] becomes

$$\frac{\partial T_{11}}{\partial x_1} = -\frac{\partial T_{12}}{\partial x_2} = -Ex_2 \frac{\partial^3 \xi}{\partial x_1^3}. \quad (11.4.19)$$

This expression can be integrated to give

$$T_{12} = \frac{Ex_2^2}{2} \frac{\partial^3 \xi}{\partial x_1^3} + g(x_1, t). \quad (11.4.20)$$

The arbitrary function $g(x_1, t)$ is evaluated by requiring that the surfaces at $x_2 = \pm b$ support no shearing stress or that

$$T_{12} = \frac{(x_2^2 - b^2)}{2} E \frac{\partial^3 \xi}{\partial x_1^3}. \quad (11.4.21)$$

The x_2 -component of (11.1.4), the force equation, is

$$\rho \frac{\partial^2 \delta_2}{\partial t^2} = \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2}. \quad (11.4.22)$$

The desired equation of motion can now be found by integrating this equation over an arbitrary cross section of the bar:

$$\rho \frac{\partial^2}{\partial t^2} \int_{-b}^{+b} \delta_2 dx_2 = \int_{-b}^{+b} \frac{\partial T_{21}}{\partial x_1} dx_2 + [T_{22}]_{x_2=b} - [T_{22}]_{x_2=-b}. \quad (11.4.23)$$

As a fourth (and last) approximation, the left-hand side of (11.4.23) is approximated by the product of the cross-sectional thickness $2b$ and the displacement of the bar center. Hence, making use of (11.4.21) and the fact that $T_{12} = T_{21}$,

$$2b\rho \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial^4 \xi}{\partial x_1^4} E \int_{-b}^{+b} \left(\frac{x_2^2 - b^2}{2} \right) dx_2 + T_2, \quad (11.4.24)$$

where T_2 is the sum of the forces per unit area acting on the x_2 -surfaces of the bar and defined by

$$T_2 = [T_{22}]_{x_2=b} - [T_{22}]_{x_2=-b}. \quad (11.4.25)$$

After the integration indicated by (11.4.24) the equation of motion for transverse displacements of the bar becomes

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{Eb^2}{3\rho} \frac{\partial^4 \xi}{\partial x_1^4} = \frac{T_2}{2b\rho}. \quad (11.4.26)$$

The independent variables in this expression are (x_1, t) ; hence we have formulated the dynamics in terms of a quasi-one-dimensional model. Beam deflections can be determined from this last equation, given four boundary conditions which arise because the ends of the beam are clamped in a certain fashion or because the ends are free of shear or normal stresses. To write boundary conditions on δ_1 , T_{11} , and T_{12} in terms of ξ we can use (11.4.16), (11.4.18), and (11.4.21).

The dependence of the longitudinal and shear stresses on the transverse coordinate x_2 is shown in Fig. 11.4.14. Note that the x_2 -dependences of T_{11}

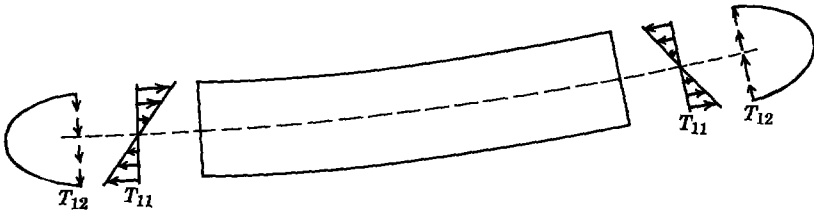


Fig. 11.4.14 Dependence of the normal and shear stresses on the transverse dimension of the beam.

and T_{12} are given by the lowest order polynomial expressions consistent with the requirements that there be no net longitudinal stress and that the shear stresses be zero at the surface of the beam.

Example 11.4.4. The situation shown in Fig. 11.4.15a provides an illustration of the role played by the boundary conditions. A thin beam is clamped at $x = 0$, so that both the transverse and longitudinal displacements of the material at this point are zero. The

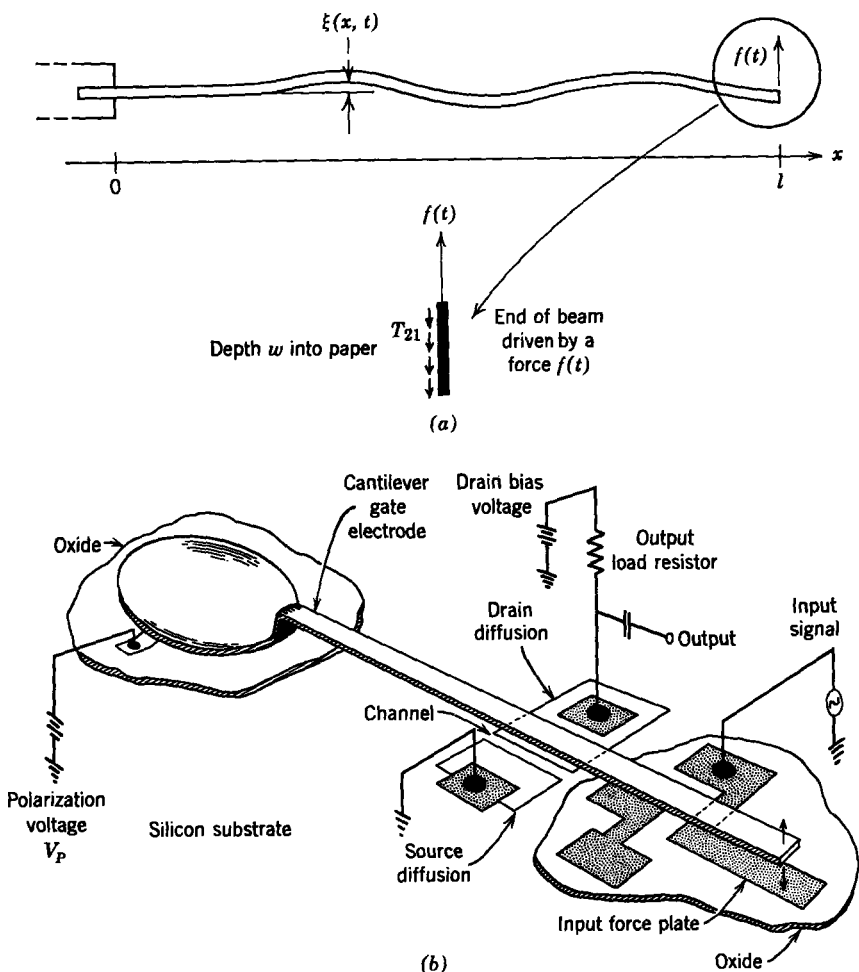


Fig. 11.4.15 (a) A thin elastic beam is driven to vibrate in a direction transverse to its smallest dimension by a force $f(t)$ applied at $x = l$. The end at $x = l$ is free of longitudinal stresses T_{11} , whereas the beam is clamped at $x = 0$. (b) One electromechanical application of the thin elastic beam is illustrated by the "Resonant Gate Transistor" (See W. E. Newell, "Ultrasonics in Integrated Electronics," *Proc. IEEE*, October 1965) A high Q integrated circuit incorporates an electrostatically driven beam. The elastic beam provides an inherently stable resonant element of extremely small proportions (see Fig. 11.4.15c).

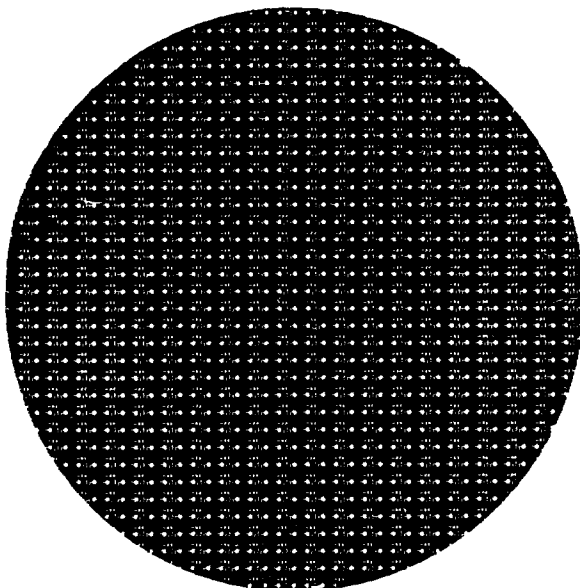


Fig. 11.4.15c Silicon wafer of 1-in. diameter containing nearly 500 resonant gate transistors of the type shown in (b). (Courtesy of Westinghouse Electric Corp.)

opposite end at $x = l$ is free of longitudinal stresses and is set into vibration by a force $f(t)$, which is sinusoidal:

$$f(t) = \text{Re} [\hat{f} e^{i\omega t}]. \quad (\text{a})$$

The sinusoidal steady-state deflections $\xi(x, t)$ are to be found. Of course, to find the driven response, we shall also find the natural frequencies of the beam. In an experiment such as that shown in Fig. 11.4.11 the bar vibrates at these eigenfrequencies. Hence the dependence of the lowest eigenfrequency on the length l will also be found.

The equation of motion is (11.4.26) with $T_2 = 0$. Because the drive assumes a sinusoidal form, we shall guess solutions:

$$\xi = \text{Re} [\hat{\xi}(x) e^{i\omega t}], \quad (\text{b})$$

in which case (11.4.26) requires that

$$\frac{d^4 \hat{\xi}}{dx^4} - \alpha^4 \hat{\xi} = 0, \quad (\text{c})$$

where

$$\alpha = \left[\omega^2 \left(\frac{3\rho}{Eb^2} \right) \right]^{1/4}.$$

The spatial dependence is found from (c) by again guessing exponential solutions $\hat{\xi} = e^{-jkx}$, in which substitution shows that $k^4 = \alpha^4$ or that there are four solutions for the spatial dependence:

$$k = \alpha, -\alpha, j\alpha, \text{ and } -j\alpha. \quad (\text{d})$$

Note that we have defined α as a positive real constant. We have assumed solutions of the form $e^{j(\omega t - kx)}$ and found [from (d)] a pair of waves propagating in each direction on the

beam and a pair of evanescent waves. The evanescent waves are required in addition to the ordinary waves to satisfy the four boundary conditions imposed on the beam. This is in contrast to the situation in Section 10.1.2, in which propagating waves became evanescent at a frequency below some cutoff frequency. Here, all four waves are present simultaneously.

As we have seen, a boundary value problem of this kind is more conveniently solved in terms of trigonometric and hyperbolic functions, rather than complex exponentials (traveling waves). Hence we use linear combinations of the four exponential solutions to write the solution in the form

$$\xi = A \sin \alpha x + B \cos \alpha x + C \sinh \alpha x + D \cosh \alpha x, \quad (e)$$

where A , B , C , and D are to be evaluated by using the boundary conditions.

Because there is no longitudinal or transverse motion of the material at $x = 0$, two boundary conditions are

$$\xi(0) = 0, \quad (f)$$

$$\frac{d\xi}{dx}(0) = 0, \quad (g)$$

where (g) follows from the expression for δ_1 given by (11.4.16). Because $T_{11} = 0$ at $x = l$, (11.4.18) shows that

$$\frac{d^2\xi}{dx^2}(l) = 0. \quad (h)$$

The fourth boundary condition arises from the transverse force equilibrium of the beam at $x = l$. The force $f(l)$ acts on a thin element of the beam at $x = l$, as shown in Fig. 11.4.15a. This force is held in equilibrium by the shear stress T_{21} . Hence, since the volume of material within the element is vanishingly small (there is no singularity of mass at the end of the beam), we can write (note that $T_{12} = T_{21}$)

$$f = w \int_{-b}^b T_{12} dx_2. \quad (i)$$

The boundary condition at $x = l$ in terms of ξ follows by using (11.4.21) to find

$$\hat{f} = \frac{wE}{2} \frac{d^3\xi}{dx^3} \int_{-b}^b (x_2^2 - b^2) dx_2 \quad (j)$$

or

$$\hat{f} = -\frac{2w}{3} Eb^3 \frac{d^3\xi}{dx^3}(l). \quad (k)$$

The first two boundary conditions show that

$$D = -B, \quad (l)$$

$$C = -A. \quad (m)$$

These two relations, together with the second two boundary conditions, give the equations

$$\begin{aligned} A[\sin \alpha l + \sinh \alpha l] + B[\cos \alpha l + \cosh \alpha l] &= 0, \\ A[\cos \alpha l + \cosh \alpha l] - B[\sin \alpha l - \sinh \alpha l] &= 2\hat{F}, \end{aligned} \quad (n)$$

where

$$\hat{F} = \frac{3\hat{f}}{wE4b^3\alpha^3}.$$

Equations (n) provide simultaneous expressions for A and B which are solved to provide

$$A = \frac{\hat{F}(\cos \alpha l + \cosh \alpha l)}{1 + \cosh \alpha l \cos \alpha l}, \quad (o)$$

$$B = \frac{-\hat{F}(\sin \alpha l + \sinh \alpha l)}{1 + \cosh \alpha l \cos \alpha l}. \quad (p)$$

To make this manipulation we have used the identities $\cos^2 x + \sin^2 x = 1$ and $\cosh^2 x - \sinh^2 x = 1$. The constants D and C follow from (l) and (m) to complete the solution for ξ given by (e).

$$\xi = \hat{F} \left[\frac{(\cos \alpha l + \cosh \alpha l)(\sin \alpha x - \sinh \alpha x) - (\sin \alpha l + \sinh \alpha l)(\cos \alpha x - \cosh \alpha x)}{(1 + \cosh \alpha l \cos \alpha l)} \right]. \quad (q)$$

The force f might be of electrical origin, in which case it might also depend on ξ . For now we assume that the forcing function is independent of ξ , that is, that $\hat{F}$ is a given complex constant. Then (b) provides $\xi(x, t)$.

When the denominator of (q) is zero, the response to the forcing function $\hat{F}$ is infinite. This resonance condition results when the frequency (remember that α is determined by the frequency) is such that

$$\cos \alpha l = -\frac{1}{\cosh \alpha l}. \quad (r)$$

The solutions to this equation are the points at which the curves shown in Fig. 11.4.16 intersect. Hence the first four modes have frequencies such that αl is as shown in Table 11.4.2. Given the value of αl , the resonance frequency follows from Eq. (c)* as

$$\omega = \frac{(\alpha l)^2}{l^2} \left(\frac{Eb^3}{3\rho} \right)^{1/2}. \quad (s)$$

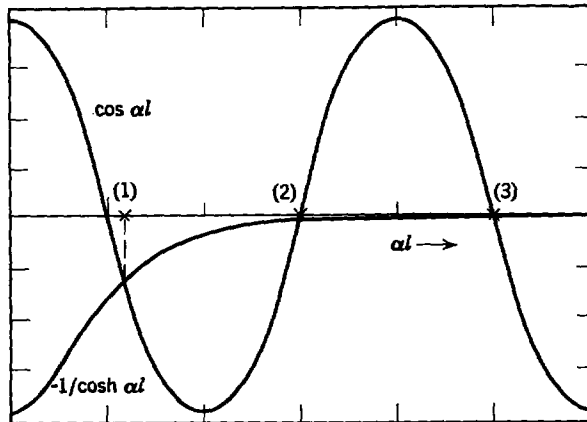


Fig. 11.4.16 Plot of the right- and left-hand sides of (r) (Example 11.4.4). The numbers indicate the solutions for the eigenvalues of the lowest three natural modes given in Table 11.4.2.

* An account of the theory of vibrating bars is given by Rayleigh in *The Theory of Sound*, 1st ed., 1877; Dover edition, 1945, p. 255, Vol. 1.

Table 11.4.2 Lowest Eigenmodes of the Beam Shown in Fig. 11.4.15*a*

Mode	(αl)
1	1.875
2	4.694
3	7.855
4	10.996

Note that the resonance frequency of any given mode varies inversely as the square of the beam length l , a fact that is easily verified by the experiment in Fig. 11.4.11. The numerator of (q) is plotted in Fig. 11.4.17 to show the instantaneous spatial variation of the deflection at frequencies close to the eigenfrequencies. The role played by the evanescent wave portion of the solution is clear from these deflections. In the lowest mode the deflection appears to have an “exponential” character, which indicates that the evanescent solutions dominate.

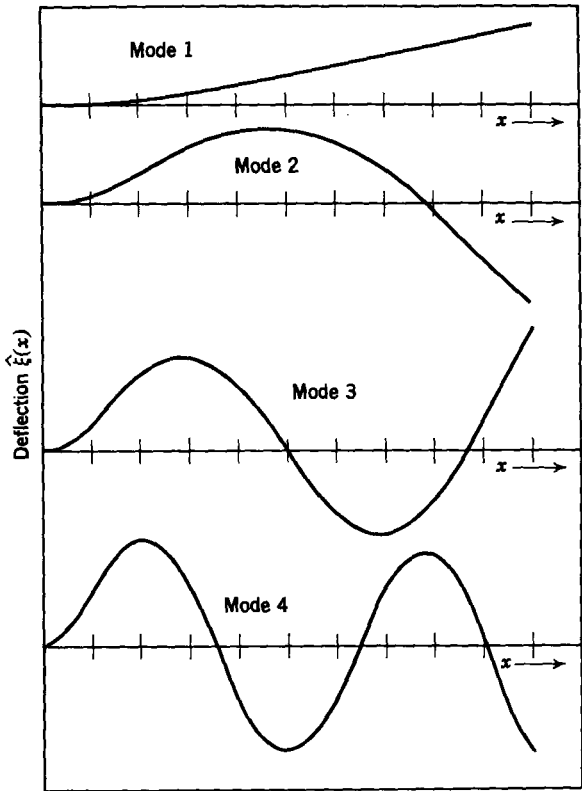


Fig. 11.4.17 Deflection of the beam as a function of the longitudinal position at an instant in time. The first four natural modes are shown, with αl as given in Table 11.4.2. The amplitude is exaggerated, with a different normalization for each mode.

By contrast the higher modes are dominated by the sinusoidal deflections of the ordinary wave solutions, with the evanescent solutions becoming apparent near the ends. This trend is also seen in Fig. 11.4.16, which shows that the higher modes (large αl) are given essentially by zeros of the function $\cos \alpha l$. These results are consistent with the notion that the evanescent waves are excited by the boundary conditions and affect only that region in the vicinity of the boundary.

The longitudinal and transverse modes considered in this section have been described in terms of quasi-one-dimensional models. As the frequency is increased, the longitudinal wavelengths take on the same magnitude as the transverse dimensions of the elastic structure. Under this condition the effect of higher order transverse modes cannot be ignored, as is illustrated in Section 11.4.3.

11.4.3 Elastic Vibrations of a Simple Guiding Structure

As mentioned in Section 11.4, the effect of boundaries is usually to couple shearing and dilatational motions of the material. As a result, the higher order modes, which become significant as the frequency is raised, are often mathematically complicated. We can, however, illustrate the basic physical effects by considering a particular class of modes composed of a purely shearing and rotational motion.*

Figure 11.4.18 shows a slab of elastic material with a thickness d . We

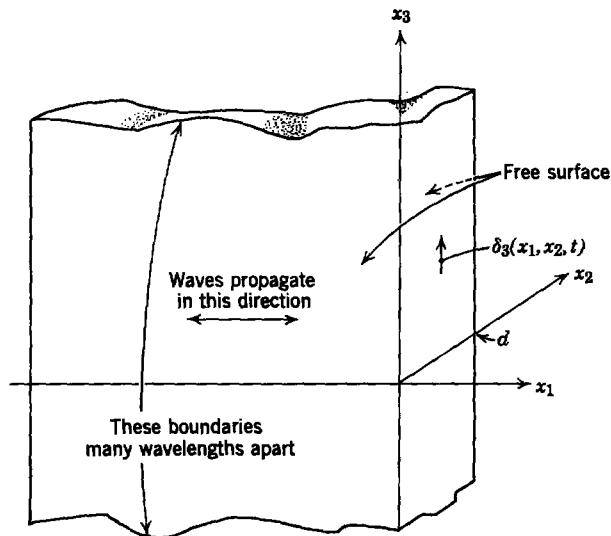


Fig. 11.4.18 Slab of elastic material with thickness d and extending to infinity in the x_3 -direction. Shearing motions of the material in the x_3 -direction are considered as they propagate in the x_1 -direction.

* For a discussion of the general modes present in elastic plates and cylinders see W. P. Mason, *Physical Acoustics*, *loc. cit.*

consider motions of the material in the x_3 -direction under the assumption that the slab has an infinite extent in the x_3 -direction. Hence displacements $\delta_3 = \delta_3(x_1, x_2, t)$ are assumed at the outset, with $\delta_1 = \delta_2 = 0$ and $\partial/\partial x_3 = 0$. These assumptions are justified if we can find solutions that satisfy (11.4.1) and the boundary conditions. The surfaces of the slab at $x_2 = 0$ and $x_2 = d$ are free; hence we require that there be no shear stresses on these surfaces:

$$T_{32}(x_1, d, t) = 0, \quad (11.4.27)$$

$$T_{32}(x_1, 0, t) = 0. \quad (11.4.28)$$

With our assumptions, (11.4.1) reduces to

$$\rho \frac{\partial^2 \delta_3}{\partial t^2} = G \left(\frac{\partial^2 \delta_3}{\partial x_1^2} + \frac{\partial^2 \delta_3}{\partial x_2^2} \right). \quad (11.4.29)$$

The boundary conditions are written in terms of δ_3 by recognizing that (11.2.32)

$$T_{32} = G \frac{\partial \delta_3}{\partial x_2}. \quad (11.4.30)$$

Except for the boundary conditions, the mathematical problem is now identical to that described in Section 10.4.1, where the two-dimensional motions of a membrane were considered; that is, (11.4.29) has a variable separable solution

$$\delta_3 = \text{Re} [X(x_1) Y(x_2) e^{j\omega t}], \quad (11.4.31)$$

and substitution shows that

$$\frac{d^2 X}{dx_1^2} + k^2 X = 0 \quad (11.4.32)$$

and

$$\frac{d^2 Y}{dx_2^2} + \alpha^2 Y = 0, \quad (11.4.33)$$

with k^2 and α^2 related by

$$k^2 + \alpha^2 = \frac{\omega^2 \rho}{G}. \quad (11.4.34)$$

The solution to (11.4.33), which satisfies the boundary conditions, is $\cos \alpha x_2$, with $\alpha = n\pi/d$, $n = 0, 1, 2, \dots$. Solutions to (11.4.32) are $e^{\pm jkx_1}$. Hence it follows that the solution (11.4.31) can be written as

$$\delta_3 = \text{Re} \cos \frac{n\pi x_2}{d} [\delta_+ e^{j(\omega t - kx_1)} + \delta_- e^{j(\omega t + kx_1)}], \quad (11.4.35)$$

where δ_+ and δ_- are complex constants determined by the longitudinal boundary conditions. For each value of n we have found solutions composed

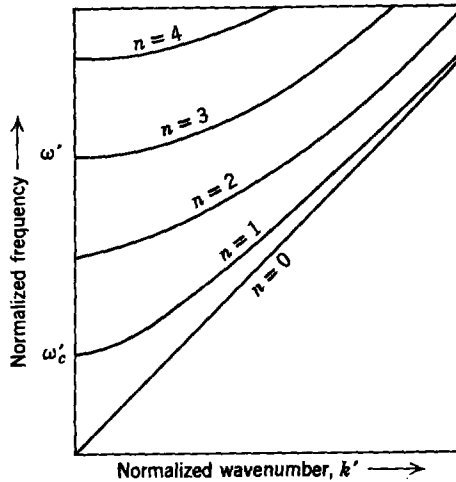


Fig. 11.4.19 Normalized frequency versus normalized wavenumber for shear modes in the elastic slab of Fig. 11.4.18: $\omega' = (\omega d/\pi)\sqrt{\rho/G}$ and $k' = kd/\pi$.

of waves that propagate along the x_1 -axis. Given the frequency ω , the wavenumber k follows from (11.4.34) as

$$k = \left[\frac{\omega^2 \rho}{G} - \left(\frac{n\pi}{d} \right)^2 \right]^{1/2}, \quad n = 0, 1, 2. \quad (11.4.36)$$

At a given frequency each of the modes has a different wavenumber and a different dependence on the transverse (x_2)-dimension. The relationship between frequency and wavenumber is shown graphically in Fig. 11.4.19. At frequencies less than $\omega' = \omega'_c$ all modes except one decay in the x_1 -direction or are evanescent in character, as we found for the membrane in Section 10.4.1. By contrast with the membrane, however, a principal mode now propagates without dispersion, even as the frequency approaches zero.

The spatial dependence of the first two modes is illustrated in Fig. 11.4.20, in which we have assumed that the frequency is below cutoff. The evanescent modes arise because of the “stiffness” introduced by the walls. The principal mode is not affected by the transverse boundary conditions, hence does not possess a cutoff frequency.

From (11.4.36) only the principal mode propagates if

$$\omega \left(\frac{\rho}{G} \right)^{1/2} < \frac{\pi}{d}. \quad (11.4.37)$$

The wavelength of the principal mode is $2\pi/k = (2\pi/\omega\sqrt{\rho/G})$; hence condition (11.4.37) can also be stated as

$$\frac{2\pi}{k} > 2d. \quad (11.4.38)$$

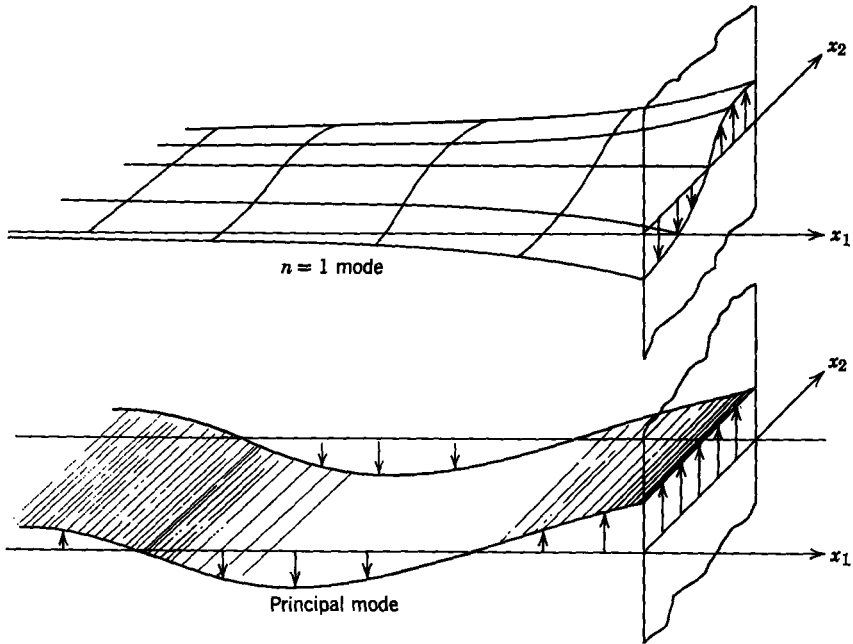


Fig. 11.4.20 Spatial distribution of the principal and $n = 1$ shear modes in an elastic slab. For the case shown the frequency is below the cutoff frequency of the $n = 1$ mode and the evanescent wave that decays in the $-x_1$ -direction is shown.

This condition illustrates the general relationship between the principal modes discussed in Section 11.4.2 and higher order modes. As long as the wavelength is long compared with the thickness, only the principal modes propagate and need be considered far from the point of excitation. As we saw in Section 10.4.1 for the membrane and in Section 11.4.2b for the thin beam, the evanescent modes are present to satisfy boundary conditions.

Modes of the kind described here are often used in delay lines. The higher modes are dispersive, hence lead to a distortion of the transmitted signal. For this reason the cutoff frequency often represents an upper limit on the frequency spectrum that can be transmitted without distortion.

11.5 ELECTROMECHANICS AND ELASTIC MEDIA

Many electromechanical interactions with elastic media can be modeled in terms of terminal pairs. This was illustrated in Chapter 9, where, even though portions of the mechanical system required continuum descriptions, the effect of electrical forces could be accounted for by means of boundary conditions. In this chapter we have confined ourselves to the three-dimensional

dynamics of elastic solids in the absence of electromechanical bulk forces. We can now readily imagine using electromechanical transducers to excite or detect the waves discussed in Section 11.4.1. At least in simple situations a discussion in this regard would parallel that given in Section 9.1.2., in which mechanical waves propagated on a thin rod. In Example 11.4.4 vibrations of an elastic beam were reduced to a terminal-pair representation that provides a convenient model for coupling to a lumped-parameter device. In a similar manner we could use a transducer to excite or detect shear waves propagating through the slab of elastic material shown in Fig. 11.4.18. By contrast, in this section we highlight a few illustrative situations in which continuum coupling with elastic media is important, but even in these cases the terminal pair concept is useful.

11.5.1 Electromagnetic Stresses and Mechanical Design

The design of electromechanical systems is often intimately concerned with material stresses produced by electromagnetic forces. A case in point is the design of large rotating machines, such as in Chapter 4. Here the energy conversion process depends on a large magnetic torque being transmitted between the rotor and stator. Because action equals reaction, the rotor and stator materials are necessarily under significant stress due to the magnetic forces; for example, this is the primary reason that conductors are placed in slots. With the conductor imbedded in a highly permeable material, the bulk of the magnetic force is on the magnetic material rather than on the conductor. If this were not the case, it would be difficult to hold the conductors down in many machines. In fact, a significant number of machine failures have been traced to fatigue of conductors and their support structures stressed by magnetic forces.

In a less obvious class of situations in which electromagnetic stresses are a major design consideration the objective is not to convert energy electromechanically. Rather the forces of electrical origin are a necessary evil. Examples in which this is the case are transformers and magnets.

In an ordinary transformer, electromechanical effects come into play in at least three mechanisms, two of which involve magnetization forces on the laminated magnetic core of the transformer. These forces arise because of inhomogeneities of the core introduced with the laminations and because of changes in the volume of the magnetic material (magnetostriction). These forces were discussed in Section 8.5.2* and are responsible for much of the noise (hum or, in transformers used for speech amplification, “transformer talk”) heard in the vicinity of an operating transformer.

A third mechanism for electromechanical effects is simply the $\mathbf{J} \times \mathbf{B}$ force density on the individual conductors in a transformer. This design consideration deserves critical attention because copper that is desirable from the point

* Summarized in Table 8.1, Appendix G.

of view of electrical conductivity tends to be lacking in mechanical strength. Transformers must be designed to withstand 25 or more times their rated currents in power applications to prevent mechanical damage under short circuit conditions. Figure 11.5.1a shows the primary and secondary windings of a distribution transformer which was intentionally subjected to currents in excess of its peak ratings. This is a step-down transformer with large

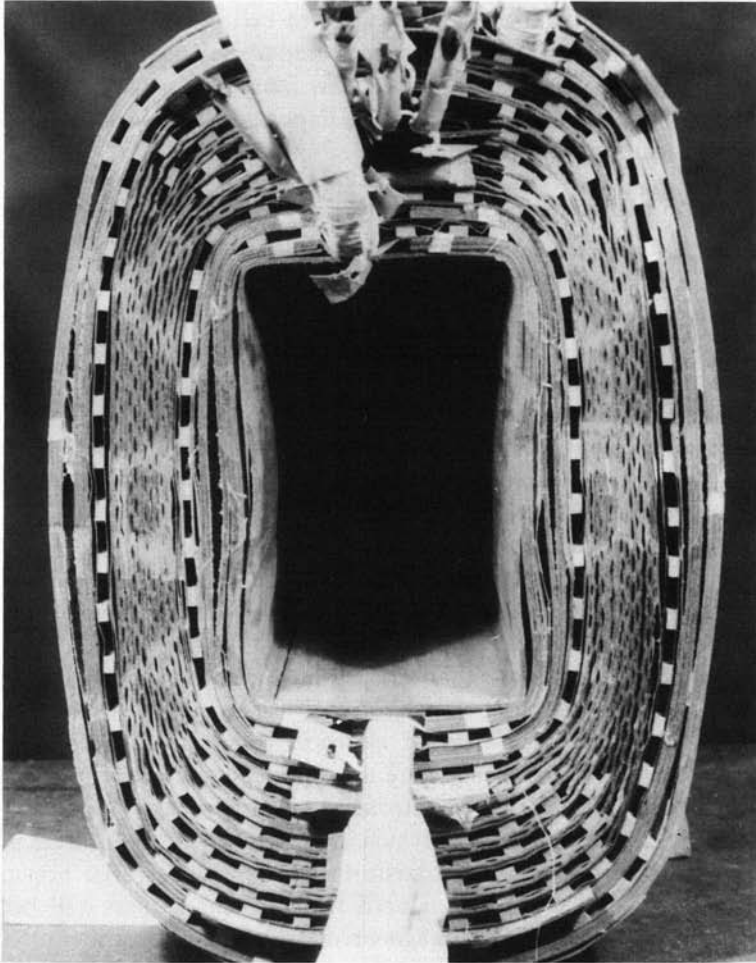


Fig. 11.5.1a End view of rectangular distribution transformer coils with core removed after being subjected to short-circuit currents in excess of design capability. Note how reaction forces on the inner secondary coil have buckled it inward on the long sides of the rectangle. Also note that forces on the outer secondary coil have rounded it outward on the long sides. Original shape of the coils on the long sides was flat. (Courtesy of the General Electric Co.)

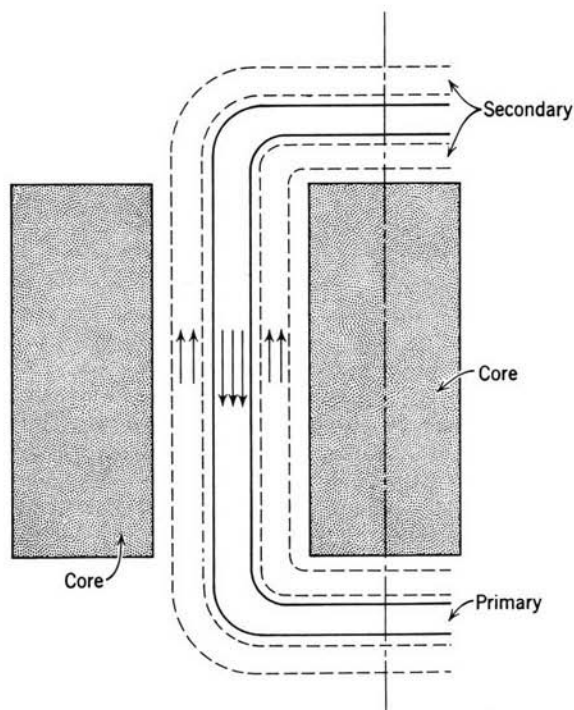


Fig. 11.5.1b Sketch of primary and secondary windings in relation to the magnetic transformer core. With the secondary short-circuited, the ampere turns in the secondary are essentially equal to those in the primary.

secondary conductors on the outside and inside and primary windings sandwiched between. The arrangement of the core and windings is sketched in Fig. 11.5.1b. The secondary windings are constructed of sheets of aluminum which were originally wound in an essentially rectangular shape. As shown in Fig. 11.5.1a, the excessive currents have distorted the secondary windings away from the primary windings. The copper secondary turns bulge inward on the inside and outward on the outside. Although, in this case, the result is not a gross mechanical failure of the structure, significant deformation of the insulation causes local damage that can lead to electrical breakdown. Also, the deformation increases the leakage reactance of the transformer. Increased leakage reactance increases regulation (voltage drops as load current increases) and this decreases the transformer efficiency, a crucial factor in distribution transformers.

So far in this chapter we have emphasized the elastic behavior of solid materials. Our main objective in this section is to draw attention to the fact that in many situations it is the *inelastic* behavior of a solid that is of interest.

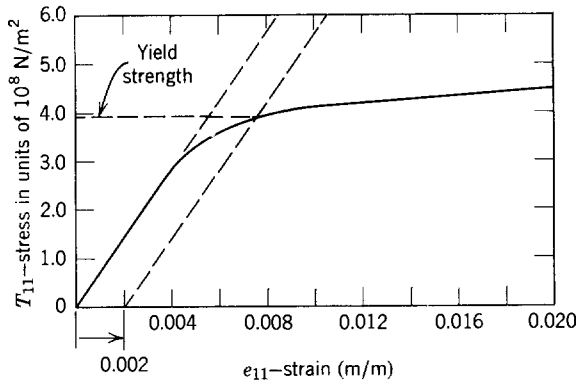


Fig. 11.5.2 Stress-strain for annealed aluminum* under tension showing definition of “yield strength” stress when limiting value of permanent strain is defined as $e_{11} = 0.002$.

If we wish to use solids to synthesize transducers, we must be careful to ensure that stresses are not so large that permanent or inelastic deformations will occur. Even more, in many situations like the one shown in Fig. 11.5.1 limiting stresses are an essential design consideration. We are then faced with the problem of defining meaningful limits on the stress that can be supported by the material. Because the inelastic behavior is an upper bound on the elastic deformation of the material, we can use the elastic theory developed in earlier sections as a starting point for computing limiting stresses.

A typical stress-strain relation for a polycrystalline metal is shown in Fig. 11.5.2. For small values of the stress and strain the relationship is essentially linear. As the stress is raised, however, a point is reached at which the resulting material strain increases more rapidly. Above this point, if the material is unloaded, it is likely that it will retain a permanent deformation. An index of the degree of this permanent set is the *yield strength* of the material, which is defined in Fig. 11.5.2. After the material has been loaded to the yield strength (the limiting stress) it is *assumed* that if it were unloaded it would return to the zero stress condition along a straight line parallel to the loading curve in the elastic range. To fix the yield strength of a material we must define the hypothetical permanent set (the strain) taken by the material when the stress is returned to zero. (In practice this might be 0.002 for metals in tension.*)

If the material has an elastic regime, it is possible to obtain an approximate prediction of material stresses that will lead to inelastic behavior by first predicting the stresses by means of the elastic model and then comparing the maximum stress to the yield strength. Generally such calculations are used to

* See S. H. Crandall and N. C. Dahl, *An Introduction to the Mechanics of Solids*, McGraw-Hill, New York, 1959, p. 173.

compute an upper bound on loading the material, with a margin of safety included in the design. The following example illustrates this procedure.

Example 11.5.1. In this example we illustrate how the simple model of an elastic beam can be used to provide insight into the limiting stresses that can be supported by current-carrying conductors in the situation illustrated in Fig. 11.5.1. We assume that the primary winding (sandwiched between the two secondary windings) will remain essentially rigid but that the secondary windings can be modeled by thin beams of the nature discussed in Section 11.4.2*b*. The problem then reduces to that illustrated in Fig. 11.5.3*a*, in which only the secondary conductors to the right of the primary are shown.

Under short circuit conditions the ampere turns in the secondary and primary are essentially equal. This means that the magnetic field between the conductors is essentially uniform and given by

$$H = \frac{I}{2w}, \quad (a)$$

for only half the secondary ampere turns are in the part of the windings shown in Fig. 11.5.3*a*.

For simplicity we assume that the section of secondary conductor can be considered as being clamped at $x_1 = 0$ and $x_1 = l$. Hence we have as boundary conditions

$$\xi(0) = 0; \quad \xi(l) = 0, \quad (b)$$

$$\frac{d\xi}{dx_1}(0) = 0; \quad \frac{d\xi}{dx_1}(l) = 0. \quad (c)$$

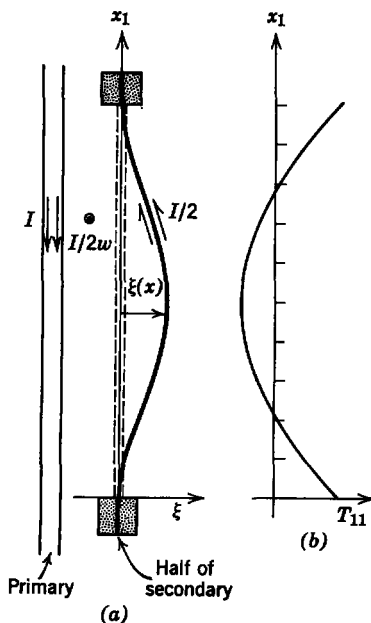


Fig. 11.5.3 (a) A simple model is used to predict elastic failure of the sheet secondary conductor. This example is a first approximation to the situation in Fig. 11.5.1; the primary is assumed to be rigid and the system has a width w into the paper. (b) Distribution of longitudinal stress T_{11} evaluated on the inside surface of the secondary.

Of course, in a transformer the currents, hence the magnetic forces, are not constant. In a distribution transformer the current alternates at 60 Hz, and the magnetic forces that depend on the square of the current are therefore composed of constant and second harmonic (120-Hz) parts. Now the conductors can respond, transducer fashion, to the alternating component of the $\mathbf{J} \times \mathbf{B}$ force.* Here, however, we are interested only in the deformations of the conductors that result over many cycles of the current. Hence we regard the magnetic force on the conductors as being constant and having its average value. Because this force varies as the square of the current, it amounts to using the rms value of the current in evaluating the magnetic force on the secondary conductors. In what follows it is assumed that I is the total rms ampere turns through the width w of the primary conductors.

Under steady conditions (11.4.26), which expresses the transverse force balance for the secondary conductors modeled as a single elastic beam, becomes

$$\frac{d^4 \xi}{dx_1^4} = \frac{3}{2b^3 E} T_2. \quad (d)$$

Here $2b$ is the thickness of the combined secondary conductors, E is the "equivalent" modulus of elasticity based on the combined conductors and insulation, and T_2 is the force per unit area acting in the transverse direction. Given T_2 , it is a simple matter to compute the deflection ξ of the beam model.

It follows from the magnetic stress tensor† that the surface force T_2 is constant and that

$$T_2 = \frac{1}{2} \mu_0 \left(\frac{I}{2w} \right)^2, \quad (e)$$

which combines with (d) to provide a simple fourth-order ordinary equation that can be integrated directly:

$$\frac{d^4 \xi}{dx_1^4} = a, \quad (f)$$

where

$$a = \frac{3\mu_0}{4b^3 E} \left(\frac{I}{2w} \right)^2.$$

Four succeeding integrations lead to a solution that involves four constants $C_1 \cdots C_4$.

$$\xi = \frac{ax_1^4}{24} + \frac{C_1 x_1^3}{6} + \frac{C_2 x_1^2}{2} + C_3 x_1 + C_4. \quad (g)$$

From boundary conditions (b) and (c) these constants are evaluated to obtain

$$\xi = \frac{al^4}{24} \left(\frac{x_1}{l} \right)^2 \left(\frac{x_1}{l} - 1 \right)^2 \quad (h)$$

for the deflection as a function of the longitudinal position x_1 . This is the deflection plotted in Fig. 11.5.3a.

So far our calculations have been based on an elastic model for the beam. The objective is to determine the values of the current that lead to permanent deformations of the secondary winding. This is done by evaluating the maximum longitudinal stress T_{11} and comparing it to that required to give elastic failure of the material according to the preceding discussions of this section.

* In fact, under conditions of extreme loading the conductors of a large transformer can be seen to "breathe" in and out at 120 Hz.

† See Table 8.1 of Appendix G.

Remember that the longitudinal stress varies linearly over the cross section of the beam (e.g., see Fig. 11.4.14). In the thin beam model this stress is related to the deflection by (11.4.18), which becomes

$$T_{11} = -x_2 E \frac{d^2 \xi}{dx_1^2} = \frac{-x_2 E a l^2}{2} \left[\left(\frac{x_1}{l} \right)^2 - \left(\frac{x_1}{l} \right) + \frac{1}{6} \right], \quad (i)$$

where x_2 is the transverse coordinate. The maximum stress is obtained at the beam surfaces, where $x_2 = \pm b$; for example, on the inside (left) surface of the beam

$$T_{11} = \left(\frac{l}{b} \right)^2 \frac{3}{8} \mu_0 \left(\frac{l}{2w} \right)^2 \left[\left(\frac{x_1}{l} \right)^2 - \left(\frac{x_1}{l} \right) + \frac{1}{6} \right]. \quad (j)$$

The manner in which this function depends on the longitudinal position is shown in Fig. 11.5.3b. At the center of the beam ($x_1 = l/2$) the stress $T_{11}(x_2 = -b)$ is negative, indicating that the material is under compression. The maximum longitudinal stress is obtained at the ends, where the material on the left side of the beam is under tension. This maximum stress on the beam is

$$T_{11}(x_2 = -b, x_1 = 0) = \left(\frac{l}{b} \right)^2 \frac{1}{16} \mu_0 \left(\frac{l}{2w} \right)^2. \quad (k)$$

Now the beam is also subject to shear stresses T_{12} , which should also be considered in determining the maximum stress. The shear stress is related to the beam deflection by (11.4.21), which shows that if $b \ll l$ the shear stress will be small compared with the longitudinal stress. It is just this fact that the beam is thin that makes the mechanical stress T_{11} much greater than the magnetic pressure. The stress T_{11} acts over the cross section $2b$ of the beam through a lever arm that is less than b to hold in equilibrium the magnetic pressure $\frac{1}{2} \mu_0 (l/2w)^2$ acting over the length l through a lever arm that is on the order of l (see Fig. 11.4.13). This is why (k) is proportional to the magnetic pressure amplified by $(l/b)^2$.

An order of magnitude calculation helps us to appreciate the significance of (k). In magnetic circuits, such as the transformer of Fig. 11.5.1, a magnetic flux density of 10 kG (1 Wb/m^2) is commonly induced. This corresponds to a magnetic pressure of

$$\frac{B^2}{2\mu_0} = \frac{1}{(2)(4\pi \cdot 10^{-7})} \approx 4 \cdot 10^5 \text{ N/m}^2.$$

If we use this number to replace the magnetic pressure $\frac{1}{2} \mu_0 (l/2w)^2$ in (k) and let $l/b = 20$, it follows that

$$T_{11}(x_2 = -b, x_1 = 0) = 2 \cdot 10^7 \text{ N/m}^2.$$

This is just above the 0.2% yield strength of annealed aluminum,* but considerably below the value in Fig. 11.5.2. The strength of aluminum can be increased considerably by cold working and alloying it with other substances. For example, considering the ability of the coil to withstand the mechanical forces imposed by short-circuit currents, a transformer designer is faced with the problem of balancing the mechanical strength of the core and coil against the cost and electrical characteristics. His problem is complicated because coil conductors that are most desirable in terms of their electrical characteristics are relatively low in mechanical strength.

A critical review of this model will show that we have ignored many facets of the problem that could be of major importance; for example, the secondary winding of the actual transformer is not a homogeneous solid but rather is composed of layers of conducting and

* A. E. Knowlton, *Standard Handbook for Electrical Engineers*, McGraw-Hill, New York, 1957, Section 4, p. 695.

insulating sheets. In practice we would probably measure an "equivalent" modulus of elasticity for this combination, although to be rigorous account should be taken of the anisotropic material in the basic model of the elastic beam.

Also, the inelastic behavior of materials is more complicated than might be deduced from our comments so far. The material is subject to repeated loading and unloading due to the second-harmonic force. This can result in a type of failure analogous to that found when a wire is bent back and forth repeatedly until it breaks. It depends on the number of cycles as well as the maximum stress and is therefore referred to as *fatigue* failure.

To complicate the picture still further, when materials are subjected to a constant stress over a long period of time, it is found that the strain has an initial value that can be predicted from the stress-strain relation but continues to increase with time. This *creep* phenomenon can eventually lead to the failure of the material. Copper is an example of a material that displays creep. Further discussion of the inelastic behavior of materials is beyond the scope of this book but should be recognized as required for the understanding of how materials are used in electromechanical systems.

11.5.2 Simple Continuum Transducers

This chapter is concluded with examples that show how quasi-one-dimensional models of elastic structures can be used in the design of electro-mechanical transducers.

11.5.2a Variable Capacitance Coupling

We begin with a situation that involves an electromechanical coupling similar to that studied back as far as Chapter 3—variable capacitance coupling. The object is to develop a simple and reliable low-frequency notch filter. It is required that the frequency of the notch be tuned by varying a voltage.

Example 11.5.2. An electromechanical filter, having as its basic element a simple cantilevered beam, is shown in Fig. 11.5.4. The beam, which is at the potential V_o , is free to vibrate between plane-parallel electrodes, and the input signal is imposed on the left electrode. Because v_i is much less than V_o , this produces a force on the beam proportional to the input signal. The beam deflections lead to a change in capacitance between the beam and the plate to the right. The resulting current through the resistance R is therefore proportional to the input signal with an amplitude determined by the response of the beam to the input.

It is assumed that the resistance R is small enough that the electrode to the right can be considered as grounded. Further, it is assumed that the capacitive reactance due to C is small compared with R , so that v_o can be taken as the voltage drop across the resistance.

The equation of motion for the beam is (11.4.26):

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{Eb^2}{3\rho} \frac{\partial^4 \xi}{\partial x_1^4} = \frac{T_2}{2b\rho}. \quad (a)$$

We assume that $d \ll l$ so that the transverse force T_2 is simply the difference in Maxwell stress* acting on the opposite surfaces of the beam:

$$T_2 = T_{22}^a - T_{22}^b = \frac{1}{2}\epsilon_0 \left[\frac{(V_o + v_i)^2}{(d - \xi)^2} - \frac{V_o^2}{(d + \xi)^2} \right]. \quad (b)$$

* Table 8.1, Appendix G.

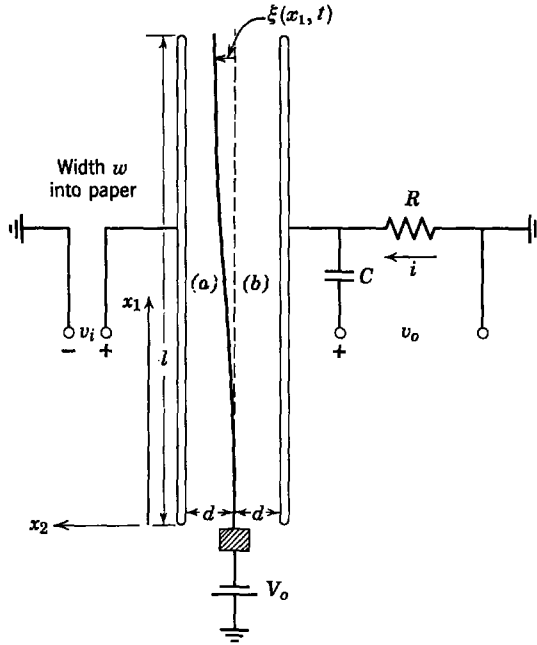


Fig. 11.5.4 A cantilevered beam has the potential V_o relative to plane-parallel driving and detecting electrodes. This device might be used as a low-frequency voltage tunable electro-mechanical filter.

For small deflections ξ and input voltage v_i this becomes

$$T_2 = 2\epsilon_0 \left(\frac{V_o}{d} \right)^2 \frac{\xi}{d} + \frac{\epsilon_0 V_o}{d^2} v_i. \quad (c)$$

The equation of motion (a) is then augmented by two additional forces, one having the nature of a spring with a negative spring constant and the other a driving force proportional to the driving voltage:

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{Eb^2}{3\rho} \frac{\partial^4 \xi}{\partial x_1^4} = \frac{\epsilon_0 V_o^2}{b\rho d^3} \xi + \frac{\epsilon_0 V_o}{2b\rho d^2} v_i. \quad (d)$$

We confine attention to the sinusoidal steady-state response of the system and so assume that the drive and response have the form

$$\begin{aligned} v_i &= \text{Re } \hat{v}_i e^{j\omega t}, \\ \xi &= \text{Re } \hat{\xi}(x_1) e^{j\omega t}. \end{aligned} \quad (e)$$

Then (d) becomes

$$\frac{d^4 \hat{\xi}}{dx_1^4} - \alpha^4 \hat{\xi} = P \hat{v}_i, \quad (f)$$

where

$$\alpha^4 = \left[\frac{\epsilon_0 V_o^2}{d^3 b \rho} + \omega^2 \right] \frac{3 \rho}{E b^2}$$

$$P = \frac{3}{2} \frac{\epsilon_0 V_o}{d^2 b^3 E}.$$

This inhomogeneous ordinary equation has a homogeneous solution which is identical in form to that studied in Example 11.4.4 of Section 11.4.2*b*. In addition, there is now an inhomogeneous solution which, because the right-hand side of (f) is a constant, is simply a constant. The complete solution is

$$\xi = A \sin \alpha x_1 + B \cos \alpha x_1 + C \sinh \alpha x_1 + D \cosh \alpha x_1 - \frac{P \hat{v}_i}{\alpha^4}. \quad (g)$$

The boundary conditions on the beam which determine the constants A , B , C , and D require that the clamped end of the beam be constrained so that no longitudinal or transverse displacement is there and that the free end of the beam at $x_1 = l$ is free of shear and longitudinal stress. In terms of the transverse displacement ξ of the beam, these conditions are

$$\begin{aligned} \xi(0) &= 0; & \frac{d^2 \xi}{dx_1^2}(l) &= 0, \\ \frac{d \xi}{dx_1}(0) &= 0; & \frac{d^3 \xi}{dx_1^3}(l) &= 0. \end{aligned} \quad (h)$$

These conditions require that the following four simultaneous equations be satisfied:

$$\begin{aligned} A(0) + B(1) + C(0) + D(1) &= P \hat{v}_i / \alpha^4, \\ A(1) + B(0) + C(1) + D(0) &= 0, \\ A(-\sin \alpha l) + B(-\cos \alpha l) + C(\sinh \alpha l) + D(\cosh \alpha l) &= 0, \\ A(-\cos \alpha l) + B(\sin \alpha l) + C(\cosh \alpha l) + D(\sinh \alpha l) &= 0. \end{aligned} \quad (i)$$

The constants A , B , C , and D follow from these equations and the deflection of the beam is now known.

To compute the output voltage it is necessary first to recognize that the surface charge density on the right plate is

$$\sigma_f(x_1, t) = \frac{\epsilon_0 V_o}{d + \xi} \approx \frac{\epsilon_0 V_o}{d} - \frac{\epsilon_0 V_o \xi}{d^2} \quad (j)$$

Then it follows that because the current through the resistance is the time rate of change of the total charge on the plate to the right

$$\hat{v}_o = -Ri = j\omega R \frac{\omega \epsilon_0 V_o}{d^2} \int_0^l \xi dx_1. \quad (k)$$

It is a straightforward matter to carry out this integration, since ξ is given by (g). Note from (i) that each of the constants is proportional to $\hat{v}_i$ and inversely proportional to the determinant of the coefficients $\Delta(\omega)$. Hence (k) for the transfer response has the form

$$\hat{v}_o = \frac{H(\omega)}{\Delta(\omega)} \hat{v}_i, \quad (l)$$

and the poles of the transfer function are given by

$$\Delta(\omega) = 0. \quad (\text{m})$$

These are the same poles found for the beam in Example 11.4.4; that is, the determinant of the coefficients is zero if (remember $\alpha = \alpha(\omega)$)

$$1 + \cosh \alpha l \cos \alpha l = 0. \quad (\text{n})$$

The roots of this expression are given in Table 11.4.2.

If we call the roots of (n) $(\alpha l)_n$, it follows that the resonance frequencies are given by

$$\omega = \pm \left(\omega_n^2 - \frac{\epsilon_0 V_o^2}{d^3 b \rho} \right)^{1/2}, \quad (\text{o})$$

where

$$\omega_n = \frac{(\alpha l)_n^2}{l^2} \left(\frac{E b^3}{3 \rho} \right)^{1/2} \quad (\text{p})$$

are the resonance frequencies of the beam without electromechanical coupling. At the frequencies given by (o) there is a resonance in the transfer function unless $H(\omega)$ happens to be zero. Note that these resonance frequencies can be tuned by varying the voltage V_o . As we might have expected at the outset, (o) shows that the beam has an unstable equilibrium at $\xi = 0$ when the lowest resonance frequency is reduced to zero and these lowest eigenfrequencies become imaginary. From (o) the condition for instability is

$$\frac{\epsilon_0 V_o^2}{d^3 b \rho} = \omega_1^2. \quad (\text{q})$$

11.5.2b Magnetostrictive Coupling

The subject of magnetostriction in solids is sufficiently complex that a comprehensive treatment is inappropriate here. We can, however, gain a considerable qualitative insight into the subject by considering one-dimensional motions of a thin rod subject to magnetostrictive forces. In this context these forces can be viewed as described by the force density developed in Section 8.5.2.* There are two reasons why the force density developed in Chapter 8 is not entirely adequate. First of all, there is no guarantee that a solid remains isotropic after a magnetic field is applied, even though it may be isotropic in the absence of a magnetic field. Second, solids that exhibit significant magnetostrictive behavior tend to be magnetic; for example nickel and nickel iron alloys are commonly used in magnetostrictive transducers.† In these materials $\mathbf{B}$ is a linear function of $\mathbf{H}$ only over a limited range of $\mathbf{H}$. Hence the permeability μ relates $\mathbf{B}$ and $\mathbf{H}$ only so long as $\mathbf{B}$ is much less than its saturation value.

By limiting ourselves to one-dimensional motions and sufficiently small magnetic field intensities that $\mathbf{B} = \mu \mathbf{H}$ we can use the results of Chapter 8

* Table 8.1, Appendix G.

† A discussion of magnetostriction, including material characteristics and applications to the design of electronic devices is given in W. P. Mason, *Electromechanical Transducers and Wave Filters*, Van Nostrand, Princeton, New Jersey, 1958, 2nd. ed., p. 215.

to gain a meaningful understanding of the basic magnetostrictive interaction. Actually, most transducers are modeled as one-dimensional, and nonlinear effects are accounted for empirically by straightforward extensions of the linear model.

As discussed in Section 8.5, we can think of magnetostrictive interactions as resulting because dilatational motions of the material, which lead to local changes in the material density, also lead to a change in the local magnetic energy storage. This makes it possible to exert a magnetic force on a volume where material is initially homogeneous. An example in which this is a desirable attribute is given in Fig. 11.5.5a. There, a magnetic wire constitutes the propagating structure for an acoustic delay line. The device, which might be used as either an input or an output transducer, is easily moved along the wire to effect a change in the delay time. Now, if the wire were capable of only rigid body motion, there could be no longitudinal force produced by the input signal. The material must change its volume in order to effect any change in the magnetic energy stored in the system as a function of material displacements. This should be contrasted with the type of magnetization

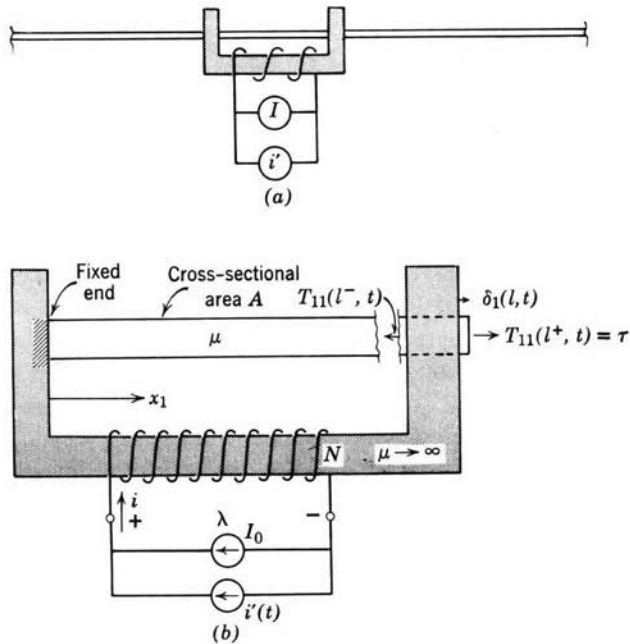


Fig. 11.5.5 Magnetostrictive transducer. Compressional motions are excited by the input current $i'(t)$: (a) transducer placed at a variable position along a magnetic wire which might be used as a delay line; (b) detail of a transducer in which the wire is fixed at one end of a magnetic circuit.

forces used in the delay line of Section 9.1.2, where the force resulted because of inhomogeneities (air gaps) in the material. The inconvenience of this mode of coupling is made apparent in Figure 9.1.14.

The following example illustrates the modeling of a magnetostrictive transducer.

Example 11.5.3. The transducer shown in Fig. 11.5.5*b* characterizes devices that have seen wide application. An input signal $i'(t)$ is transduced into a force τ that acts through a displacement $\delta_1(l, t)$, we wish to find the mechanical terminal relation between τ and $\delta_1(l, t)$. To simplify our discussion, it is assumed that the magnetostrictive material takes the form of a rod with cross-sectional area A and one end fixed at $x_1 = 0$.

According to (8.5.38), the rod is subject to the force density

$$\mathbf{F} = -\frac{1}{2}\mathbf{H} \cdot \mathbf{H} \nabla \mu + \nabla \left(\frac{1}{2}\mathbf{H} \cdot \mathbf{H} \frac{\partial \mu}{\partial \rho} \right) \quad (\text{a})$$

In the rod it is reasonable to view μ and ρ as being uniquely related, $\mu = \mu(\rho)$. Note that if a material is inhomogeneous this is not a meaningful statement; for example, a material could have a uniform density but be composed of regions occupied by materials of different permeabilities μ . On the basis of the restriction that the force is valid only in the interior of the rod so that $\mu = \mu(\rho)$, we can write

$$\nabla \mu = \frac{\partial \mu}{\partial \rho} \nabla \rho. \quad (\text{b})$$

Then (a) reduces to

$$\mathbf{F} = \rho \nabla \left(\frac{1}{2} \frac{\partial \mu}{\partial \rho} \mathbf{H} \cdot \mathbf{H} \right). \quad (\text{c})$$

In what follows we make the assumption that insofar as the force is concerned variations in ρ can be ignored so that the mass density multiplying the gradient term in (c) is replaced by ρ_0 . Note that this does not say that ρ is actually a constant, but simply that it can be approximated as constant in (c). Then the longitudinal equation of motion for the rod becomes

$$\rho_0 \frac{\partial^2 \delta_1}{\partial t^2} = \frac{\partial}{\partial x_1} \left(E \frac{\partial \delta_1}{\partial x_1} + \frac{\rho_0}{2} \frac{\partial \mu}{\partial \rho} \cdot \mathbf{H} \right). \quad (\text{d})$$

It is a good approximation to ignore the effect of mechanical deformation on the field. This means that $\mathbf{H}$ is uniform over the length of the rod between $x_1 = 0$ and $x_1 = l$. Over this range, material displacements are then governed by the simple wave equation for the thin rod

$$\frac{\partial^2 \delta_1}{\partial t^2} = \left(\frac{E}{\rho_0} \right)^{1/2} \frac{\partial^2 \delta_1}{\partial x_1^2}. \quad (\text{e})$$

The influence of the magnetostrictive force is felt through the boundary condition at $x_1 = l$. Force equilibrium for a section of the rod in the neighborhood of $x_1 = l$ is shown in Fig. 11.5.6.

The quantity in brackets on the right-hand side of (d) is the longitudinal stress transmitted along the rod. Hence the left face of the section of material shown in Fig. 11.5.6 is subject to this stress acting over the cross section A of the rod. Within the length h of the material section the lines of magnetic field intensity are shunted into the magnetic circuit. The stress on the right surface is simply τA , where τ is the mechanical stress due to the system

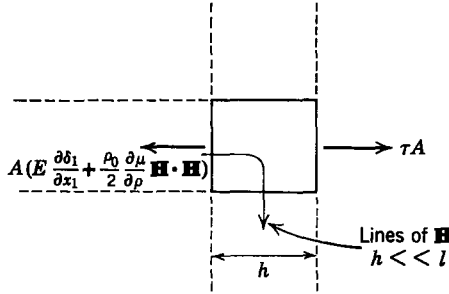


Fig. 11.5.6 A section of the rod shown in Fig. 11.5.5 in the neighborhood of $x_1 = l$. This section is assumed to have negligible length h compared with l .

being driven. We can argue that there are no shear stresses acting on the volume by recognizing from (c) that the magnetostriction force is capable only of producing normal stresses. Hence at $x = l$,

$$E \frac{\partial \delta_1}{\partial x_1}(l, t) + \rho_0 \frac{1}{2} \frac{\partial \mu}{\partial \rho} \mathbf{H}^2(l, t) = \tau. \quad (f)$$

This becomes a useful boundary condition once $\mathbf{H}$ is evaluated in terms of the current i . For illustrative purposes we assume that the magnetic circuit is of much greater permeability than the magnetostrictive wire. Then

$$H = \frac{Ni}{l} = \frac{N}{l} (I_0 + i'). \quad (g)$$

The force equilibrium represented by (f) has a constant part due to the bias current I_0 and a dynamic part due to small perturbations $i'(t)$ in the transducer current. We assume that the constant part is balanced out by a constant part of τ due to the system to the right. Then the linearized dynamic part of (f) is

$$\frac{\partial \delta'}{\partial x_1}(l, t) + \gamma i' = \frac{\tau'}{E}, \quad \text{where } \gamma = \frac{\rho_0}{E} \frac{\partial \mu}{\partial \rho} \left(\frac{N}{l} \right)^2 I_0, \quad (h)$$

and τ' and δ' are, respectively, the time-varying parts of the stress acting on the right surface of the transducer rod and displacement $\delta_1(x_1, t)$.

Our objective is to characterize the transducer by the relation between τ' and $\delta'(l, t)$, given the input signal $i'(t)$. This is easily accomplished for sinusoidal steady-state solutions in the form of

$$\begin{aligned} i' &= \text{Re } i e^{j\omega t}, \\ \delta' &= \text{Re } \delta(x_1) e^{j\omega t}, \\ \tau' &= \text{Re } \hat{\tau} e^{j\omega t} \end{aligned} \quad (i)$$

by recognizing that the solution to (e), which satisfies the condition that there be no displacement at $x_1 = 0$, is

$$\delta = C \sin kx_1, \quad k = \omega \left(\frac{\rho_0}{E} \right)^{1/2}. \quad (j)$$

The complex amplitudes of (i) must satisfy a further condition represented by (h),

$$Ck \cos kl + \gamma i = \frac{\hat{\tau}}{E}. \quad (k)$$

Finally, it follows from (j) that $C = \delta(l)/\sin kl$ and that this last expression becomes the required terminal relation between the mechanical variables $\delta'(l, t)$ and $\tau'(t)$ and the driving signal $i'(t)$.

$$\gamma i = \frac{\hat{\tau}}{E} - \delta(l)k \cot kl. \quad (l)$$

This terminal relation is all that is required to represent the magnetostrictive transducer as it affects the medium being driven; for example, if the transducer were used to drive a rod to the right (l) would constitute a boundary condition to be used at $x_1 = l$.

The constant γ can be positive or negative depending on the properties of the rod. As made familiar by preceding chapters, the transducer has a linear response only if it is biased by an external source such as I_0 .

11.5.2c Piezoelectric Coupling

A salient feature of all the mechanisms for electromechanical coupling so far discussed has been that electromagnetic forces depend on the square of the applied currents, potentials, or other electrical excitations. This has meant that to obtain an electromagnetic force proportional to the applied signal a bias field is required. It has also been necessary to provide a bias field in situations in which a mechanical motion is to be transduced into an electrical signal. The magnetostrictive interaction discussed in Section 11.5.2b illustrates this point. The bias current I_0 is required to make the force a linear function of the input signal $i'(t)$. This bias current is also required if the transducer is to be used to detect the motion of the magnetized rod, as, for example, at the output end of a delay line.

Piezoelectric and piezomagnetic forms of electromechanical coupling are of interest because in effect they provide their own internal bias. The dielectric bar shown in Fig. 11.5.7 is an example of a piezoelectric transducer. That there are new ingredients to this physical situation is apparent from two simple experiments. First, suppose that a voltage is applied between the upper and lower electroded surfaces of the bar. The result is an expansion or contraction of the bar in the x -direction, depending on the sign of the applied voltage. The mechanical response reflects the sign of the applied signal. Second, suppose that the bar is stretched or compressed along the x -axis. A proportionate voltage will be developed across the terminals. These electrical-to-mechanical and mechanical-to-electrical effects are similar to those found in a transducer with an internal bias. In piezoelectric materials the effect of the bias is intrinsic to the material.

Materials that display piezoelectric properties can be either single crystals, for example, quartz, or polycrystalline ferroelectrics such as barium titanate ceramics. In the latter materials the "bias" referred to previously is provided

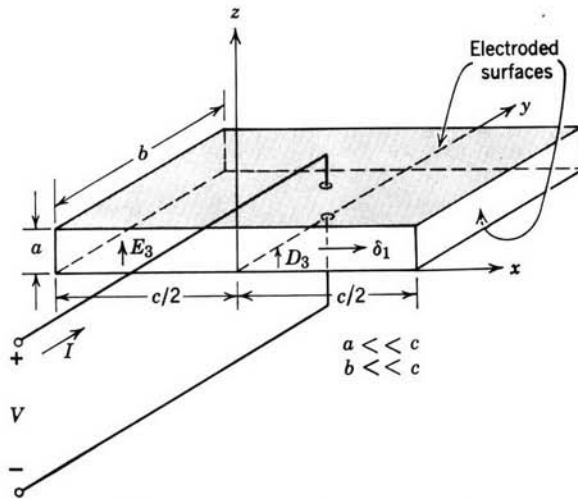


Fig. 11.5.7 Piezoelectric length expander bar.

by a permanent polarization. In single crystals the piezoelectric phenomenon is brought about by asymmetry in the crystal structure. In general, this subject therefore involves the elastic and electrical behavior of anisotropic solids. In the following introduction to this class of electromechanical interaction attention is confined to a particular one-dimensional type of interaction which allows us to develop some insight into the subject without becoming involved with general statements about the dynamics of anisotropic media.*

As might be imagined from the analogy between the piezoelectric transducer and the biased transducer, it is impossible to distinguish between electrical and mechanical forces in piezoelectric materials. If we refer to the total stress in the material as T_{11} , the electrical constitutive law relating D_3 and E_3 is

$$D_3 = \epsilon E_3 + \gamma T_{11}. \quad (11.5.1)$$

Here we have confined attention to quasi-one-dimensional motions of the bar along the x -axis and a crystal configuration such that the induced electric field is in the z -direction. Thus

$$\delta = \delta_1(x, t)\mathbf{i}_1, \quad \mathbf{E} = E_3(x, t)\mathbf{i}_3. \quad (11.5.2)$$

The mechanical constitutive law, which represents a generalization of the stress-strain relation, is

$$e_{11} = \gamma E_3 + S T_{11}. \quad (11.5.3)$$

* For a more general discussion, see W. P. Mason, *Physical Acoustics* Vol. 1, part A, Academic, New York, 1964 p. 170.

The parameter S will be recognized as the reciprocal of the modulus of elasticity. Note that the same constant γ appears in (11.5.1) and (11.5.3) to account for the electromechanical coupling. This is a consequence of a reciprocity condition, based on conservation of energy in much the same spirit discussed in Section 3.1.2c. To see this consider a section of the bar with length Δx which is subject to a slowly varying stress T_{11} , as shown in Fig. 11.5.8. T_{11} is the total stress (mechanical plus electrical), hence for slow variations it is constant over the length Δx of the section. The work done on the sample as it undergoes the incremental displacement $d\delta_1$ is

$$ab[T_{11} d\delta_1(x + \Delta x) - T_{11} d\delta_1(x)] \simeq abT_{11} \Delta x d\left(\frac{\partial \delta_1}{\partial x}\right) = abT_{11} \Delta x de_{11}. \quad (11.5.4)$$

Energy can also be added to the sample through the electrical terminals. The charge on the upper electrode is $q = -\Delta x b D_3$ and the voltage between the electrodes is $-aE_3$. A change in the charge dq on the upper plate corresponds to an addition of energy through the electrical terminals given by

$$v dq = ab \Delta x E_3 dD_3. \quad (11.5.5)$$

It is now possible to write a conservation of energy equation by defining the energy density (mechanical and electrical) within the element as w :

$$ab \Delta x E_3 dD_3 + ab \Delta x T_{11} de_{11} = ab \Delta x dw \quad (11.5.6)$$

or

$$E_3 dD_3 + T_{11} de_{11} = dw. \quad (11.5.7)$$

Note that the thermodynamic subsystem described by this conservation of energy equation includes energy storage in the elastic deformation of the material. This is necessary because we cannot distinguish between mechanical and electrical stresses as we can in Chapter 3, where we consider forces f^e that are zero with the electrical terminals unexcited. It is appropriate to think of D_3 and e_{11} in (11.5.7) as thermodynamically independent variables. This representation is similar to that used in Chapter 3, in which D_3 would be the charge q and e_{11} would be the mechanical displacement. To make E_3 and T_{11} (which are analogous to the voltage and force) the independent variables we use Legendre's dual transformation (see Section 3.1.2b) to write (11.5.7) as

$$D_3 dE_3 + e_{11} dT_{11} = dw', \quad (11.5.8)$$

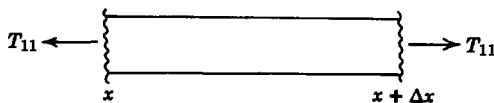


Fig. 11.5.8 Incremental length of bar shown in Fig. 11.5.7.

where the coenergy density w' is defined as

$$w' = E_3 D_3 + T_{11} e_{11} - w. \quad (11.5.9)$$

Because $w' = w'(E_3, T_{11})$, we can also write

$$\frac{\partial w'}{\partial E_3} dE_3 + \frac{\partial w'}{\partial T_{11}} dT_{11} = dw'. \quad (11.5.10)$$

Then comparison of (11.5.8) and (11.5.10) shows that

$$D_3 = \frac{\partial w'}{\partial E_3}, \quad e_{11} = \frac{\partial w'}{\partial T_{11}}. \quad (11.5.11)$$

It finally follows from this last pair of equations that

$$\frac{\partial D_3}{\partial T_{11}} = \frac{\partial e_{11}}{\partial E_3}, \quad (11.5.12)$$

which is the desired reciprocity condition. The same coefficient γ appears in (11.5.1) and (11.5.3) because the electromechanical coupling is conservative. The following example indicates how these constitutive laws can be the basis for describing the electromechanical dynamics of the bar.

Example 11.5.4. We wish to determine the electrical input admittance to the device shown in Fig. 11.5.7. In this case the electric field intensity E_3 is related to the potential V by

$$E_3 = -\frac{V}{a}. \quad (a)$$

Because E_3 is independent of x , the equation of motion in the bulk of the material does not involve electromechanical coupling; that is, the force equation in the x -direction is

$$\rho \frac{\partial^2 \delta_1}{\partial t^2} = \frac{\partial T_{11}}{\partial x}, \quad (b)$$

and from (11.5.3), in which E_3 is independent of x , this becomes

$$\rho \frac{\partial^2 \delta_1}{\partial t^2} = \frac{1}{S} \frac{\partial^2 \delta_1}{\partial x^2}. \quad (c)$$

The boundary conditions at the free ends of the bar, however, do reflect the effect of the electrical input. It follows from (11.5.3) that because $T_{11}(-c/2, t) = 0$ and $T_{11}(c/2, t) = 0$

$$\frac{\partial \delta_1}{\partial x} \left(\frac{-c}{2}, t \right) = \frac{-\gamma V}{a}, \quad \frac{\partial \delta_1}{\partial x} \left(\frac{c}{2}, t \right) = \frac{-\gamma V}{a} \quad (d)$$

The input admittance is defined as

$$Y = \frac{\hat{I}}{\hat{V}}, \quad V = \operatorname{Re} \hat{V} e^{j\omega t}, \quad (e)$$

$$I = \operatorname{Re} \hat{I} e^{j\omega t},$$

where by conservation of charge on the upper plate

$$\hat{I} = -bj\omega \int_{-c/2}^{c/2} D_3 dx. \quad (f)$$

In view of (11.5.1), (11.5.3), and (a), this expression can also be written as

$$\hat{I} = -j\omega b \int_{-c/2}^{c/2} \left[\frac{\gamma}{S} \frac{d\delta}{dx} - \frac{\epsilon V}{a} (1 - K^2) \right] dx, \quad K^2 = \frac{\gamma^2}{\epsilon S}. \quad (g)$$

To proceed in the computation of the input current we require a knowledge of the distribution of the strain e_{11} over the length c of the transducer, which is obtained by solving the bulk equation (c) subject to boundary conditions (d). Solutions take the form $\delta = \operatorname{Re} \delta(x) \exp j\omega t$, where

$$\delta = A \sin kx + B \cos kx \quad \begin{cases} k = \frac{\omega}{\sqrt{\rho S}}, \\ \delta_1 = \operatorname{Re} \delta(x) e^{j\omega t}. \end{cases} \quad (h)$$

The boundary conditions require that

$$A \cos \left(\frac{kc}{2} \right) + B \sin \left(\frac{kc}{2} \right) = -\frac{\gamma \hat{V}}{ak}, \quad (i)$$

$$A \cos \left(\frac{kc}{2} \right) - B \sin \left(\frac{kc}{2} \right) = -\frac{\gamma \hat{V}}{ak}.$$

These conditions show that $B = 0$ unless $kc/2$ is a multiple of π . In what follows we assume that the driving frequency does not coincide with one of these natural frequencies of the even modes. Only the odd modes are excited by the electrical input. Then by adding the two equations of (i)

$$A = \frac{-\gamma \hat{V}}{ak \cos (kc/2)}. \quad (j)$$

It is now possible to use (h) and (j) to evaluate the current as given in (g). Division of this expression by the voltage $\hat{V}$ gives the required input admittance.

$$Y = j\omega \left(\frac{bc}{a} \right) \epsilon \left[(1 - K^2) + \frac{K^2 \tan (kc/2)}{(kc/2)} \right]. \quad (k)$$

In the absence of piezoelectric coupling the coupling coefficient K is zero and (k) reduces to the admittance of a parallel plate capacitor with a dielectric of permittivity ϵ . Even with the coupling the system appears as a simple capacitance at low frequencies. (Remember that k is proportional to the frequency so that, in the limit in which $\omega \rightarrow 0$, $kc \rightarrow 0$ and the last term in brackets reduces to K^2 .)

As might be expected from the fact that the bar supports elastic waves, there are resonances in the response to a driving potential. The admittance is infinite at frequencies such that $\cos(kc/2) = 0$. For this reason the transducer is often used as a resonator with a single electrical terminal pair. Operation is then limited to frequencies in the neighborhood of one of the infinite admittance points. In this case the electromechanical system can be modeled by the electrical circuit shown in Fig. 11.5.9, which has the admittance

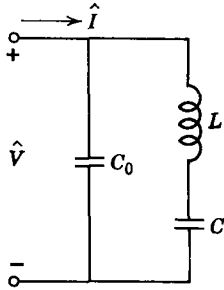


Fig. 11.5.9 Equivalent circuit for expander bar piezoelectric resonator.

$$Y = j\omega \left[C_0 + \frac{1/L}{(1/\sqrt{LC} + \omega)(1/\sqrt{LC} - \omega)} \right], \quad (l)$$

which for ω approximately equal to $1/\sqrt{LC}$ can also be written as

$$Y = j\omega \left[C_0 + \frac{\sqrt{C/L}}{2} \left(\frac{1}{\sqrt{LC}} - \omega \right) \right]. \quad (m)$$

In the neighborhood of the first resonance (k) can be written in this same form by expanding the second term in brackets about the first resonance frequency:

$$\omega = \omega_0 + \omega', \quad \omega_0 = \frac{\pi}{c\sqrt{\rho S}}. \quad (n)$$

Then

$$\frac{kc}{2} \approx \omega_0 \sqrt{\rho S} \frac{c}{2}, \quad (o)$$

$$\cot\left(\frac{kc}{2}\right) \approx -\frac{c}{2} \sqrt{\rho S} \omega',$$

and (k) becomes

$$Y = j\omega \left(\frac{bc}{a} \right) \epsilon \left[(1 - K^2) + \frac{4K^2}{c\pi\sqrt{\rho S}(\omega_0 - \omega)} \right]. \quad (p)$$

Comparison of terms in (m) and (p) shows that the equivalent parameters in the electrical circuit of Fig. 11.5.9 are

$$C_0 = \frac{bc}{a} \epsilon (1 - K^2),$$

$$L = \frac{cS^2 \rho a}{8\gamma^2 b},$$

$$C = \frac{8K^2 \epsilon bc}{a\pi^2}.$$

Of course, even though we have represented the device by an electrical equivalent circuit, it is apparent from the expressions for L and C that the resonance is electromechanical in

nature. The transducer is one way of obtaining an extremely large equivalent L . In practice effects of damping would come into play. The effects of losses would introduce an equivalent resistance into the L - C branch of the equivalent circuit.

The simple piezoelectric resonator discussed in the preceding example can provoke only a small awareness of the wide variety of uses to which piezoelectric phenomena can be put. Much of the attractiveness of the devices based on this interaction is related to their small size and great reliability. Figure 11.5.10 shows a pair of devices that involve the same expander modes as discussed two-dimensionally in the example. Here a thin slab of lead titanate zirconate has several electroded regions, hence constitutes a multi-terminal pair system capable of performing logic and modulator functions. The relative size of the devices is apparent from the figure.

11.6 DISCUSSION

In this chapter we have extended the concepts of Chapters 9 and 10, which were developed by using one-dimensional elastic models, to obtain mathematical models for more complex situations.

This chapter completes our introduction to electromechanical interactions with elastic media. We now proceed to a consideration of electromechanical interactions with fluids.

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Fig. 11.5.10 A pair of piezoelectric devices with several electrical terminal pairs. Here the working material is a thin sheet of lead titanate zirconate which undergoes mechanical deformations essentially in the plane of the paper. Note the several electroded regions and the small size. (Courtesy of Sandia Corporation, Albuquerque, New Mexico.)

PROBLEMS

11.1. In Fig. 11P.1 a static elastic material is constrained along its vertical sides so that

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_3} = 0.$$

In the absence of a gravitational field, the material has surfaces at $x_1 = 0$ and $x_1 = L$.

- Compute the material displacement $\delta_1(x_1)$ caused by the gravitational field.
- Find all components of the stress T_{ij} .

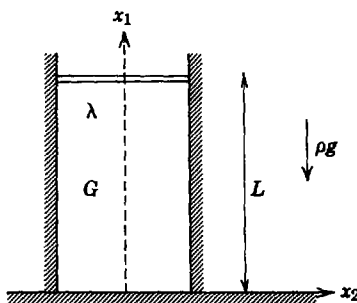


Fig. 11P.1

11.2. In Fig. 11P.2 a slab of elastic solid with constants ρ , G , λ and a thickness L is attached on one side to a rigid wall at $x_1 = 0$. A perfectly conducting thin plate of mass M is attached to the other side of the solid. A second perfectly conducting plate is fixed at $x_1 = -L - d$. Assume that $\partial/\partial x_2 = \partial/\partial x_3 = 0$ and $\delta_1(-L, t) \ll d$.

- If the voltage between the two capacitor plates is $V(t) = V_0 + V_1 \cos \omega t$, find $\delta_1(-L, t)$ $V_1 \ll V_0$.
- For what frequency range does the *mechanical* part of the system appear lumped?
- Give the mechanical lumped parameters for the frequency range defined in (b).

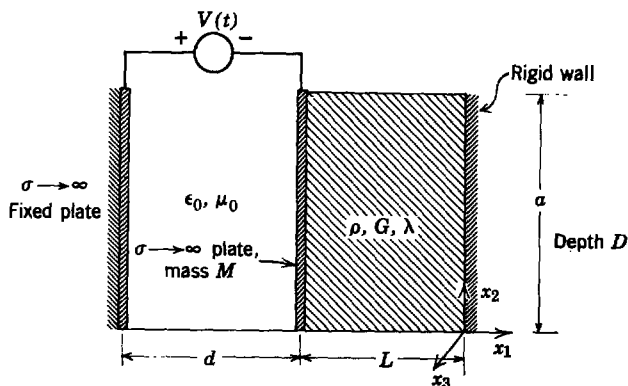


Fig. 11P.2

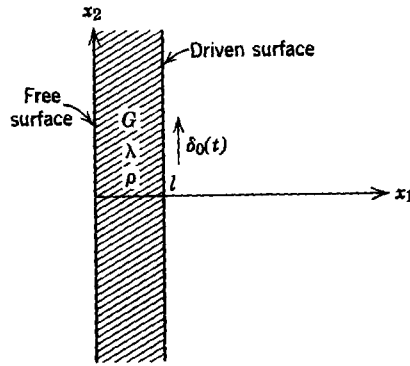


Fig. 11P.3

11.3. A slab of elastic material of length l in the x_1 -direction and infinite in extent in the x_2 - and x_3 -directions has the elastic constants G and λ and mass density ρ . Its surface at $x_1 = l$ is driven in the x_2 -direction uniformly by a displacement source $\delta_0(t)$. The surface at $x_1 = 0$ is free to move in the x_2 -direction without restraint.

Assume that $\delta_0(t) = \text{Re}(\delta_0 e^{j\omega t})$, where δ_0 and ω are given constants. Neglect the force of gravity and assume that

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_3} = 0.$$

- Find the stress and displacement in the slab.
- In the limit of low frequency to what lumped mechanical element does the slab correspond?
- Find the lowest frequency for which the slab may be said to “resonate.”

11.4. In a coordinate system (x_1, x_2, x_3) a surface with the normal vector $\mathbf{n}$ and supporting the stress T_{ij} is subject to the traction (see Section 8.2.2)* $\tau_i = T_{ij}n_j$. Assume that the stress components T_{ij} are known and that there is a surface with an orientation such that the traction is in the same direction as the normal vector; that is, $\tau_i = \alpha \delta_{ij}n_j$, where α is the stress acting normal to the surface.

- Write three equations in the three unknowns (n_1, n_2, n_3) .
- Because these equations are homogeneous, their compatibility requires that the determinant of the coefficients vanish. Show that this gives an expression for α .
- Consider the case in which $T_{12} = T_{21} = T_0$ and all other components are zero. What are the possible values of the normal stress α ? Compare your result with that found in Example 11.2.1.

11.5. In Example 11.2.1 it was shown that the three elastic constants (G, E, ν) must be related if a perfectly elastic material is isotropic [(g) of that example]. This was done by considering the transformation of a particular case of stress and strain from one coordinate system to a second with the same x_3 -axis but a 45° rotation in the x_1 - x_2 plane. Follow the arguments presented in Example 11.2.1 to show that the relation is implied for an arbitrary stress condition and an arbitrary rotation of coordinates. Remember that the a_{ij} that determine the rotation of coordinates are related by (8.2.23)* and that if $T'_{pq} = a_{pr}a_{qs}T_{rs}$ then $T_{rs} = a_{pr}a_{qs}T'_{pq}$.

* Appendix G.

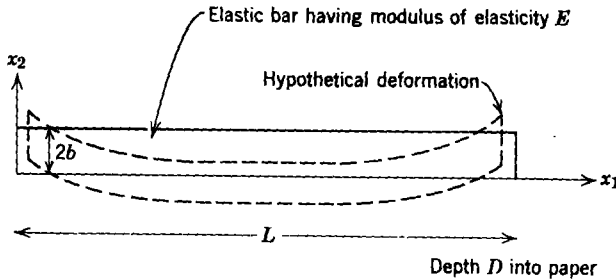


Fig. 11P.6

11.6. An elastic bar is often used in musical instruments as a source of audio-frequency tone. The bar is suspended by strings, attached to it at such points that the transverse (x_2) motion of the elastic material is not appreciably inhibited. (Examples are the vibraharp and marimba.) If the bar is struck by a mallet, it vibrates at one or more of its resonance frequencies. We consider here the problem of finding these frequencies, under the assumption that the bar is as shown in Fig. 11P.6. The bar is supported so that transverse motions are uninhibited, that is, both ends are free.

- Find an equation of the form $\cos \beta \cosh \beta = 1$ [$\beta = \beta(\omega)$] which stipulates the resonance frequencies.
- Use a graphical solution of the equation found in (a) to determine the two lowest resonance frequencies in terms of E and the dimensions of the bar.
- Sketch the transverse deflection as a function of x_1 for the lowest nontrivial mode.

11.7. A thin elastic beam of thickness $2b$, density ρ , and modulus of elasticity E is clamped on both ends to rigid walls. The total length of the beam is L , as shown in Fig. 11P.7.

- If the beam is suddenly struck from above, what is the lowest (nonzero) frequency at which it will "ring"; that is, what is its lowest natural frequency?
- Give a numerical answer for (a) in Hertz if the beam is steel with length $L = 50$ cm and thickness $2b = 0.10$ cm.
- What is the numerical value, again in Hertz, of the next higher resonance frequency?

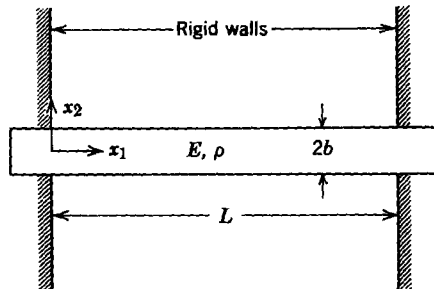


Fig. 11P.7

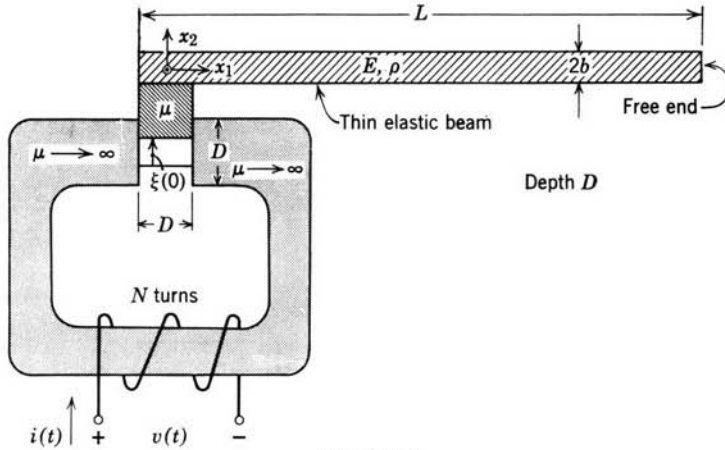


Fig. 11P.8

11.8. The electromechanical system shown in Fig. 11P.8 consists of a long thin elastic beam attached to the plunger of an electromagnet. The plunger has permeability μ and is free to slide between the faces of the electromagnet. Treat the plunger as a rigid body with mass M . Assume that $D \ll L$. The coil on the electromagnet is now excited with a current $i(t) = I_0 + i_1 \cos \omega t$, where $|i_1| \ll I_0$. You may assume that an externally applied force F_0 holds the plunger in equilibrium against the current I_0 . Also in equilibrium, the displacement of the beam $\xi(0) = 0$.

- What is the value of F_0 required for equilibrium?
- Find an expression for the electrical impedance $Z(j\omega)$ seen at the terminals of the coil, where $Z(j\omega) = \hat{v}(j\omega)/i_1$, and $\hat{v}(j\omega)$ is the complex amplitude of the steady-state voltage developed at the terminals.
- What is the expression that determines the poles of the impedance $Z(j\omega)$?

11.9. A thin beam clamped to two rigid walls is shown in Fig. 11P.9. Suppose that the beam is perfectly conducting and that it is placed between two perfectly conducting rigid

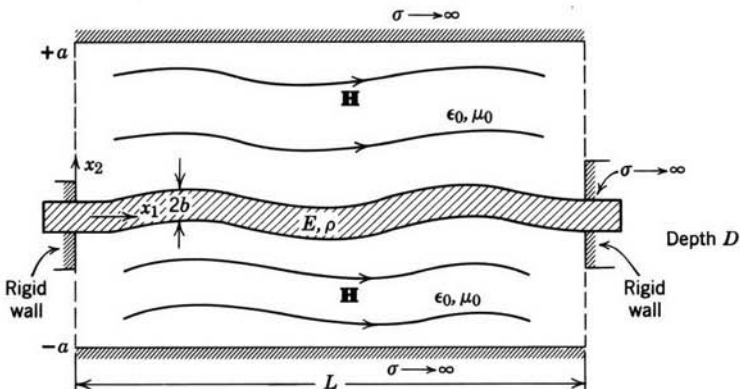


Fig. 11P.9

plates at $x_2 = \pm a$. Assume that there is a magnetic field trapped between the beam and the plates, so that *when the beam is flat* $\mathbf{H} = H_0 \mathbf{i}_1$ on both sides of the beam. (In the perfect conductor $\mathbf{H} = 0$.) Make the approximations that wavelengths of a disturbance on the beam are *long* compared with a and that the magnetic field is always uniform in the x_2 - and x_3 -directions.

- Write the equation of motion for the beam.
- Compute the first resonance frequency of the beam.
- Compare the result of (b) with Problem 11.7 and give a physical explanation for any differences which occur.
- Can the system be unstable? Explain.

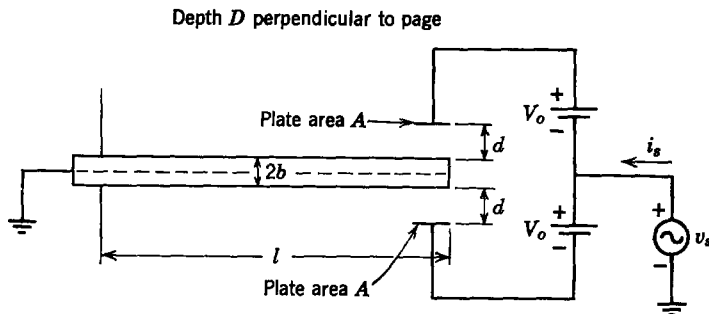


Fig. 11P.10

11.10. The system shown schematically in Fig. 11P.10 is similar to that discussed in Section 11.4.2b. The material of the beam is steel and the system constants and dimensions are

$$\begin{aligned} E &= 2.2 \times 10^{11} \text{ N/m}^2 & A &= 10^{-4} \text{ m}^2 \\ \rho &= 7.9 \times 10^3 \text{ kg/m}^3 & D &= 10^{-2} \text{ m} \\ l &= 10^{-1} \text{ m} & b &= 10^{-3} \text{ m} \\ V_o &= 1000 \text{ V} & d &= 10^{-3} \text{ m} \end{aligned}$$

We are interested in investigating the impedance seen by the signal source v_s for values of exciting frequency near the first resonance of the elastic bar. This type of information would be essential if we planned to use this system to control the frequency of an oscillator. For sinusoidal excitation $v_s = \text{Re} [\hat{v}_s e^{j\omega t}]$ and small-signal, steady-state operation:

- Find a *literal* expression for the input impedance $Z(j\omega) = \hat{v}_s / \hat{i}_s$, where $\hat{i}_s$ is the complex amplitude of the input current.
- For the numerical values given find a *numerical value* for the lowest frequency ω_0 at which the impedance $Z(j\omega)$ has a zero.
- Assume operation at frequencies near ω_0 by setting $\omega = \omega_0 + \Delta\omega$, where $|\Delta\omega| \ll \omega_0$ and $\Delta\omega$ can be either positive or negative. For this restriction the impedance Z appears as a series LC circuit. Find numerical values for the equivalent capacitance C and equivalent inductance L .

11.11. Consider the planar elastic waveguide of Fig. 11.4.18 but with the walls at $x_2 = 0$ and $x_2 = d$ fixed.

- Find the dispersion equation for waves in the form of

$$\delta_3 = \text{Re} \delta(x_2) \exp j(\omega t - kx_1).$$

- (b) Sketch the results of part (a) as an ω - k plot and compare with Fig. 11.4.19. Is there a principal mode of propagation?

11.12. A cylindrical, circular elastic section of material with the shear modulus G , density ρ , and radius R is embedded in a perfectly rigid solid so that the material at $r = R$ is fixed. This structure is to be used as a waveguide for elastic shear waves. To find the dispersion equation for these waves, we confine interest to material displacements in the form of $\delta = \delta_\theta(r, z, t)\mathbf{i}_\theta$. Find the dispersion equation for all modes in this form. (A discussion of Bessel's functions is given on p. 207 of S. Ramo, J. Whinnery, and T. Van Duzer, *Fields and Waves in Communication Electronics*, Wiley, New York, 1965.)

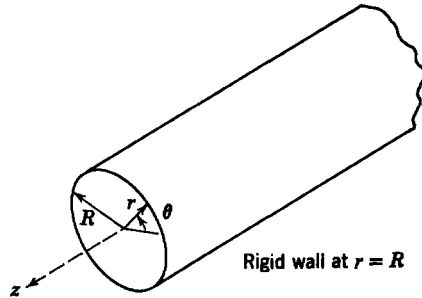


Fig. 11P.12

Chapter 12

ELECTROMECHANICS OF INCOMPRESSIBLE, INVISCID FLUIDS

12.0 INTRODUCTION

We are all familiar with the distinctions between the three pure states of matter: solids, liquids, and gases. A solid body possesses a definite shape and size that is retained unless the body is acted on by outside forces. A given mass of liquid possesses a definite size (volume) but conforms in shape to its container. A particular mass of gas possesses neither definite size (volume) nor shape because it will deform to fill completely whatever vessel it occupies.

Liquids and gases are grouped together and called fluids when their dynamic behavior is to be studied. The essential difference between a solid and a fluid is that the force necessary to deform a solid is a function of the deformation (strain), whereas in a fluid the force necessary to produce a deformation is a function of the *rate* of deformation (strain rate) and a hydrostatic pressure. A fluid left to itself in a force-free environment will relax to a state that has no internal stresses except an isotropic (hydrostatic) pressure balanced by the surface forces exerted by the container or by surface tension.

Although because of some similarities liquids and gases are classified together as fluids, they also exhibit striking differences. Moderate changes in temperature and pressure cause very small fractional changes in the density of a liquid but the corresponding changes in a gas are quite large.

All real fluids exhibit internal friction that is described mathematically by the property called *viscosity*. The effects of viscosity can be large or small, depending on the physical situation being studied. It is standard practice for an electrical engineer to represent a real coil of wire mathematically by an

ideal, lossless circuit element called inductance. Similarly, the fluid dynamicist often uses an idealization of a fluid in which viscosity is neglected. Such an idealization is called an *inviscid* fluid.

In most electromechanical systems involving fluids the principal effects of viscosity result from the contact between the fluid and a solid boundary. As in most continuum problems, the effect of the boundary becomes less pronounced at greater distances from the boundary. Thus, when the behavior of a fluid is desired far from a boundary, an inviscid model is often adequate. How a distance that is adequate for the neglect of viscosity is determined is a rather complex subject and depends quite naturally on the system to be analyzed and the accuracy desired. Much experimental and theoretical data are available to answer this question.* We address ourselves to a few simple cases in which viscosity is important in Chapter 14.

Our purpose in this book is to present models and do analyses of systems in which electromechanical interactions are important. This means essentially that for coupling with a fluid the electromechanical forces must dominate the viscous forces. It is fortuitous that many situations exist in which this occurs, notably magnetohydrodynamic pumps and generators and plasma accelerators.† Consequently, our use of an inviscid fluid model is realistic with respect to the dominant electromagnetic forces and viscous effects can be added later as perturbations.

When a fluid flows past a solid boundary, the fluid friction makes the fluid particles that are in contact with the boundary remain at rest with respect to the boundary. This makes the flow of fluid parallel to the boundary vary with distance from the boundary and introduces a shear rate into the flow. At low velocities each fluid particle flows along a smooth path (a streamline), and the flow is said to be *laminar*. At high velocities the shearing effect of the boundary makes the flow unstable and each fluid particle has a significant random motion in addition to its average motion in the direction of flow. This flow regime is said to be *turbulent*. When a flow becomes turbulent, its internal friction (viscous) losses increase. In spite of this, we can often represent a fluid in turbulent flow in terms of a steady flow at the average velocity and obtain a good model for electromechanical interactions.

Compressibility is a property of a fluid that describes the fact that when the hydrostatic pressure on the fluid is increased the density increases. Every fluid exhibits this property to some extent. Liquids are only slightly compressible, whereas gases are highly compressible. Compressibility to fluids is

* H. Schlichting, *Boundary Layer Theory*, 4th ed., McGraw-Hill, New York, 1960, pp. 1-41.

† These situations are illustrated graphically in the film entitled "Magnetohydrodynamics" produced for the National Committee on Fluid Mechanics Films by Education Development Center, Newton, Mass.

what elastic modulus is to elastic solids. Thus we expect a compressible fluid to transmit longitudinal (sound) waves just as an elastic solid does. When we are interested in the flow of a liquid, the compressibility can often be neglected. This is analogous to the treatment of the gross motion of an elastic solid as the motion of a rigid body. Even though a gas is highly compressible, we can sometimes treat gas flow by using an incompressible fluid model, especially at very low flow velocities. In other cases the compressibility of a gas will have a marked effect on the flow, and we must account for it in our mathematical model.

Our interest here is in electromechanical interactions; in each case we select the simplest mathematical model that illustrates the physical phenomena of interest in a realistic way. Thus in many cases we use a simple fluid model that adequately illustrates the electromechanical interactions but ignores some fluid-mechanical phenomena. The inclusion of such phenomena is beyond the scope of this book. For further information on these topics the reader can consult a good treatise on fluid mechanics.* In this chapter we investigate various phenomena that result from electromechanical interactions with incompressible, inviscid fluids. In Chapter 13 we treat compressible inviscid fluids and in Chapter 14 introduce viscosity.

12.1 INVISCID, INCOMPRESSIBLE FLUIDS

An incompressible inviscid fluid model lends itself to simple mathematical analysis and to an understanding of many fluid-mechanical phenomena. Moreover, it provides considerable insight into the fundamental interactions of magnetohydrodynamics (MHD) and often gives an accurate description of MHD interactions with liquid conductors such as liquid metals.

In what follows we first introduce the equations of motion for an incompressible inviscid fluid and then consider some simple, fluid-mechanical examples. Finally, we investigate the important electromechanical interactions appropriate for study with this model.

12.1.1 The Substantial Derivative

In the study of fluid mechanics we are concerned with describing the fluid motion and relating it to the applied forces and boundary conditions. Most often the desired information consists of determining a flow pattern in a region of space at a given instant of time. Because of this desired result, fluid dynamicists have focused their attention on fluid variables at a given position in relation to a fixed reference frame. Since the fluid is moving past this point, different material elements occupy the point at different instants in time. This

* See, for example, Schlichting, *op. cit.*

method of representing fluid properties (such as velocity) in terms of a fixed point in space is called an Eulerian or field description. An alternative method, called the Lagrangian description, gives the velocity and other properties of the individual particles.

The best-known example of the use of the Lagrangian description is in particle dynamics (or the rigid-body mechanics of Chapter 2) in which it is conventional to ascribe to each particle (or mechanical node) a velocity $\mathbf{v}$ which is a function of the initial position (a, b, c) of the particle and of time t . Thus $\mathbf{v}(a, b, c, t)$ describes the velocity of a particular particle. This same method is carried over into continuum mechanics by describing the velocity $\mathbf{v}(a, b, c, t)$ of the grain of matter at position a, b, c at $t = 0$. This Lagrangian description was used in Chapter 11, in which the displacement of a grain of elastic material was written as a function of the unstrained (initial) position.

For electrical engineering students the best-known example of the use of an Eulerian description is in electromagnetic field theory. We usually describe the electromagnetic field and source quantities as functions of space and time. Thus for a cartesian coordinate system (x_1, x_2, x_3) we give the electric field intensity as $\mathbf{E}(x_1, x_2, x_3, t)$. This prescribes the field intensity at the point (x_1, x_2, x_3) at any instant of time t . Using the Eulerian description, we can describe a velocity field $\mathbf{v}(x_1, x_2, x_3, t)$ that ascribes a velocity to a position in space rather than to a particular grain of matter. At the point (x'_1, x'_2, x'_3) the velocity $\mathbf{v}(x'_1, x'_2, x'_3, t')$ specifies the velocity of that grain of matter that occupies the point (x'_1, x'_2, x'_3) at the instant of time t' . If at a later time t'' this grain of matter is at point (x''_1, x''_2, x''_3) , its velocity will be $\mathbf{v}(x''_1, x''_2, x''_3, t'')$. The Eulerian system is normally used in the study of fluid mechanics and is also used here.*

Later in this chapter we shall need the time derivative of an Eulerian function as experienced by a particular grain of matter. The acceleration of a grain of matter is such a derivative and we shall need it to write Newton's second law.

Consider a system of moving matter with an Eulerian or field description of the velocity, $\mathbf{v}(x_1, x_2, x_3, t)$ and of the quantity $f(x_1, x_2, x_3, t)$. It is necessary to find the time rate of change of f experienced by a grain of matter. Consider the grain of matter that occupies position (x_1, x_2, x_3) at time t and has velocity $\mathbf{v}(x_1, x_2, x_3, t)$ with components v_1, v_2 , and v_3 . At time $(t + \Delta t)$ the grain will occupy a new position, given to first order in (Δt) by $(x_1 + v_1 \Delta t, x_2 + v_2 \Delta t, x_3 + v_3 \Delta t)$. Thus in the interval (Δt) the grain has experienced a change in f of

$$\Delta f = f(x_1 + v_1 \Delta t, x_2 + v_2 \Delta t, x_3 + v_3 \Delta t, t + \Delta t) - f(x_1, x_2, x_3, t) \quad (12.1.1)$$

* For a more thorough discussion of these alternative representations, see, for example, H. Lamb, *Hydrodynamics*, 6th ed., Dover, New York, 1945, Chapter I, Articles, 4 to 9, 13, and 14.

The first term in this expression is expanded in a Taylor series about the point (x_1, x_2, x_3, t) and second- and higher order terms in Δt are discarded to obtain

$$\Delta f = \frac{\partial f}{\partial t} \Delta t + \frac{\partial f}{\partial x_1} v_1 \Delta t + \frac{\partial f}{\partial x_2} v_2 \Delta t + \frac{\partial f}{\partial x_3} v_3 \Delta t. \quad (12.1.2)$$

The desired time rate of change is defined as

$$\frac{Df}{Dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta f}{\Delta t}. \quad (12.1.3)$$

Substitution of (12.1.2) into (12.1.3) yields

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + v_1 \frac{\partial f}{\partial x_1} + v_2 \frac{\partial f}{\partial x_2} + v_3 \frac{\partial f}{\partial x_3}, \quad (12.1.4)$$

which is written in the compact form

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla)f. \quad (12.1.5)$$

The function f may be considered to be one component of a cartesian vector $\mathbf{f}$. Equation 12.1.5 holds for each component of the vector; consequently, the time rate of change of a vector field quantity $\mathbf{f}(x_1, x_2, x_3, t)$ experienced by a grain of matter is given by

$$\frac{D\mathbf{f}}{Dt} = \frac{\partial \mathbf{f}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{f}. \quad (12.1.6)$$

This derivative is variously called the Stokes, total, particle, material, substantial, or convective derivative.

The interpretation of the physical meaning of (12.1.5) or (12.1.6) is quite simple. It merely states that an observer moving with the velocity $\mathbf{v}$, relative to the coordinate system (x_1, x_2, x_3) in which the quantity $f(x_1, x_2, x_3, t)$ is defined, will detect a time rate of change of f made up of two parts: $(\partial f / \partial t)$ is the rate of change of f at a fixed point and $(\mathbf{v} \cdot \nabla)f$ is the change in f that results from the motion of the observer through a fixed (in time) distribution of f . In fact, $(\mathbf{v} \cdot \nabla)f$ is simply the space derivative of f taken in the direction of $\mathbf{v}$ and weighted by the magnitude of $\mathbf{v}$.

An example of the application of (12.1.6), which will occur in Section 12.1.3 is the acceleration of a grain of matter moving in a velocity field $\mathbf{v}(x_1, x_2, x_3, t)$. According to (12.1.6),

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}. \quad (12.1.7)$$

Example 12.1.1. As an example of the calculation of an acceleration, consider the velocity

$$\mathbf{v} = \frac{V_o}{a} (\mathbf{i}_1 x_2 - \mathbf{i}_2 x_1), \quad (\text{a})$$

where V_o and a are positive constants. This will be recognized as the velocity of a fluid undergoing a rigid-body rotation about the x_3 -axis. In fact, the angular velocity of the fluid is V_o/a , where $r = \sqrt{x_1^2 + x_2^2}$ is the radial distance from the x_3 -axis. Note that $\partial \mathbf{v} / \partial t = 0$. Yet we know that the fluid is accelerating (centrifugal acceleration), and it is this acceleration that is given by the second term in (12.1.7), which becomes

$$\frac{D\mathbf{v}}{Dt} = (\mathbf{v} \cdot \nabla) \mathbf{v} = \left(v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} \right) \mathbf{i}_1 + \left(v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} \right) \mathbf{i}_2, \quad (\text{b})$$

because v_3 and $\partial/\partial x_3$ are zero. Substitution of (a) into (b) gives

$$\frac{D\mathbf{v}}{Dt} = \left(\frac{V_o}{a} \right)^2 [-x_1(1)] \mathbf{i}_1 + [x_2(-1)] \mathbf{i}_2 \quad (\text{c})$$

as the acceleration of the fluid. This acceleration is directed radially inward toward the x_3 -axis and has the expected magnitude $(V_o/a)^2 r$ (the centrifugal acceleration).

We now obtain differential equations of motion that are appropriate for studying the dynamical behavior of incompressible inviscid fluids. We obtain the desired equations from two postulates:

1. Conservation of mass.
2. Conservation of momentum (Newton's second law).

The validity of these postulates has been verified by a variety of experiments.

12.1.2 Conservation of Mass

The conservation of mass states that mass can be neither created nor destroyed and thus must be conserved. To apply this postulate to a particular system consider the system of Fig. 12.1.1 in which an arbitrary volume V enclosed by the surface S is defined in a region containing material with a mass

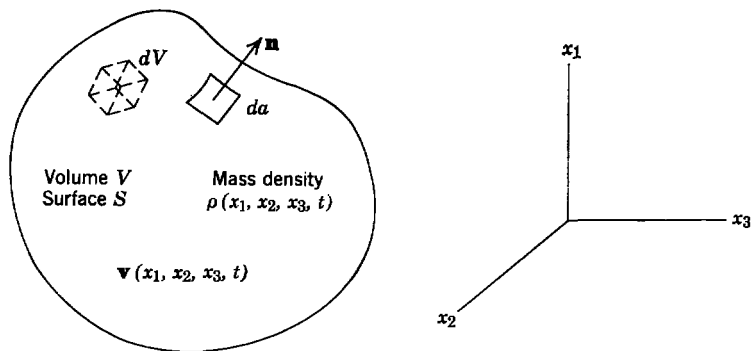


Fig. 12.1.1 Definition of system for writing conservation of mass.

density $\rho(x_1, x_2, x_3, t)$ (kg/m^3) and a velocity $\mathbf{v}(x_1, x_2, x_3, t)$ (m/sec). A differential volume element is dV , a differential surface element is da , and the normal vector $\mathbf{n}$ is normal to the surface and directed outward from the volume.

Because mass must be conserved, we can write the expression for the system in Fig. 12.1.1:

$$\oint_S (\rho \mathbf{v} \cdot \mathbf{n}) da = - \frac{d}{dt} \int_V \rho dV. \quad (12.1.8)$$

The left side of this expression evaluates the net rate of mass flow (kg/sec) out of the volume V across the surface S . The right side indicates the rate at which the total mass within the volume decreases. Note the similarity between (12.1.8) and the conservation of charge described by (1.1.26)* in Chapter 1.

Example 12.1.2. The system in Fig. 12.1.2 consists of a pipe of inlet area A_i and outlet area A_o . A fluid of constant density ρ flows through the pipe. The velocity is assumed to be uniform across the pipe's cross section. The instantaneous fluid velocity at the inlet is

$$\mathbf{v}_i = \mathbf{i}_1 v_i$$

and is known. We wish to find the velocity $\mathbf{v}_o$ at the outlet.

We use the closed surface S indicated by dashed lines in Fig. 12.1.2 with the conservation of mass (12.1.8) to find $\mathbf{v}_o$. Because the density ρ is constant,

$$\oint_S (\mathbf{v} \cdot \mathbf{n}) da = 0.$$

The only contributions to this integral come from the portions of the surface that coincide with the inlet and outlet. The result is

$$\oint_S (\mathbf{v} \cdot \mathbf{n}) da = [\mathbf{v}_i \cdot (-\mathbf{i}_1)] A_i + (\mathbf{v}_o \cdot \mathbf{i}_1) A_o = 0$$

from which

$$\mathbf{v}_o = \mathbf{i}_1 v_o = \mathbf{i}_1 \frac{A_i}{A_o} v_i.$$

This expresses the intuitively apparent fact that in the steady state as much fluid leaves the closed surface S as enters it.

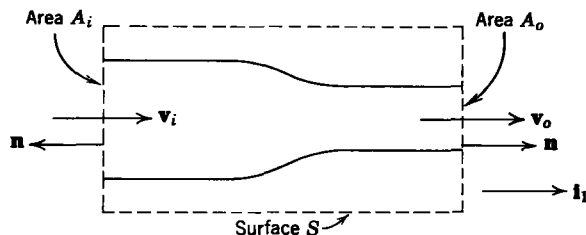


Fig. 12.1.2 Example for application of conservation of mass.

* Table 1.2, Appendix G.

We now write (12.1.8) in differential form by using the divergence theorem*

$$\oint_S (\mathbf{A} \cdot \mathbf{n}) da = \int_V (\nabla \cdot \mathbf{A}) dV$$

to change the surface integral in (12.1.8) to a volume integral

$$\int_V (\nabla \cdot \rho \mathbf{v}) dV = - \int_V \frac{\partial \rho}{\partial t} dV. \quad (12.1.9)$$

The time derivative has been taken inside the integral sign because we assume that the volume V is stationary. This expression holds for any arbitrary volume V ; therefore it must hold for a differential volume. Thus

$$\nabla \cdot \rho \mathbf{v} = - \frac{\partial \rho}{\partial t}, \quad (12.1.10)$$

which is the partial differential equation that describes the conservation of mass.

The left side of (12.1.10) can be expanded and the terms rearranged to obtain

$$\rho(\nabla \cdot \mathbf{v}) = - \frac{D\rho}{Dt}, \quad (12.1.11)$$

where the derivative on the right is the substantial derivative defined by (12.1.5). Equation 12.1.11 relates the rate of density decrease in a grain of matter to the divergence of the velocity and is in a form particularly useful when studying incompressible fluids because then the time rate of change of the density as viewed by a particle of fluid is zero, that is, $D\rho/Dt = 0$. Equation 12.1.11 indicates that in this case the velocity field has no divergence ($\nabla \cdot \mathbf{v} = 0$).

12.1.3 Conservation of Momentum (Newton's Second Law)

The second postulate of fluid mechanics is that Newton's second law of motion (conservation of momentum) must hold for each grain of matter. To express this postulate mathematically we assume that in the coordinate system (x_1, x_2, x_3) there exists a fluid of density $\rho(x_1, x_2, x_3, t)$ moving in a velocity field $\mathbf{v}(x_1, x_2, x_3, t)$. The mass of a grain of matter occupying the differential volume element $dx_1 dx_2 dx_3$ is $\rho dx_1 dx_2 dx_3$. We multiply this mass by the instantaneous acceleration found in (12.1.7) and equate the result to

* F. B. Hildebrand, *Advanced Calculus for Engineers*, Prentice-Hall, New York, 1948, pp. 312-315.

the total force $\mathbf{f}$ applied to the grain of matter*

$$\rho(dx_1 dx_2 dx_3) \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = \mathbf{f}. \quad (12.1.12)$$

We now divide both sides of this expression by the volume element and define the force density $\mathbf{F}$ as

$$\mathbf{F} = \frac{\mathbf{f}}{dx_1 dx_2 dx_3} \quad (12.1.13)$$

to obtain the result

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{F}. \quad (12.1.14)$$

This is the differential form of the conservation of momentum equation that we use most often in our treatment of continuum electromechanics.

The force density $\mathbf{F}$ in (12.1.14) can be written as

$$\mathbf{F} = \mathbf{F}^e + \rho \mathbf{g} + \mathbf{F}^m, \quad (12.1.15)$$

where $\mathbf{F}^e$ represents the electromagnetic forces that were expressed in various forms in Sections 8.1 and 8.3 of Chapter 8†, $\rho \mathbf{g}$ represents the force density resulting from gravity, and $\mathbf{F}^m$ represents mechanical forces applied to the grain of matter by adjacent material. This latter force density $\mathbf{F}^m$ depends on the physical properties of the fluid and will thus be described in Section 12.1.4 (on constituent relations).

Equation 12.1.14 can be expressed in a particularly simple and often useful form when we recognize that the force density on the right can be expressed as the space derivative of a stress tensor. We have already shown in Sections 8.1 and 8.3 of Chapter 8 that this is true. The i th component of the electromagnetic force density $\mathbf{F}^e$ is

$$F_i^e = \frac{\partial T_{ij}^e}{\partial x_j}, \quad (12.1.16)$$

where T_{ij}^e is the Maxwell stress tensor given for magnetic-field systems by (8.1.11)† and for electric field systems by (8.3.10)†. Because the gravitational field is conservative, we can write the gravitational force as the negative

* Newton's second law, written as $\mathbf{f} = M\mathbf{a}$, applies only for a mass M of fixed identity. Because $D\mathbf{v}/Dt$ is a derivative following a grain of matter, it is the acceleration of a set of mass particles ($\rho dx_1 dx_2 dx_3$) of fixed identity. Thus (12.1.12) is a valid description of Newton's second law written as $\mathbf{f} = M\mathbf{a}$ and is valid even when ρ is changing with space and time.

† See Table 8.1, Appendix G.

gradient of a scalar potential. We define the gravitational potential as U and write

$$\rho \mathbf{g} = -\nabla U, \quad (12.1.17)$$

or, in index notation, the i th component is

$$\rho g_i = -\frac{\partial U}{\partial x_i} = -\frac{\partial}{\partial x_j} (\delta_{ij} U). \quad (12.1.18)$$

We obtain the force density $\mathbf{F}^m$ of mechanical origin as the derivative of a stress tensor in Section 12.1.4 and therefore assume that the i th component of the mechanical force density $\mathbf{F}^m$ is

$$F_i^m = \frac{\partial T_{ij}^m}{\partial x_j}, \quad (12.1.19)$$

where T_{ij}^m is the mechanical stress tensor to be calculated later.

Now the total stress tensor T_{ij} for the system is

$$T_{ij} = T_{ij}^e - \delta_{ij} U + T_{ij}^m, \quad (12.1.20)$$

and we can express the i th component of (12.1.14) simply as

$$\rho \frac{Dv_i}{Dt} = \frac{\partial T_{ij}}{\partial x_j}. \quad (12.1.21)$$

This form is particularly useful in applying boundary conditions.

Equation 12.1.14 is often useful when it is expressed in integral form. To achieve this end we multiply the conservation of mass (12.1.11) by the velocity $\mathbf{v}$ and add it to (12.1.14) to obtain

$$\rho \frac{D\mathbf{v}}{Dt} + \mathbf{v} \frac{D\rho}{Dt} + \rho \mathbf{v}(\nabla \cdot \mathbf{v}) = \mathbf{F}. \quad (12.1.22)$$

Because zero has been added to the left side of (12.1.14), (12.1.22) still expresses Newton's second law. Combination of the first two terms of (12.1.22) into the derivative of the product $(\rho \mathbf{v})$ and use of the definition of (12.1.6) leads to

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + (\mathbf{v} \cdot \nabla) \rho \mathbf{v} + \rho \mathbf{v}(\nabla \cdot \mathbf{v}) = \mathbf{F}. \quad (12.1.23)$$

The i th component of this expression is

$$\frac{\partial(\rho v_i)}{\partial t} + (\mathbf{v} \cdot \nabla) \rho v_i + \rho v_i(\nabla \cdot \mathbf{v}) = F_i. \quad (12.1.24)$$

Combination of the second two terms on the left side of this expression yields

$$\frac{\partial(\rho v_i)}{\partial t} + (\nabla \cdot \rho v_i \mathbf{v}) = F_i. \quad (12.1.25)$$

We now integrate (12.1.25) throughout a volume V to obtain

$$\int_V \frac{\partial(\rho v_i)}{\partial t} dV + \int_V (\nabla \cdot \rho v_i \mathbf{v}) dV = \int_V F_i dV. \quad (12.1.26)$$

The divergence theorem is used to change the second term on the left to an integral over the surface S that encloses the volume V and has the outward directed normal $\mathbf{n}$; thus

$$\int_V \frac{\partial(\rho v_i)}{\partial t} dV + \oint_S \rho v_i (\mathbf{v} \cdot \mathbf{n}) da = \int_V F_i dV. \quad (12.1.27)$$

Using the definition of the total force density in terms of a stress tensor* in (12.1.21), we can also write (12.1.27) as

$$\int_V \frac{\partial(\rho v_i)}{\partial t} dV + \oint_S \rho v_i (v_j n_j) da = \oint_S T_{ij} n_j da. \quad (12.1.28)$$

Equation 12.1.27 can be written for each of the three components and then combined to obtain the vector form

$$\int_V \frac{\partial(\rho \mathbf{v})}{\partial t} dV + \oint_S \rho \mathbf{v} (\mathbf{v} \cdot \mathbf{n}) da = \int_V \mathbf{F} dV. \quad (12.1.29)$$

This is the integral form of the equation that expresses conservation of momentum (Newton's second law).

The momentum density of the fluid is $\rho \mathbf{v}$; consequently, the first term on the left of (12.1.29) represents the time rate of increase of momentum density of the fluid that is instantaneously in the volume V . The second term gives the net rate at which momentum density is convected by the flow out of the volume V across the surface S . Thus the left side of (12.1.29) represents the *net* rate of increase of momentum in the volume V . The right side of (12.1.29) gives the net force applied to all the matter instantaneously in the volume V .

12.1.4 Constituent Relations

To complete the mathematical description of a fluid we must describe mathematically how the physical properties of the fluid affect the mechanical behavior. The physical properties of a fluid are described by constituent relations (equations of state), and the form of the equations depends on the fluid model to be used.

* See (8.1.13) and (8.1.17) of Appendix G.

A homogeneous, incompressible fluid, which is the model we are considering at present, has constant mass density, independent of other material properties (density and temperature) and of time. Thus one constituent relation is

$$\rho = \text{constant.} \quad (12.1.30)$$

This constituent relation is normally expressed in a different form by substituting (12.1.30) into (12.1.11) to obtain the equation

$$\nabla \cdot \mathbf{v} = 0, \quad (12.1.31)$$

which is the mathematical description normally used to express the property of incompressibility. Note, however [from (12.1.11)], that ρ does not have to be constant for (12.1.31) to hold. The fluid could be inhomogeneous and still be incompressible.

The next step in the description of physical properties is to determine how the mechanical force density $\mathbf{F}^m$ of (12.1.15) arises in a fluid.

First, consider a fluid at rest. By definition, a fluid at rest can sustain no shear stresses. Moreover, a fluid at rest can sustain only compressive stresses and a homogeneous, isotropic fluid will sustain the same compressive stress across a plane of arbitrary orientation. This isotropic compressive stress is defined as a positive hydrostatic pressure p .

We can define a mechanical stress tensor for the fluid at rest in the nomenclature of Sections 8.2 and 8.2.1*. Thus, because there are no shear stresses,

$$T_{ij}^m = 0, \quad \text{for } i \neq j. \quad (12.1.32)$$

The normal stresses are all given by

$$T_{11}^m = T_{22}^m = T_{33}^m = -p. \quad (12.1.33)$$

The information contained in (12.1.32) and (12.1.33) can be written in compact form by using the Kronecker delta defined in (8.1.7) of Chap. 8*; therefore

$$T_{ij}^m = -\delta_{ij}p. \quad (12.1.34)$$

We can verify that the stress tensor in (12.1.34) describes an isotropic, normal compressive stress by calculating the traction* $\boldsymbol{\tau}^m$ applied to a surface of arbitrary orientation. To do this assume a surface with normal vector

$$\mathbf{n} = n_1 \mathbf{i}_1 + n_2 \mathbf{i}_2 + n_3 \mathbf{i}_3. \quad (12.1.35)$$

Now use (8.2.2) of Chapter 8 with (12.1.34) and (12.1.35) to calculate the i th component of $\boldsymbol{\tau}^m$,

$$\tau_i^m = T_{ij}^m n_j = -p \delta_{ij} n_j = -p n_i \quad (12.1.36)$$

The vector traction then is

$$\boldsymbol{\tau}^m = -p(n_1 \mathbf{i}_1 + n_2 \mathbf{i}_2 + n_3 \mathbf{i}_3) = -p\mathbf{n}. \quad (12.1.37)$$

* Appendix G.

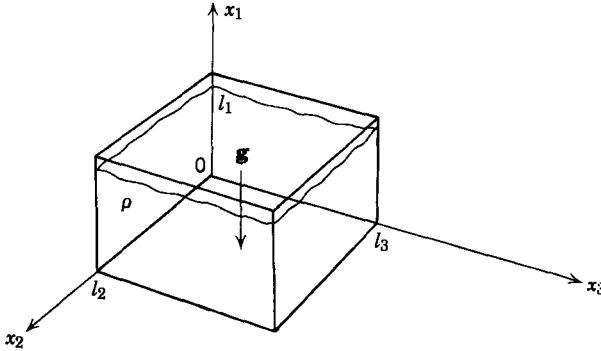


Fig. 12.1.3 Example for the application of stress tensor to a fluid at rest.

This traction is normal to the surface and in the direction opposite to the normal vector $\mathbf{n}$. Thus the stress tensor of (12.1.34) describes an isotropic compressive stress.

The pressure p may be a function of position; consequently, a volume force density can result from a space variation of pressure. To find this force density we use (8.2.7)* to evaluate the i th component

$$F_i^m = \frac{\partial T_{ij}^m}{\partial x_j} = -\delta_{ij} \frac{\partial p}{\partial x_j} = -\frac{\partial p}{\partial x_i}. \quad (12.1.38)$$

When the three components are combined, the vector force density becomes

$$\begin{aligned} \mathbf{F}^m &= -\left(\frac{\partial p}{\partial x_1} \mathbf{i}_1 + \frac{\partial p}{\partial x_2} \mathbf{i}_2 + \frac{\partial p}{\partial x_3} \mathbf{i}_3\right) \\ \mathbf{F}^m &= -\nabla p. \end{aligned} \quad (12.1.39)$$

Example 12.1.3. As an example of the application of this mechanical force density, consider the system shown in Fig. 12.1.3 which consists of a container of lateral dimensions l_2 and l_3 and filled to a height l_1 with a fluid of constant mass density ρ . The acceleration of gravity $\mathbf{g}$ acts in the negative x_1 -direction. The fluid is open to atmospheric pressure p_o at the top. We wish to find the hydrostatic pressure at any point in the fluid.

The fluid is at rest, so the acceleration is zero. Moreover, the only forces applied to the material are the force of gravity and the mechanical force from adjacent material. Thus the conservation of momentum (12.1.14) and (12.1.15) yields for this system

$$0 = -\mathbf{i}_1 \rho g - \nabla p.$$

In component form this equation becomes

$$\begin{aligned} 0 &= -\rho g - \frac{\partial p}{\partial x_1}, \\ 0 &= -\frac{\partial p}{\partial x_2}, \\ 0 &= -\frac{\partial p}{\partial x_3} \end{aligned}$$

* See Appendix G.

We integrate these three equations to find that p is independent of x_2 and x_3 and is given in general by

$$p = -\rho g x_1 + C.$$

The integration constant C is determined by the condition that in the absence of surface forces the pressure must be continuous at $x_1 = l_1$. Thus

$$p = p_o + \rho g(l_1 - x_1).$$

Equations 12.1.34 and 12.1.39 describe mechanical properties of a fluid at rest. In a real fluid, motion will result in internal friction forces that add to the pressure force. In an *inviscid* fluid, however, motion results in no additional mechanical forces other than the forces of inertia already included in the momentum equation (12.1.14). Consequently, in the inviscid model the only mechanical force density [$\mathbf{F}^m$ in (12.1.15)] results from a space variation of pressure expressed by (12.1.39).

For an incompressible inviscid fluid the physical properties are completely specified by (12.1.31) and (12.1.39). Therefore, when boundary conditions and applied force densities (electrical and gravity) are specified, these constituent relations and (12.1.14) can be used to determine the motion of the fluid. We treat first some of the purely fluid-mechanical problems to identify the kinds of flow phenomena to be expected from this fluid model and then add electromechanical coupling terms.

12.2 MAGNETIC FIELD COUPLING WITH INCOMPRESSIBLE FLUIDS

An important class of electromechanical interactions is describable by irrotational flow; that is,

$$\nabla \times \mathbf{v} = 0. \quad (12.2.1)$$

When such an approximation is appropriate, the equations of motion can be solved quite easily because a vector whose curl is zero can be expressed as the gradient of a potential. Thus we define the class of problems for which (12.2.1) holds as *potential flow* problems and we define a *velocity potential* ϕ such that

$$\mathbf{v} = -\nabla\phi. \quad (12.2.2)$$

For incompressible flow $\nabla \cdot \mathbf{v} = 0$ from (12.1.31) and the potential ϕ must satisfy Laplace's equation

$$\nabla^2\phi = 0. \quad (12.2.3)$$

A solution of a potential flow problem then reduces to a solution of Laplace's equation that fits the boundary conditions imposed on the fluid.

We can now establish some important properties of potential flow. The momentum equation (12.1.14) takes the form

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p - \nabla U + \mathbf{F}^e, \quad (12.2.4)$$

where we have used the definition of the substantial derivative in (12.1.6) and the definition of the gravitational potential U in (12.1.17). The use of the vector identity

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{2}\nabla(v^2) - \mathbf{v} \times (\nabla \times \mathbf{v}),$$

where $v^2 = \mathbf{v} \cdot \mathbf{v}$, and (12.2.1) yields (12.2.4) in the alternative form

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2}\rho \nabla(v^2) = -\nabla p - \nabla U + \mathbf{F}^e. \quad (12.2.5)$$

We now use the facts that ρ is constant, that the space (∇) and time ($\partial/\partial t$) operators are independent, and that the velocity is expressed by (12.2.2) to write (12.2.5) in the form

$$\nabla \left(\rho \frac{\partial \phi}{\partial t} + \frac{1}{2}\rho v^2 + p + U \right) = \mathbf{F}^e. \quad (12.2.6)$$

By taking the curl of both sides of (12.2.6) we find that potential flow is possible only when

$$\nabla \times \mathbf{F}^e = 0. \quad (12.2.7)$$

If this condition is not satisfied, the assumption that $\nabla \times \mathbf{v} = 0$ is not valid.

Thus we restrict the treatment of the present section to electromechanical interactions in which the force density of electrical origin has no curl (12.2.7). In view of (12.2.7), we express the force density $\mathbf{F}^e$ as

$$\mathbf{F}^e = -\nabla \psi, \quad (12.2.8)$$

where ψ is an electromagnetic force potential, and write (12.2.6) as

$$\nabla \left(\rho \frac{\partial \phi}{\partial t} + \frac{1}{2}\rho v^2 + p + U + \psi \right) = 0. \quad (12.2.9)$$

The most general solution for this differential equation is

$$\rho \frac{\partial \phi}{\partial t} + \frac{1}{2}\rho v^2 + p + U + \psi = H(t); \quad (12.2.10)$$

that is, this expression can be a function of time but not a function of space.

When the flow is steady, $\partial \phi / \partial t = 0$ and none of the other quantities on the left of (12.2.10) is a function of time. Then (12.2.10) reduces to

$$\frac{1}{2}\rho v^2 + p + U + \psi = \text{constant}. \quad (12.2.11)$$

This result, known as *Bernoulli's equation*, expresses a constant of the motion and is useful in the solution of certain types of problem.

Example 12.2.1. As an example of the application of Bernoulli's equation, consider the system in Fig. 12.2.1. This system consists of a tank that is open to atmospheric pressure p_o and filled to a height h_1 with an inviscid, incompressible fluid. The fluid discharges through a small pipe at a height h_2 with velocity v_2 . The area of the tank is large compared with the area of the discharge pipe; thus we assume that the tank empties so slowly that we can neglect the vertical velocity of the fluid and consider this as a steady flow problem.

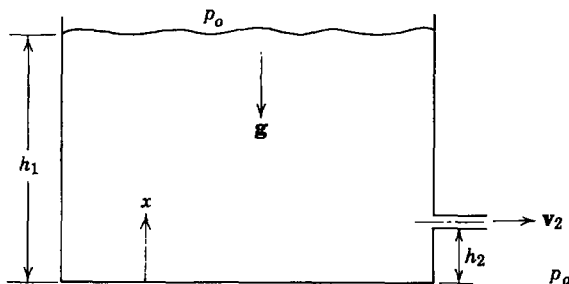


Fig. 12.2.1 Example of application of Bernoulli's equation.

There are no externally applied forces other than pressure and gravity, which has a downward acceleration g . We wish to find the discharge speed v_2 .

The gravitational potential U is

$$U = \rho g x,$$

where we assume that x is measured from the bottom of the tank. (We could choose any other convenient reference point.)

Application of Bernoulli's equation (12.2.11) with $\psi = 0$ (there are no electromagnetic forces) at the top of the fluid and at the outlet of the discharge pipe yields

$$p_0 + \rho g h_1 = p_0 + \rho g h_2 + \frac{1}{2} \rho v_2^2,$$

from which

$$v_2 = \sqrt{2g(h_1 - h_2)}.$$

We now apply the equations of motion for potential flow to examples involving electromechanical coupling.

12.2.1 Coupling with Flow in a Constant-Area Channel

We first consider the flow of an incompressible inviscid fluid in a horizontal channel with the dimensions and coordinate system defined in Fig. 12.2.2. At the channel inlet ($x_1 = 0$) the fluid velocity is constrained to be

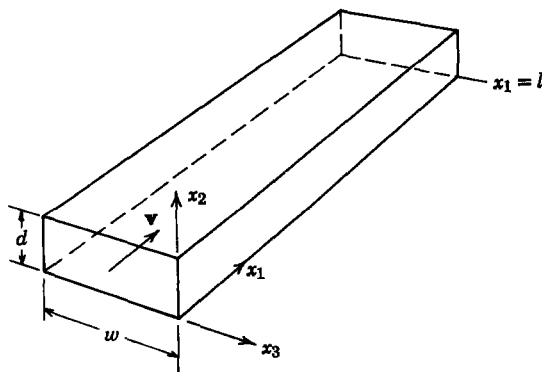


Fig. 12.2.2 A channel of constant cross-sectional area.

uniform and in the x_1 -direction

$$\mathbf{v}(0, x_2, x_3, t) = \mathbf{i}_1 v_o(t). \quad (12.2.12)$$

At a fixed channel wall, the normal component of velocity must be zero and the tangential component is unconstrained (for an inviscid fluid); consequently, the velocity of flow throughout the channel is

$$\mathbf{v}(x_1, x_2, x_3, t) = \mathbf{i}_1 v_o(t) \quad (12.2.13)$$

and the velocity potential is

$$\phi(x_1, x_2, x_3, t) = -x_1 v_o(t). \quad (12.2.14)$$

Note that this potential satisfies Laplace's equation (12.2.3) and the boundary conditions.

Equation 12.2.13 is the velocity distribution in the constant-area channel with the boundary condition specified (12.2.12) regardless of the space distributions or time variations of applied force densities but with the restriction that these force densities be irrotational (12.2.7).

12.2.1a Steady-State Operation

In this section we analyze a simple coupled system that is the basic configuration for illustrating the most important phenomena in magnetohydrodynamic (MHD) conduction machines. In spite of the myriad factors (viscosity, compressibility, turbulence, etc.) that affect the properties of real devices, the model presented is used universally for making initial estimates of electromechanical coupling in MHD conduction machines of all types.

The basic configuration is illustrated in Fig. 12.2.3 and consists of a rectangular channel of length l , width w , and depth d , through which an electrically

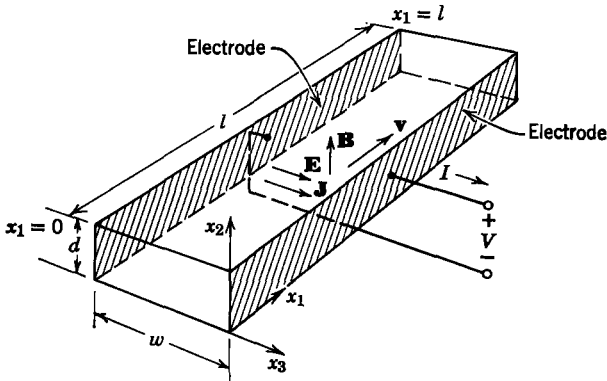


Fig. 12.2.3 Conduction-type, MHD machine.

conducting nonmagnetizable fluid flows with velocity $\mathbf{v}$ in the x_1 -direction. The two channel walls perpendicular to the x_2 -direction are insulators and the two walls perpendicular to the x_3 -direction are highly conducting electrodes from which terminals are connected to an external circuit. The flux density $\mathbf{B}$ is in the x_2 -direction and is produced by external coils or magnets not shown. The electrical conductivity σ of the fluid is high enough that the system can be modeled as a quasi-static magnetic field system.

We are considering an inviscid fluid model and we assume that the inlet ($x_1 = 0$) velocity is uniform as expressed by (12.2.12); thus the velocity is uniform throughout the channel as expressed by (12.2.13). We neglect fringing magnetic fields and the magnetic field due to current in the fluid* and assume that $\mathbf{B}$ is uniform:

$$\mathbf{B} = \mathbf{i}_2 B, \quad (12.2.15)$$

where B is constant. Because we are dealing with a steady-flow problem with time-invariant boundary conditions, $\partial/\partial t = 0$ and Faraday's law yields

$$\nabla \times \mathbf{E} = 0. \quad (12.2.16)$$

Once again we neglect fringing fields at the ends of the channel† and obtain the resulting solution

$$\mathbf{E} = -\mathbf{i}_3 \frac{V}{w}, \quad (12.2.17)$$

where V is the potential difference between the electrodes with the polarity defined in Fig. 12.2.3.

We now use Ohm's law for a moving conductor of conductivity σ (6.3.5),

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (12.2.18)$$

to write the current density for the system of Fig. 12.2.3 as

$$\mathbf{J} = \mathbf{i}_3 \sigma \left(-\frac{V}{w} + v_o B \right). \quad (12.2.19)$$

Note that this current density is uniform and therefore satisfies the conservation of charge condition $\nabla \cdot \mathbf{J} = 0$. Because the current density is uniform, it can

* The neglect of the self-field due to current in the fluid is justified for MHD generators when the magnetic Reynolds number based on channel length is much less than unity (see Section 7.1.2a).

† This assumption is quite good provided the l/w ratio of the channel is large (five or more). This result has been obtained in a detailed analysis of end effects by using a conformal mapping technique. The results of this analysis are presented in "Electrical and End Losses in a Magnetohydrodynamic Channel Due to End Current Loops," G. W. Sutton, H. Hurwitz, Jr., and H. Poritsky, Jr., *Trans. AIEE (Comm. Elect.)*, **81**, 687-696 (January 1962).

be related to the terminal current by the area of an electrode; thus

$$\mathbf{J} = \mathbf{i}_s \frac{I}{ld}. \quad (12.2.20)$$

To obtain the electrical terminal characteristics of this machine, we combine (12.2.19) and (12.2.20) to obtain

$$IR_i = -V + v_o Bw, \quad (12.2.21)$$

where we have defined the internal resistance R_i as

$$R_i = \frac{w}{\sigma ld}. \quad (12.2.22)$$

Equation 12.2.21 can be represented by the equivalent circuit of Fig. 12.2.4. The open-circuit voltage ($v_o Bw$) is generated by the motion of the conducting fluid through the magnetic field and has the same physical nature as speed voltage generated in conventional dc machines using solid conductors (see Section 6.4). This speed voltage can supply current to a load through the internal resistance R_i which is simply the resistance that would be measured between electrodes with the fluid at rest. From an electrical point of view the electromechanical interaction occurs in the equivalent battery ($v_o Bw$) in Fig. 12.2.4.

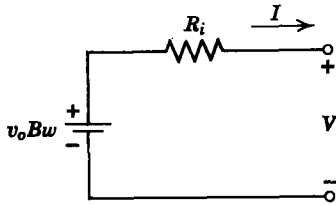


Fig. 12.2.4 Electrical equivalent circuit of conduction-type MHD machine.

To describe the properties of the MHD machine of Fig. 12.2.3, viewed from the electrical terminals, we have obtained a relation between terminal voltage and terminal current (12.2.21). From a mechanical point of view a similar relation is that between the pressure difference over the length of the channel and the velocity through the channel. This mechanical terminal relation is obtained from the x_1 -component of the momentum equation (12.2.4):

$$0 = -\frac{\partial p}{\partial x_1} - \frac{IB}{ld}. \quad (12.2.23)$$

Integration of this equation over the length of the channel yields

$$\Delta p = -\frac{IB}{d}, \quad (12.2.24)$$

where the pressure rise Δp is defined by

$$\Delta p = p(l) - p(0). \quad (12.2.25)$$

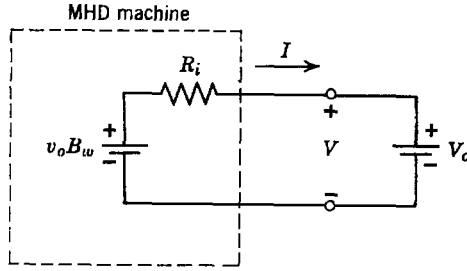


Fig. 12.2.5 MHD conduction machine with a constant-voltage constraint on the electrical terminals.

Equation 12.2.24 indicates that for this system the pressure rise along the channel is a function of the terminal current only and independent of the fluid velocity. This is reasonable because the pressure gradient is balanced by the $\mathbf{J} \times \mathbf{B}$ force density, regardless of the velocity. For an arbitrary electrical source or load the pressure rise will vary with velocity because the current depends on velocity through (12.2.21).

To study the energy conversion properties of the machine in Fig. 12.2.3 we constrain the electrical terminals with a constant-voltage source V_o as indicated in Fig. 12.2.5 and study the behavior of the device as a function of the fluid velocity v_o . For this purpose we use (12.2.21) to find the current I as

$$I = \frac{v_o B_w - V_o}{R_i} \quad (12.2.26)$$

Substitution of this result into (12.2.24) yields for the pressure rise

$$\Delta p = -\frac{B}{dR_i} (v_o B_w - V_o). \quad (12.2.27)$$

The current and pressure rise are shown plotted as functions of velocity v_o in Fig. 12.2.6.

To determine the nature of the device we define the electric power output P_e which, when positive, indicates a flow of electric energy from the MHD machine into the source V_o :

$$P_e = IV_o. \quad (12.2.28)$$

We also define the mechanical power out P_m , which represents power flow from the MHD machine into the velocity source v_o :

$$P_m = \Delta p w d v_o. \quad (12.2.29)$$

For the range of velocities

$$v_o > \frac{V_o}{Bw}$$

we have

$$P_e > 0, \quad P_m < 0$$

and the device is a generator; that is, mechanical power input is in part converted to electric power. For the velocity range

$$0 < v_o < \frac{V_o}{Bw}$$

we have

$$P_e < 0, \quad P_m > 0$$

and the device is a pump. Electric power input is converted in part to mechanical power. For the velocity range

$$v_o < 0, \\ P_e < 0, \quad P_m < 0;$$

that is, both mechanical and electrical power are into the MHD machine. All of this input power is dissipated in the internal resistance of the machine. In this region the machine acts as an electromechanical brake because electric

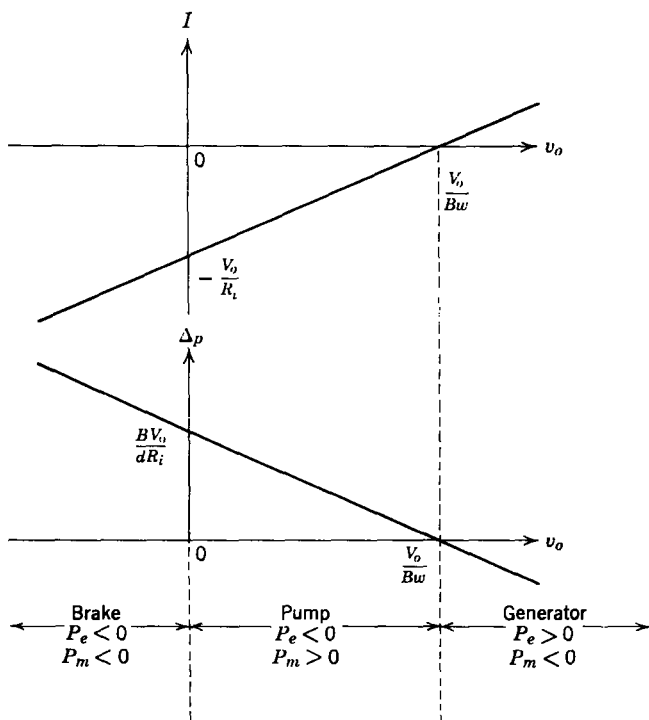


Fig. 12.2.6 Terminal characteristics of an MHD conduction-type machine with constant terminal voltage.

power is put in, and the only electromechanical result is to retard the fluid flow.

The properties of the MHD machine, as indicated by the curves of Fig. 12.2.6, can be interpreted in terms of the equivalent circuit of Fig. 12.2.5. We substitute (12.2.24) into (12.2.29) to find that the mechanical output power is expressible as

$$P_m = -I(v_o B w). \quad (12.2.30)$$

Reference to Fig. 12.2.5 shows that this is the power input to the battery that represents the speed voltage. Thus, when the battery ($v_o B w$) absorbs power, energy is being supplied to the velocity source by the MHD machine. When the battery ($v_o B w$) supplies power, energy is being supplied to the external voltage source by the MHD machine. When the battery ($v_o B w$) supplies power, energy is being extracted from the velocity source. Thus, when the two batteries of Fig. 12.2.5 have opposing polarities, energy can flow from one battery to the other and the machine can operate as a pump or a generator, the operation being determined by the relative values of the two battery voltages. When the polarities of both batteries are in the same direction ($v_o < 0$ in Fig. 12.2.5), the two batteries supply energy to the resistance R_i , and the MHD machine acts as a sink for both electrical and mechanical energy. This is operation as a brake.

This analysis has been done for a particular set of terminal constraints. Essentially the same techniques can be used for other constraints. It is worthwhile to point out that (12.2.24) indicates that if the machine is constrained mechanically by a constant pressure source the electrical output will be at constant current.

The analysis just completed provides the basic model used in any examination of the electromechanical coupling process in conduction-type MHD devices, regardless of whether they are pumps or generators and whether the working fluid is a liquid or gas. The model and its consequences should be compared with those of commutator machines (Section 6.4.1) and of homopolar machines (Section 6.4.2). The similarities are evident and the opportunity of using the results of the analysis of one device for interpreting the behavior of another will broaden our understanding of electromechanical interactions of this kind.

An alternative method of achieving electromechanical coupling between an electrical system and a conducting fluid is to use a system that is analogous to the squirrel-cage induction machine analyzed in Section 4.1.6b. We shall not analyze this type of system here, but the analysis is a straightforward extension of concepts and techniques already presented. The system consists basically of a channel of flowing conducting fluid that is subjected to a transverse magnetic field in the form of a wave traveling in the direction of

flow. This wave is most often established by a distributed polyphase winding (Sections 4.1.4 and 4.1.7). When the wave of magnetic field travels faster than the fluid, the fluid is accelerated by the field and pumping action results. When the fluid travels faster than the magnetic field wave, the fluid is decelerated and electric power is generated. In the analysis of an induction machine magnetic diffusion and skin effect are important (Section 7.1.4).

Both conduction- and induction-type MHD machines are used for pumping liquid metals*; they are proposed for power generation with liquid metals† and used to accelerate ionized gases for space propulsion systems‡; both are proposed for power generation with ionized gases,§ although the conduction-type machine appears more attractive by far for this purpose.

12.2.1b Dynamic Operation

We now consider the kinds of phenomena that can result from electromechanical coupling with an incompressible fluid of time-varying velocity. We start by considering the fluid dynamic behavior of a simple example, which will then be the basis for a study of electromechanical transient effects.

The configuration to be studied is shown in Fig. 12.2.7. The system consists of a rigid tube of rectangular cross section bent into the form of a U. The depth d of the tube is small compared with the radius of the bends. The tube is filled with an incompressible inviscid fluid to a length l measured along the center of the tube. The two surfaces are open to atmospheric pressure p_o and gravity acts downward as shown.

It is clear that for static equilibrium the two surfaces of the fluid are at the same height. The displacement of the two surfaces from the equilibrium positions are designated x_a and x_b .

To study the dynamic behavior of this system we displace the fluid from equilibrium, release it from rest, and study the ensuing fluid motions.

The equations for solving this problem express conservation of mass and force equilibrium. Conservation of mass (12.1.31) used with the irrotational flow condition (12.2.1) and the fact that the channel has constant cross-sectional area leads to the conclusion that the flow velocity is uniform across the channel. (Here we ignore effects due to the channel curvature.)

* L. R. Blake, "Conduction and Induction Pumps for Liquid Metals," *Proc. Inst. of Elec Engrs. (London)*, **104A**, 49 (1957).

† D. G. Elliott, "Direct-Current Liquid Metal MHD Power Generation," *AIAA J.*, 627-634 (1966). M. Petrick and K. V. Lee, "Performance Characteristics of a Liquid Metal MHD Generator," *Intl. Symp. MHD Elec. Power Gen.*, Vol 2, pp. 953-965, Paris, July 1964.

‡ E. L. Resler and W. R. Sears, "The Prospects for Magnetohydrodynamics," *J. Aerospace Sci.*, **25**, No. 4, 235-245 (April 1958).

§ H. H. Woodson, "Magnetohydrodynamic AC Power Generation," *AIEE Pacific Energy Conversion Conf. Proc.*, pp. 30-1-30-2, San Francisco, 1964.

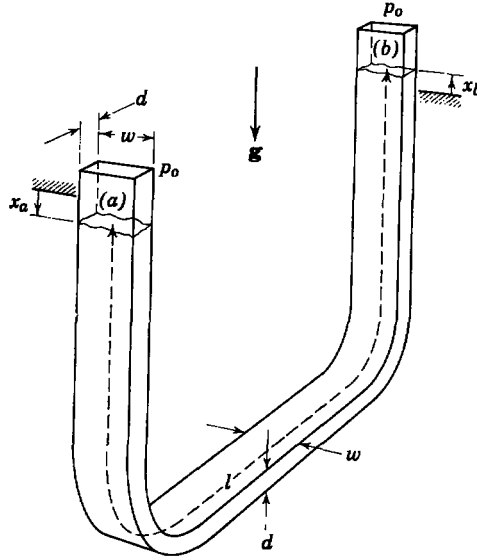


Fig. 12.2.7 Configuration for transient flow problem.

Furthermore, the displacements of the two surfaces are equal

$$x_a = x_b. \quad (12.2.31)$$

The form of the momentum equation that is most useful for this example is (12.2.5) with $\mathbf{F}^e = 0$.

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla \left(p + \rho \frac{v^2}{2} + U \right), \quad (12.2.32)$$

where U is the gravitational potential. We now do a line integration of (12.2.32) from the surface at (a) to the surface at (b) along the center of the tube to obtain

$$\int_a^b \rho \frac{\partial \mathbf{v}}{\partial t} \cdot d\mathbf{l} = \int_a^b -\nabla \left(p + \rho \frac{v^2}{2} + U \right) \cdot d\mathbf{l} \quad (12.2.33a)$$

$$\rho l \frac{\partial v}{\partial t} = -2\rho g x_a. \quad (12.2.33b)$$

This result could have been obtained by using (12.2.10), a fact that is not surprising because the steps leading from (12.2.32) to (12.2.33) parallel those used in Section 12.2.

The velocity v is given by

$$v = \frac{dx_a}{dt};$$

thus we rewrite (12.2.33) as

$$l \frac{d^2 x_a}{dt^2} + 2g x_a = 0, \quad (12.2.34)$$

which is a convenient expression for the surface displacement x_a . It shows that the dynamics are those of an undamped second-order system.

We now displace the fluid surface at (a) to the position

$$x_a(0) = X_o \quad (12.2.35)$$

and release it from rest

$$\frac{dx_a}{dt}(0) = 0. \quad (12.2.36)$$

The solution of (12.2.34) with the initial conditions of (12.2.35) and (12.2.36) is

$$x_a(t) = u_{-1}(t) X_o \cos \omega t, \quad (12.2.37)$$

where $u_{-1}(t)$ is the unit step and the frequency ω is given by

$$\omega = \left(\frac{2g}{l} \right)^{1/2}. \quad (12.2.38)$$

Note that this lossless, fluid-mechanical system has the basic property of a simple pendulum in that the natural frequency depends only on the acceleration of gravity and the length of fluid in the flow direction and is independent of the mass density of the fluid.

We now couple electromechanically to the system of Fig. 12.2.7 with an MHD machine of the kind analyzed in Section 12.2.1a placed in the U tube as shown in Fig. 12.2.8. The total length of fluid between the surfaces at (a) and (b) is still l and the length of the MHD machine in the flow direction is l_1 . The flux density B is uniform over the length of the MHD machine and is again produced by a system not shown. As in Section 12.2.1a, we neglect the magnetic field due to current in the fluid as well as the end and edge effects. The terminals of the MHD machine are loaded with a resistance R .

In this analysis we are interested in the fluid dynamical transient that will usually be much slower than purely electrical transients whose time constant depends on the inductance of the electrode circuit. Thus we neglect the inductance of the electrode circuit and the electric terminal relation is obtained from (12.2.21) by setting

$$V = IR. \quad (12.2.39)$$

The resulting relation between current and velocity is

$$I = \frac{vBw}{R_i + R}, \quad (12.2.40)$$

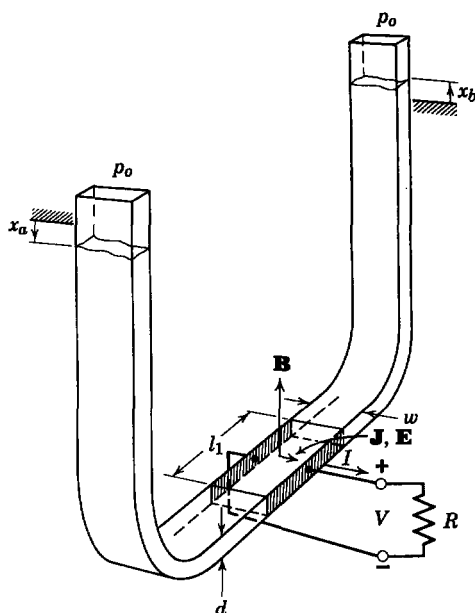


Fig. 12.2.8 Transient-flow problem with electromechanical coupling.

where the internal resistance is

$$R_i = \frac{w}{\sigma l_1 d}$$

and σ is the electrical conductivity of the fluid.

The addition of the electrical force term to the momentum equation (12.2.32) yields

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla \left(p + \rho \frac{v^2}{2} + U \right) + \mathbf{J} \times \mathbf{B}. \quad (12.2.41)$$

Integration of this expression between the two fluid surfaces in the manner of (12.2.33) yields

$$\rho l \frac{\partial v}{\partial t} = -2\rho g x_a - \frac{IB}{d}. \quad (12.2.42)$$

Note that the last term on the right is simply the pressure rise through the MHD machine due to the electromagnetic force density (12.2.24).

Substitution of (12.2.40) and $v = dx_a/dt$ into (12.2.42) yields the differential equation in x_a

$$\rho l \frac{d^2 x_a}{dt^2} + \frac{B^2 w}{d(R_i + R)} \frac{dx_a}{dt} + 2\rho g x_a = 0. \quad (12.2.43)$$

Comparison of (12.2.43) with (12.2.34) shows that the electromechanical coupling with a resistive load has added a damping term to the differential equation. This is easily understandable in terms of the analysis of the MHD machine in Section 12.2.1a. The fluid motion produces a voltage proportional to speed, a resistive load on this voltage produces a current proportional to speed, and the current in the fluid interacts with the applied flux density to produce a retarding force proportional to speed. Thus the electrical force appears as a damping term in the differential equation.

To consider the kind of behavior that can result in a real system of this kind we assume that the fluid is mercury, which has the following constants

$$\rho = 13,600 \text{ kg/m}^3, \quad \sigma = 10^6 \text{ mhos/m.}$$

The system dimensions are chosen to be

$$\begin{aligned} l &= 1 \text{ m}, & l_1 &= 0.1 \text{ m}, \\ w &= 0.02 \text{ m}, & d &= 0.01 \text{ m}. \end{aligned}$$

We set the load resistance R equal to the internal resistance R_i

$$R = R_i = 2 \times 10^{-5} \Omega.$$

For these given constants the differential equation (12.2.43) reduces to

$$\frac{d^2 x_a}{dt^2} + 3.68 B^2 \frac{dx_a}{dt} + 19.6 x_a = 0. \quad (12.2.44)$$

When the fluid is released from rest with the initial conditions of (12.2.35) and (12.2.36), the resulting transients in fluid position and electrode current are shown in Fig. 12.2.9. It is clear that with attainable flux densities the electro-mechanical coupling force can provide significant damping for the system.*

Some properties of the curves of Fig. 12.2.9 are worth noting. First, for very small time ($t < 0.1$ sec) the response in fluid position is essentially unaffected by the force of electric origin. This occurs because the initial velocity is zero and it takes velocity to generate voltage and drive current. Thus the initial increase in velocity is independent of the value of flux density and the initial current buildup is proportional to flux density.

The resistive load on the electrodes of the MHD machine in Fig. 12.2.8 can be replaced by an electrical source and the fluid displacement can be driven electrically. In such a case, when the fluid motion is of interest, (12.2.21) and (12.2.42) are adequate for the study.

* An experiment to demonstrate this effect is complicated by the fact that the contact resistance between the liquid metal and the electrodes is likely to be appreciable.

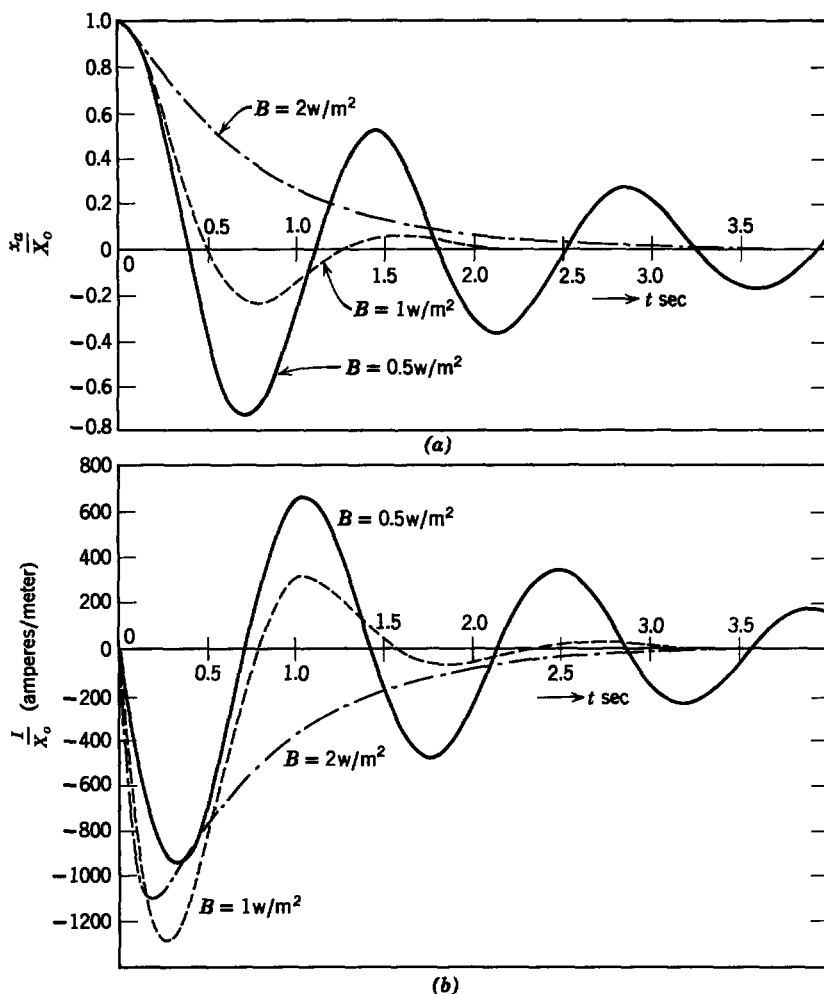


Fig. 12.2.9 Transient response of MHD-damped system: (a) fluid position; (b) electrode current.

12.2.2 Coupling with Flow in a Variable-Area Channel

To establish some insight into the properties of potential flow in two dimensions, consider the flow around a corner in the configuration of Fig. 12.2.10. The fluid container has constant depth in the x_3 -direction and the fluid is incompressible and inviscid. There are no electrical forces, and we neglect gravity effects (assume gravity to act in the x_3 -direction).

For potential flow the velocity is given by (12.2.2) as $\mathbf{v} = -\nabla\phi$ and the velocity potential ϕ satisfies Laplace's equation ($\nabla^2\phi = 0$). The boundary

condition is that the normal component of velocity must be zero along the rigid surfaces. The solution of Laplace's equation which satisfies these boundary conditions is

$$\phi = \frac{v_o}{\sqrt{2}a} (x_2^2 - x_1^2), \quad (12.2.45)$$

where $2v_o$ is the speed of the fluid at $x_1 = x_2 = a$. The velocity is thus given by

$$\mathbf{v} = \mathbf{i}_1 \sqrt{2} v_o \frac{x_1}{a} - \mathbf{i}_2 \sqrt{2} v_o \frac{x_2}{a}. \quad (12.2.46)$$

Equipotential lines and streamlines are shown in Fig. 12.2.10. This solution is valid, even if v_o is time-varying.

We now restrict our attention to a steady-flow problem ($v_o = \text{constant}$) and find that Bernoulli's equation (12.2.11) yields

$$\frac{1}{2} \rho v^2 + p = \text{constant}. \quad (12.2.47)$$

We note from (12.2.46) that at $x_1 = x_2 = 0$ the velocity $\mathbf{v} = 0$. Because the velocity is zero, this is called a *stagnation point*. If we designate the pressure at the stagnation point as p_o , (12.2.47) becomes

$$\frac{1}{2} \rho v^2 + p = p_o. \quad (12.2.48)$$

Thus with a knowledge of the stagnation point pressure and the velocity distribution we can find the pressure at any other point in the fluid. The use of

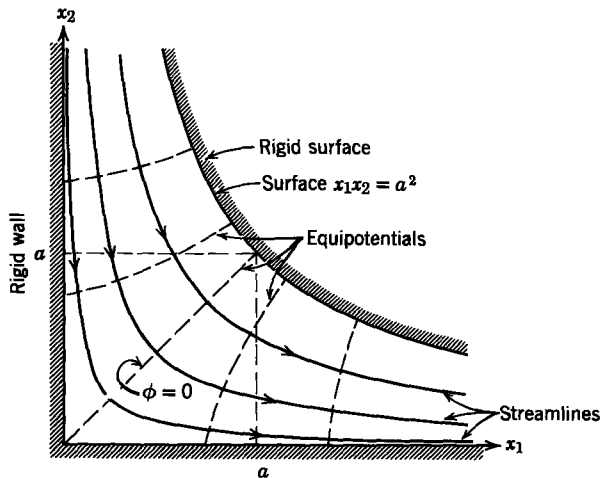


Fig. 12.2.10 Example of potential flow.

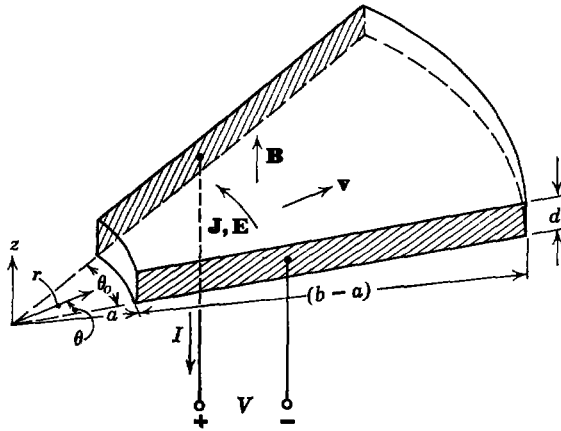


Fig. 12.2.11 MHD conduction machine with variable area.

(12.2.46) in (12.2.48) yields for the pressure at point (x_1, x_2)

$$p = p_o - \frac{\rho v_o^2}{a^2} (x_1^2 + x_2^2). \quad (12.2.49)$$

From this result we conclude that in a flowing incompressible fluid the highest pressure occurs at the stagnation point. Moreover, for a given flow the higher the local fluid speed, the lower the local pressure.*

This example indicates that the pressure can be changed by changing the velocity and vice versa. Variations of velocity are obtained by varying the cross-sectional area of the fluid flow. We now do an example of an MHD interaction with a two-dimensional fluid flow in which the geometry of the channel can be adjusted to vary the relation between input pressure and velocity and output pressure and velocity. Such freedom is desirable in many MHD applications. Here it allows us to extend the basic ideas introduced in Section 12.2.1a to a case in which the fluid is accelerating but the flow is steady ($\partial/\partial t = 0$).

The system to be considered is the conduction machine shown schematically in Fig. 12.2.11. The channel forms a segment of a cylinder. The inlet is at radius $r = a$ and the outlet is at radius $r = b$. The insulating walls perpendicular to the z -direction are separated by a distance d . The electrodes are in radial planes separated by the angle θ_o . We use a cylindrical coordinate system r, θ, z , defined in Fig. 12.2.11. There is an applied flux density $\mathbf{B}$ in

* Even though (12.2.49) indicates that the pressure p can go negative, in fact it cannot. As long as we use an incompressible model, the pressure appears in only one place in the equations of motion, and they remain unaltered if an arbitrary constant is added to (or subtracted from) p . Other effects, such as compressibility, depend on an equation of state that is sensitive to the absolute magnitude of the pressure. If these effects are included, a negative pressure is not physically possible.

the z -direction. The electrodes are connected to electrical terminals at which the voltage V and current I are defined.

The velocity at the inlet ($r = a$) and the velocity at the outlet ($r = b$) are assumed to be radial and constant in magnitude. We assume solutions with cylindrical symmetry. These solutions are quite accurate, provided the angle θ_0 is reasonably small. Again the magnetic field generated by current in the fluid is neglected (low magnetic Reynolds number).

As already assumed, the fluid is incompressible and inviscid with electrical conductivity σ and permeability μ_0 . The velocity is radial

$$\mathbf{v} = \mathbf{i}_r v_r, \quad (12.2.50)$$

and the electric field intensity and current density are azimuthal

$$\mathbf{E} = \mathbf{i}_\theta E_\theta, \quad (12.2.51)$$

$$\mathbf{J} = \mathbf{i}_\theta J_\theta. \quad (12.2.52)$$

We have already specified that the total flux density is

$$\mathbf{B} = \mathbf{i}_z B_z, \quad (12.2.53)$$

where B_z is a constant.

We first assume that at the inlet ($r = a$) the radial component of velocity is

$$v_r = v_a. \quad (12.2.54)$$

Next, conservation of mass for incompressible flow requires that

$$\oint_S \mathbf{v} \cdot \mathbf{n} \, da = 0. \quad (12.2.55)$$

The value of v_r at any radius r follows as

$$v_r = \frac{a}{r} v_a. \quad (12.2.56)$$

Steady-state operation yields $\nabla \times \mathbf{E} = 0$ and the z -component of $\nabla \times \mathbf{E} = 0$ [assuming that $\mathbf{E}$ takes the form of (12.2.51)] is

$$\frac{1}{r} \frac{\partial(rE_\theta)}{\partial r} = 0. \quad (12.2.57)$$

This yields the result that

$$E_\theta = \frac{A}{r}, \quad (12.2.58)$$

where A is a constant to be determined from the boundary conditions. To evaluate the constant A , the definition of the terminal voltage

$$-\int_0^{\theta_0} E_\theta r \, d\theta = V \quad (12.2.59)$$

is used to obtain

$$E_\theta = -\frac{V}{r\theta_o}. \quad (12.2.60)$$

Substitution of (12.2.53), (12.2.56), and (12.2.60) into the θ -component of Ohm's law for a moving, conducting medium (12.2.18) yields

$$J_\theta = \sigma \left(-\frac{V}{r\theta_o} + \frac{a}{r} v_a B_z \right). \quad (12.2.61)$$

Note that this expression satisfies $\nabla \cdot \mathbf{J} = 0$.

A relation between current density and terminal current can be obtained from the expression

$$I = \int_a^b J_\theta d r. \quad (12.2.62)$$

Performance of this integration yields

$$IR_i = -V + a\theta_o v_a B_z, \quad (12.2.63)$$

where we have defined the internal resistance R_i as

$$R_i = \frac{\theta_o}{\sigma d \ln(b/a)} \quad (12.2.64)$$

Note the similarity between (12.2.63) and (12.2.21) for the simpler geometry in Fig. 12.2.3.

The radial component of the momentum equation (12.2.4) for steady-state conditions is

$$\rho v_r \frac{\partial v_r}{\partial r} = -\frac{\partial p}{\partial r} + J_\theta B_z. \quad (12.2.65)$$

Multiplication of the expression by dr , integration from $r = a$ to $r = b$, and use of (12.2.56) and (12.2.62) yields

$$\frac{1}{2} \rho v_a^2 \left[\left(\frac{a}{b} \right)^2 - 1 \right] = -\Delta p - \frac{IB_z}{d}, \quad (12.2.66)$$

where the pressure rise Δp is defined as

$$\Delta p = p(b) - p(a). \quad (12.2.67)$$

Note the similarity between (12.2.66) and (12.2.24) for the constant-area channel. The difference lies in the first term on the left of (12.2.66) which results from the changing area and therefore changing velocity in the channel of Fig. 12.2.11.

Equation 12.2.66 could have been obtained from Bernoulli's equation (12.2.11); in a simple case like this, however, it is more informative to obtain the result from first principles.

To study some of the properties of the system with varying area consider first the case in which the electrical terminals are open-circuited. The terminal voltage, as obtained from (12.2.63) is

$$V = a\theta_0 v_a B_z \quad (12.2.68)$$

and the pressure rise obtained from (12.2.66) is

$$\Delta p = \frac{1}{2} \rho v_a^2 \left[1 - \left(\frac{a}{b} \right)^2 \right]. \quad (12.2.69)$$

Because $a < b$, this pressure rise is positive, which indicates that the outlet pressure is higher than the inlet pressure. This results because the fluid velocity decreases as r increases and this fluid deceleration must be balanced by a pressure gradient as indicated by the momentum equation (12.2.65). Thus the variable area channel by itself acts as a kind of "fluid transformer" that can increase pressure as it decreases velocity or vice versa.

The electrical terminal relation (12.2.63) for the machine with variable area (Fig. 12.2.11) has the same form as the electrical terminal relation (12.2.21) for the machine with constant area (Fig. 12.2.3). Thus, if the inlet velocity v_a is the independent mechanical variable, the analysis of the electric terminal behavior is exactly the same as that of the constant-area machine; that is, from an electrical point of view the machine appears to have an open-circuit voltage ($a\theta_0 v_a B_z$) in series with an internal resistance R_i (12.2.64), as illustrated in Fig. 12.2.12. This equivalent circuit can be connected to any combination of active and passive loads, and the electrical behavior can be predicted correctly within the limitations of the assumptions made in arriving at the model.

To study the energy conversion properties of the variable-area machine we must generalize the concept of mechanical input power that was used in (12.2.29) for the constant-area machine. No longer is the mechanical input power simply equal to the pressure difference times the volume flow rate of fluid because the difference in inlet and outlet velocities indicates that there is

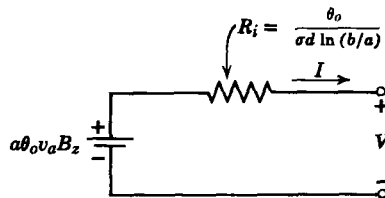


Fig. 12.2.12 Electric equivalent circuit for a variable-area MHD machine.

a net transport of kinetic energy into or out of the volume of the channel by the fluid. To illustrate this concept consider the system operating with the electrical terminals open-circuited. There is clearly no electrical output power and no I^2R_i losses in the fluid. Moreover, the fluid is inviscid, so there can be no mechanical losses. Thus we expect the mechanical input power to be zero, although there is a nonzero pressure difference between inlet and outlet of the channel.

To determine the mechanical energy interchange between the MHD device and the energy source which makes the fluid flow through the device we use the *conservation of energy* which states, in general,

$$\left[\begin{array}{l} \text{total power input} \\ \text{to channel volume} \end{array} \right] = \left[\begin{array}{l} \text{rate of increase of} \\ \text{energy stored in volume} \end{array} \right] \quad (12.2.70)$$

For the steady-state problem being considered the energy stored in the volume is constant and the right side of (12.2.70) is zero. We thus define the mechanical *output* power from the channel as P_m and the power converted to electrical form as P_{em} and write (12.2.70) for conservation of *mechanical* energy* as

$$-P_m - P_{em} = 0. \quad (12.2.71)$$

For open-circuit conditions the electromechanical power P_{em} is zero and

$$P_m = 0. \quad (12.2.72)$$

To calculate P_m , which has been defined as the work done by the fluid in the channel *on* the fluid mechanical source, we must specify how work is done on the fluid in the channel and how energy is stored and transported by the fluid.

At a surface of a fluid (this can be an imaginary surface in a fluid) with outward directed normal vector $\mathbf{n}$, as illustrated in Fig. 12.2.13, there will be a pressure force on the fluid enclosed by the surface of magnitude p and directed opposite to the normal vector ($-\mathbf{p}\mathbf{n}$) [see (12.1.37)]. If the fluid is moving with velocity $\mathbf{v}$ at the surface, the rate at which the pressure force ($-\mathbf{p}\mathbf{n} da$) does work on the fluid inside the volume V is

$$\left[\begin{array}{l} \text{power input from} \\ \text{pressure forces} \end{array} \right] = \oint_S -\mathbf{p}\mathbf{n} \cdot \mathbf{v} da. \quad (12.2.73)$$

A fluid can store kinetic energy with a density $\frac{1}{2}\rho v^2$. At each point along the surface of Fig. 12.2.13 fluid flow across the surface will transport kinetic energy into or out of the volume V . The volume of fluid crossing the surface

* Even though electrical losses in the fluid (I^2R_i) occur within the volume of the channel, they are not included in this energy expression. This is possible here because these losses do not affect the mechanical properties of an incompressible, inviscid fluid. When we consider gaseous conductors in Chapter 13, the electrical losses must be included because they will affect the mechanical properties of the conducting fluid.

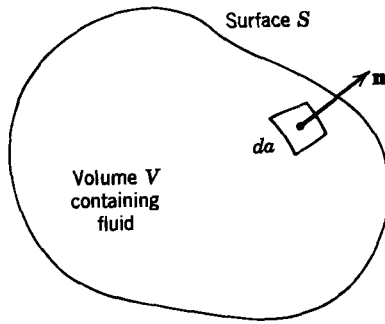


Fig. 12.2.13 Geometry for writing conservation of energy for a fluid.

element da in unit time is $\mathbf{v} \cdot \mathbf{n} da$. Thus the total kinetic energy transported out of the volume in unit time is

$$\left[\begin{array}{c} \text{power output from kinetic} \\ \text{energy transport} \end{array} \right] = \oint_S \frac{1}{2} \rho v^2 \mathbf{v} \cdot \mathbf{n} da. \quad (12.2.74)$$

For an incompressible inviscid fluid (12.2.73) and (12.2.74) represent the only mechanisms for interchanging mechanical energy with a fluid; thus the mechanical output power P_m defined in (12.2.71) is given by

$$P_m = \oint_S p \mathbf{n} \cdot \mathbf{v} da + \oint_S \frac{1}{2} \rho v^2 \mathbf{v} \cdot \mathbf{n} da. \quad (12.2.75)$$

To apply (12.2.75) to the variable-area channel of Fig. 12.2.11 we must define the surface that encloses the fluid in the channel. This surface consists of the four channel walls and the two concentric cylindrical surfaces at $r = a$ and $r = b$. The velocity is nonzero only along the last two surfaces; consequently, (12.2.75) integrates to

$$P_m = -p(a)v_r(a)a\theta_0 d + p(b)v_r(b)b\theta_0 d - \frac{1}{2}\rho v_r^3(a)a\theta_0 d + \frac{1}{2}\rho v_r^3(b)b\theta_0 d. \quad (12.2.76)$$

The assumption that $v_r(a) = v_a$ (12.2.54) and the use of (12.2.56) to write

$$v_r(b) = \frac{a}{b} v_a \quad (12.2.77)$$

allows us to write (12.2.76) in the simplified form

$$P_m = a\theta_0 d v_a \left[\Delta p - \frac{1}{2} \rho v_a^2 \left(1 - \frac{a^2}{b^2} \right) \right], \quad (12.2.78)$$

where the pressure rise Δp has been defined in (12.2.67) as $\Delta p = p(b) - p(a)$.

To apply (12.2.78) we first note that for the open-circuit condition $I = 0$, and (12.2.66) yields

$$\Delta p_{oc} = \frac{1}{2} \rho v_a^2 \left(1 - \frac{a^2}{b^2} \right). \quad (12.2.79)$$

Substitution of this result into (12.2.78) yields for open-circuit conditions

$$P_m = 0.$$

This is in agreement with our intuitive physical prediction made at the start of this development. Next, for any arbitrary load $I \neq 0$ (12.2.66) yields

$$\Delta p = \frac{1}{2} \rho v_a^2 \left(1 - \frac{a^2}{b^2} \right) - \frac{IB_z}{d}. \quad (12.2.80)$$

Substitution of this result into (12.2.78) and simplification yield

$$P_m = -a\theta_o v_a B_z I. \quad (12.2.81)$$

From (12.2.71) the power converted electromechanically is

$$P_{em} = -P_m = a\theta_o v_a B_z I. \quad (12.2.82)$$

Reference to the equivalent circuit of Fig. 12.2.12 shows that this converted power is simply the power supplied to the electric circuit by the battery representing the open-circuit voltage.

This interpretation leads to the conclusion that for conversion of energy the variable-area machine has exactly the same properties as the constant-area machine analyzed earlier. The only difference arises when we are interested in the details of the pressure and velocity distributions and in the nature of the fluid mechanical source that provides the fluid flow through the machine. As we shall see in Chapter 13, however, these are essential considerations if the velocity is large enough (compared with that of sound) to make the effects of compressibility important.

12.2.3 Alfvén Waves

So far in the treatment of electromechanical coupling with incompressible inviscid fluids we have considered problems in which there has been gross motion of the fluid. All of these examples have been analyzed by using potential flow. In this section we consider electromechanical coupling that results in no gross motion of the fluid but rather involves the propagation of a signal through a fluid. Moreover, the fluid velocity has a finite curl and a potential flow model is inappropriate. Our discussion is pertinent to an understanding of MHD transient phenomena.

As discussed in Section 12.1.4, an inviscid, incompressible fluid can, by itself, support no shear stresses; but when such a fluid with very high

electrical conductivity is immersed in a magnetic field the magnetic field provides shear stiffness such that transverse waves, called Alfvén waves and very much akin to the shear waves in elastic media, can be propagated. They play an essential role in determining the dynamics of a highly conducting liquid or gas (plasma) interacting with a magnetic field.

To introduce the essential features of Alfvén waves we use a rectangular system in which variables are functions of only one dimension. It is difficult to realize physically the boundary conditions necessary for this model. Thus, after the ideas are introduced, we extend the example to cylindrical geometry, where all boundary conditions can be imposed realistically.

The magnetohydrodynamic system is shown in Fig. 12.2.14. An incompressible, inviscid, highly conducting ($\sigma \rightarrow \infty$) fluid is contained between rigid parallel walls. An external magnet is used to impose a magnetic flux density B_0 in the x_1 -direction. It is the effect of this flux density on the motions of the fluid transverse to the x_1 -axis that is of interest.

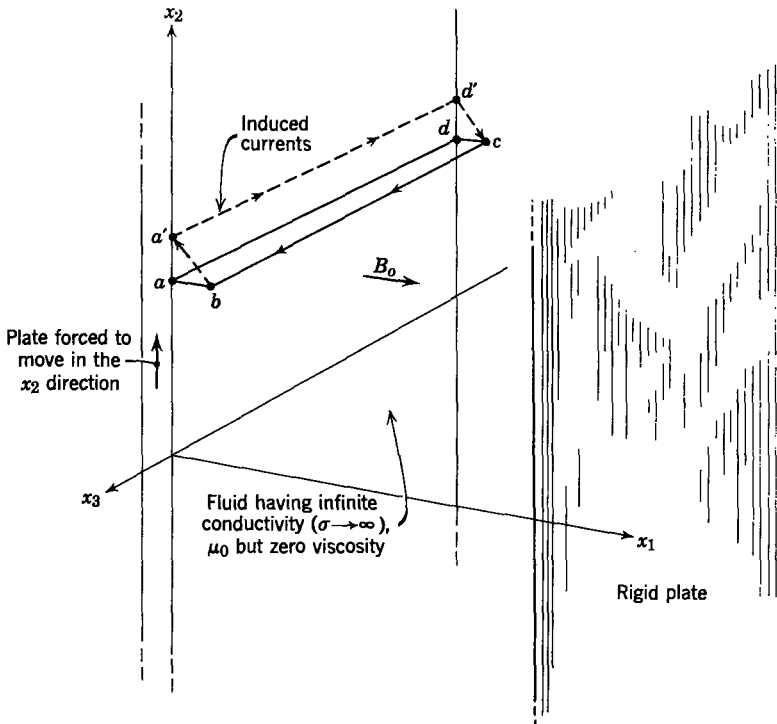


Fig. 12.2.14 Fluid contained between rigid parallel plates and immersed in a magnetic induction B_0 . Motions of the fluid are induced by transverse motions of the left-hand plate, which, like the fluid, is assumed to be highly conducting.

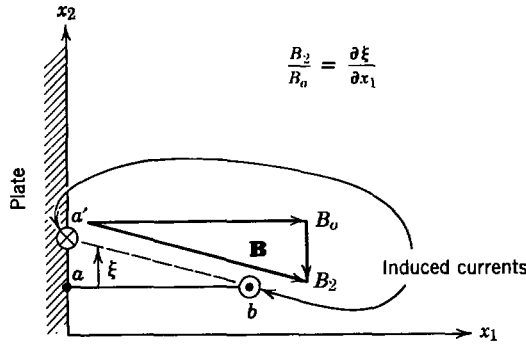


Fig. 12.2.15 End view of the loop $abcd$ shown in Fig. 12.2.14. The initial loop formed by conducting fluid and the plate links zero flux λ . To conserve the flux, density $\mathbf{B}$ remains tangential to the loop with the additional magnetic flux density B_2 created by an induced current.

Suppose that in the absence of a magnetic field the rigid plate is set in motion in the x_2 -direction. Because the fluid is inviscid, there is no shearing stress imposed on the fluid and the plate will transmit no motion to the fluid. In fact, if any sheet of fluid perpendicular to the x_1 -axis is set into transverse motion, the adjacent sheets of fluid remain unaffected because of the lack of shearing stresses.

Now consider the effect of imposing a magnetic field. The fluid is highly conducting, and this means that the electric field in the frame of the fluid is essentially zero. The law of induction can be written for a contour C attached to the fluid particles:

$$\oint_C \mathbf{E}' \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da \equiv -\frac{d\lambda}{dt}, \quad (12.2.83)$$

where $\mathbf{E}'$ is the electric field measured in the frame of the fluid.* Because the first integral is zero, the flux λ linked by a conduction path always made up of the same fluid particles remains constant.

This is an important fact for the situation shown in Fig. 12.2.14, as can be seen by considering the conduction path $abcd$ intersecting the fluid and the edge of the rigid plate at $x_1 = 0$. Initially the surface enclosed by this path is in the x_2 - x_3 plane, hence links no flux ($\lambda = 0$). When the plate is forced to move in the x_2 -direction this surface, which is always made up of the same material particles, moves to $a'bcd'$. Because the surface is tilted, there is now a flux from B_0 that contributes to λ . Because λ must remain zero, however, there is a current induced around the loop in such a direction that it cancels the flux contributed by B_0 . There is then an addition to the magnetic field (induced by this current) along the x_2 -axis (Fig. 12.2.15) that makes the net

* See (1.1.23), Table 1.2, Appendix G.

magnetic field remain tangential to the surface of the deformed loop. This is necessary if λ is to remain zero.

The current, returning along the path cb in the fluid flows transverse to the field B_0 ; hence there is a magnetic force on the fluid ($\mathbf{J} \times B_0 \mathbf{i}_1$) in the x_2 -direction. The result of moving the highly conducting plate in the x_2 -direction is a motion of the fluid adjacent to the plate in the same direction. The motion of the plate creates a magnetic shearing stress on the fluid. This stress is transmitted through the fluid in the x_1 -direction because the magnetic force sets the fluid in the plane of bc into motion, and this sheet of fluid now plays the role of the plate in inducing motions in the neighboring sheets of fluid.

In our arguments we have assumed that motions of the fluid are the same at all points in a given x_2 - x_3 plane. To provide an analytical picture of the dynamics consistent with this assumption it is assumed that all variables are independent of x_2 and x_3 . As an immediate consequence of this assumption, the condition that $\nabla \cdot \mathbf{B} = 0$ requires that B_0 be independent of x_1 . If, in addition, B_0 is imposed by an external magnet driven by a constant current, it follows that $B_1 = B_0 = \text{constant}$, regardless of the fluid motions. By similar reasoning the incompressible nature of the fluid ($\nabla \cdot \mathbf{v} = 0$), together with the rigid walls that do not permit flow along the x_1 -axis, require that $v_1 = 0$ everywhere in the fluid. Hence both the fluid motions and additions to the magnetic flux density occur transverse to the x_1 -axis.

From the discussion that has been given it is clear that three essential ingredients in the fluid motions are of interest here. First, a mathematical model must account for the law of induction. In particular, since the magnetic field is induced in the x_2 -direction, we write the x_2 -component of the induction equation

$$\frac{\partial E_3}{\partial x_1} = \frac{\partial B_2}{\partial t}. \quad (12.2.84)$$

The second important effect comes from the high conductivity of the fluid. In order that the conduction current may remain finite in the limit in which the conductivity σ becomes large, we must require that $\mathbf{E}' = 0$. This in turn means that $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$, and it is the x_3 -component of this equation that is of interest to us:

$$E_3 = v_2 B_0. \quad (12.2.85)$$

Substitution of this expression for E_3 into (12.2.84) gives an equation that expresses the effect of the fluid deformation on the magnetic field.

$$B_0 \frac{\partial v_2}{\partial x_1} = \frac{\partial B_2}{\partial t} \quad (12.2.86)$$

Note that if we define a transverse particle displacement in the fluid such that $v_2 = \partial \xi / \partial t$ (12.2.86) simply requires that the magnetic flux density remain

tangential to the deformed surface of fluid initially in a given x_1 - x_3 plane. Equation 12.2.86 shows that the lines of magnetic field intensity are deformed as though they were “frozen” to the particles of fluid (see Fig. 12.2.15).

The third input to our analytical description comes from the effect of the magnetic field on the fluid motions. Because the fluid moves in the x_2 -direction, we write the x_2 -component of the force equation (12.1.21)

$$\rho \frac{\partial v_2}{\partial t} = \frac{\partial T_{21}}{\partial x_1} = \frac{B_o}{\mu_0} \frac{\partial B_2}{\partial x_1}. \quad (12.2.87)$$

Note that the absence of a velocity component v_1 and the one-dimensional character of the motions under consideration eliminate the spatial derivatives from the substantial derivative (the first term) in this expression. The only component of the Maxwell stress tensor* that enters on the right is T_{21} because variables do not depend on x_2 or x_3 and we have made use of the fact that B_o is a constant in writing Eq. 12.2.87.

The last two equations can be used to write an expression for either B_2 or v_2 ; for example, we eliminate B_2 between the time derivative of (12.2.87) and the space derivative of (12.2.86) to obtain

$$\frac{\partial^2 v_2}{\partial t^2} = \frac{B_o^2}{\mu_0 \rho} \frac{\partial^2 v_2}{\partial x_1^2}, \quad (12.2.88)$$

where

$$a_b = \left(\frac{B_o^2}{\mu_0 \rho} \right)^{1/2}.$$

This is the wave equation, considered in some detail in Chapters 9 and 10. The velocity a_b with which waves propagate in the x_1 -direction is called the Alfvén velocity.†

To develop further a physical feel for the nature of an Alfvén wave, consider the propagation in the positive x_1 -direction of the pulse illustrated in Fig. 12.2.16. The pulse, as drawn, represents what happens along the x_1 -axis; but, because in our model the variables are independent of x_2 and x_3 , the figure applies to all elements having the same coordinate x_1 . With reference to Fig. 12.2.16, we can easily show that the variables as sketched satisfy (12.2.86) and (12.2.87) with $\mathbf{J}$ found by Ampère’s law. Moreover, (12.2.88) is satisfied when the waveforms maintain constant shape and propagate in the x_1 -direction with the Alfvén velocity a_b .

We note from Fig. 12.2.16 that the force density $\mathbf{J} \times \mathbf{B}$ has an x_2 -component equal to $J_3 B_o$ and that this force density is in the positive x_2 -direction in the leading half of the wave and in the negative x_2 -direction in the trailing half of the wave. Thus, as the wave propagates in the x_1 -direction, the fluid at the

* Table 8.1, Appendix G.

† Alfvén waves are named after the man who first recognized their significance for astrophysics. See H. Alfvén, *Cosmical Electrodynamics*, Oxford, 1950.

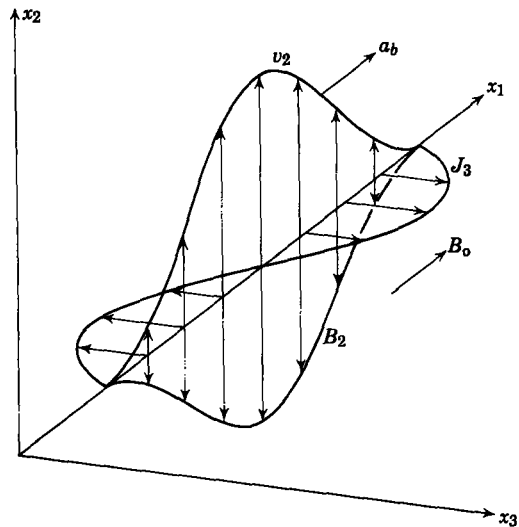


Fig. 12.2.16 The variables associated with an Alfvén wave.

leading edge is accelerated upward by the electrical force and at the trailing edge the fluid is decelerated.

It is instructive to use the pulse of Fig. 12.2.16 to construct the curves of Fig. 12.2.17 which show the displacement of the fluid particles that were initially on the x_1 -axis. Fluid particles and magnetic flux lines are displaced in the same way by the passage of the Alfvén wave. For a highly conducting ($\sigma \rightarrow \infty$) fluid the fluid particles and magnetic flux lines are “frozen” together and any motion of the fluid causes a distortion of the flux lines.

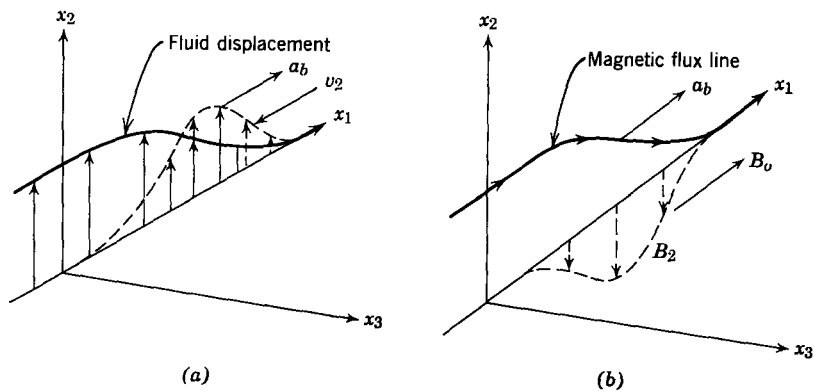


Fig. 12.2.17 Fluid displacement and flux-line distortion in an Alfvén wave: (a) fluid displacement; (b) magnetic flux line.

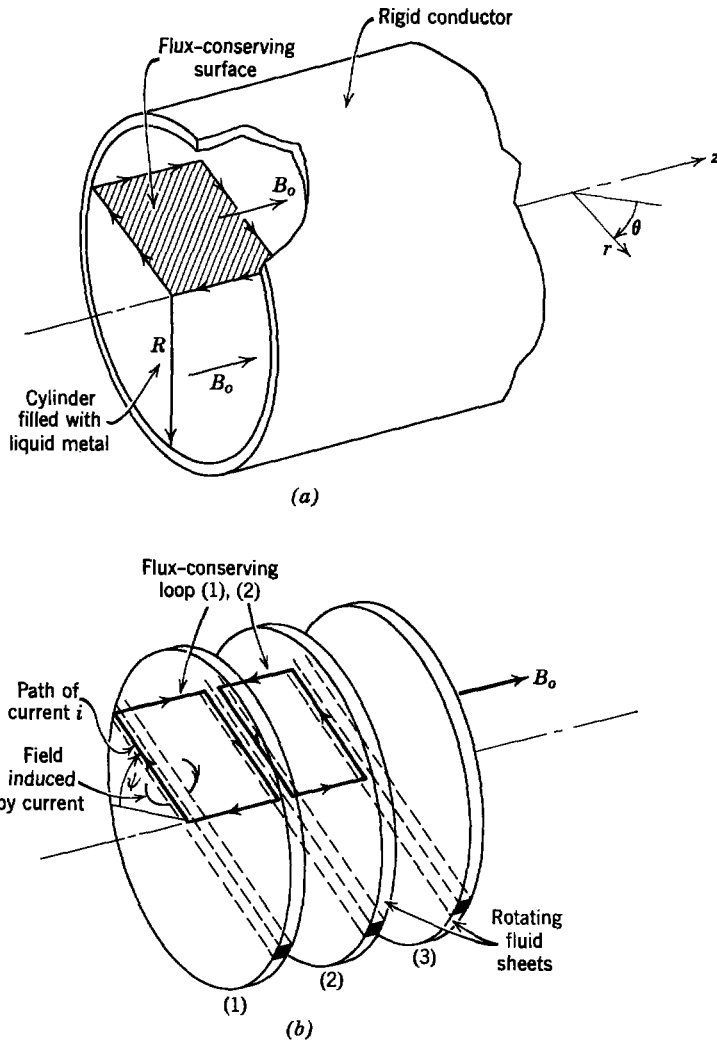


Fig. 12.2.18 (a) Experimental arrangement for producing torsional Alfvén waves in highly conducting cylindrical container; (b) conduction paths represented as “spokes” in adjacent wheels of perfectly conducting fluid.

It would be difficult to generate Alfvén waves in the cartesian geometry of Fig. 12.2.14 for two reasons. First, fluid motions in the x_2 -direction have been assumed independent of x_2 and this implies that container boundaries in x_1 - x_3 planes must not inhibit the velocity v_2 . Second, currents that flow along the x_3 -axis must have a return path ($\nabla \cdot \mathbf{J} = 0$), and this implies that conducting walls are provided by the container in x_1 - x_2 planes. We can satisfy both requirements by using the cylindrical container shown in Fig. 12.2.18. Here

we expect that Alfvén waves will appear as torsional motions of the fluid about the axis of the cylinder. These motions, like those just considered, are transverse to the imposed magnetic field B_0 (which has the same direction as the axis of the cylinder).

Again it is helpful to think of the fluid as composed of sheets, as shown in Fig. 12.2.18. Now the sheets take the form of wheels that can execute torsional motions about the cylinder axis. Currents can flow radially outward along “spokes” of a “wheel” through the outer cylinder wall, inward along another “spoke,” and finally complete the loop along the cylinder axis (Fig. 12.2.18). In fact, these loops provide a simple picture of the electromechanical mechanism responsible for the propagation of waves along the magnetic field B_0 .

Suppose that the first slice of fluid is forced to rotate to the positive angle ψ (Fig. 12.2.18*b*). The loop formed by the conducting path through the neighboring sheet initially links no flux. To conserve this condition in spite of the rotation a current i is induced which tends to cancel the flux caused by B_0 . This current returns to the center through the neighboring sheet. In doing so it produces a force density $\mathbf{J} \times \mathbf{B}_0$ which tends to rotate this second sheet in the positive ψ -direction. Of course, as the second sheet rotates, a current must flow around a loop through the third sheet to conserve the zero flux condition in the second loop of Fig. 12.2.18. Hence the third sheet of fluid is set into motion and the initial rotation propagates along the cylinder axis. These arguments can be repeated for motions that propagate in the opposite direction. The waves have no polarity and can propagate in either direction along the lines of magnetic field B_0 . The propagation is not instantaneous because each sheet has a finite mass and time is required to set the fluid in motion

The magnetic field has the same effect on the fluid as if the fluid sheets were interconnected by taut springs (Fig. 12.2.19). Wave propagation occurs

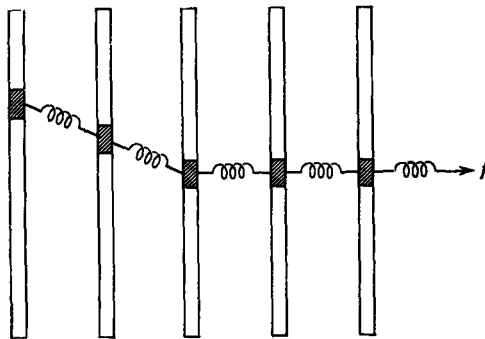


Fig. 12.2.19 Side view of the circular sheets of fluid in Fig. 12.2.18 showing equivalent interconnecting springs under the tension f .

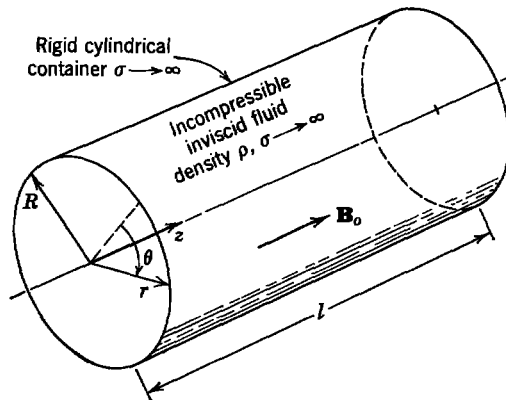


Fig. 12.2.20 Cylindrical geometry for the study of Alfvén waves.

very much as it does on a string (Section 9.2.). In the string the wave velocity was proportional to the square root of the tension f^* . Here the tension is apparently proportional to B^2 , as can be seen by comparing (12.2.88) and (9.2.4). This would be expected from a simple experiment: hold one sheet fixed and twist the next sheet and there is a restoring torque proportional to f . With the magnetic field the restoring torque is caused by $\mathbf{J} \times \mathbf{B}_0$, but since $\mathbf{J}$ is induced in proportion to B_0 this magnetic restoring torque is proportional to B_0^2 . Hence we can think of B_0^2 as producing a magnetic tension in a perfectly conducting fluid.

To be precise about the fluid velocity and electrical current distribution, we now consider a specific analytical example. The system, illustrated in Fig. 12.2.20, consists of a rigid, cylindrical container made of highly conducting material, filled with a highly conducting fluid, and immersed in an equilibrium axial flux density $\mathbf{B}_0$ produced externally. The ends of the cylinder are also rigid and may be insulators or conductors, depending on the boundary conditions desired. The fluid is modeled as incompressible and inviscid with mass density ρ , permeability μ_0 , and high electrical conductivity ($\sigma \rightarrow \infty$). The fluid in the cylinder has axial length l and radius R . We use the cylindrical coordinate system illustrated in Fig. 12.2.20.

We specify that any drive will be applied at the ends and will have cylindrical symmetry; that is, there will be no variation with the angle θ and $\mathbf{v} = \mathbf{i}_\theta v_\theta(r, z, t)$. In this case we can require that the relevant variables have only the following components, defined in terms of the cylindrical coordinate system (r, θ, z) in Fig. 12.2.20.

$$\mathbf{B} = \mathbf{i}_z B_0 + \mathbf{i}_\theta B_\theta, \quad (12.2.89)$$

$$\mathbf{J} = \mathbf{i}_r J_r + \mathbf{i}_z J_z, \quad (12.2.90)$$

$$\mathbf{E} = \mathbf{i}_r E_r + \mathbf{i}_z E_z. \quad (12.2.91)$$

* See Table 9.2, Appendix G.

The variables v_θ , B_θ , J_r , J_z , E_r , and E_z can be functions of r , z , and time t .

To analyze this system we must write the necessary equations in cylindrical coordinates by recognizing that $(\partial/\partial\theta) = 0$. For the basic equations refer to Table 1.2*, and for their forms in cylindrical coordinates refer to any standard text on electromagnetic theory.† The use of the constituent relation $\mathbf{B} = \mu_0\mathbf{H}$ with Ampère's law (1.1.1)* in cylindrical coordinates yields

$$-\frac{1}{\mu_0} \frac{\partial B_\theta}{\partial z} = J_r, \quad (12.2.92)$$

$$\frac{1}{\mu_0 r} \frac{\partial}{\partial r} (r B_\theta) = J_z. \quad (12.2.93)$$

We obtain from Faraday's law (1.1.5)

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\frac{\partial B_\theta}{\partial t}. \quad (12.2.94)$$

Ohm's law (12.2.18) yields

$$J_r = \sigma(E_r + v_\theta B_\theta), \quad (12.2.95)$$

$$J_z = \sigma E_z, \quad (12.2.96)$$

with J_r and J_z related by the condition of conservation of charge (1.1.3)*

$$\frac{1}{r} \frac{\partial}{\partial r} (r J_r) + \frac{\partial J_z}{\partial z} = 0. \quad (12.2.97)$$

The θ -component of the momentum equation (12.1.14) with $\mathbf{F}^e = \mathbf{J} \times \mathbf{B}$ is

$$\rho \frac{\partial v_\theta}{\partial t} = -J_r B_\theta. \quad (12.2.98)$$

We now assume high conductivity ($\sigma \rightarrow \infty$), which, coupled with the fact that $\mathbf{J}$ remains finite, reduces (12.2.95) and (12.2.96) to

$$E_r = -v_\theta B_\theta, \quad (12.2.99)$$

$$E_z = 0. \quad (12.2.100)$$

These expressions are used in 12.2.94 to write

$$B_\theta \frac{\partial v_\theta}{\partial z} = \frac{\partial B_\theta}{\partial t}. \quad (12.2.101)$$

* See Table 1.2, Appendix G.

† See, for example, R. M. Fano, L. J. Chu, and R. B. Adler, *Electromagnetic Fields, Energy, and Forces*, Wiley, New York, 1960, p. 510.

Equations 12.2.92, 12.2.98, and 12.2.101 are combined to obtain the wave equation

$$\frac{\partial^2 v_\theta}{\partial t^2} = \frac{B_o^2}{\mu_0 \rho} \frac{\partial^2 v_\theta}{\partial z^2}. \quad (12.2.102)$$

This equation indicates that waves can propagate in the z -direction with the Alfvén velocity [see (12.2.88)].

$$a_b = \left(\frac{B_o^2}{\mu_0 \rho} \right)^{1/2}. \quad (12.2.103)$$

Note that (12.2.102) has no derivatives with respect to the radius r although the variables may be functions of r as indicated by (12.2.93) and (12.2.97). Variations with r are determined by boundary conditions; for instance, the general solution of (12.2.102) can be written in the separable form as

$$v_\theta = A(r) f(z, t). \quad (12.2.104)$$

The function $A(r)$ is then set by boundary conditions and automatically satisfies all the differential equations.

To consider a specific example of boundary conditions we assume that the end of the container at $z = 0$ is rigid, fixed, and made of insulating material ($\sigma \rightarrow 0$). The end at $z = l$ is highly conducting ($\sigma \rightarrow \infty$) and is rotated about its axis with a velocity

$$v_p = \text{Re} (\Omega r e^{i\omega t}). \quad (12.2.105)$$

These constraints impose the following boundary conditions:

$$\text{at } z = 0, \quad J_z = 0 \quad (12.2.106)$$

$$\text{at } z = l, \quad v_\theta = \text{Re} (\Omega r e^{i\omega t}). \quad (12.2.107)$$

This last boundary condition reflects the fact that there can be no slip between the perfectly conducting moving wall and the adjacent fluid because of the magnetic field; that is, the electric field must remain continuous across this boundary. Since $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ and the normal $\mathbf{B}$ is continuous across the boundary, it follows that the fluid velocity must also be continuous.

The solution for v_θ can now be assumed to have the form

$$v_\theta = \text{Re} [A(r) \hat{v}_\theta(z) e^{j\omega t}]. \quad (12.2.108)$$

Substitution of this assumed solution into (12.2.102) yields the differential equation

$$\frac{d^2 \hat{v}_\theta}{dz^2} = -k^2 \hat{v}_\theta, \quad (12.2.109)$$

where

$$k = \frac{\omega}{a_b}.$$

The solution of this equation is, in general,

$$\hat{v}_\theta = C_1 \cos kz + C_2 \sin kz. \quad (12.2.110)$$

Imposing the boundary condition at $z = l$, (12.2.107) yields

$$\Omega r = A(r)(C_1 \cos kl + C_2 \sin kl). \quad (12.2.111)$$

To maintain $A(r)$ nondimensional as indicated by (12.2.108), while satisfying this last equation for all values of r , we set

$$A(r) = \frac{r}{R} \quad (12.2.112)$$

and rewrite (12.2.111) as

$$\Omega R = C_1 \cos kl + C_2 \sin kl. \quad (12.2.113)$$

To apply the boundary condition at $z = 0$ we need to find an expression for J_z . We accomplish this by first substituting (12.2.108) into (12.2.101) to obtain

$$\frac{\partial B_\theta}{\partial t} = \text{Re} \left[B_o A(r) \frac{d\hat{v}_\theta}{dz} e^{j\omega t} \right]. \quad (12.2.114)$$

If we assume that

$$B_\theta = \text{Re} [A(r) \hat{B}_\theta(z) e^{j\omega t}], \quad (12.2.115)$$

then, using (12.2.114), we obtain

$$j\omega \hat{B}_\theta(z) = B_o \frac{d\hat{v}_\theta}{dz}, \quad (12.2.116)$$

which, by using (12.2.110), yields

$$\hat{B}_\theta(z) = \frac{B_o k}{j\omega} (-C_1 \sin kz + C_2 \cos kz). \quad (12.2.117)$$

Now we use (12.2.93) to evaluate J_z as

$$J_z = \text{Re} \left[\frac{2B_o k}{j\omega \mu_0 R} (-C_1 \sin kz + C_2 \cos kz) e^{j\omega t} \right]. \quad (12.2.118)$$

The boundary condition at $z = 0$ (12.2.106) now requires

$$C_2 = 0. \quad (12.2.119)$$

We use this result with (12.2.113) to find

$$C_1 = \frac{\Omega R}{\cos kl}. \quad (12.2.120)$$

The resulting solutions are

$$v_\theta = \operatorname{Re} \left(\Omega r \frac{\cos kz}{\cos kl} e^{j\omega t} \right), \quad (12.2.121)$$

$$B_\theta = \operatorname{Re} \left(- \frac{\Omega r B_o k}{j\omega} \frac{\sin kz}{\cos kl} e^{j\omega t} \right), \quad (12.2.122)$$

$$J_z = \operatorname{Re} \left(- \frac{2\Omega B_o k}{\omega\mu_0} \frac{\sin kz}{\cos kl} e^{j\omega t} \right), \quad (12.2.123)$$

$$J_r = \operatorname{Re} \left(- \frac{j\omega\rho\Omega r}{B_o} \frac{\cos kz}{\cos kl} e^{j\omega t} \right). \quad (12.2.124)$$

Study of these solutions indicates that there are standing, torsional waves in the system. The fluid motion is azimuthal and the flux line distortion is azimuthal. The details of the phenomena involved in the wave propagation are as described in connection with Fig. 12.2.18. Now, however, we see that the current loops are distributed throughout the fluid.

Because Alfvén waves are reflected from both ends of the container, the system exhibits an infinite number of resonances whose frequencies are defined by

$$\cos kl = 0. \quad (12.2.125)$$

The boundary condition at the insulated end of the cylinder (12.2.106) is essentially a free end condition. This is true because no current can flow in the insulator and no electrical forces are available at the boundary to perturb the fluid motion. Also, because the fluid is inviscid, there can be no tangential mechanical force applied to the fluid by the end plate. At the perfectly conducting end plate ($z = l$) the fluid “sticks” to the end plate because of electrical forces. A small radial current loop with one side in the end plate and the other side in the fluid will keep the flux linking it at zero. This produces the currents that interact with B_o to allow no slippage of the fluid at a perfectly conducting boundary that is perpendicular to the equilibrium flux density.

To ascertain the kinds of numbers that would be involved in an experimental system of this sort, consider a container with the dimensions

$$l = 0.1 \text{ m}, \quad R = 0.1 \text{ m}.$$

Assume the fluid to be liquid sodium (sometimes used as a coolant for nuclear reactors) which has a mass density, at 100°C , of

$$\rho = 930 \text{ kg/m}^3.$$

If we assume a flux density of

$$B_o = 1 \text{ Wb/m}^2,$$

which is easily obtainable with iron-core electromagnets, we obtain an Alfvén velocity of

$$a_b = 31 \text{ m/sec.}$$

The lowest resonance frequency of this system is given from (12.2.125) by

$$k = \frac{\pi}{2l},$$

which yields

$$\omega = 490 \text{ rad/sec}$$

or

$$f = 78 \text{ Hz.}$$

From these results we can see that Alfvén waves propagate at low velocities in liquid metals and that for devices of reasonable size the resonance frequencies also are low.

In our treatment of Alfvén waves we have assumed that the electrical conductivity of the fluid is infinite. In such a case we wonder how the flux density B_0 can exist in the fluid. The answer is simply that the conductivity is large but finite, and in establishing the equilibrium conditions sufficient time was allowed for the flux density B_0 to be established by diffusion into the fluid. In the analysis of the waves the assumption $\sigma \rightarrow \infty$ means simply that the diffusion time of the magnetic field through the fluid is much longer than the time required for the wave to propagate through the fluid* (see Section 7.1).

We have introduced Alfvén waves by using an incompressible fluid model. These waves can also propagate in compressible, highly conducting fluids such as gases. The analysis is essentially the same in both cases; however, more complex waves are possible in compressible fluids. Thus we must exercise care to ensure that only Alfvén waves are driven by a particular excitation in a compressible fluid.

12.2.4 Ferrohydrodynamics

Attention has been confined so far in this section to coupling with fluids that carry free currents. As pointed out in Section 8.5.2, magnetization forces can also be the basis for interaction with liquids. Commonly found fluids have no appreciable permeability. Ferromagnetic fluids, however, can be synthesized by introducing a colloidal suspension of magnetizable particles into a carrier fluid. Colloidal suspensions tend to settle out over long periods of time, and in the presence of a magnetic field the magnetized particles tend

* For an example of the experimental conditions necessary see A. Jameson, "A Demonstration of Alfvén Waves, I: Generation of Standing Waves," *J. Fluid Mech.*, **19**, 513–527 (August 1964).

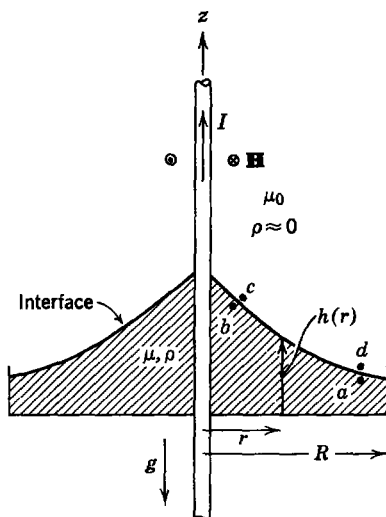


Fig. 12.2.21 A dish of magnetizable fluid is subjected to the magnetic field induced by current I .

to flocculate. Recent research efforts have led to the synthesis of colloidal suspensions (e.g., submicron-sized ferrite particles in a carrier fluid of kerosene), which are stable over indefinite periods of time.* We have no intention of delving into this topic in depth here; rather we confine ourselves to one simple example that illustrates this class of phenomena.

Although the ferrofluid is easily magnetizable, it can be made to be highly insulating against electrical conduction. In a magnetic field system the electric field is important because it determines the conduction current (through Ohm's law). In the region occupied by a magnetic insulator the conduction current is negligible and the equations for the magnetic field are simply

$$\nabla \times \mathbf{H} = 0, \quad (12.2.126)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (12.2.127)$$

These are the equations used to describe the magnetic field, even in a dynamic situation. At any instant in time the magnetic field, at least insofar as it is determined by the magnetized fluid, has the same distribution as if the system were static.

As an illustration of the nature of the magnetization force consider the experiment shown in Fig. 12.2.21. A constant current I is imposed along the z -axis by means of a conductor. This conductor passes vertically through

* R. E. Rosensweig, "Magnetic Fluids," *International Sci. Technol.* **55**, 48–66, 90 (July 1966).

a dish containing the magnetic fluid. We wish to compute the static equilibrium of the fluid that results after the current I has been turned on. This amounts to determining the altitude h of the fluid interface above the bottom of the dish. We can expect that, because a force density, $-\mathbf{H} \cdot \mathbf{H} \nabla \mu$, tends to pull the fluid upward, the depth will be greatest where the magnetic field intensity is greatest.

We assume that the magnetic field induced by the return current can be ignored. Then, under the assumption of axial symmetry, Ampère's law requires that the current I induce an azimuthally directed magnetic field intensity

$$\mathbf{H} = \mathbf{i}_\theta \frac{I}{2\pi r}. \quad (12.2.128)$$

This problem is relatively simple because the magnetizable fluid has no effect on the distribution of $\mathbf{H}$; that is, because the physical system is axially symmetric, we can argue that the fluid deformations are also axially symmetric and $h = h(r)$. It is clear that (12.2.128) satisfies the field equations (12.2.126) and (12.2.127) in the region occupied by the fluid, and because the interface is axisymmetric it also satisfies the boundary conditions. The tangential component of $\mathbf{H}$ is continuous and there is no normal component of $\mathbf{B}$ at the liquid interface. Hence we know the magnetic field intensity at the outset, and this makes finding $h(r)$ straightforward.

The magnetic force acting on the fluid (from Section 8.5.2)* is

$$\mathbf{F} = -\frac{1}{2}\mathbf{H} \cdot \mathbf{H} \nabla \mu + \nabla \left(\frac{1}{2}\rho \frac{\partial \mu}{\partial \rho} \mathbf{H} \cdot \mathbf{H} \right). \quad (12.2.129)$$

In the bulk of the liquid, μ is constant. Hence the force density can be written as

$$\mathbf{F} = -\nabla \psi, \quad \psi = -\frac{\rho}{2} \frac{\partial \mu}{\partial \rho} \mathbf{H} \cdot \mathbf{H}. \quad (12.2.130)$$

This is the form assumed in deriving Bernoulli's equation (12.2.11) which, in the case of a static fluid, becomes

$$p + \rho g z - \frac{\rho}{2} \frac{\partial \mu}{\partial \rho} \mathbf{H} \cdot \mathbf{H} = \text{constant}. \quad (12.2.131)$$

Remember that this equation is valid in the bulk of the fluid. It can therefore be used to relate the pressures and heights at the points (a) and (b) in Fig. 12.2.21. These points are just beneath the interface, where pressures are p_a and p_b , respectively, and the altitudes are h_a and h_b . From (12.2.131)

$$p_a + \rho g h_a - \frac{\rho}{2} \frac{\partial \mu}{\partial \rho} (H^a)^2 = p_b + \rho g h_b - \frac{\rho}{2} \frac{\partial \mu}{\partial \rho} (H^b)^2. \quad (12.2.132)$$

* See Table 8.1, Appendix G.

Similar reasoning shows that the pressures p_c and p_a just across the interface from points (b) and (a), respectively, are related by

$$p_c = p_a. \quad (12.2.133)$$

Here we have assumed that the density of the air above the liquid can be ignored.

Now, if we could relate the pressures at adjacent points on opposite sides of the interface, we would have four equations that would make it possible to relate all four of the pressures p_a , p_b , p_c , and p_d . At the interface there is a jump in μ , and we must be careful to include the effect of the first term in (12.2.129) [which was not accounted for in (12.2.131)]. The stress tensor representation of the force density is convenient for determining the jump in pressure at the interface [see (8.5.41)].* A thin volume is shown in Fig. 12.2.22, as it encloses the region of interface between points b and c. To make use of the stress tensor in cartesian coordinates we erect a set of orthogonal coordinates (u , v , w) at the interface, with w in the θ -direction. Force equilibrium then requires that the sum of the surface forces balance

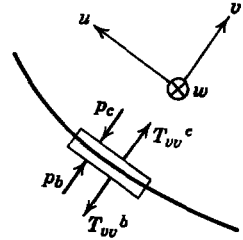


Fig. 12.2.22 A small volume element encloses the interface between points c and b in Fig. 12.2.21.

$$p_b - p_c = T_{vv}^b - T_{vv}^c = \frac{1}{2}(H^b)^2 \left[\mu_0 - \mu \left(1 - \frac{\rho}{\mu} \frac{\partial \mu}{\partial \rho} \right) \right]. \quad (12.2.134)$$

Similarly, at the interface between points (d) and (a)

$$p_d - p_a = -\frac{1}{2}(H^a)^2 \left[\mu_0 - \mu \left(1 - \frac{\rho}{\mu} \frac{\partial \mu}{\partial \rho} \right) \right]. \quad (12.2.135)$$

Now addition of these last four equations eliminates the pressures and gives an expression for the difference in surface elevation at points a and b as a function of the magnetic field intensities.

$$\rho g(h_a - h_b) = \frac{1}{2}(\mu_0 - \mu)[(H^b)^2 - (H^a)^2]. \quad (12.2.136)$$

Until now we have not specified the field intensity at points a and b . It has been known all along, however, because of (12.2.128). In particular, if we take the point a as being at $r = R$ (which could be the outside radius of the pan), (12.2.136) becomes an expression for the dependence of interface altitude on the radius r .

$$\rho g(h_b - h_a) = \frac{1}{2}(\mu - \mu_0) \frac{I^2}{(2\pi)^2} \left(\frac{1}{r^2} - \frac{1}{R^2} \right). \quad (12.2.137)$$

This result is sketched in Fig. 12.2.23. We have assumed that the density of the liquid is constant. This means that the total volume of the liquid must be

* Table 8.1, Appendix G.

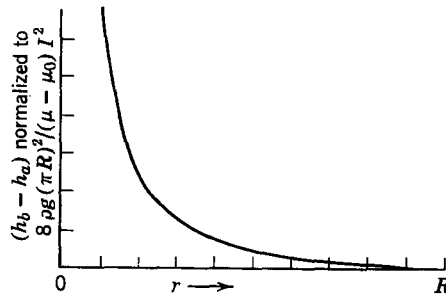


Fig. 12.2.23 Sketch of the liquid interface contour predicted by (12.2.137).

conserved, a fact that could be used to find the distance from a to the bottom of the pan.

An experiment with essentially the same ingredients as this example is shown in Fig. 12.2.24. In actuality, a significant magnetic saturation of the liquid makes the electrically linear model used here ($\mathbf{B} = \mu\mathbf{H}$) only approximately correct. As we know from Chapters 3 and 8, energy methods can also be used to calculate magnetization forces for electrically nonlinear systems, and this is what is required to make a careful comparison of theory and experiment.

Finally, it is worthwhile to observe that the magnetostriction force density has no observable effect on the surface deformation. This will always be the case as long as interest is confined to situations in which the fluid density remains essentially constant.

12.3 ELECTRIC FIELD COUPLING WITH INCOMPRESSIBLE FLUIDS

There is a wide range of mechanisms by which an electric field can produce a force on a fluid. In this section examples are used to illustrate two of the most commonly encountered types of interaction.

12.3.1 Ion-Drag Phenomena

Electrical forces can be produced in highly insulating gases and liquids by injecting charged particles and using an electric field to pull them through the fluid. Here we assume that these charged particles are ions that might be emitted by the corona discharge in the neighborhood of sharply pointed electrodes placed at a high potential (several kilovolts). These ions move through a liquid or gas under the influence of an applied electric field. Their motion, however, is retarded by friction, and momentum is imparted to the fluid. Therefore the ion-drag effect can be used to pump or accelerate the fluid. Similarly, if the ion is transported by the fluid against the retarding

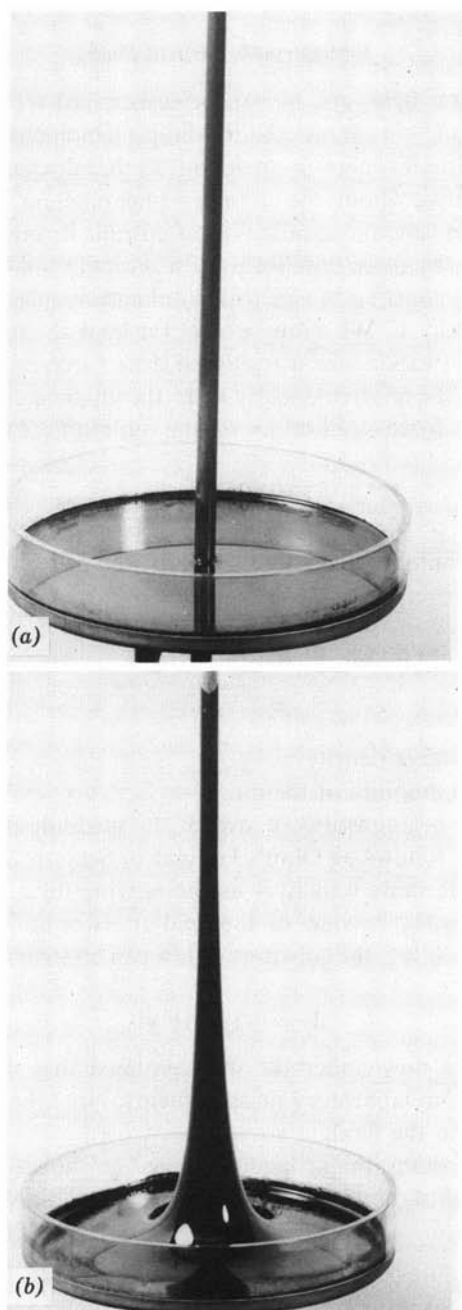


Fig. 12.2.24 (a) A conductor passes along the axis of symmetry through a pan containing the magnetizable liquid, $I = 0$. (b) The current I has been turned on. The result is a force density $-\mathbf{H} \cdot \mathbf{H} \nabla \mu$ that tends to lift the fluid upward, as predicted by (12.2.137). (Courtesy of AVCO Corporation, Space Systems Division.)

force of the electric field, energy can be transferred from the flow into an electrical circuit. In this context the ion-drag phenomenon is the basis for a gaseous Van de Graaff generator analogous to that discussed in Section 7.2.2.

To be quantitative about the ion-drag phenomenon we require a constitutive law to describe the conduction of current through the fluid. Here a simple picture of the force equilibrium for a single ion is helpful. Suppose that an ion with a charge q moves with a velocity $\mathbf{v}_r$ relative to the gas in an electric field intensity $\mathbf{E}$. We would expect (at least at atmospheric pressure) that the ion would experience a frictional drag force proportional (say, by the constant γ) to the relative velocity $\mathbf{v}_r$. In the absence of appreciable effects from acceleration, force equilibrium on the ion requires that

$$\mathbf{v}_r = \frac{q\mathbf{E}}{\gamma}. \quad (12.3.1)$$

If we let n be the number density of the ions, then the current density is

$$\mathbf{J}_f = nq\mathbf{v}_r, \quad (12.3.2)$$

which, in view of (12.3.1), can also be written

$$\mathbf{J}_f = \rho_f \mu \mathbf{E}, \quad (12.3.3)$$

where $\rho_f = \text{free charge density} = nq$,
 $\mu = q/\gamma = \text{mobility of the ion}.$

Equation 12.3.3 is a constitutive law for the medium at rest and plays the same role in what follows as Ohm's law did in Section 7.2. This law holds in a frame with the same velocity $\mathbf{v}$ as the moving fluid, where it would be written as $\mathbf{J}'_f = \rho_f \mu \mathbf{E}'$. In view of the field transformations for an electric field system (Table 6.1)*, the constitutive law can be written in the laboratory frame as

$$\mathbf{J}_f = \rho_f(\mu \mathbf{E} + \mathbf{v}). \quad (12.3.4)$$

In the example we now undertake it is assumed that the mobility μ is a constant, found from laboratory measurements. Note that $\mu \mathbf{E}$ is the velocity of an ion relative to the fluid.

An electrostatic pump might be constructed as shown in Fig. 12.3.1. The system consists of a nonpolarizable ($\epsilon = \epsilon_0$) gas flowing with constant velocity

$$\mathbf{v} = \mathbf{i}_z v_0 \quad (12.3.5)$$

through a cylindrical insulating tube of cross-sectional area A . At $z = 0$ and $z = l$, plane conducting screens are placed perpendicular to the axis. We assume that the screens do not affect the gas flow but make electrical contact with the gas. The screens are connected to external terminals that are excited

* Appendix G.

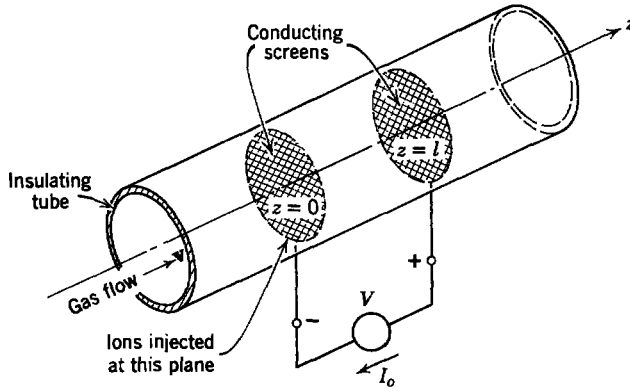


Fig. 12.3.1 Configuration for a gaseous electrostatic pump or generator.

by the constant-current source I_0 as shown. At the plane $z = 0$ positive ions are injected into the gas by a source of ions.

For now we assume that this source supplies ions at a rate necessary to maintain the charge density ρ_0 at the inlet screen:

$$\text{at } z = 0, \quad \rho_f = \rho_0. \quad (12.3.6)$$

The current density and electric field intensity are assumed to have only z -components,

$$\begin{aligned} \mathbf{J}_f &= \mathbf{i}_z J, \\ \mathbf{E} &= \mathbf{i}_z E, \end{aligned} \quad (12.3.7)$$

and to be functions of z alone. Attention is confined to steady-state operation.

In addition to the boundary condition of (12.3.6), the equations we need to solve this problem are the z -component of (12.3.4)

$$J = \rho_f(\mu E + v_0), \quad (12.3.8)$$

Gauss's law written as

$$\epsilon_0 \frac{dE}{dz} = \rho_f, \quad (12.3.9)$$

and the conservation of charge for steady-state conditions

$$\frac{dJ}{dz} = 0. \quad (12.3.10)$$

The area over which current flows is A ; consequently, the current density and the source current are related by

$$J = \frac{I_0}{A}. \quad (12.3.11)$$

We first solve (12.3.8) for the electric field intensity to obtain

$$E = -\frac{v_o}{\mu} + \frac{J}{\rho_f \mu}. \quad (12.3.12)$$

Next, this expression is differentiated with respect to z and (12.3.9) is used to eliminate E :

$$\frac{\rho_f}{\epsilon_o} = \frac{d}{dz} \left(\frac{J}{\rho_f \mu} \right). \quad (12.3.13)$$

Expansion of the derivative and use of (12.3.10) yields

$$\frac{d\rho_f}{dz} = -\frac{\mu}{\epsilon_o J} \rho_f^3. \quad (12.3.14)$$

Integration of this expression, use of the boundary condition of (12.3.6), and some manipulation yield

$$\frac{\rho_f}{\rho_o} = \left[1 + \frac{2}{R_e} \left(\frac{\rho_o v_o}{J} \right) \frac{z}{l} \right]^{-1/2}, \quad (12.3.15)$$

where $R_e = \epsilon_o v_o / \rho_o \mu l$ is the electric Reynolds number.

The plot of the free charge density shown in Fig. 12.3.2 makes it evident that the rate of decay down the channel is decreased as the electric Reynolds number is increased.

Substitution of (12.3.15) into (12.3.12) gives the electric field intensity between the grids

$$E = \frac{v_o}{\mu} \left\{ -1 + \frac{J}{\rho_o v_o} \left[1 + \frac{2}{R_e} \left(\frac{\rho_o v_o}{J} \right) \left(\frac{z}{l} \right) \right]^{1/2} \right\}. \quad (12.3.16)$$

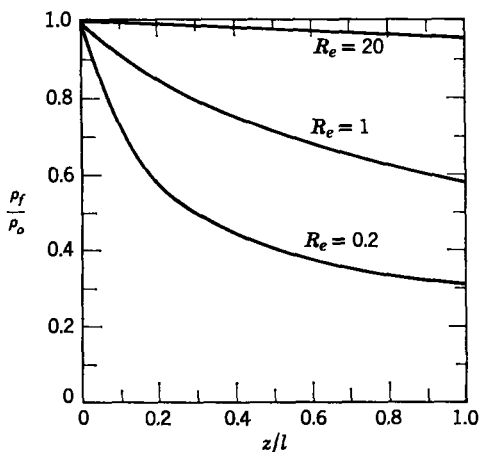


Fig. 12.3.2 Charge-density distribution between grids. $J/\rho_o v_o = 1$.

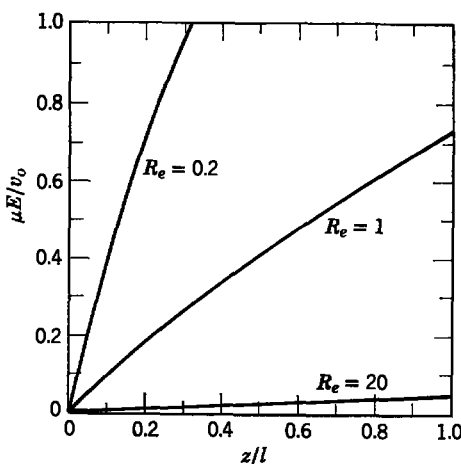


Fig. 12.3.3 Distribution of electric field intensity between the grids. $J/\rho_0 v_0 = 1$.

If the ion source at $z = 0$ is essentially limited by space charge, it will emit just enough charge to make the electric field at $z = 0$ vanish.* From (12.3.16) this requires that

$$\frac{J}{\rho_0 v_0} = 1. \quad (12.3.17)$$

In our discussion it is assumed that J , ρ_0 , and v_0 are positive. Remember that μE is the velocity of the ions relative to the fluid. This condition requires that the ions have the same velocity as the fluid at $z = 0$. Then the electric field is positive everywhere between the grids, as shown in Fig. 12.3.3. This means that the ions move more rapidly than the fluid and, as we shall see, the system operates as a pump.

As for the MHD machine discussed in Section 12.2.1, two “terminal” characteristics of the electrohydrodynamic flow interaction are of interest—the pressure change from inlet to outlet and the terminal voltage. The first can be computed from the electric field intensity by making use of the Maxwell stress tensor†. The pressure forces acting on the fluid in the channel section between $z = 0$ and $z = l$ are just balanced by the Maxwell stresses acting over the surface enclosing this section. Because there are no electrical shear forces,

$$A[p(l) - p(0)] = A[T_{11}(l) - T_{11}(0)] = \frac{1}{2} A \epsilon_0 [E^2(l) - E^2(0)]. \quad (12.3.18)$$

Since we have constrained $E(0)$ to vanish, it is clear from this statement that

* For a discussion of this model for the ion source see O. M. Stuetzer, “Ion Drag Pumps,” *J. Appl. Phys.*, **31**, 136 (January 1960).

† Section 8.3 or Appendix G.

$p(l) > p(0)$. In fact, from (12.3.16)

$$p(l) - p(0) = \frac{\epsilon_0}{2} \left(\frac{v_0}{\mu} \right)^2 \left[-1 + \left(1 + \frac{2}{R_e} \right)^{1/2} \right]^2. \quad (12.3.19)$$

This result indicates that the larger R_e , the smaller the pressure rise between the grids. This is misleading because both the electric Reynolds number R_e and the first factor in (12.3.19) depend on v_0 . If we think of holding v_0 fixed, however, and recall that R_e is inversely proportional to l , (12.3.19) shows that the pressure rise increases as l increases.

To obtain the terminal voltage V we integrate the negative of the electric field intensity from $z = 0$ to $z = l$:

$$V = \frac{lv_0}{\mu} \left\{ 1 - \frac{R_e}{3} \left[\left(1 + \frac{2}{R_e} \right)^{3/4} - 1 \right] \right\}. \quad (12.3.20)$$

This voltage is negative, as must be the case if power is supplied by the current source to the fluid. The fact that there is a pressure rise in the direction of flow indicates that work is done on the fluid as it passes through the region between the grids.

Ion-drag interactions can be used not only to pump slightly conducting fluids but also for conversion of energy from mechanical to electrical form.* In gases the mobility of ions is so great that such devices tend to lack efficiency. This shortcoming can be obviated either by using liquids, in which the mobility of ions tends to be much lower, or by replacing the ions with larger charged particles of liquid or solid. In any case, the electric pressure $\frac{1}{2}\epsilon_0 E^2$ tends to be small compared with the magnetic pressure $\frac{1}{2}\mu_0 H^2$ because the electric field intensity is limited by the breakdown strength of the dielectric medium. Hence *for a given size of device* the amount of energy converted in an electric field interaction is much less than that found for a magnetic field interaction.

One of the most significant reasons for our discussion of the ion-drag phenomenon is that it is commonly (and altogether too easily) encountered in high voltage systems, in which it accompanies corona discharge. A simple laboratory demonstration of the effect is shown in Fig. 12.3.4, in which two wire grids are placed at a potential difference of about 25 kV. Perpendicular segments of wire are mounted on the lower electrode to form a "bed of nails," and when this grid is electrified the tips of these segments provide sites for corona discharge. This discharge is the source of ions at $z = 0$ in Fig. 12.3.1.

* B. Kahn and M. C. Gourdine, "Electrostatic Power Generation," *AIAA J.*, 2, No. 8, 1423-1427 (August 1964). Also, A. Marks, E. Barreto, and C. K. Chu, "Charged Aerosol Energy Converter," *AIAA J.*, 2, No. 1, 45-51 (January 1964).

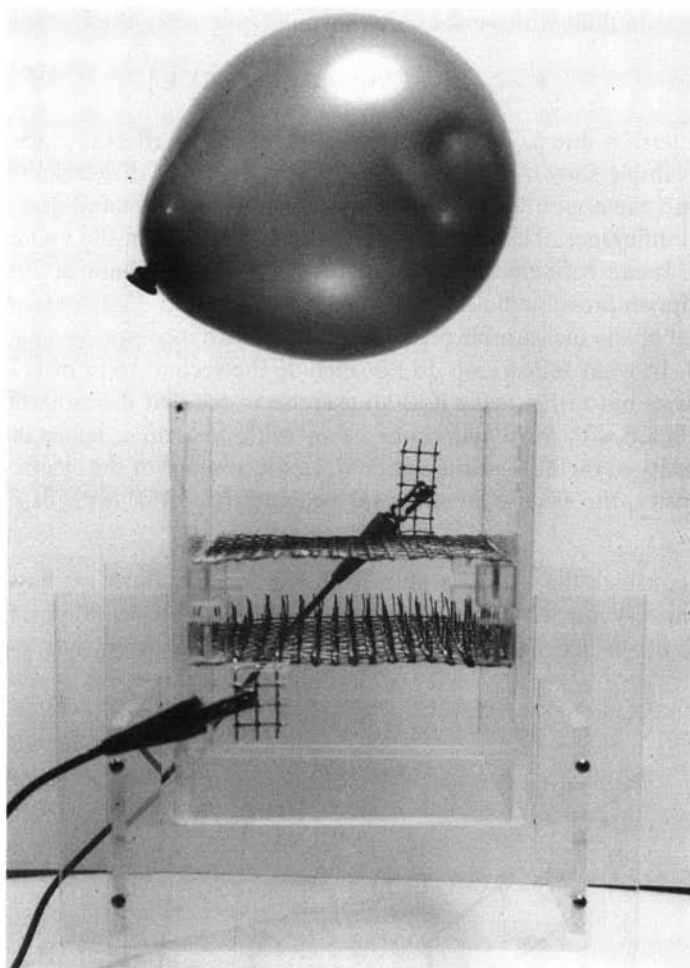


Fig. 12.3.4 Simple laboratory demonstration of ion-drag effect. In the absence of an applied voltage the balloon rests on the plastic enclosure. With voltage, it is pushed upward by ions being conducted between the grids.

In the absence of an applied voltage, the balloon rests on the plastic enclosure. With voltage, it is pushed upward by the pumping action between the grids.

12.3.2 Polarization Interactions

The analog to the magnetization interactions with fluids, discussed in Section 12.2.4, is the polarization interaction with electric fields—sometimes referred to as “dielectrophoresis.” The polarization force density for fluids was developed in Section 8.5, in which it was found that in the absence of

free charges a fluid with permittivity ϵ would experience the force density*

$$\mathbf{F} = -\frac{1}{2}\mathbf{E} \cdot \nabla \epsilon + \nabla \left(\frac{1}{2} \rho \frac{\partial \epsilon}{\partial \rho} \mathbf{E} \cdot \mathbf{E} \right). \quad (12.3.21)$$

The first term is due to inhomogeneity of the fluid. Its effect is made familiar by the example shown in Fig. 8.5.6, in which a slab of dielectric material is pulled into the region between plane-parallel electrodes placed at a constant potential difference. The second term is due to changes in the volume of the material. It can be included in the analysis of electromechanical interactions with an incompressible fluid, but as we saw in Section 12.2.4, its effect will cancel out of any measurable prediction based on an incompressible model for the fluid. In what follows we do not include the second term of (12.3.21) in our analysis but rather leave it as an exercise to see that it has no effect. We are concerned with the dynamics of a fluid with uniform ϵ ; hence there is no force density in the bulk of the material. In the absence of the electrostriction force density, the electric stress tensor becomes [(8.5.46)* with $\partial \epsilon / \partial \rho = 0$]

$$T_{ij} = \epsilon E_i E_j - \frac{1}{2} \delta_{ij} \epsilon E_k E_k. \quad (12.3.22)$$

Now consider the example shown in Fig. 12.3.5. Here we have a fluid pendulum very much like that shown in Fig. 12.2.7. This pendulum, however, is upside down because g is directed upward. The problem has a practical

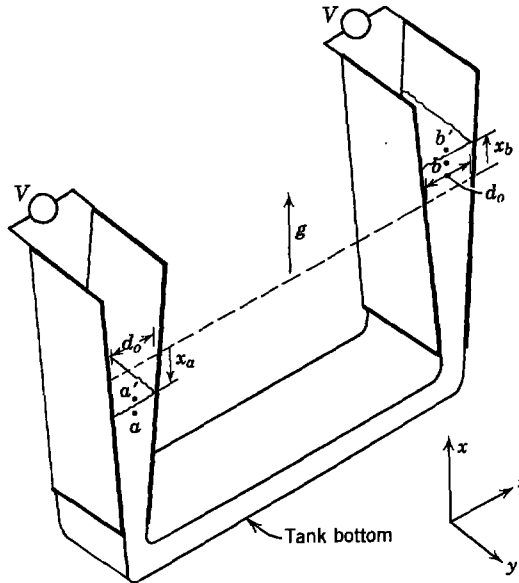


Fig. 12.3.5 A liquid pendulum containing dielectric fluid. Slightly diverging plates are used to impose a spatially varying electric field that tends to maintain the liquid in the bottom of the tank in spite of the acceleration g .

* See Table 8.1, Appendix G.

basis. Suppose that we wish to use the electric field to provide an artificial “gravity” to bottom the fluid within a tank under the near-zero gravity conditions of outer space; for example, the fluid might be the cryogenic liquid fuel used to propel a spacecraft. The electric field then provides fluid at a drain placed at the “bottom” of the tank. In this case g represents the effect of acceleration of the vehicle, as, for example, that which would occur during attitude control maneuvers. We have chosen g to be upward because this appears to be the worst possible situation in terms of removing the fluid from the bottom of the tank.

The U-shaped tank is considered in this example because it is easily analyzed with the tools developed in this chapter. Even though the example may seem academic, it has practical significance in the design of fluid orientation systems.

Because there are no electrical forces in the bulk of the liquid, we can use Bernoulli's equation derived in Section 12.2.1*b*. Again we carry out an integration of the momentum equation, as indicated by (12.2.33*a*), between points a and b , defined in Fig. 12.3.5. Now, however, the interfaces are subject to surface forces [due to the first term in (12.3.21)], and we cannot claim that the pressures p_a and p_b (just below the respective interfaces) are equal. In carrying out the integral of (12.2.33*a*) we retain the pressures evaluated at the points a and b to obtain

$$\rho l \frac{\partial v}{\partial t} = 2\rho g x_a + p_a - p_b. \quad (12.3.23)$$

Here v is the velocity of the fluid directed from a to b so that

$$\frac{\partial v}{\partial t} = \frac{d^2 x_a}{dt^2}. \quad (12.3.24)$$

We have approximated the velocity as being the same along a streamline connecting the points a and b . The cross-sectional area of the pendulum varies somewhat because the vertical legs are constructed with side walls composed of slightly diverging electrodes. Insofar as the fluid velocity is concerned, the effect of the diverging plates represents a nonlinear effect equivalent to slight changes in the length l of the pendulum; this effect is ignored here.

The fundamental difficulty in keeping the liquid in the bottom of the tank, with no electric field, is illustrated by combining the last two equations. With no applied voltage, $p_a = p_b$, and it is clear that the equilibrium represented by $x_a = 0$ is unstable. It is the purpose of the electric field to stabilize this equilibrium.

Before completing the mathematical representation of the dynamics consider physically how the polarization force density [the first term in

(12.3.21)] can stabilize the equilibrium at $x_a = 0$. This force is finite only at the two interfaces, where it is singular (infinite in magnitude over an infinitely thin region of space); that is, it comprises a surface force directed in the positive x -direction on each of the interfaces in proportion to the square of the electric field intensity. With the pendulum in equilibrium, the electrical forces on each of the interfaces just balance. Suppose that the system is perturbed to the position shown in Fig. 12.3.5. Then the upward-directed force on the interface at a is increased (the plates are closer together at this point; therefore E is greater), whereas that at b is decreased. This tends to return the pendulum to its equilibrium position. We expect that if we can make this stabilizing electrical effect large enough it will outweigh the de-stabilizing effect of gravity.

To provide a quantitative statement of the condition for stability we complete the equation of motion by relating the pressures p_a and p_b . Force balance on the interfaces, in view of the force diagrams shown in Fig. 12.3.6, requires

$$p_{a'} - p_a = T_{11}^{a'} - T_{11}^a = -\frac{1}{2}(\epsilon_0 - \epsilon)\left(\frac{V}{d_a}\right)^2, \quad (12.3.25)$$

$$p_{b'} - p_b = T_{11}^{b'} - T_{11}^b = -\frac{1}{2}(\epsilon_0 - \epsilon)\left(\frac{V}{d_b}\right)^2. \quad (12.3.26)$$

Of course, the spacing d used in these expressions is evaluated at the instantaneous locations of the respective interfaces.

$$d_a = d_o - cx_a, \quad (12.3.27)$$

$$d_b = d_o + cx_b. \quad (12.3.28)$$

Here c is determined by the rate at which the electrodes diverge. Then, to linear terms, the combination of (12.3.25) and (12.3.26) (remember, $x_a = x_b$) gives

$$p_a - p_b + p_{b'} - p_{a'} = -2c(\epsilon - \epsilon_0)\left(\frac{V}{d_o}\right)^2\left(\frac{x_a}{d_o}\right). \quad (12.3.29)$$

Formally, we can see that $p_{b'} = p_{a'}$ by joining points a' and b' with a streamline passing through the fluid above the interfaces (where the vapor phase is present and density is negligible). Then, by combining (12.3.25), (12.3.26),

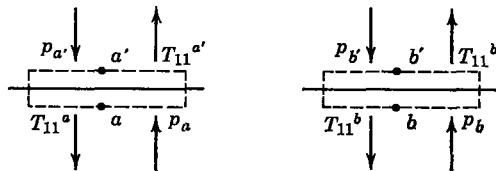


Fig. 12.3.6 Force equilibrium for each of the interfaces shown in Fig. 12.3.5.

and this last expression, we obtain the required equation of motion for the pendulum:

$$\rho l \frac{d^2 x_a}{dt^2} + x_a \left[-2\rho g + 2c(\epsilon - \epsilon_0) \frac{V^2}{d_o^3} \right] = 0. \quad (12.3.30)$$

From this it is clear that the equilibrium is stable if the voltage is made large enough to satisfy the condition

$$c(\epsilon - \epsilon_0) \frac{V^2}{d_o^3} > \rho g. \quad (12.3.31)$$

Liquid being oriented under near-zero gravity conditions is shown in Fig. 12.3.7. Each pair of adjacent plates has a potential difference. The zero gravity situation was created by flying the experiment within a KC-135 in a near-zero gravity trajectory. The liquid is Freon 113 with aniline dye added for purposes of observation. The basic mechanism for orienting the liquid is the same as that for the example considered in this section. Any two pairs of diverging plates can be considered as constituting a fluid pendulum with the essential behavior of that shown in Fig. 12.3.5. The stability condition of (12.3.31) guarantees that the equilibrium with the fluid in the tank "bottom" will be stable. Of course, a more complete representation of the dynamics requires a continuum model,* for instability may develop in the region between a single pair of electrodes.

We have stated from the outset that free charge forces are of negligible importance. In practice, this is guaranteed by making the applied voltages V of alternating polarity with sufficiently high frequency that free charges cannot relax into the fluid. If the frequency is high compared with typical mechanical frequencies, it is possible to use the same mathematical model as that developed here, except that V is the rms value of the voltage.

12.4 DISCUSSION

In this chapter we have introduced some of the fundamental laws and analytical techniques that are used in the study of electromechanical interactions with conducting, magnetizable and polarizable fluids. We have applied these laws and techniques to the analysis of systems in which an incompressible, inviscid fluid model is appropriate. Even though the incompressible, inviscid fluid model may seem quite restrictive, it provides an understanding of the basic electromechanical interactions that occur in all sorts of magneto-hydrodynamic and electrohydrodynamic systems, including those with gaseous conductors.

* In fact, a description of this mode of instability is given in Section 10.1.

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Fig. 12.3.7 Orientation system for storing liquids within a tank in the zero gravity environment of space. The tank used in this test was spherical and transparent, with circular electrodes which are seen here edge-on. The electrodes converge toward the bottom of the picture; thus this is the region in which the electric field should provide an artificial "bottom." (a) With one g acting toward the *top* of the picture and no electric field, the liquid is in the upper half of tank; (b) liquid oriented at artificial "bottom" of tank under near-zero-gravity conditions created by flying the tank within a KC-135 in a zero gravity trajectory. The electrodes are at alternate polarities and can be viewed as a combination of pendulums with the basic configuration shown in Fig. 12.3.5. (Courtesy of Dynatech Corp., Cambridge, Mass.)

In Chapter 13 the restriction of an incompressible fluid is relaxed, and the effects of compressibility on electromechanical interactions are studied, although the restriction to inviscid fluid models is still retained.

PROBLEMS

12.1. The mechanism shown in Fig. 12P.1 is to be used as an electrically driven rocket. An insulating fluid of constant density ρ is compressed by a piston. The fluid is then ejected through a slit (nozzle) with a velocity V_p ; because $dD \ll LD$, V_p is approximately a constant, and the system is approximately in a steady-state condition ($\partial/\partial t = 0$):

- What is the pressure p at the inside surface of the piston? (Assume that $p = 0$ outside the rocket.)
- Under the assumption that $d \ll L$, what is V_p ?
- What is the total thrust of the rocket in terms of the applied voltage V_o and other constants of the system?

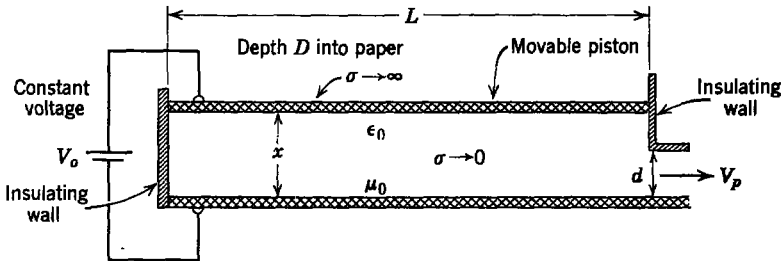


Fig. 12P.1

12.2. A magnetic rocket is shown in Fig. 12P.2. A current source (distributed over the width W) drives a circuit composed in part of a movable piston. This piston drives an incompressible fluid through an orifice of height d and width W . Because $D \gg d$, the flow is essentially steady.

- Find the exit velocity V as a function of I .
- What is the thrust developed by the rocket? (You may assume that it is under static test.)
- Given that $I = 10^3$ A, $d = 0.1$ m, $W = 1$ m, and the fluid is water, what are the numerical values for V and the thrust? Would you prefer to use water or air in your rocket?

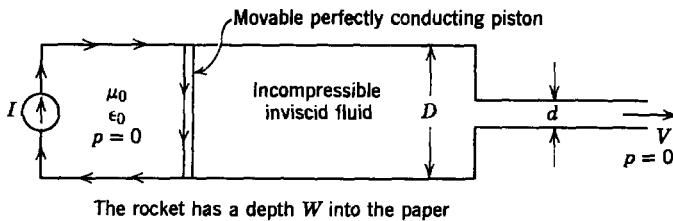


Fig. 12P.2

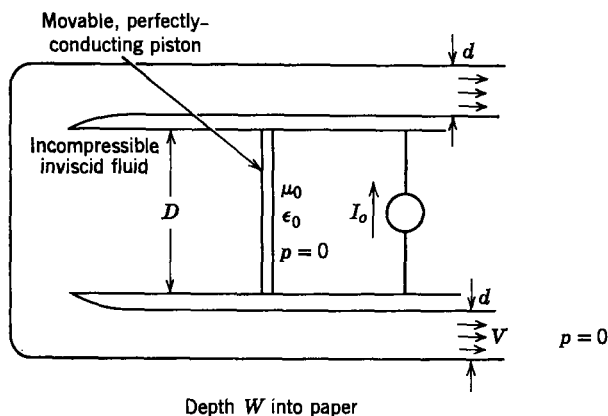


Fig. 12P.3

12.3. A magnetic rocket is shown in Fig. 12P.3. A current source I_0 (distributed over the depth W) drives a circuit composed in part of a movable piston. This piston drives an incompressible, inviscid, nonconducting fluid through two orifices, each of height d and depth W . Because $D \gg d$, the flow is essentially steady.

- Find the exit velocity V as a function of I_0 .
- What is the thrust developed by the rocket? (You may assume that it is under static test.)

12.4. An incompressible, inviscid fluid of density ρ flows between two parallel walls as shown in Fig. 12P.4. The bottom wall has a small step of height d in it at $x_1 = 0$. At $x_1 = -L$, the velocity of the fluid is $\mathbf{v} = V_0 \mathbf{i}_1$ and the pressure is p_0 . Also, at $x_1 = +L$ the velocity of the fluid is uniform with respect to x_2 and is in the x_1 -direction, since $d \ll L$. Assuming that the flow is steady ($\partial/\partial t = 0$) and irrotational, find the x_1 -component of the force per unit depth on the bottom wall. The system is uniform in the x_3 -direction and has the x_2 dimensions shown in Fig. 12P.4.

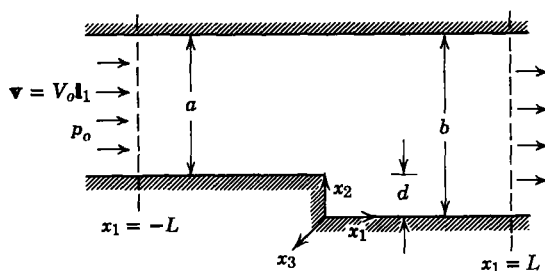


Fig. 12P.4

12.5. Far away from the rigid cylinder shown in Fig. 12P.5 the velocity of a fluid with density ρ is a constant $\mathbf{V} = V_0 \mathbf{i}_1$ and its pressure is p_0 . Assume that the fluid is incompressible and that the flow is steady and irrotational:

- Find the velocity of the fluid everywhere.
- Sketch the velocity field.
- Find the pressure everywhere.
- Use the stress tensor to find the total pressure force (give magnitude and direction) exerted by the fluid on the rigid cylinder. Assume that $\partial/\partial x_3 = 0$.

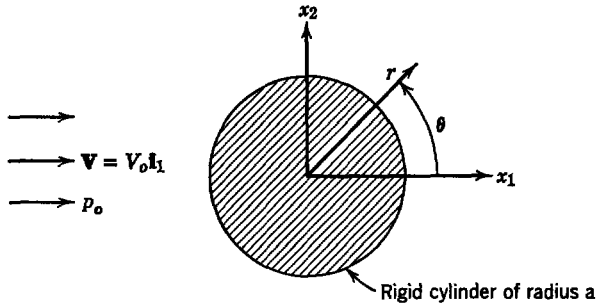


Fig. 12P.5

12.6. An inviscid incompressible fluid flows around a rigid sphere of radius a , as shown in Fig. 12P.6. At $x_1 = \pm\infty$ the fluid velocity becomes $\mathbf{v} = V_0 \mathbf{i}_1$.

- Compute the velocity distribution $\mathbf{v}(x_1, x_2, x_3)$.
- Find the pressure $p(x_1, x_2, x_3)$. [Assume that the pressure is zero at $(x_1, x_2, x_3) = (-a, 0, 0)$.]
- Use the results of (b) to compute the force exerted on the sphere in the x_1 -direction by the fluid.

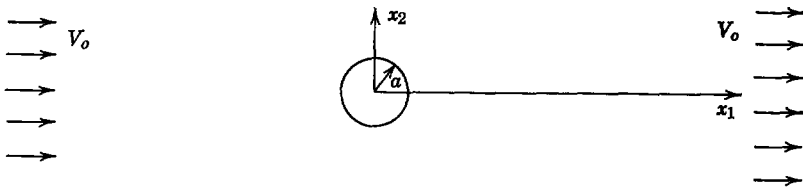


Fig. 12P.6

12.7. The velocity distribution of an inviscid fluid is given as $\mathbf{v} = -\nabla\phi$, where $\phi = (V_0/a)x_1x_2$ and V_0 and a are constants.

- Show by means of a sketch the direction and magnitude of the velocity in the x_1 - x_2 plane.
- Compute the fluid acceleration. Sketch the direction and magnitude of the acceleration in the x_1 - x_2 plane.
- In what physical situation would you expect the flow to have this distribution?

12.8. In the configuration of Fig. 12P.8 an incompressible, inviscid fluid of mass density ρ flows without rotation ($\nabla \times \mathbf{v} = 0$), between two rigid surfaces shown, with velocity

$$\mathbf{v} = \mathbf{i}_1 v_0 \frac{x_2}{a} + \mathbf{i}_2 v_0 \frac{x_1}{a},$$

where v_0 and a are positive constants. Neglect gravity.

- Find the fluid acceleration at all points in the flow.

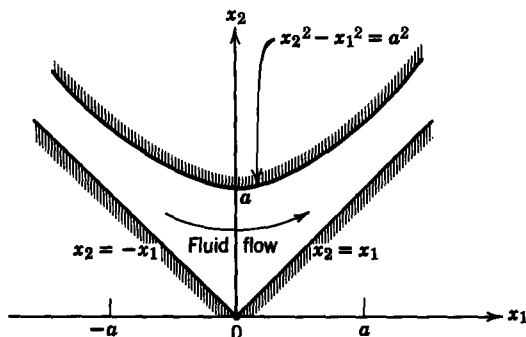


Fig. 12P.8

- (b) The pressure is constrained at the origin ($x_1 = x_2 = 0$) to be p_o . Find the pressure at all other points in the fluid.

12.9. Consider the situation of Prob. 12.8, but now with gravity acting in the $-x_2$ direction.

- Find the velocity between the rigid walls.
- Show that the boundary conditions are satisfied at the walls.

12.10. Figure 12P.10 shows an irrotational flow in a corner formed by a rigid wall.

- Let $\mathbf{v} = -\nabla\phi$. What are the boundary conditions on ϕ ? Sketch the contour of constant ϕ in the x - y plane.
- What function $\phi(x, y)$ satisfies both $\nabla \cdot \mathbf{v} = 0$ and $\nabla \times \mathbf{v} = 0$ and the boundary conditions of part (a)?
- Assuming that the pressure $p = p_o$ at $(x, y) = 0$ and that $p = p_o$ to the left and below the wall, what is the force exerted by the fluid on the section of the wall between $x = c$ and $x = d$?
- Compute the fluid acceleration. Make a sketch to show the magnitude and direction of the acceleration in the x - y plane.

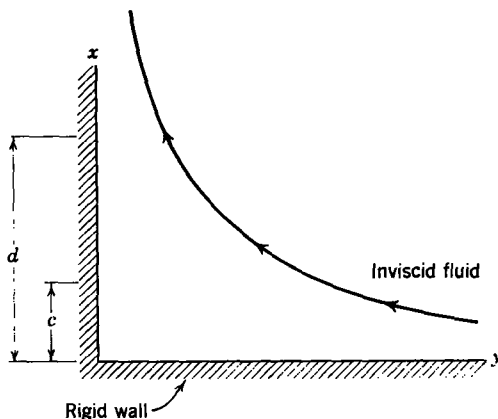


Fig. 12P.10

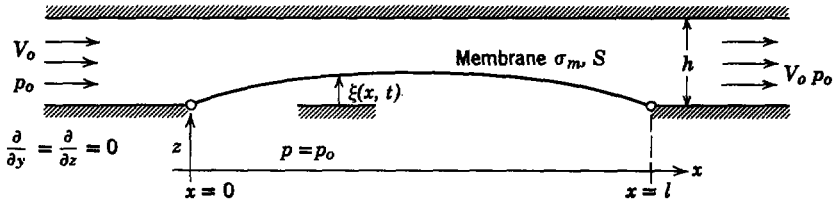


Fig. 12P.11

12.11. An inviscid incompressible fluid enters the channel shown in Fig. 12P.11, with the velocity V_0 and pressure p_0 . One wall of the channel includes a section of length l composed of a taut membrane with the deflection ξ .

- Assume that spatial variations in the membrane deflection occur slowly so that the velocity v_x is independent of z . Relate $v_x(x)$ to V_0 , ξ , and h .
- Determine the pressure on the upper surface of the membrane using the fact that $p = p_0$ at the inlet where $v = V_0$.
- Find an expression of the form $T_z = C\xi$ for the force per unit area T_z on the membrane as a function of ξ and a constant C . To do this assume that perturbations in ξ are small and use the fact that the pressure below the membrane is p_0 .
- Now assume that the dynamics occur slowly enough that the result of part (c) will remain true even if $v = v(x, t)$ and $\xi = \xi(x, t)$. The membrane has a tension S and mass per unit area σ_m . For what values of V_0 will the static equilibrium of the membrane at $\xi = 0$ be stable?
- Explain physically why the instability of part (d) occurs.

12.12. A perfectly conducting membrane with tension S and mass per unit area σ_m is fixed at $x = 0$ and $x = L$. An inviscid, incompressible fluid with mass density ρ_0 flows underneath the membrane (Fig. 12P.12). An electric field exists in the region above the membrane. The upper region is filled with a light gas that is at atmospheric pressure p_0 everywhere.

- Find the value of the pressure p_1 in the fluid for $-d \leq y \leq 0$ in terms of given parameters if $\xi(x, t) = 0$ is a state of equilibrium.
- Under what conditions can a small signal sinusoidal oscillation exist about the equilibrium position $\xi(x, t) = 0$?

Note. The gravitational field affects both the equilibrium and small signal solutions.

- Make a dimensioned ω - k plot for a real wavenumber.
- Justify the validity of imposing boundary conditions at $x = 0$ and $x = L$ such that these conditions affect the membrane for $0 \leq x \leq L$.

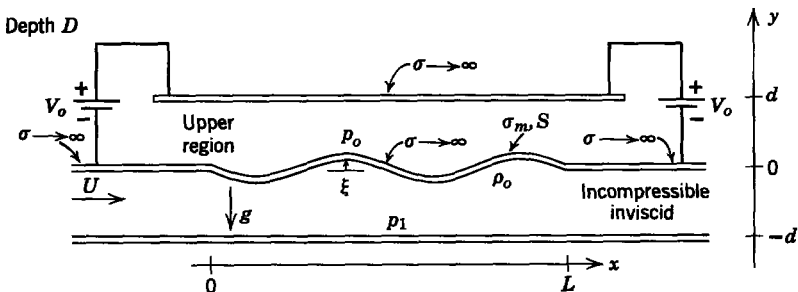


Fig. 12P.12

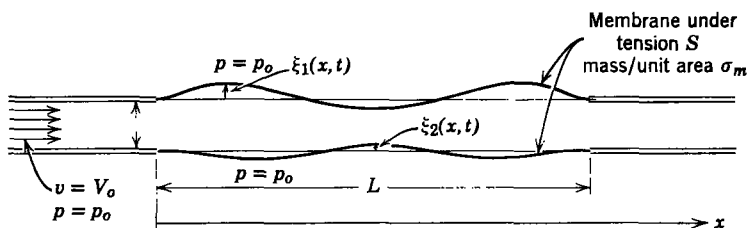


Fig. 12P.13

12.13. A duct is formed by stretching membranes between plane parallel rigid plates, as shown in Fig. 12P.13. An inviscid fluid flows through this duct, entering at the left with velocity V_o . The pressure outside the duct is p_o , so that the membrane can be in static equilibrium with $\xi_1 = \xi_2 = 0$.

- What is the largest velocity V_o that can be used and have the membranes remain in a state of stable equilibrium?
- What would the appearance of the membranes be if V_o were just large enough to make the equilibrium $\xi_1 = \xi_2 = 0$ unstable?

12.14. An inviscid, incompressible fluid rests on a rigid bottom, as shown in Fig. 12P.14. In the absence of any disturbances it is static and has a depth a . If disturbed, the surface of the fluid has the position $\xi(x, t)$. As is obvious to anyone who has observed ocean waves, disturbances of the interface propagate as waves. It is our object here to derive an equation for the propagation of these gravity waves, which have “wavelengths” that are long compared with the depth a of the fluid. To do this we make the following assumptions:

- The effects of inertia in the y -direction are negligible. Hence the force equation in the y -direction is

$$\frac{\partial p}{\partial y} = -\rho g.$$

Because $y = \xi(x, t)$ is a free surface, the pressure there is constant (say zero). What is p in terms of y and ξ ?

- Because the fluid is very shallow, we can assume that $v_x = v_x(x, t)$; that is, the horizontal fluid velocity is independent of y . On the basis of this assumption, use the conservation of mass equation for the incompressible fluid ($\nabla \cdot \mathbf{v} = 0$) to find $v_y(x, y, t)$ in terms of v_x .
- Use the result of (a) to write the horizontal component of the force equation as one equation in $v_x(x, t)$ and $\xi(x, t)$.
- Use the result of (b) to write an additional equation in ξ and v_x (assume $\xi \ll a$ so that only linear terms need be retained).
- Combine equations from parts (c) and (d) to obtain the wave equation for gravity waves. What is the phase velocity of these waves?

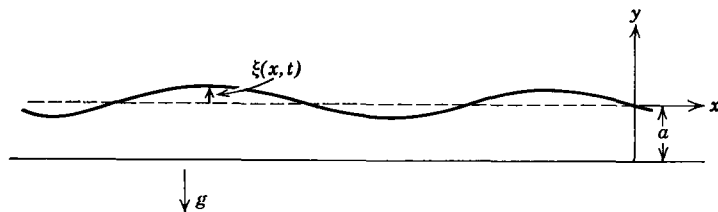


Fig. 12P.14

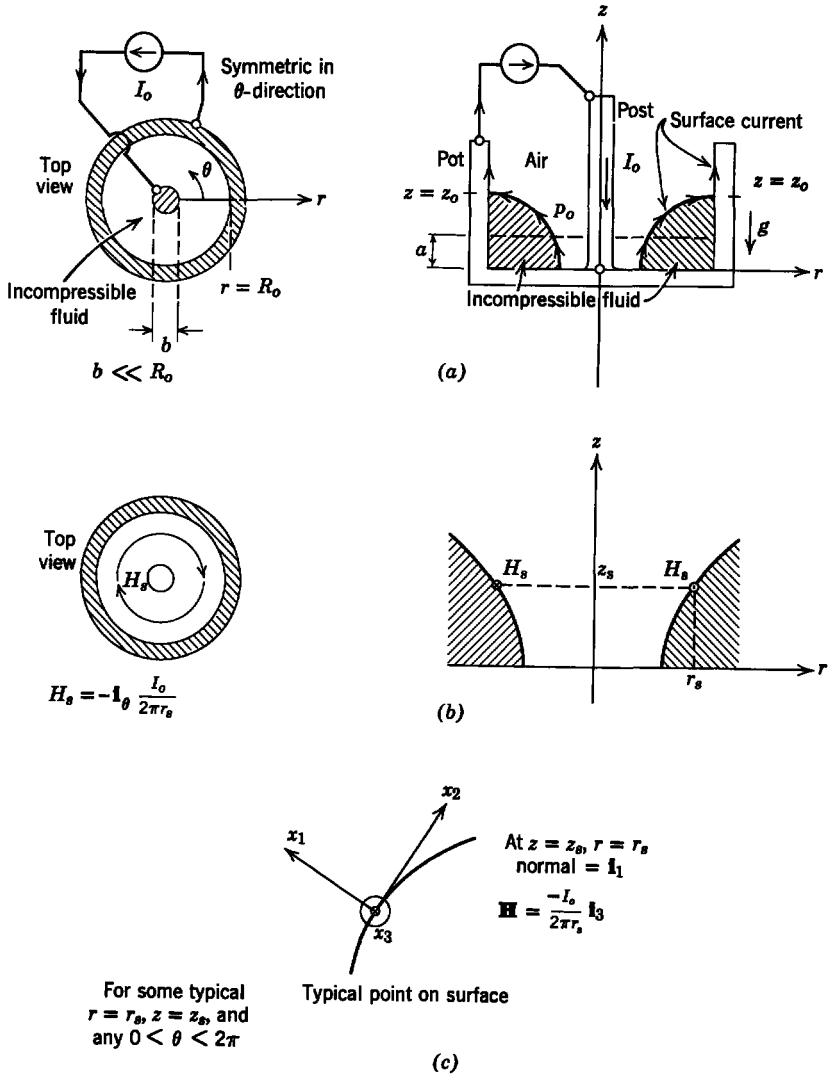


Fig. 12P.15

12.15*. A perfectly conducting, cylindrical pot contains a perfectly conducting fluid. A center coaxial post is placed inside the cylinder. A current source is attached between the center post and the outer wall of the pot to cause a current to flow on the perfectly conducting surfaces, as shown in Fig. 12P.15a. When the current source is turned off, the fluid comes to rest with its surface at $z = a$. When the current source is turned on, the magnetic field pressure (normal surface traction) causes the surface to deform (e.g. as shown).

* Colgate, Furth, and Halliday, *Rev. Mod. Phys.*, 32, No. 4, 744 (1960).

- (a) At any typical point on the surface (θ -direction symmetry exists; see Fig. 12P.15b), the normal traction on the surface can be found by using the Maxwell Stress Tensor and a coordinate system arranged for the sample point, as shown in Fig. 12P.15c. Since the MST result is good for any surface point (r, z) , the normal traction is known everywhere as a function of r , the radial position of the surface point. Find the normal traction due to the magnetic field as a function of r .
- (b) The hydrostatic pressure of the dense fluid varies appreciably with z , due to gravity, whereas the pressure of the light gas (air) may be assumed to be p_o (atmospheric) everywhere. Using the fact that the forces acting normal to the fluid-air interface must balance, find an equation for the surface. Neglect surface tension and call the top point on the surface $z = z_o$, $r = R_o$. *Hint.* Remember that at $z = z_o$, $r = R_o$, the magnetic traction $+ p_o$ exists on the air side of the interface and is counterbalanced by the hydrostatic pressure on the fluid side of the interface. No magnetic field exists in the fluid; hydrostatic pressure exerts a normal force on the interface.
- (c) In part (b) the value of z_o remains unknown. Because the total mass of the fluid (or volume for an incompressible fluid) must be conserved, *set up* an expression that will determine z_o . Integration need not be carried out.

12.16. The MHD machine for which dimensions and parameters are defined in Fig. 12P.16 can be assumed to operate with incompressible, uniform flow velocity v in the z -direction. The fluid has constant, scalar conductivity. There is a uniform applied flux density B in the

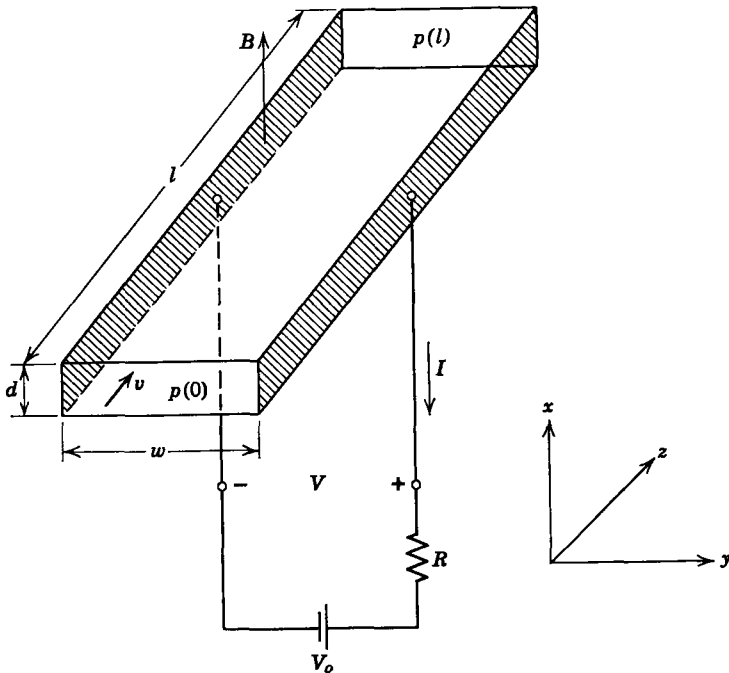


Fig. 12P.16

x -direction and the magnetic field due to current through the fluid may be neglected. The electrical terminals are connected to a battery of constant voltage V_o and a constant resistance R in series. For steady-state operation calculate and sketch the electric power out of the generator $P_e = VI$ and the mechanical power into the generator $P_m = [p(0) - p(l)]w dv$ as functions of the fluid velocity v . Specify the range of velocity over which the system operates as a generator, pump, and brake.

12.17. From a conformal-mapping analysis of end effects in an MHD generator,* for a generator having

channel width	w ,	channel depth	d ,
electrode length	l ,	uniform velocity	v_o ,
fluid conductivity	σ ,	flux density	B_o over length of electrodes,

the electrical output power is

$$P_{\text{out}} = \frac{1}{R_i} (V_{oc} - V)V - \frac{1}{R_i} aV^2,$$

where

$$R_i = \frac{w}{\sigma ld},$$

$$V_{oc} = v_o B_o w,$$

$$a = \frac{2}{\pi} \left(\frac{w}{l} \right) \ln 2$$

V = terminal voltage.

(a) Show that the mechanical power input is given by

$$P_m = \int_0^l \int_0^w dJ_y B_o dy dz = \frac{1}{R_i} (V_{oc} - V)V_{oc}$$

by direct integration. *Note.* All that is needed for this integration is the facts that $\mathbf{E} = -\nabla\phi$ and the difference in potential between the electrodes is V .

(b) Defining efficiency as $\eta = P_{\text{out}}/P_m$, find the efficiency at maximum power output and the maximum efficiency and plot them as functions of $1/a$ for $0 < 1/a < 10$.

12.18. For the MHD machine with solid electrodes, for which parameters, dimensions, and variables are defined in Fig. 12P.18a, assume that the fluid is incompressible, inviscid, and has a constant, scalar conductivity σ . Neglect the magnetic field due to current in the fluid. The source that supplies the pressure $\Delta p = p_i - p_o$ has the linear characteristic $\Delta p = \Delta p_o(1 - v/v_o)$, where Δp_o and v_o are positive constants. The machine with this mechanical source can be represented electrically by the equivalent circuit of Fig. 12P.18b. Find the open-circuit voltage V_{oc} and the internal resistance R_i in terms of the given data.

* G. W. Sutton and A. Sherman, *Engineering Magnetohydrodynamics*, McGraw-Hill, New York, 1965, Section 14.6.1.

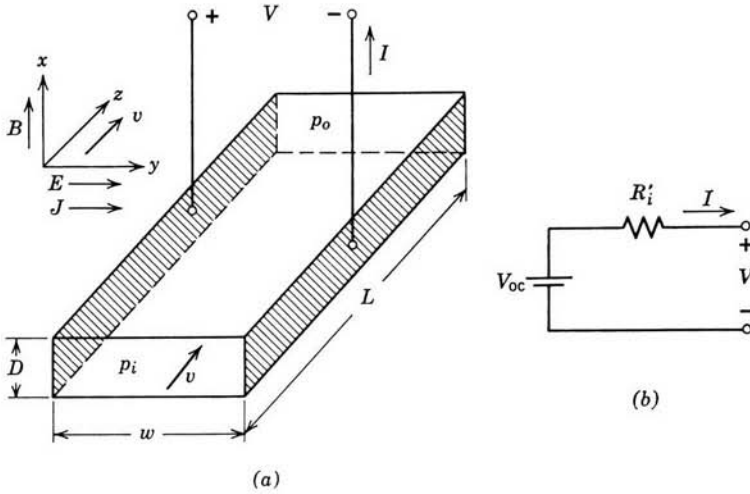


Fig. 12P.18

12.19. An MHD conduction generator has the configuration and dimensions defined in Fig. 12P.19. The fluid is inviscid, has conductivity σ , and is flowing with a uniform, constant velocity v in the x -direction. The field intensity H_o , in the y -direction, is produced by the system shown, which consists of a magnetic yoke with two windings; one (N_o) carries a constant current I_o and the other (N_L) carries the load current I_L . The resistance of winding N_L , fringing effects at the ends and sides of the channel, and the magnetic field due to current in the fluid may be neglected (the magnetic Reynolds number based on length l is small). For steady-state operating conditions find the number of turns N_L necessary to make the terminal voltage V_L independent of load current I_L .

System has length l perpendicular to the paper

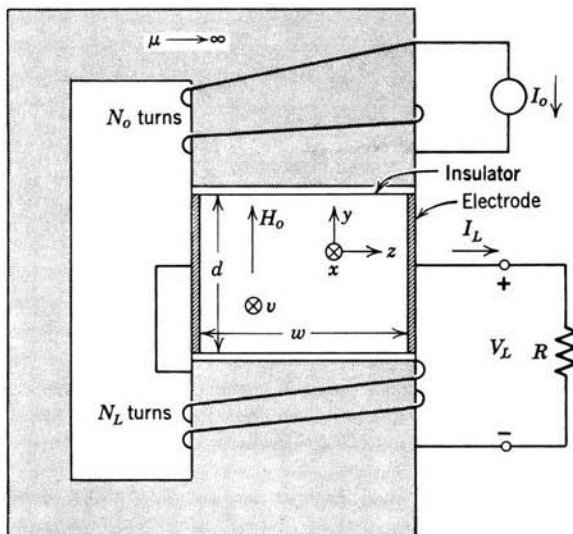


Fig. 12P.19

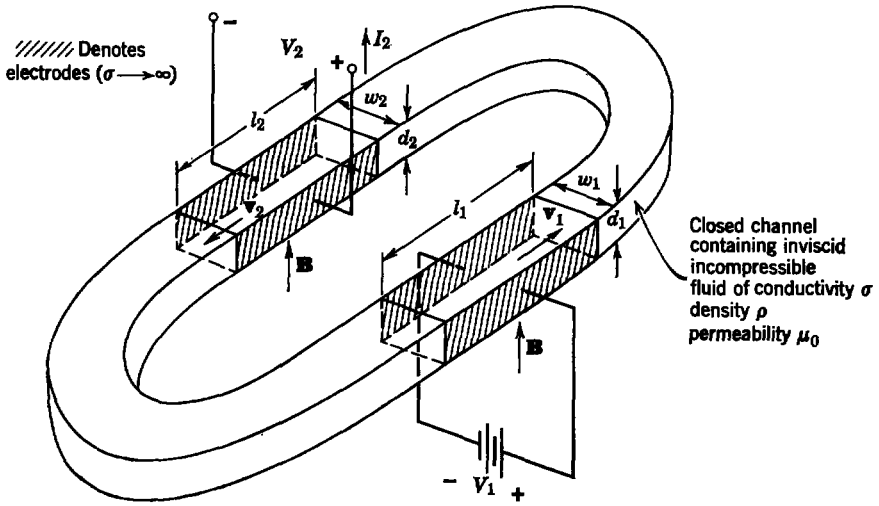


Fig. 12P.20

12.20. A dc transformer is to be made by using a closed channel of incompressible, inviscid fluid of conductivity σ , permeability μ_0 and density ρ , and two MHD conduction machines, as illustrated in Fig. 12P.20. Both machines have the same applied uniform flux density $\mathbf{B}$, but their dimensions are different, as indicated. Make the usual assumptions of uniform velocity in the machines, neglect the magnetic fields induced by current in the fluid, and neglect end and edge effects and fringing. For steady-state conditions find a relation between V_2 and I_2 in terms of input voltage V_1 , conductivity σ , and the dimensions. Draw the Thevenin equivalent circuit this relation implies.

12.21. A conducting liquid flows with a constant velocity v in the closed channel shown in Fig. 12P.21. The motion is produced by an MHD pump which provides a pressure rise

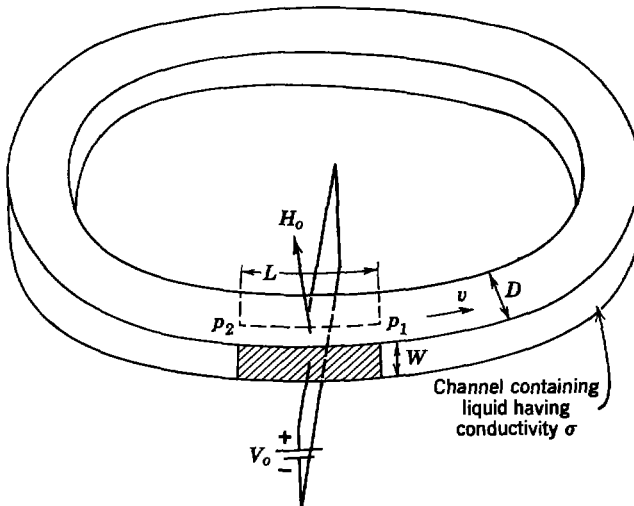


Fig. 12P.21

$p_1 - p_2 > 0$ where p_1 and p_2 are the outlet and inlet pressures. The fluid, as it flows through the remainder of the channel, undergoes the pressure drop $p_1 - p_2 = kv$, where k is a known constant. Determine the velocity v in terms of the imposed magnetic field H_o and the other constants of the system.

12.22. The rectangular channel with the dimensions shown in Fig. 12P.22 is to be used in an MHD pump for a highly conducting liquid of conductivity σ . The channel has two sides which are perfectly conducting electrodes, and an electrical circuit is connected to them. An

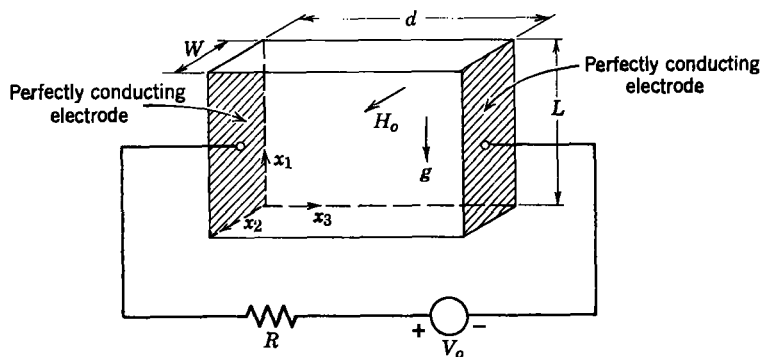


Fig. 12P.22

external magnet produces a constant magnetic field H_o in the x_2 -direction. The device is to pump against a gravitational field (which is in the $(-x_1)$ -direction):

- What range of values for the voltage source V_o will make the liquid flow upward in the $+x_1$ -direction? Assume that the pressure at $x_1 = 0$ is the same as the pressure at $x_1 = L$.
- Under the conditions of part (a) show clearly that the voltage source is supplying power to the liquid.

12.23. Two large reservoirs of water are connected by a large duct, as shown in Fig. 12P.23a. Over a length l of this duct the walls are highly conducting electrodes short-circuited together by an external circuit, as shown in Fig. 12P.23b. A uniform, constant magnetic field B_o is imposed perpendicular to the direction of flow. Because the water has a conductivity σ , there is a current through the water between the electrodes. Assume that the reservoirs are so large that h_1 and h_2 remain constant and that the fluid is incompressible and inviscid. What is the velocity v of the fluid between the electrodes?

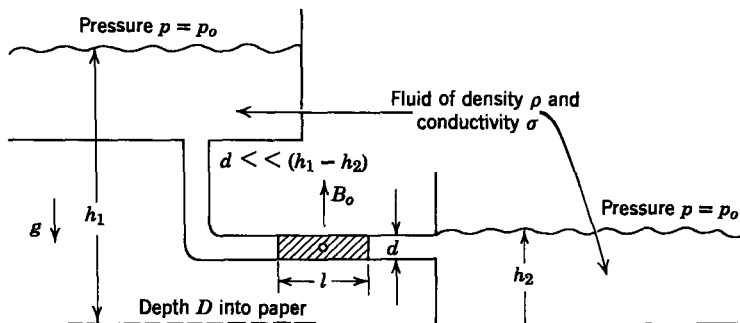


Fig. 12P.23 (a)

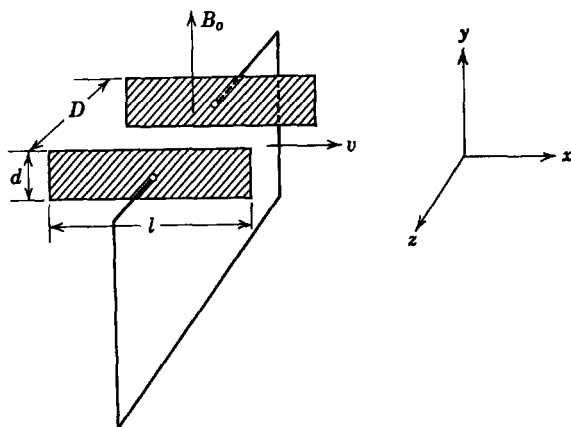
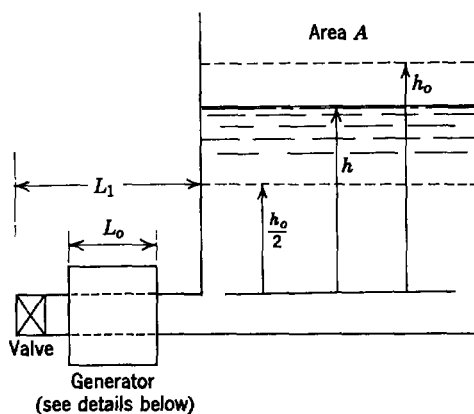


Fig. 12P.23 (b)



Generator details

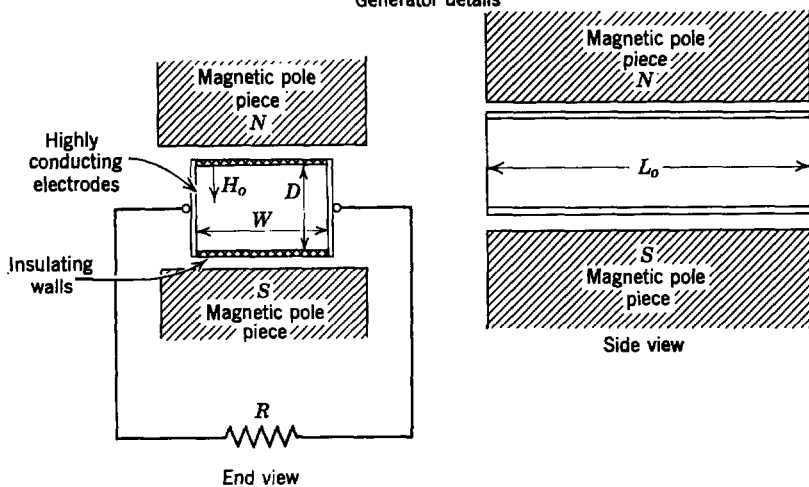


Fig. 12P.24

12.24. The system shown in Fig. 12P.24 consists of a storage tank of horizontal cross-sectional area A , open to the atmosphere at the top, and an MHD generator through which fluid stored in the tank can flow to atmospheric pressure. The generator is loaded by resistance R as shown. The tank is initially filled to a height h_0 with mercury. At $t = 0$ the valve is opened to allow mercury to discharge through the generator. When the height has decreased to $h_0/2$ the valve is closed again. Do the following calculations, using the numerical data given below. Make any approximations that are justified by the numerical data. Assume all flow to be incompressible ($\nabla \cdot \mathbf{v} = 0$) and irrotational ($\nabla \times \mathbf{v} = 0$).

- Calculate the height h of mercury in the tank as a function of time.
- Calculate the current in the load resistance R as a function of time.

Numerical Data

Mercury		Generator	
Density:	$1.35 \times 10^4 \text{ kg/m}^3$	$H_0 = 1.6 \times 10^5 \text{ A/m}$	
		$D = 0.1 \text{ m}$	
Conductivity:	10^6 mhos/m	$W = 0.2 \text{ m}$	
		$L_0 = 1 \text{ m}$	
Tank Dimensions		$L_1 = 2 \text{ m}$	
	$A = 100 \text{ m}^2$	$R = 2 \times 10^{-5} \Omega$	
	$h_0 = 10 \text{ m}$	Acceleration of Gravity	
		$g = 9.8 \text{ m/sec}^2$	

12.25. A simple, bulk-coupled MHD system is used to pump mercury from one storage tank to another, as shown in Fig. 12P.25a. Figure 12P.25b shows the details of the MHD system. The MHD system is driven with a voltage source V_0 in series with a resistance R .

Each storage tank has area A and is open to atmospheric pressure at the top. Consider all flow to be incompressible and irrotational ($\nabla \cdot \mathbf{v} = 0$ and $\nabla \times \mathbf{v} = 0$). Use the following numerical data for your solutions.

Mercury		MHD System	
Density:	$1.35 \times 10^4 \text{ kg/m}^3$	$H_0 = 5 \times 10^5 \text{ A/m}$	
Conductivity:	10^6 mhos/m	$D = 0.01 \text{ m}$	
		$W = 0.02 \text{ m}$	
Tank Area:	$A = 0.1 \text{ m}^2$	$L_1 = 0.1 \text{ m}$	
Acceleration of Gravity:	$g = 9.8 \text{ m/sec}^2$	$L_2 = 1.9 \text{ m}$	
		$R = 10^{-5} \Omega$	

- What voltage V_0 is required to maintain the levels in the tanks $h_1 = 0.4 \text{ m}$ and $h_2 = 0.5 \text{ m}$. How much current and power does the voltage source supply in this case?
- With the equilibrium conditions of (a) established, the voltage V_0 is doubled at $t = 0$. Find $h_2(t)$ and the source current $i(t)$ for $t > 0$. Sketch and label curves of these time functions. To solve this problem exactly it is necessary to know the three-dimensional flow pattern in the tanks. For this problem, however, it is sufficient to make the following approximations. In writing dynamical equations neglect the acceleration of the fluid in the tanks compared with the acceleration of the fluid in the pipe. Be sure to estimate the error caused by making this assumption *after* the solution has been completed. Also neglect the magnetic field due to current in the mercury.

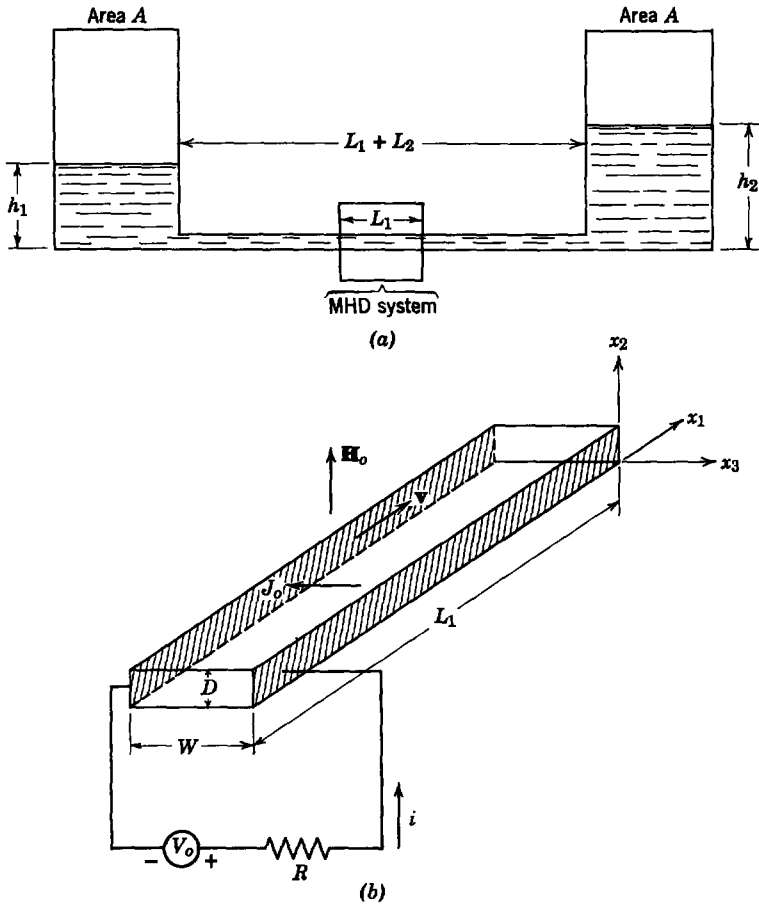


Fig. 12P.25

12.26. Figure 12P.26 shows schematically an ac, series-excited, liquid metal conduction pump. The liquid metal has mass density ρ and electrical conductivity σ . The excitation winding has N turns ($N \gg 1$) and the magnetic path is closed externally by infinitely permeable, nonconducting magnetic material. The dimensions are given in the figure. The electrical terminals are driven by an alternating current source: $i(t) = I \sin \omega t$, where I and ω are positive constants. The pump works against a velocity dependent pressure rise $p(l) - p(0) = \Delta p_0(v/v_0)$, where Δp_0 and v_0 are positive constants. For steady-state operation complete the following:

- Find the velocity v as a function of time.
- Evaluate the ratio of the amplitudes of the ac and dc components of velocity.

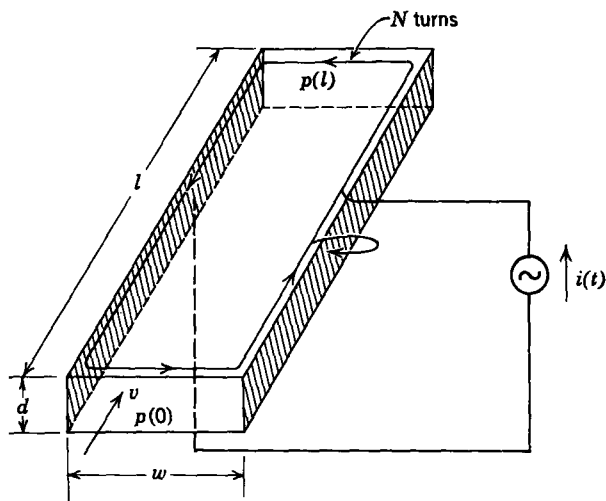


Fig. 12P.26

12.27. A pair of magnetohydrodynamic conduction generators is shown in Fig. 12P.27. In each generator a conducting fluid flows through a channel of width w and height a with approximately uniform constant velocity V . A magnetic field is applied to each generator by means of a magnetic circuit that produces (approximately) a uniform magnetic field in the vertical direction. The magnetic circuits can be considered as having infinite permeability with all the drop in mmf across the channels. Currents are passed through the channels by means of highly conducting electrodes of height a and length b , as shown in Fig. 12P.27. The current i_1 (amperes) through the left generator is used to produce a magnetic field in the left generator and in the right generator, as shown, and to deliver power to the load R_L . The current i_2 through the right generator follows a similar path, except that the turns N and N_m have different directions. We wish to establish the dynamics that would be expected for two generators interconnected in this way, with the objective of producing ac rather than dc power delivered to the loads R_L .

- Find the pair of ordinary differential equations in $i_1(t)$ and $i_2(t)$ that defines the system dynamics under the given conditions.
- Determine the condition, in terms of the given system parameters, that the generators be stable.
- Under what condition will the system operate in the sinusoidal steady state? Given that $R_L = 0$, $\sigma = 50$ mhos/m, $V = 4000$ m/sec, and $N = 1$ turn, what is the length b required to meet this condition?
- Compute the frequency at which the system will operate in the sinusoidal steady state under the above conditions, given that $N_m = 1$ turn.

12.28. The system shown in Fig. 12P.28 has been proposed for an ac, self-excited, MHD power generator. It consists of a *single* channel with length l , width D , and height W , through which an incompressible, inviscid, highly conducting liquid with conductivity σ and permeability μ_0 flows. The velocity of the liquid is *constrained* externally to be a *constant* and always in the x_1 -direction; that is, $\mathbf{v} = V\mathbf{i}_1$ everywhere in the channel. (The scalar V is a known fixed constant.) The sides of the channel at $x_2 = 0$ and $x_2 = W$ are insulating, but the other two sides, namely those at $x_3 = 0$ and $x_3 = D$, are perfectly conducting electrodes.

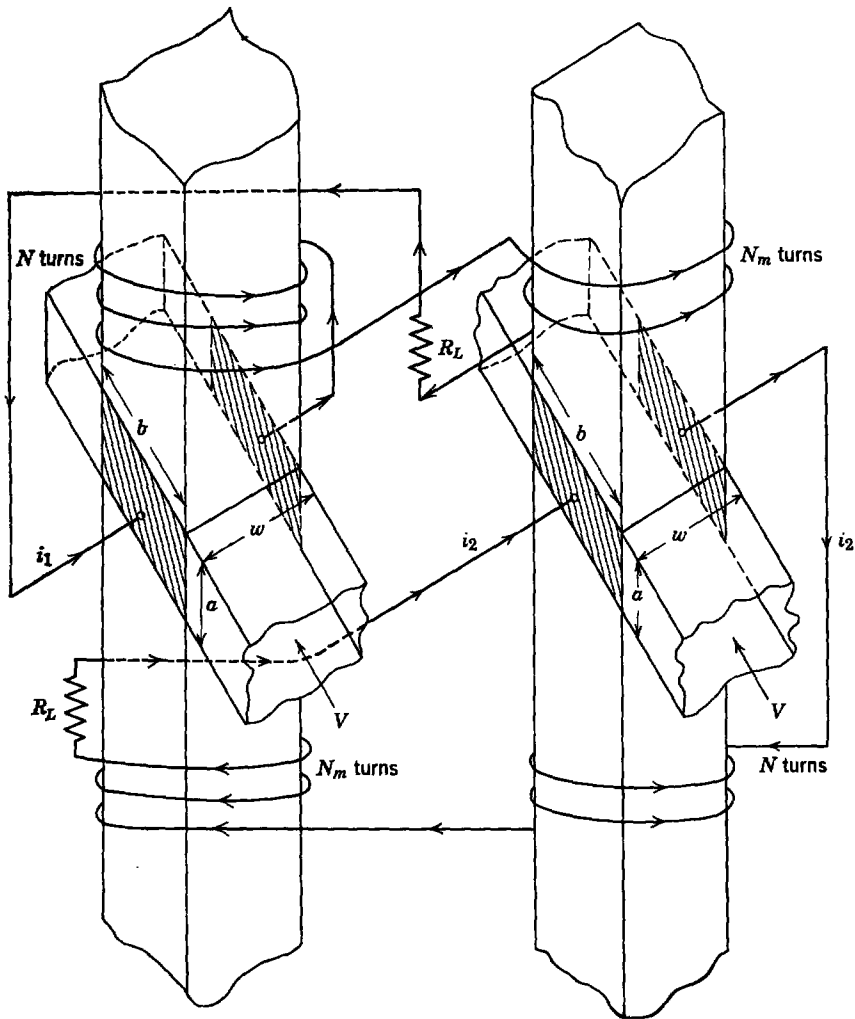


Fig. 12P.27

An external circuit connected to the electrodes consists of three elements in series. The first two elements are a load resistance R_L and a capacitance C . The last element is a *lossless* coil of N turns which is wound on the top ($x_2 = W$) surface of the channel to produce a *uniform* x_2 -directed magnetic field everywhere in the channel. (Reasonable assumptions may be made about this field; namely, that it is equivalent to the field produced as if the N turns were distributed uniformly through the depth W of the channel but on the outer perimeter of the channel.)

- Find the value of the load resistance R_L that will make the device act as an ac generator (so that the current i is a pure sinusoid). Note that ac power will then be continuously dissipated in the load resistance R_L .
- What is the frequency of the resulting sinusoid of part (a)?

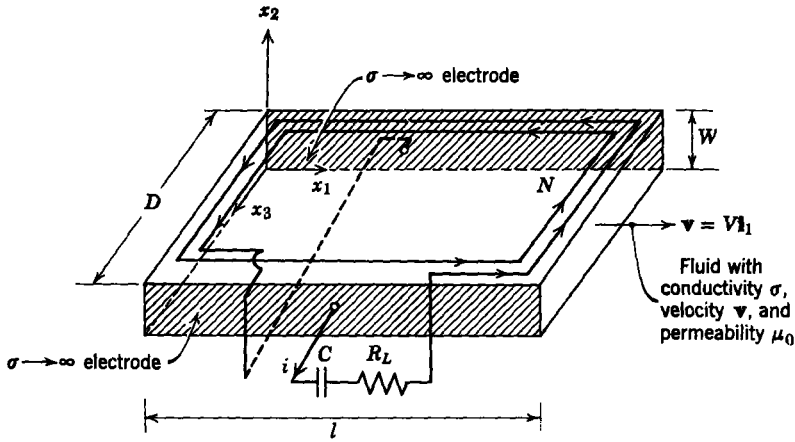


Fig. 12P.28

12.29. The MHD generator of Fig. 12P.29 has a channel of length l , width w , and depth d . An external system not shown establishes the magnetic flux density $\mathbf{B}_0 = B_0\mathbf{i}_x$. Two equal resistances of $R \Omega$ each are connected in series between the electrodes, with a switch S in parallel with one resistance. The channel contains an inviscid, incompressible fluid of mass density ρ flowing under the influence of a pressure difference $\Delta p = p_i - p_o$, which is positive and maintained constant by external means. The fluid has an electrical conductivity σ and the magnetic field due to current flow in the fluid can be neglected.

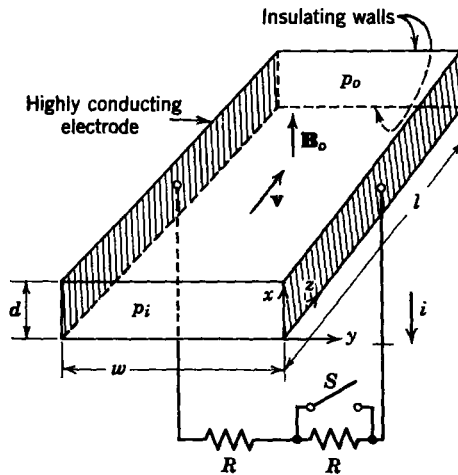


Fig. 12P.29

- (a) For steady-state conditions with switch S open, find the velocity v and load current i in terms of given data.
- (b) With the system operating in the steady state as defined in part (a), the switch S is closed at $t = 0$. Find the velocity v and load current i as functions of the given data and time for $t > 0$.

12.30. This problem concerns the self-excitation of a dc generator. The variables and dimensions are given in Fig. 12P.30. The channel has a constant cross-sectional area, the fluid is incompressible and inviscid, and the conductivity is constant and scalar. Assume

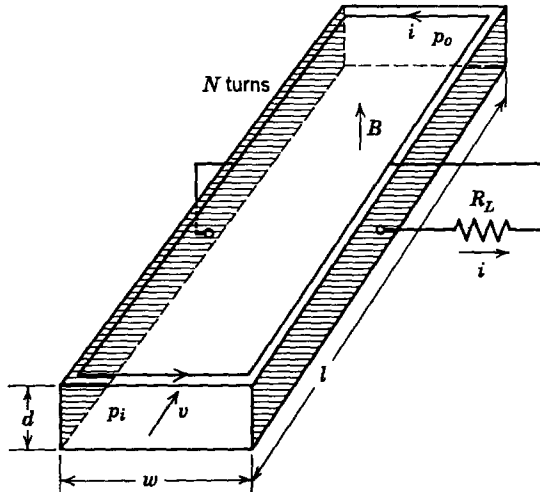


Fig. 12P.30

that the magnetic circuit is closed outside the channel with infinitely permeable iron. The mechanical source providing power has the pressure-velocity characteristic $\Delta p = p_i - p_o = \Delta p_o(1 - v/v_o)$, where Δp_o and v_o are positive constants. The volume, geometry, and space factor of the field winding are constants so that the field coil resistance varies as the square of the number of turns $R_c = N^2 R_{co}$. The numerical constants of the system are

$$\begin{array}{lll} \Delta p_o = 2 \times 10^5 \text{ n/m}^2 & v_o = 10^3 \text{ m/sec} & d = 0.2 \text{ m} \\ l = 2 \text{ m} & w = 0.4 \text{ m} & R_L = 2.5 \times 10^{-2} \Omega \\ \sigma = 40 \text{ mhos/m} & R_{co} = 10^{-6} \Omega & \end{array}$$

- (a) Find the number of turns necessary to produce a load power in R_L of $1.5 \times 10^6 \text{ W}$. If there is more than one solution, pick the most efficient.
- (b) For the number of turns in part (a), find the start up transient in current and plot it as a function of time. Assume an initial current of 10 A, provided by external means.
- (c) For the number of turns found in part (a) find the steady-state load power as a function of R_L . Plot the curve.

12.31. The system of Fig. 12P.31 represents an MHD transverse-current generator with continuous electrodes. We make the usual assumptions about incompressible, inviscid, uniform flow. The fluid has mass density ρ and conductivity σ . The pressure drop along the

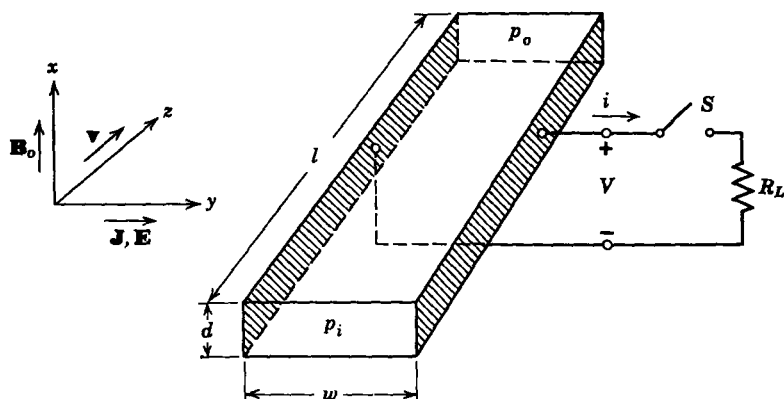


Fig. 12P.31

length of the channel is constrained by a mechanical source to be

$$p_i - p_o = \Delta p = \Delta p_o \left(1 - \frac{v}{v_o} \right),$$

where Δp_o and v_o are positive constants. The flux density B_o is uniform and constant and is supplied by a system not shown. Neglect the magnetic field due to current in the fluid. The switch S is open initially and the system is in the steady state:

- Find the terminal voltage V .
- At $t = 0$ switch S is closed. Find and sketch the ensuing transients in fluid velocity and load current.

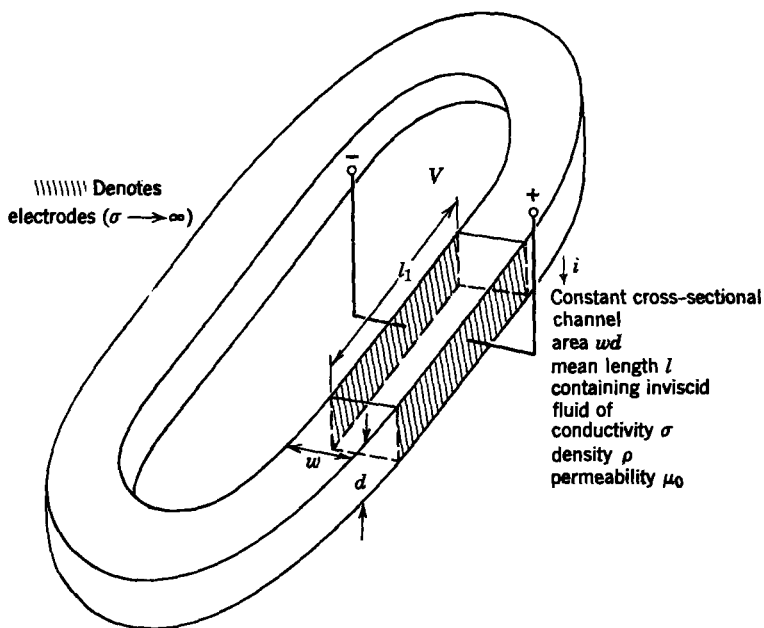


Fig. 12P.32

12.32. An energy storage element is to be made by using a closed channel of incompressible inviscid fluid of conductivity σ , permeability μ_0 , and mass density ρ , with an MHD conduction machine for coupling (Fig. 12P.32). The channel has constant cross-sectional area wd and mean length l and the radius of the bends is large compared with the channel width w . The flux density $\mathbf{B}$ is supplied by a system not shown. Assume that the velocity is uniform across the channel and neglect end and edge effects and the magnetic field induced by current in the fluid. Find an equivalent electric circuit as seen from the electrical terminals and evaluate the circuit parameters.

12.33. In the system of Fig. 12P.33 an MHD generator is to be used to charge a capacitor. The MHD generator has a channel of constant cross-sectional area with the dimensions and arrangements shown in Fig. 12P.33. The working fluid has electrical conductivity σ and is incompressible and inviscid; it is constrained by external means to flow through the channel with a constant velocity v_0 that is uniform across the cross section. The constant uniform flux density B_0 is established by an external magnet not shown. Neglect the magnetic field due to current in the fluid and neglect fringing effects at the ends.

- Find the capacitor voltage V_c as a function of time and evaluate the final energy stored in capacitance C .
- Find the pressure difference supplied by the fluid source as a function of time and evaluate the total energy supplied by the fluid source.
- Account for the difference between your answers for energy in parts (a) and (b).

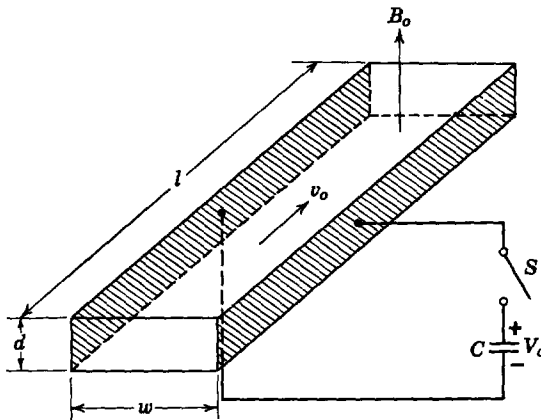


Fig. 12P.33

12.34. This problem is similar to the example worked in Section 12.2.1b. A U-tube of constant, rectangular cross section contains an inviscid, incompressible conducting fluid of mass density ρ and conductivity σ . The fluid has a total length l between the two surfaces which are open to atmospheric pressure as illustrated in Fig. 12P.34a. A conduction type MHD machine of length l_1 is inserted at the bottom of the U-tube. The details of the MHD channel are illustrated in Fig. 12P.34b. Neglect end effects and the magnetic field due to current flow in the fluid. The system parameters are

$$l = 1 \text{ m}, \quad l_1 = 0.1 \text{ m}, \quad w = 0.01 \text{ m}, \quad d = 0.01 \text{ m}, \\ B_0 = 2 \text{ Wb/m}^2, \quad V = 0.001 \text{ V}, \quad g = 9.8 \text{ m/sec}^2.$$

The fluid is mercury with constants $\sigma = 10^6$ mhos/m, $\rho = 1.36 \times 10^4$ kg/m³. With the system in equilibrium, with switch S open, $x_a = x_b = 0$; switch S is closed at $t = 0$. Calculate the ensuing transients in fluid position x_a and electrode current i . Sketch and label curves of these transients.

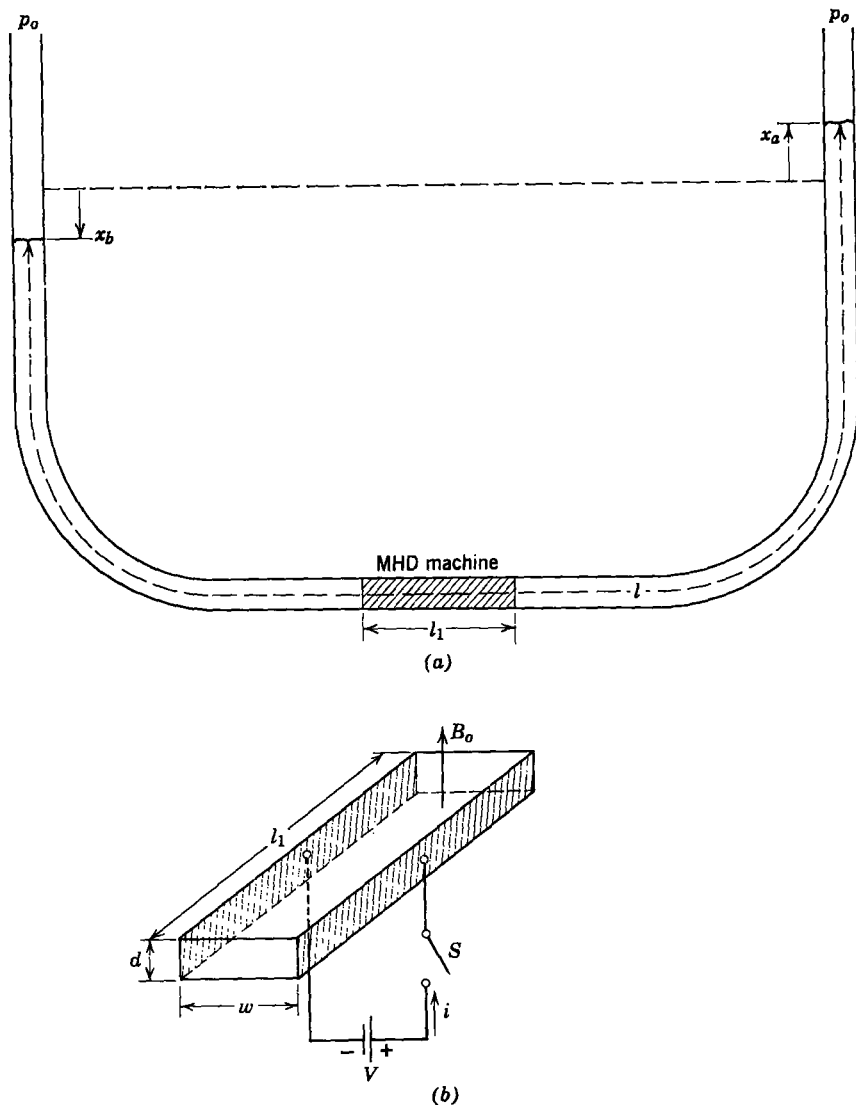


Fig. 12P.34

12.35. An incompressible inviscid conducting liquid fills the conduit shown in Fig. 12P.35. A current density J_o (known constant) flows through the fluid over a length L of the channel. This section of the fluid is also subjected to a magnetic field produced by a magnetic circuit. (The gap D is the only portion of the circuit where $\mu \neq \infty$.) This magnetic field is produced by two N turn coils wound as shown. The liquid has two free surfaces denoted by x and y . When the fluid is stationary, $x = L/2$ and $y = L/2$. In the regions of the free surfaces, electrodes carrying the currents I_1 and I_2 are arranged as shown. It is seen that the resistance

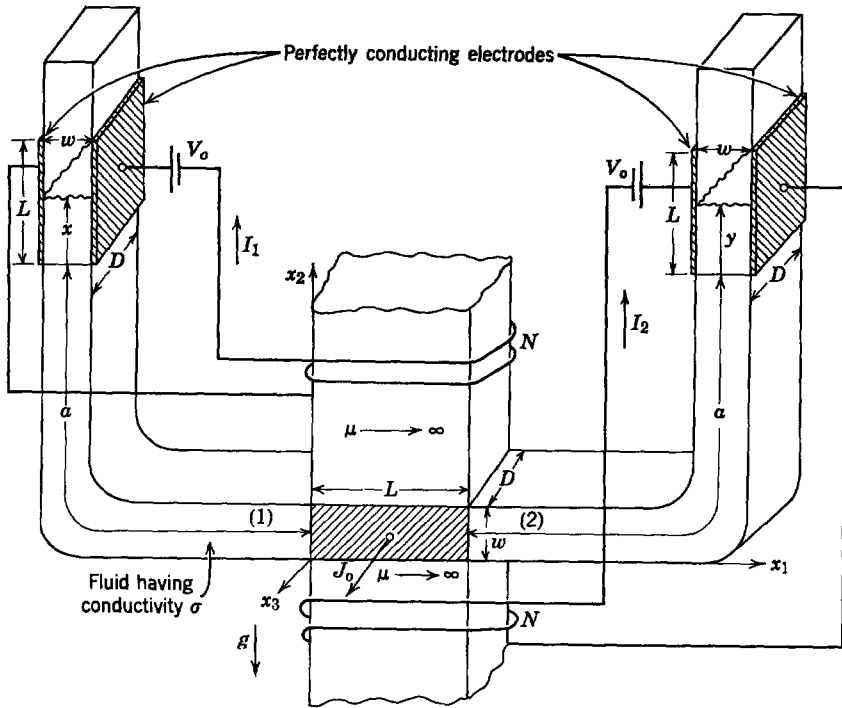


Fig. 12P.35

of the circuits is related to the heights of the interfaces. Find an equation for $x(t)$ that describes the system. Specifically, when $t = 0$, $x = L$, and $dx/dt = 0$, what is $x(t)$?

Assumptions

- (a) v is directed along the channel and is independent of cross-sectional position.
- (b) Ignore $L(dI_1/dt)$ and $L(dI_2/dt)$ (the rates of change with time are slow because of the fluid inertia).
- (c) Ignore the magnetic field produced by J_0 .

12.36. Rework the Alfvén-wave problem in cylindrical geometry in Section 12.2.3 with the following change in the boundary condition at $z = 0$. The end of the tube at $z = 0$ is fixed, rigid, and perfectly conducting. Your answer should consist of solutions for v_θ , B_θ , J_z , and J_r which are similar to (12.2.121)–(12.2.124). The functional dependences on z will be different in your answer because of the different boundary conditions.

12.37. A perfectly conducting inviscid fluid is bounded on the right by a perfectly conducting rigid wall, as shown in Fig. 12P.37. On the left a perfectly conducting plate also makes electrical contact with the liquid while being forced to execute an oscillatory motion in the x -direction. The system is immersed in a magnetic field so that when there are no motions $\mathbf{B} = B_0 \mathbf{i}_z$, where B_0 is a constant.

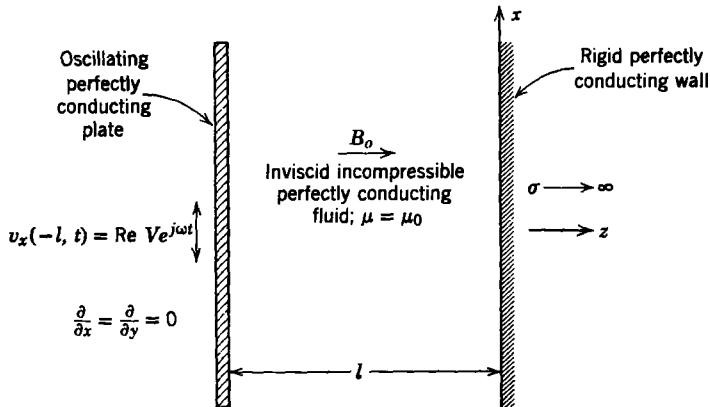


Fig. 12P.37

- Find the equations of motion, which together predict the transverse fluid velocity $v_x(z, t)$ and field intensity $H_x(z, t)$.
- Use appropriate boundary conditions to find $H_x(z, t)$ in the sinusoidal steady state.
- Compute the current density implied by (b). If you were to do this experiment, how would you construct the walls of the container that are parallel to the x - z plane? Explain in words why the fluid can transmit shearing motions even though it lacks viscosity.

Chapter 13

ELECTROMECHANICS OF COMPRESSIBLE, INVISCID FLUIDS

13.0 INTRODUCTION

In this chapter we introduce the additional law (conservation of energy) and constituent relations necessary to describe mathematically a compressible, inviscid fluid. This more general model is then used to study electromechanical interactions. Attention is focused on the effects of compressibility on the MHD machine analyzed in Chapter 12 and on how magnetic fields can affect the propagation of longitudinal disturbances (sound waves) in a compressible fluid.

13.1 INVISCID, COMPRESSIBLE FLUIDS

Cases of electromechanical coupling with fluids that have appreciable compressibility are found in MHD systems which use ionized gases as working fluids. We have chosen a perfect gas as our model of a compressible fluid. Although alternative models can be used, the principal phenomena that we shall study also occur in systems for which other models are appropriate.

It is a well-known fact that when work is done to compress a gas the temperature increases. This is an indication that the mechanical work of compression has been stored as internal (thermal) energy in the gas. The strong coupling between thermal and mechanical energy in a gas will necessitate the inclusion of the conservation of energy as one of the fundamental equations; and it will also require that we specify thermal and mechanical equations of state as constituent relations for the fluid.

The compressible fluids we deal with will obey the conservation of mass as

derived and discussed in Section 12.1.2. The differential form of the conservation of mass is (12.1.11)

$$\frac{D\rho}{Dt} = -\rho(\nabla \cdot \mathbf{v}), \quad (13.1.1)$$

where (D/Dt) is the substantial derivative defined in (12.1.5)

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla). \quad (13.1.2)$$

The integral form expressing conservation of mass is (12.1.8)

$$\oint_S (\rho \mathbf{v} \cdot \mathbf{n}) da = - \frac{d}{dt} \int_V \rho dV. \quad (13.1.3)$$

The surface S encloses the volume V and $\mathbf{n}$ is the outward-directed unit normal vector.

The derivation of the conservation of momentum (Newton's second law) in Section 12.1.3 was done without assuming that the mass density ρ was constant. Consequently, the resulting equations are equally applicable to compressible fluids. The differential form of the momentum equation is (12.1.14)

$$\rho \frac{D\mathbf{v}}{Dt} = \mathbf{F}, \quad (13.1.4)$$

where $\mathbf{F}$ is the force density applied to the fluid by all sources—mechanical, gravity, and electrical. The integral form of the momentum equation is (12.1.29)

$$\int_V \frac{\partial(\rho \mathbf{v})}{\partial t} dV + \oint_S \rho \mathbf{v}(\mathbf{v} \cdot \mathbf{n}) da = \int_V \mathbf{F} dV, \quad (13.1.5)$$

where the surface S encloses the volume V and $\mathbf{n}$ is the outward-directed unit normal vector.

After deriving the conservation of energy equation for a compressible fluid, we describe the appropriate constituent relations. These equations, along with the conservation of mass, the conservation of momentum, and appropriate boundary conditions, will allow us to solve problems in which there is electromechanical coupling with compressible fluids.

13.1.1 Conservation of Energy

In accounting for the conservation of energy we are concerned only with thermal and mechanical energy storage in a fluid. There will be energy input to the fluid from electromechanical conversion. The Poynting theorem can be written as a separate electromagnetic energy conservation equation; in

this system, however, which is quasi-static electromagnetically, this is unnecessary.

When a fluid is in motion, its kinetic energy density (joules per cubic meter) is $\frac{1}{2}\rho v^2$ and its kinetic energy per unit mass (joules per kilogram) is $\frac{1}{2}v^2$. This kinetic energy represents energy storage in the ordered or average motion of fluid particles. In a gas the particles also have random motion. The kinetic energy stored because of random motion is called thermal or internal energy. The internal energy per unit mass (joules/kilogram) is designated as u . The internal energy, like the velocity $\mathbf{v}$, is an Eulerian variable; thus the internal energy of the fluid in the vicinity of a point is specified by the value of u at that point. The internal energy density (joules per cubic meter) is ρu . The total energy per unit mass (kinetic and thermal) of the fluid at a point is $(u + \frac{1}{2}v^2)$; the energy density at any point in space is $\rho(u + \frac{1}{2}v^2)$.

Consider now a volume V enclosed by the surface S with outward-directed unit normal vector $\mathbf{n}$. The conservation of energy for the fluid within the volume is written

$$\int_V \frac{\partial}{\partial t} \rho(u + \frac{1}{2}v^2) dV + \oint_S \rho(u + \frac{1}{2}v^2) \mathbf{v} \cdot \mathbf{n} da = [\text{power input to fluid}]. \quad (13.1.6)$$

The first term on the left specifies the time rate of increase of energy stored by thermal and kinetic energy in the fluid that occupies the volume V at the instant of time in question. The second term on the left specifies the rate at which thermal and kinetic energy is transported across the surface S and out of the volume V . Thus the left side of (13.1.6) represents the energy that must be supplied by the total power input to the fluid in the volume V . This power input can be supplied by volume force densities, such as those of gravity and of electromagnetic origin, by volume heat generation, such as joule losses (J^2/σ) and viscous losses, by forces due to pressure that do work, and by heat conduction and radiation. An inviscid fluid model is being used, and viscous effects are ignored. Heat conduction and radiation will also be ignored because they have very small effects in practical situations on the electromechanical phenomena to be studied.

Before (13.1.6) can be specified in more detail and before a useful differential form can be obtained it is necessary to use the physical properties of the fluid to describe constituent relations.

13.1.2 Constituent Relations

A homogeneous, isotropic, compressible fluid at rest can sustain no shear stresses. Moreover, an inviscid fluid in motion can sustain no shear stresses.

Consequently, the mechanical stresses transmitted by an inviscid incompressible fluid are always normal and compressive; thus we define a pressure p exactly as we did in Section 12.1.4 with the result that the mechanical stress tensor is (12.1.34)

$$T_{ij}^m = -\delta_{ij}p. \quad (13.1.7)$$

The traction applied to a surface whose normal vector is $\mathbf{n}$ (12.1.37) is

$$\boldsymbol{\tau}^m = -p\mathbf{n} \quad (13.1.8)$$

and the mechanical force density (12.1.39) is

$$\mathbf{F}^m = -\nabla p. \quad (13.1.9)$$

We model the compressible fluid as a perfect gas. The mechanical equation of state for a perfect gas is

$$p = \rho RT, \quad (13.1.10)$$

where T is the temperature in degrees Kelvin and R is the gas constant for the particular gas in question with units joules per kilogram-°K. The gas constant R is obtained from the universal gas constant R_g as follows. The universal gas constant is

$$R_g = 8.31 \text{ J/mole-}^\circ\text{K}. \quad (13.1.11)$$

The gas constant R in mks units is obtained from

$$R = \frac{R_g}{M}, \quad (13.1.12)$$

where M is the mass of one mole of the gas in kilograms. This is simply the molecular weight multiplied by 10^{-3} ; for example, consider Argon, which has a molecular weight of 39.9. The gas constant for Argon is thus

$$R = \frac{8.31}{39.9 \times 10^{-3}} = 208 \text{ J/kg-}^\circ\text{K}. \quad (13.1.13)$$

Equation 13.1.10 is conventionally called a mechanical equation of state. Because we must consider internal energy storage in the gas, we must also specify a thermal equation of state that relates the internal energy storage to the variables of the system.* For a perfect gas the internal energy is a function of temperature alone and is conventionally expressed as

$$du = c_v dT, \quad (13.1.14)$$

where c_v is the specific heat capacity at constant volume with units joules per kilogram-°K. Equation 13.1.14 is expressed in differential form because,

* For a more thorough discussion see, for instance, W. P. Allis and M. A. Herlin, *Thermodynamics and Statistical Mechanics*, McGraw-Hill, New York, 1952, pp. 16–20 and 62–65.

over the range of temperatures of interest to us, c_v can be assumed constant; but over a wider range of temperature c_v is not constant and the variation must be accounted for in evaluating internal energy. Our purpose of examining electromechanical interaction phenomena will be served adequately by assuming that the specific heat capacity is constant.

Another specific heat capacity often useful and that we assume is constant in our treatment is the specific heat capacity at constant pressure c_p , which is related to c_v by the expression

$$c_p = c_v + R. \quad (13.1.15)$$

Yet another useful parameter is the ratio of specific heat capacities

$$\gamma = \frac{c_p}{c_v}. \quad (13.1.16)$$

In the ranges of temperature and pressure and for the gases of interest in this treatment the specific heat capacities vary appreciably but the ratio of specific heat capacities remain essentially constant.* Our assumption that all three parameters are constant is adequate for describing the phenomena resulting from electromechanical interactions.

Now that we have described the physical properties of inviscid, compressible fluids by the constituent relations of (13.1.9), (13.1.10), and (13.1.14) we shall recast the momentum and energy equations in more useful forms. We are concerned primarily with pressure and electromagnetic forces and we neglect the force of gravity.

The use of (13.1.9) for the mechanical force density in (13.1.4) yields the momentum equation in the form

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \mathbf{F}^e, \quad (13.1.17)$$

where $\mathbf{F}^e$ is the force density of electrical origin. To rewrite the integral form of the momentum equation we use

$$\int_V -\nabla p \, dV = \oint_S -p \mathbf{n} \, da \quad (13.1.18)$$

to write (13.1.5) in the form

$$\int_V \frac{\partial(\rho \mathbf{v})}{\partial t} \, dV + \oint_S \rho \mathbf{v}(\mathbf{v} \cdot \mathbf{n}) \, da = \oint_S -p \mathbf{n} \, da + \int_V \mathbf{F}^e \, dV. \quad (13.1.19)$$

* For a thorough discussion of the properties of gases, see, for example, H. B. Callen, *Thermodynamics*, Wiley, New York, 1960, pp. 324–333.

To write the energy equation (13.1.6) in more precise form we must specify the power input to the fluid within the volume V from all sources. Consider first the pressure forces that can be viewed as doing net work only at the surface of the volume V . Thus, because the pressure forces are compressive and normal to any surface, the power input to the fluid from pressure forces is

$$\oint_S -p \mathbf{v} \cdot \mathbf{n} \, da.$$

The use of the divergence theorem allows us to write this quantity as

$$\oint_S -\mathbf{n} \cdot (p\mathbf{v}) \, da = \int_V -\nabla \cdot (p\mathbf{v}) \, dV. \quad (13.1.20)$$

The electrical power input to the fluid within the volume V is the total rate at which electrical work is done on charged particles. This includes both the work done by electromagnetic forces and the electrical losses due to finite conductivity in the fluid. In all cases the electrical input power density is $\mathbf{J} \cdot \mathbf{E}$ and the total electrical power input is

$$\left[\begin{array}{c} \text{electrical power} \\ \text{input} \end{array} \right] = \int_V \mathbf{J} \cdot \mathbf{E} \, dV. \quad (13.1.21)$$

To interpret $\mathbf{J} \cdot \mathbf{E}$ as the input power density to the moving gas consider first a *magnetic field system* and denote with primes the variables defined in a reference frame fixed with respect to the fluid. Using (6.1.36), (6.1.37), and (6.1.38)*, we write

$$\mathbf{J} \cdot \mathbf{E} = \mathbf{J}' \cdot (\mathbf{E}' - \mathbf{v} \times \mathbf{B}'). \quad (13.1.22)$$

Then from the vector identity

$$\mathbf{J}' \cdot \mathbf{v} \times \mathbf{B}' = -\mathbf{J}' \times \mathbf{B}' \cdot \mathbf{v}$$

it follows that

$$\mathbf{J} \cdot \mathbf{E} = \mathbf{J}' \cdot \mathbf{E}' + \mathbf{J}' \times \mathbf{B}' \cdot \mathbf{v}. \quad (13.1.23)$$

The first term on the right is the electric power density that heats up the fluid. For a linear conductor $\mathbf{J}' = \sigma \mathbf{E}'$ and

$$\mathbf{J}' \cdot \mathbf{E}' = \frac{J'^2}{\sigma}.$$

The second term on the right of (13.1.23) is simply $\mathbf{F}^e \cdot \mathbf{v}$, which is the rate at which the magnetic force density does mechanical work on the fluid.

For an *electric field system* we use (6.1.54), (6.1.56), and (6.1.58)* to write $\mathbf{J} \cdot \mathbf{E}$ in the reference frame of the fluid as

$$\mathbf{J} \cdot \mathbf{E} = (\mathbf{J} + \rho' \mathbf{v}) \cdot \mathbf{E}'. \quad (13.1.24)$$

* See Table 6.1, Appendix G.

Expansion of this expression yields

$$\mathbf{J} \cdot \mathbf{E} = \mathbf{J}' \cdot \mathbf{E}' + \rho' \mathbf{E}' \cdot \mathbf{v}. \quad (13.1.25)$$

The first term on the right is the rate of heating of the fluid and the second term is the rate at which the electric force density $\rho' \mathbf{E}'$ does mechanical work on the fluid.

The use of (13.1.20) and (13.1.21) with (13.1.6) yields

$$\begin{aligned} \int_V \frac{\partial}{\partial t} [\rho(u + \tfrac{1}{2}v^2)] dV + \oint_S \rho(u + \tfrac{1}{2}v^2) \mathbf{v} \cdot \mathbf{n} da \\ = \int_V -\nabla \cdot (p\mathbf{v}) dV + \int_V \mathbf{J} \cdot \mathbf{E} dV. \end{aligned} \quad (13.1.26)$$

The divergence theorem is used to write

$$\oint_S \rho(u + \tfrac{1}{2}v^2) \mathbf{v} \cdot \mathbf{n} da = \int_V \nabla \cdot [\rho(u + \tfrac{1}{2}v^2) \mathbf{v}] dV. \quad (13.1.27)$$

Then all terms in (13.1.26) are volume integrals. The volume is arbitrary; thus the equation must hold for the differential volume dV .

$$\frac{\partial}{\partial t} [\rho(u + \tfrac{1}{2}v^2)] + \nabla \cdot [\rho(u + \tfrac{1}{2}v^2) \mathbf{v}] = -\nabla \cdot p\mathbf{v} + \mathbf{J} \cdot \mathbf{E}. \quad (13.1.28)$$

Expansion of the derivatives in the two terms on the left and use of the conservation of mass (13.1.1) yield the simplified result

$$\rho \frac{D}{Dt} (u + \tfrac{1}{2}v^2) = -\nabla \cdot (p\mathbf{v}) + \mathbf{J} \cdot \mathbf{E}. \quad (13.1.29)$$

Equations 13.1.26 and 13.1.29 are convenient forms that express the conservation of energy for time-varying situations. Many important problems involve steady flow, in which case $(\partial/\partial t = 0)$ and (13.1.26) simplifies to

$$\oint_S \rho(u + \tfrac{1}{2}v^2) \mathbf{v} \cdot \mathbf{n} da = \int_V -\nabla \cdot (p\mathbf{v}) dV + \int_V \mathbf{J} \cdot \mathbf{E} dV \quad (13.1.30)$$

and (13.1.29) simplifies to

$$\rho(\mathbf{v} \cdot \nabla)(u + \tfrac{1}{2}v^2) = -\nabla \cdot (p\mathbf{v}) + \mathbf{J} \cdot \mathbf{E}. \quad (13.1.31)$$

This last equation is conventionally written in a different form by expanding the first term on the right

$$\nabla \cdot (p\mathbf{v}) = (\mathbf{v} \cdot \nabla)p + p(\nabla \cdot \mathbf{v}). \quad (13.1.32)$$

The use of the conservation of mass to eliminate $\nabla \cdot \mathbf{v}$ yields

$$\nabla \cdot (p\mathbf{v}) = (\mathbf{v} \cdot \nabla)p - \frac{p}{\rho}(\mathbf{v} \cdot \nabla)\rho. \quad (13.1.33)$$

Recognizing that

$$(\mathbf{v} \cdot \nabla) \frac{p}{\rho} = \frac{1}{\rho}(\mathbf{v} \cdot \nabla)p - \frac{p}{\rho^2}(\mathbf{v} \cdot \nabla)\rho.$$

We write (13.1.33) in the form

$$\nabla \cdot (p\mathbf{v}) = \rho(\mathbf{v} \cdot \nabla) \frac{p}{\rho}$$

and (13.1.31) becomes

$$\rho(\mathbf{v} \cdot \nabla) \left(u + \frac{p}{\rho} + \frac{1}{2}v^2 \right) = \mathbf{J} \cdot \mathbf{E}. \quad (13.1.34)$$

This expression is simplified further by defining the *specific enthalpy* h as

$$h = u + \frac{p}{\rho} = u + RT \quad (13.1.35)$$

or, in differential form,

$$dh = du + R dT = (c_v + R) dT = c_p dT. \quad (13.1.36)$$

Thus (13.1.34) is written as

$$\rho(\mathbf{v} \cdot \nabla)(h + \frac{1}{2}v^2) = \mathbf{J} \cdot \mathbf{E}. \quad (13.1.37)$$

This equation is in a form that emphasizes the electromechanical aspects of a problem. It shows that electrical input power goes into enthalpy or kinetic energy in the gas. Thus for steady-flow problems enthalpy plays the role of energy storage in the gas other than kinetic energy.

13.2 ELECTROMECHANICAL COUPLING WITH COMPRESSIBLE FLUIDS

Now that we have completed the description of the mathematical models we shall use for inviscid, compressible fluids, we treat some steady-state and dynamic systems that emphasize the physical consequences of electromechanical coupling. The simplest examples that illustrate the electromechanical aspects of the problems are selected. It should be clear that many other effects will be significant in an engineering system that uses the basic phenomena that we describe. The details of these other effects are outside the scope of this work but they are well-documented in the literature.*

* See, for example, G. W. Sutton and A. Sherman, *Engineering Magnetohydrodynamics*, McGraw-Hill, New York, 1965.

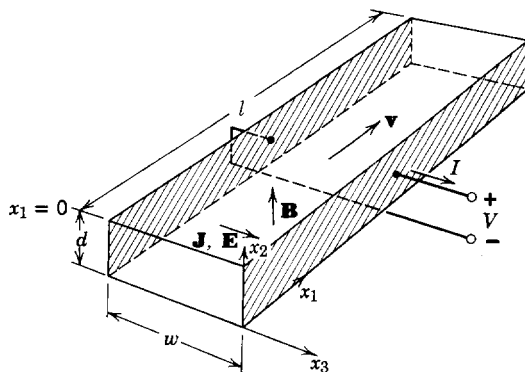


Fig. 13.2.1 A conduction-type MHD machine with constant-area channel.

13.2.1 Coupling with Steady Flow in a Constant-Area Channel

In this section we analyze the system of Fig. 13.2.1 which consists of a channel of constant cross-sectional area through which an electrically conducting gas flows with velocity $\mathbf{v}$. The electrical conductivity is high enough to justify a quasi-static magnetic field model. The two walls perpendicular to the x_2 -direction are electrical insulators and the two walls perpendicular to the x_3 -direction are highly conducting electrodes. A flux density $\mathbf{B}$ is produced in the x_2 -direction by external means not shown. The electrodes are connected to electrical terminals at which a voltage V and current I are defined. Note that this is the same configuration as that in Fig. 12.2.3 which was used in Section 12.2.1a for the analysis of electromechanical coupling with an inviscid, incompressible fluid. Thus the example in this section, when compared with that of Section 12.2.1a, highlights the effects of compressibility on the basic MHD interaction.

We assume that the l/w and l/d ratios of the channel are large enough that we can reasonably neglect end effects. Also it is assumed that the flow velocity is uniform over the cross section of the channel and that the magnetic field induced by current in the fluid is negligible compared with the applied field (low magnetic Reynolds number). Thus the magnetic flux density and electric field intensity are constant and uniform along the length of the channel

$$\mathbf{B} = \mathbf{i}_2 B_2, \quad (13.2.1)$$

$$\mathbf{E} = \mathbf{i}_3 E_3 = -\mathbf{i}_3 \frac{V}{w}, \quad (13.2.2)$$

and the velocity and current density are given by

$$\mathbf{v} = \mathbf{i}_1 v_1, \quad (13.2.3)$$

$$\mathbf{J} = \mathbf{i}_3 J_3. \quad (13.2.4)$$

The velocity v_1 , current density J_3 , and the gas variables (p, ρ, T) are functions of x_1 but not of x_2 and x_3 . We assume that the gas has a constant, scalar electrical conductivity σ and consider only a steady-flow problem.

The equations that describe this essentially one-dimensional problem are obtained by simplifying equations already presented. From (13.1.1) we obtain the equation for the conservation of mass:

$$v_1 \frac{d\rho}{dx_1} + \rho \frac{dv_1}{dx_1} = 0. \quad (13.2.5)$$

The momentum equation is obtained from (13.1.17) with $\mathbf{F}^e = \mathbf{J} \times \mathbf{B}$:

$$\rho v_1 \frac{dv_1}{dx_1} = - \frac{dp}{dx_1} - J_3 B_2. \quad (13.2.6)$$

The conservation of energy (13.1.37) yields

$$\rho v_1 \frac{d}{dx_1} \left(h + \frac{1}{2} v_1^2 \right) = J_3 E_3. \quad (13.2.7)$$

The mechanical equation of state (13.1.10) is

$$p = \rho RT \quad (13.2.8)$$

and the thermal equation of state (13.1.36) is

$$dh = c_p dT. \quad (13.2.9)$$

Finally, Ohm's law for the moving gas is $\mathbf{J}' = \sigma \mathbf{E}'$ or*

$$J_3 = \sigma(E_3 + v_1 B_2). \quad (13.2.10)$$

In these equations a total space derivative is written because x_1 is the only independent variable.

The six equations (13.2.5) to (13.2.10) have six unknowns (p, ρ, T, h, v_1, J_3) that vary with x_1 . These equations are nonlinear and direct integration in a general form is not possible. The usual method of solution is to assume that all of the variables are known at the inlet and then to integrate the equations numerically to find the variables along the length of the channel.

The equations can be put in a form convenient for interpretation and numerical integration by finding *influence coefficients*. This process is one of essentially finding each space derivative as a function of the variables themselves. In the derivation of influence coefficients it is convenient to define the velocity of sound (see Section 13.2.3)

$$a = \sqrt{\gamma RT} \quad (13.2.11)$$

* Table 6.1, Appendix G or Section 6.3.1.

and the Mach number of the flow

$$M = \frac{v_1}{a}. \quad (13.2.12)$$

By manipulating (13.2.5) to (13.2.10) and using (13.2.11), (13.2.12), and the ratio of specific heat capacities γ (13.1.16) we obtain the influence coefficients in these forms

$$\frac{1}{v_1} \frac{dv_1}{dx_1} = - \frac{1}{\rho} \frac{d\rho}{dx_1} = \frac{[(\gamma - 1)E_3 + \gamma v_1 B_2]J_3}{(1 - M^2)\gamma p v_1}, \quad (13.2.13)$$

$$\frac{1}{T} \frac{dT}{dx_1} = \frac{[(1 - \gamma M^2)E_3 - \gamma M^2 v_1 B_2](\gamma - 1)J_3}{(1 - M^2)\gamma p v_1}, \quad (13.2.14)$$

$$\frac{1}{p} \frac{dp}{dx_1} = - \frac{\{(\gamma - 1)M^2 E_3 + [1 + (\gamma - 1)M^2]v_1 B_2\}\gamma J_3}{(1 - M^2)\gamma p v_1}, \quad (13.2.15)$$

$$\frac{1}{M^2} \frac{d(M^2)}{dx_1} = \frac{\{(\gamma - 1)(1 + \gamma M^2)E_3 + \gamma[2 - (\gamma - 1)M^2]v_1 B_2\}J_3}{(1 - M^2)\gamma p v_1}. \quad (13.2.16)$$

We first use these influence coefficients to draw some general conclusions about electromechanical interactions with a conducting gas and then solve a problem in some detail to assess the consequences of compressibility.

First, with reference to Fig. 13.2.1, consider the situation in which the system is acting as a *generator* along the length of the channel. In this case

$$E_3 < 0, \quad J_3 = \sigma(E_3 + v_1 B_2) > 0.$$

It is clear from (13.2.13) to (13.2.16) that we can distinguish two cases:

$$\begin{array}{ll} \text{subsonic flow} & M^2 < 1, \\ \text{supersonic flow} & M^2 > 1. \end{array}$$

For subsonic flow ($M^2 < 1$) (13.2.13) to (13.2.16) yield the results

$$\frac{dv_1}{dx_1} > 0, \quad \frac{d\rho}{dx_1} < 0, \quad \frac{dp}{dx_1} < 0, \quad \frac{dT}{dx_1} < 0, \quad \frac{d(M^2)}{dx_1} > 0.$$

These results show the curious property that with $\mathbf{J} \times \mathbf{B}$ in a direction to decelerate the gas the flow velocity actually increases. This is a direct result of compressibility. The temperature decreases rapidly enough for the enthalpy of the gas to supply both the energy fed into the electrical circuit and the energy necessary for the increasing kinetic energy.

For supersonic flow ($M^2 > 1$) (13.2.13) to (13.2.16) yield the results

$$\frac{dv_1}{dx_1} < 0, \quad \frac{d\rho}{dx_1} > 0, \quad \frac{dp}{dx_1} > 0, \quad \frac{dT}{dx_1} > 0, \quad \frac{d(M^2)}{dx_1} < 0.$$

In this case the fluid decelerates as would at first be expected because the $\mathbf{J} \times \mathbf{B}$ force density tends to decelerate the gas. At the same time, however, the increase in temperature indicates that the kinetic energy of the gas supplies both the electrical output power and the power necessary to increase the enthalpy of the gas.

In the subsonic case the Mach number increases and in the supersonic case it decreases. Both changes make the Mach number tend toward unity. It is clear from (13.2.13) to (13.2.16) that the derivatives go to infinity at $M^2 = 1$ and our model becomes inaccurate. The treatment of the flow in the vicinity of the Mach number of one is outside the scope of our discussion. Suffice it to say that for a subsonic flow that approaches Mach one the flow chokes, and a smooth transition to supersonic flow is possible only for a very special set of circumstances. For a supersonic flow that approaches Mach one a shock wave will form. A shock wave is a narrow region in which the gas variables change rapidly and the flow velocity changes from supersonic to subsonic. A more complete model of the gas than we have used is necessary for an analysis of shock waves. The additional constraint needed is the second law of thermodynamics.*

The operation of the system in Fig. 13.2.1 as a *pump* is somewhat more complicated. By operation as a pump (or accelerator) we mean that the terminal voltage has the polarity shown, and $v_1 > 0$, $J_3 < 0$. Thus electric power is fed into the channel, and the $\mathbf{J} \times \mathbf{B}$ force density is in a direction that tends to accelerate the gas. Whether it does accelerate depends on the results obtained from (13.2.13) to (13.2.16).

Consider first the subsonic flow ($M^2 < 1$). The requirement that $J_3 < 0$ imposes through (13.2.10) the requirement that

$$E_3 < -v_1 B_2.$$

This ensures that electric power will be put into the fluid. Equations 13.2.13 and 13.2.14 yield the qualitative sketches of Fig. 13.2.2*a*. The constant γ is always in the range $1 < \gamma < 2$; thus we must distinguish two possible curves for the temperature variation. It is evident from Fig. 13.2.2*a* that a $\mathbf{J} \times \mathbf{B}$ force density applied in a direction that tends to accelerate a gas flowing with subsonic velocity may actually decelerate the flow and heat the gas to a higher temperature. The curve of (dv_1/dx_1) also indicates that when the magnitude of J_3 is made large enough the flow velocity can be increased.

For supersonic flow ($M^2 > 1$) with $J_3 < 0$ and the terminal voltage set to the polarity indicated in Fig. 13.2.1 (13.2.13) and (13.2.14) yield the qualitative curves of Fig. 13.2.1*b*. The upper curve indicates that for small magnitudes

* For a thorough and lucid description of the many fluid-mechanical phenomena that can occur in one-dimensional steady flow see A. H. Shapiro, *The Dynamics and Thermodynamics of Compressible Fluid Flow*, Vol. I, Ronald, New York, 1953, pp. 73-264.

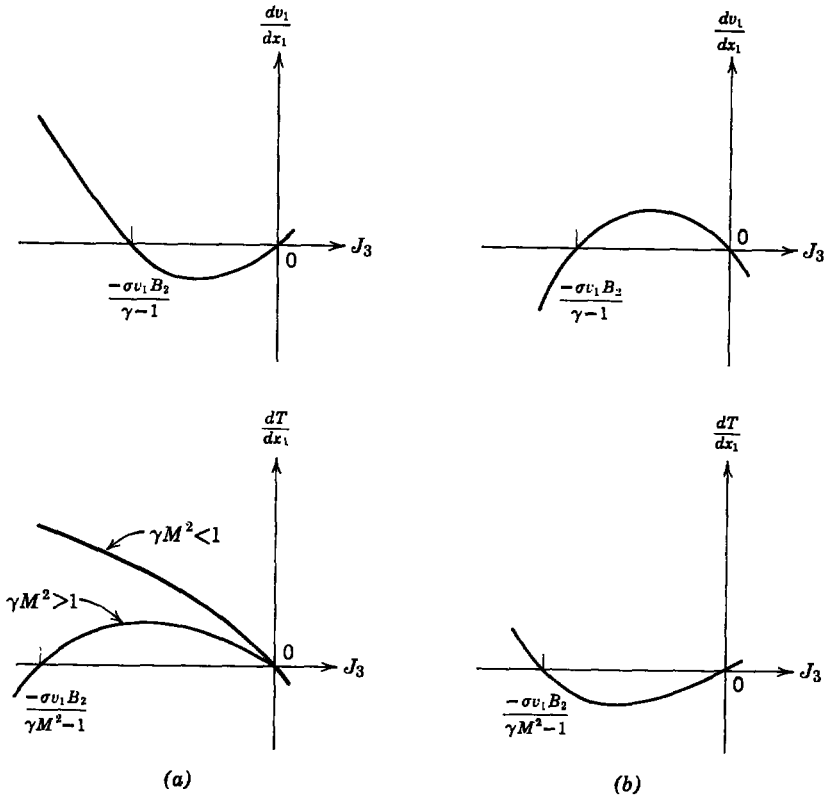


Fig. 13.2.2 Variation of velocity and temperature in a constant-area channel flow of a compressible fluid driven by a $\mathbf{J} \times \mathbf{B}$ force: (a) subsonic ($M^2 < 1$); (b) supersonic ($M^2 > 1$).

of J_3 the velocity is accelerated, but for too much driving current the velocity decreases.

Phenomena such as those demonstrated in Fig. 13.2.2 complicate the behavior of MHD devices that use compressible working fluids. Such phenomena are crucial in applications like plasma propulsion in which the object is to obtain a gas velocity as high as possible. When it is realized that these complications are predicted by an extremely simple model that neglects viscous and boundary layer effects, turbulence, and variation of electrical conductivity with temperature and is not complete enough to describe shock waves in supersonic flow, then we understand how complex the behavior of gaseous MHD systems can be and how we have to be extremely careful in obtaining the desired result from a particular model.

In order to understand how the behavior of a constant-area channel, MHD machine is affected by compressibility and to compare it with the incompressible analysis of Section 12.2.1a, a numerical example is presented.

For this example we assume gas properties typical of seeded combustion gases suitable for use in MHD generators:

$$R = 250 \text{ J/kg}^\circ\text{K}, \quad \gamma = 1.4, \quad c_p = 875 \text{ J/kg}^\circ\text{K}, \quad \sigma = 40 \text{ mhos/m}.$$

We assume that the inlet ($x_1 = 0$) conditions are known:

$$\begin{aligned} v_1(0) &= 500 \text{ m/sec}, & T(0) &= 3000^\circ\text{K}, \\ p(0) &= 4 \times 10^5 \text{ N/m}^2, & \rho(0) &= 0.534 \text{ kg/m}^3, \\ M^2(0) &= 0.238. \end{aligned}$$

The channel dimensions are assumed to be $w = 0.2 \text{ m}$, $d = 0.1 \text{ m}$, and $l = 0.95 \text{ m}$. The terminals are constrained with a constant voltage source

$$V = 150 \text{ V},$$

which constrains the electric field intensity to be constant along the length of the channel

$$E_3 = -750 \text{ V/m}.$$

The magnetic flux density is assumed to be

$$B_2 = 3 \text{ Wb/m}^2.$$

These numerical values lead to an inlet current density

$$J_3(0) = 3 \times 10^4 \text{ A/m}^2.$$

These numerical data are used with numerical integration of (13.2.13) and (13.2.14) and the mechanical equation of state and the definition of the Mach number to generate the normalized curves of Fig. 13.2.3. It is clear from these curves that the gas properties and flow velocity vary significantly over the length of the channel. Moreover, the rate of variation increases with x_1 . With reference to the curve of M^2 , it is evident that if the channel were made longer M^2 would pass through unity. Although the equations would give numerical answers, the solutions are physically impossible because the flow would choke and it would be impossible physically to make the Mach number greater than unity.

For this particular generator and these specified conditions the current density can be integrated numerically over the length of the channel to obtain the total terminal current

$$I = 4100 \text{ A}$$

Thus the generated power, that is, the power fed to the voltage source at the terminals is

$$P = 615,000 \text{ W}.$$

The total pressure drop through the channel is

$$\Delta p = 2.11 \times 10^5 \text{ N/m}^2,$$

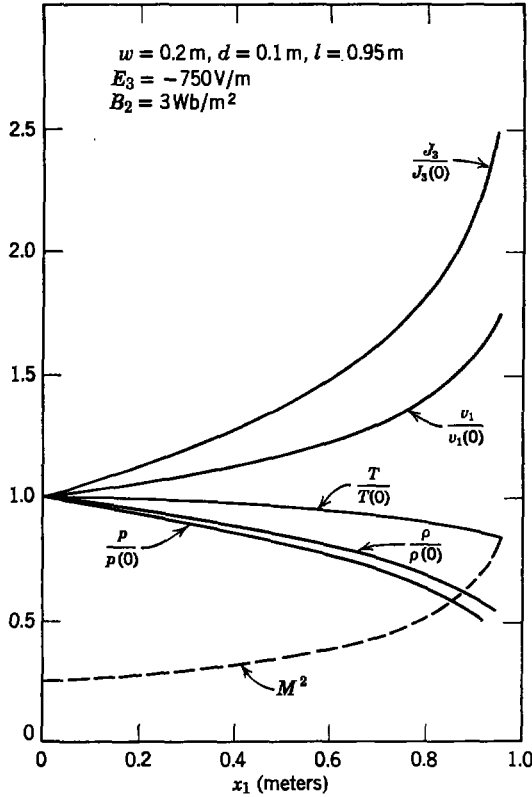


Fig. 13.2.3 Variation of properties along a constant-area channel with compressible flow acting as a generator.

or about 2.11 atm. It is interesting to compare these numbers with those of a generator that has an incompressible fluid operating with the same inlet velocity, electric field intensity, and flux density. Equations 12.2.19, 12.2.20, and 12.2.24 yield the results for the incompressible model:

$$I = 2850 \text{ A}, \quad P = 427,000 \text{ W}, \quad \Delta p = 0.95 \times 10^5 \text{ N/m}^2.$$

Comparison of these numbers with those of the compressible flow shows that with compressible flow the output current, power, and pressure drop are increased. Reference to the curves of Fig. 13.2.3 indicates that these increases are direct results of the increase in flow velocity with distance down the channel. The rather large difference in pressure drop is accounted for by the necessity to accelerate the gas flow in opposition to the decelerating $\mathbf{J} \times \mathbf{B}$ force.

This example has been presented to highlight some of the effects of compressibility. It must be emphasized that these results and the discussion hold only for *generator* operation with *subsonic* flow. For other conditions the effects can be grossly different. The techniques involved are the same, however.

13.2.2 Coupling with Steady Flow in a Variable-Area Channel

It is evident from the results of the preceding example that compressibility can limit the performance of a constant-area channel with MHD coupling; for example, with the conditions specified it would be impossible to operate the system with a larger pressure drop simply by lengthening the channel. Such limitations can be avoided by constructing the channel to make the cross-sectional area a function of distance (x_1) along the channel. When the channel area varies "slowly" enough with distance along the channel, we can use a *quasi-one-dimensional* model to describe the system with only one independent space variable. This technique is commonly employed in fluid mechanics* and magnetohydrodynamics,† and it yields quite accurate results in most applications. Its use in problems involving elastic media was introduced in Chapters 9 and 11. We present this technique in the context of a conduction-type MHD machine.

The system to be analyzed is illustrated in Fig. 13.2.4. It consists of a channel of rectangular cross-section but with the dimensions of the cross-section functions of the axial distance x_1 . A perfect gas having constant electrical conductivity flows with velocity $\mathbf{v}$ through the channel as indicated

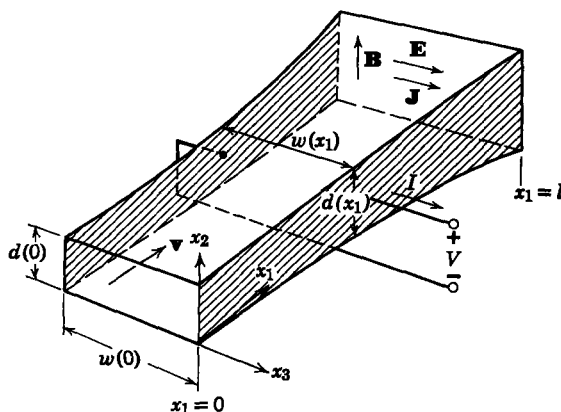


Fig. 13.2.4 MHD conduction machine with varying area.

* Shapiro, op. cit., pp. 73 and 74.

† Sutton and Sherman, op. cit., Chap. II.

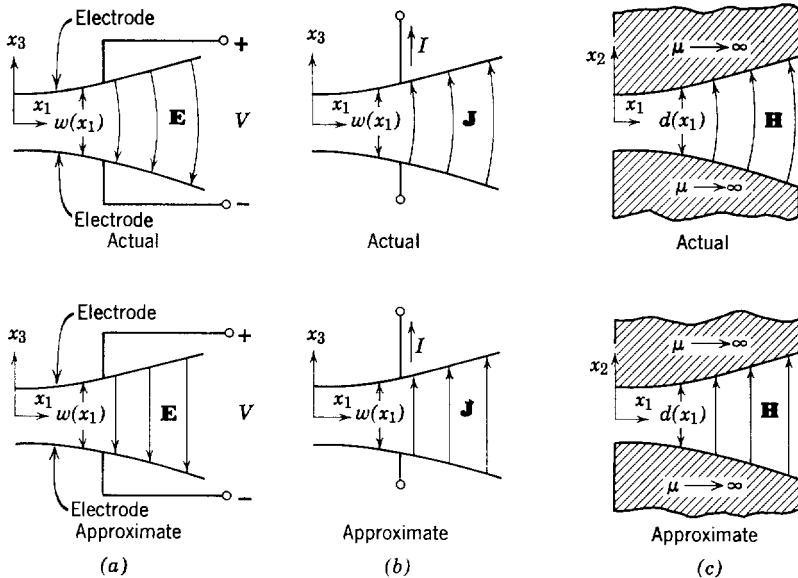


Fig. 13.2.5 Approximations for electromagnetic quantities in quasi-one-dimensional model: (a) electric field intensity; (b) current density; (c) magnetic field intensity.

in the figure. Two walls of the channel are insulators and two are electrodes that are connected to electrical terminals at which the terminal voltage V and terminal current I are defined with the polarities indicated.

We shall now develop the quasi-one-dimensional mathematical model for steady-flow in the system of Fig. 13.2.4. The derivation for non-steady flow is similar but more complex. The essential feature of the quasi-one-dimensional model is that all variables are assumed independent of x_2 and x_3 over a cross-section and they are thus functions only of x_1 , the distance along the channel. This basic assumption involves approximations that will be discussed as we proceed.

We are considering a steady-flow problem; thus $(\partial/\partial t = 0)$ and the electric field is conservative ($\nabla \times \mathbf{E} = 0$). The actual electric field lines between the electrodes will have the shapes shown qualitatively in Fig. 13.2.5a. In the quasi-one-dimensional model we assume that the field lines are only in the x_3 -direction and the field intensity has the value

$$\mathbf{E} = \mathbf{i}_3 E_3 = -\mathbf{i}_3 \frac{V}{w(x_1)}. \quad (13.2.17)$$

This approximation is also illustrated in Fig. 13.2.5a and is the same as the long-wave limit used in the treatment of elastic continua in Chapters 9 and 10. It should be evident that the quality of the approximation improves as (dw/dx_1) becomes smaller.

The current density $\mathbf{J}$ will have the actual configuration shown in Fig. 13.2.5b. In the quasi-one-dimensional model we assume that $\mathbf{J}$ is in the x_3 -direction:

$$\mathbf{J} = \mathbf{i}_3 J_3 \quad (13.2.18)$$

and that J_3 is a function of x_1 only. This approximation is illustrated in Fig. 13.2.5b.

We neglect the magnetic field induced by current flow in the gas (low magnetic Reynolds number), thus within the gas $\nabla \times \mathbf{H} = 0$. For illustration purposes we assume infinitely permeable pole pieces that conform to the insulating walls of Fig. 13.2.4; consequently, the actual magnetic field intensity appears as in Fig. 13.2.5c. In the quasi-one-dimensional approximation the magnetic field intensity (and flux density because $\mathbf{B} = \mu_0 \mathbf{H}$ in the gas) is in the x_2 -direction and given by

$$\mathbf{H} = \mathbf{i}_2 \frac{F}{d(x_1)}, \quad (13.2.19)$$

where F is the mmf (ampere-turns) applied by external means between the pole pieces. Thus

$$\mathbf{B} = \mathbf{i}_2 B_2(x_1) = \mathbf{i}_2 \frac{\mu_0 F}{d(x_1)}. \quad (13.2.20)$$

This approximation, also illustrated in Fig. 13.2.5c, improves in validity as (dd/dx_1) decreases.

Although Fig. 13.2.5c represents a reasonable method for establishing the flux density, the magnetic material may not conform to the insulating walls or the field may be excited by air-core coils. In these cases we still assume that there is only an x_2 -component of $\mathbf{B}$ and that it varies only with x_1 in a manner determined by the method of excitation. Thus $B_2(x_1)$ is most often a function independently set in the analysis of an MHD device.

It is clear from (13.2.17), (13.2.18), and (13.2.20) and Fig. 13.2.5 that fringing fields at the ends of the channel are neglected. It should also be clear that the approximate field quantities (13.2.17), (13.2.18) and (13.2.20) do not satisfy the required electromagnetic equations exactly. This is a consequence of the approximation.

In the quasi-one-dimensional model we assume that all the gas properties (p , ρ , T) are uniform over a cross section and functions only of x_1 . Moreover, we assume that the x_1 -component of the velocity is uniform over a cross section. We neglect the effects of transverse velocity components. Thus, in view of (13.2.17), (13.2.18) and (13.2.20), we write Ohm's law as

$$J_3 = \sigma(E_3 + v_1 B_2). \quad (13.2.21)$$

Use of the small volume between planes at x_1 and at $x_1 + \Delta x_1$, as illustrated in Fig. 13.2.6 with the integral form of the conservation of mass (13.1.3) and the assumption of the uniformity of v_1 over a cross section gives

$$\oint (\rho \mathbf{v} \cdot \mathbf{n}) da = \rho(x_1 + \Delta x_1)v_1(x_1 + \Delta x_1)A(x_1 + \Delta x_1) - \rho(x_1)v_1(x_1)A(x_1) = 0, \quad (13.2.22)$$

where A is the cross-sectional area given by

$$A(x_1) = w(x_1) d(x_1). \quad (13.2.23)$$

We divide (13.2.22) by Δx_1 and take the limit as $\Delta x_1 \rightarrow 0$ to obtain

$$\frac{d(\rho v_1 A)}{dx_1} = 0. \quad (13.2.24)$$

This is the differential form that expresses conservation of mass in the quasi-one-dimensional model.

In deriving the quasi-one-dimensional momentum equation it is often the practice to use a small volume, shown in Fig. 13.2.6, with the integral form of the momentum equation (13.1.5). It is more direct, however, to recognize initially the assumptions that all gas properties and the x_1 -component of velocity are uniform over a cross section and that transverse components of velocity have negligible effects and to write the x_1 -component of (13.1.4)

$$\rho v_1 \frac{dv_1}{dx_1} = -\frac{dp}{dx_1} - J_3 B_2. \quad (13.2.25)$$

In this equation we have used (13.1.9) for the mechanical force density and $\mathbf{J} \times \mathbf{B}$ for the magnetic force density.

The same comments hold true for the conservation of energy. Recognizing the assumptions made, we can write the quasi-one-dimensional energy equation from (13.1.37) as

$$\rho v_1 \frac{d}{dx_1} (h + \frac{1}{2}v_1^2) = J_3 E_3. \quad (13.2.26)$$

In the quasi-one-dimensional model the equations of state (13.1.10) and (13.1.14) or (13.1.36) are unchanged from their general forms.

The quasi-one-dimensional model of MHD interactions in the variable-area channel of Fig. 13.2.4 consists of (13.2.17), (13.2.18), (13.2.20), (13.2.21), (13.2.24), (13.2.25), (13.2.26), (13.1.10), and (13.1.36). This set of

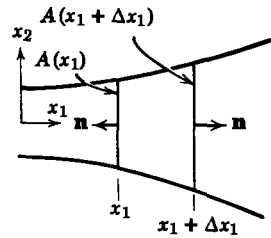


Fig. 13.2.6 Closed surface for derivation of conservation of mass equation for quasi-one-dimensional model.

coupled, nonlinear, differential equations can be used with specified boundary conditions to calculate how the gas properties, flow velocity, and electromagnetic quantities vary along the length of the channel. The equations are summarized in Table 13.2.1.

As is the case with compressible flow in a constant-area channel, (Section 13.2.1) it is useful to derive influence coefficients that express each derivative as a function of the variables themselves. These influence coefficients are useful for general interpretation of properties and for numerical integration of the equations.

By solving the equations in Table 13.2.1 for each of the derivatives separately we arrive at the following set of influence coefficients:

$$\frac{1}{v_1} \frac{dv_1}{dx_1} = \frac{1}{(1 - M^2)} \left\{ [(\gamma - 1)E_3 + \gamma v_1 B_2] \frac{J_3}{\gamma p v_1} - \frac{1}{A} \frac{dA}{dx_1} \right\}, \quad (13.2.27)$$

$$\frac{1}{\rho} \frac{d\rho}{dx_1} = \frac{1}{(1 - M^2)} \left\{ -[(\gamma - 1)E_3 + \gamma v_1 B_2] \frac{J_3}{\gamma p v_1} + \frac{M^2}{A} \frac{dA}{dx_1} \right\}, \quad (13.2.28)$$

$$\frac{1}{T} \frac{dT}{dx_1} = \frac{(\gamma - 1)}{(1 - M^2)} \left\{ [(1 - \gamma M^2)E_3 - \gamma M^2 v_1 B_2] \frac{J_3}{\gamma p v_1} + \frac{M^2}{A} \frac{dA}{dx_1} \right\}, \quad (13.2.29)$$

$$\frac{1}{p} \frac{dp}{dx_1} = \frac{\gamma}{(1 - M^2)} \left\{ -[(\gamma - 1)M^2 E_3 + \{1 + (\gamma - 1)M^2\} v_1 B_2] \right. \\ \left. \times \frac{J_3}{\gamma p v_1} + \frac{M^2}{A} \frac{dA}{dx_1} \right\}, \quad (13.2.30)$$

$$\frac{1}{M^2} \frac{dM^2}{dx_1} = \frac{\gamma}{(1 - M^2)} \left\{ [(\gamma - 1)(1 + \gamma M^2)E_3 + \gamma \{2 - (\gamma - 1)M^2\} v_1 B_2] \frac{J_3}{\gamma p v_1} \right. \\ \left. - \frac{[2 + (\gamma - 1)M^2]}{A} \frac{dA}{dx_1} \right\}. \quad (13.2.31)$$

These influence coefficients should be compared with those of (13.2.13) to (13.2.16) for the constant-area channel. It is clear that when $(dA/dx_1 = 0)$ the two sets of influence coefficients become identical.

It is also clear from (13.2.27) to (13.2.31) that for any set of conditions the derivative of any variable can be made to have either sign and any magnitude by adjusting the factor (dA/dx_1) . Thus the tendency of the flow to approach Mach one in a constant-area channel can be counteracted by letting the area of the channel vary. In fact, by adjusting the area $A(x_1)$ such that the quantity in braces in (13.2.31) is zero all along the channel the Mach number can be held constant along the channel. It is also true that any of the other influence coefficients can be used to design a channel [fix $A(x_1)$] such that one property (v_1 , ρ , p , or T) is constant along the length of the channel.

Table 13.2.1 Summary of Quasi-One-Dimensional Equations for System of Fig. 13.2.4

Channel area	$A(x_1) = w(x_1) d(x_1)$	(13.2.23)
Electric field intensity	$E = i_3 E_3 = -i_3 \frac{V}{w(x_1)}$	(13.2.17)
Current density	$J = i_3 J_3$	(13.2.18)
Magnetic flux density	$B = i_2 B_2(x_1)$	(13.2.20)
Ohm's law	$J_3 = \sigma(E_3 + v_1 B_2)$	(13.2.21)
Conservation of mass	$\frac{d(\rho v_1 A)}{dx_1} = 0$	(13.2.24)
Conservation of momentum	$\rho v_1 \frac{dv_1}{dx_1} = -\frac{dp}{dx_1} - J_3 B_2$	(13.2.25)
Conservation of energy	$\rho v_1 \frac{d}{dx_1} (h + \frac{1}{2} v_1^2) = J_3 E_3$	(13.2.26)
Mechanical equation of state	$p = \rho RT$	(13.1.10)
Thermal equation of state	$dh = c_p dT$	(13.1.36)
Local sound velocity	$a = \sqrt{\gamma RT}$	(13.2.11)
Local Mach number	$M = \frac{v_1}{a}$	(13.2.12)

Although the influence coefficients of (13.2.27) to (13.2.31) are useful for examining general properties of the variable-area MHD machine and for numerical integration when necessary, some exact solutions are possible and they are best obtained by using the basic equations summarized in Table 13.2.1.

Before proceeding with an example of an exact solution of the equations it is useful to introduce a convention used in the analysis of gaseous MHD generators. This convention defines a loading factor K as

$$K = -\frac{E_3}{v_1 B_2}. \quad (13.2.32)$$

The use of the factor K in Ohm's law (13.2.21) yields

$$J_3 = (1 - K)\sigma v_1 B_2. \quad (13.2.33)$$

Thus, when $0 < K < 1$, electric energy is being extracted from the gas; otherwise it is being put into the gas. The power density extracted electrically from the gas [see (13.2.26)] is

$$p_e = -J_3 E_3 = K(1 - K)\sigma v_1^2 B_2^2. \quad (13.2.34)$$

Thus it is evident that maximum energy is extracted locally when $K = \frac{1}{2}$ or when the electric field intensity is one half $\mathbf{v} \times \mathbf{B}$. On a continuum basis this is the maximum output condition when the external impedance is made equal to internal impedance. In general, K can be a function of x_1 ; however, to achieve maximum power extraction along the channel, K should be kept close to the optimum value of one half. It is evident from (13.2.34) that the maximum power density that can be extracted electrically from the gas is

$$P_{e(\max)} = \frac{\sigma v_1^2 B_2^2}{4}. \quad (13.2.35)$$

We now set constraints suitable for obtaining an exact solution of the quasi-one-dimensional equations that describe the variable area MHD machine in Fig. 13.2.4. A set of constraints is selected to correspond closely to those used for analyzing MHD generators for large amounts of power (more than 100 MW). We present a normalized solution in literal form and then introduce numerical constants.

It is assumed that the values of all quantities are known at the inlet ($x_1 = 0$). We select the channel dimensions to achieve constant flow velocity v_1 , constant loading factor K , and constant-channel aspect ratio $[w(x_1)/d(x_1)]$. The requirements of constant K and constant aspect ratio are satisfied only if

$$B_2 \sim \frac{1}{d(x_1)}.$$

Thus we assume that the magnetic field is excited by using infinitely permeable pole pieces that conform to the insulating walls, as illustrated in Fig. 13.2.5c. It follows that the flux density B_2 is given by (13.2.20).

For the constraints that have been specified, with the loading factor K defined by (13.2.32) and the current density J_3 given by (13.2.33), the equations of Table 13.2.1 can be simplified to the following:

$$\frac{d(\rho A)}{dx_1} = 0, \quad (13.2.36)$$

$$\frac{dp}{dx_1} = -(1 - K)\sigma v_1 B_2^2, \quad (13.2.37)$$

$$\rho v_1 c_p \frac{dT}{dx_1} = -K(1 - K)\sigma v_1^2 B_2^2, \quad (13.2.38)$$

$$p = \rho RT. \quad (13.2.39)$$

Before solving for any variable as a function of x_1 , it is convenient to obtain relations between pairs of unknowns; for example, division of (13.2.38) by

(13.2.37) and simplification of the results yield

$$\rho c_p \frac{dT}{dx_1} = K \frac{dp}{dx_1}. \quad (13.2.40)$$

This equation can be written as

$$(\rho RT) \left(\frac{c_p}{R} \right) \frac{1}{T} \frac{dT}{dx_1} = (Kp) \frac{1}{p} \frac{dp}{dx_1}. \quad (13.2.41)$$

Using (13.2.39) and the fact that

$$\frac{c_p}{R} = \frac{\gamma}{\gamma - 1},$$

we integrate (13.2.41) to obtain the result that

$$\frac{p(x_1)}{p(0)} = \left[\frac{T(x_1)}{T(0)} \right]^{\gamma/[K(\gamma-1)]} \quad (13.2.42)$$

Note from (13.2.33) that when $K = 1$ no current flows and (13.2.42) reduces to the standard isentropic relation between temperature and pressure.*

We now use (13.2.42) with (13.2.39) to obtain the relation between temperature and density as

$$\frac{\rho(x_1)}{\rho(0)} = \left[\frac{T(x_1)}{T(0)} \right]^{[\gamma - K(\gamma-1)]/K(\gamma-1)} \quad (13.2.43)$$

The use of this result with (13.2.36) yields

$$\frac{A(x_1)}{A(0)} = \left[\frac{T(x_1)}{T(0)} \right]^{[K(\gamma-1) - \gamma]/K(\gamma-1)} \quad (13.2.44)$$

Because the aspect ratio (w/d) is constant, (13.2.44) yields the result

$$\frac{d(x_1)}{d(0)} = \frac{w(x_1)}{w(0)} = \left[\frac{A(x_1)}{A(0)} \right]^{1/2} \quad (13.2.45)$$

Finally, the definition of Mach number M in Table 13.2.1 with the constraint of constant velocity yields the relation between the square of the Mach number and the temperature:

$$\frac{M^2(x_1)}{M^2(0)} = \left[\frac{T(x_1)}{T(0)} \right]^{-1}. \quad (13.2.46)$$

Now that we have relations among the unknowns it is necessary to obtain a solution for only one of the unknowns as a function of x_1 . It is easiest to

* Allis and Herlin, op. cit., p.78.

do this for the temperature by using (13.2.38), which we rewrite as

$$\frac{dT}{dx_1} = \frac{-K(1-K)\sigma v_1 B_2^2}{\rho c_p}, \quad (13.2.47)$$

From (13.2.36) $\rho A = \text{constant}$ and from (13.2.20) and (13.2.45) $B_2^2 A = \text{constant}$.

Thus (13.2.47) becomes

$$\frac{dT}{dx_1} = - \frac{K(1-K)\sigma v_1 B_2^2(0)}{\rho(0)c_p}. \quad (13.2.48)$$

The right side of this expression is constant and integration yields

$$T(x_1) - T(0) = - \frac{K(1-K)\sigma v_1 B_2^2(0)}{\rho(0)c_p} x_1. \quad (13.2.49)$$

By normalizing and rearranging this expression we obtain

$$\frac{T(x_1)}{T(0)} = 1 - \frac{(\gamma-1) K(1-K)\sigma v_1 B_2^2(0)}{\gamma p(0)} x_1. \quad (13.2.50)$$

We define the constant C_1 as

$$C_1 = \frac{(\gamma-1) K(1-K)\sigma v_1 B_2^2(0)}{\gamma p(0)}. \quad (13.2.51)$$

and rewrite (13.2.50) as

$$\frac{T(x_1)}{T(0)} = 1 - C_1 x_1. \quad (13.2.52)$$

We now use (13.2.42) to (13.2.46) to obtain the space variations of the other variables; thus

$$\frac{p(x_1)}{p(0)} = (1 - C_1 x_1)^{\gamma/[K(\gamma-1)]}, \quad (13.2.53)$$

$$\frac{\rho(x_1)}{\rho(0)} = (1 - C_1 x_1)^{[\gamma-K(\gamma-1)]/K(\gamma-1)}, \quad (13.2.54)$$

$$\frac{A(x_1)}{A(0)} = (1 - C_1 x_1)^{[K(\gamma-1)-\gamma]/K(\gamma-1)}, \quad (13.2.55)$$

$$\frac{d(x_1)}{d(0)} = \frac{w(x_1)}{w(0)} = (1 - C_1 x_1)^{[K(\gamma-1)-\gamma]/2K(\gamma-1)}, \quad (13.2.56)$$

$$\frac{M^2(x_1)}{M^2(0)} = (1 - C_1 x_1)^{-1}. \quad (13.2.57)$$

To complete the description of this generator we note from (13.2.17) and (13.2.32) that the terminal voltage with polarity defined in Fig. 13.2.4 is

$$V = Kv_1 B_2(x_1) w(x_1) \quad (13.2.58)$$

and constant. From (13.2.33) the current density is

$$J_3 = (1 - K)\sigma v_1 B_2(x_1). \quad (13.2.59)$$

The total terminal current is

$$I = \int_0^l J_3 d(x_1) dx_1 = \int_0^l (1 - K)\sigma v_1 B_2(x_1) d(x_1) dx_1. \quad (13.2.60)$$

From (13.2.20) we have

$$B_2(x_1) d(x_1) = B_2(0) d(0); \quad (13.2.61)$$

thus (13.2.60) is written as

$$I = \int_0^l (1 - K)\sigma v_1 B_2(0) d(0) dx_1. \quad (13.2.62)$$

In this expression the integrand is constant, which indicates that each element dx_1 along the length makes the same contribution to the total current. Integration of (13.2.62) yields

$$I = (1 - K)\sigma v_1 B_2(0) d(0)l. \quad (13.2.63)$$

It is interesting to note by reference to Section 12.2.1 that this is the same as the current output from a constant-area channel of depth $d(0)$, width $w(0)$, and length l , using an incompressible fluid with conductivity σ and velocity v_1 in the presence of a uniform flux density of value $B_2(0)$.

It will be instructive to make the input dimensions and variables the same as those of the constant-area channel in Section 13.2.1 and to compare the performance of the variable area and constant-area channels. Thus we set

$$\begin{array}{lll} R = 250 \text{ J/kg}^\circ\text{K}, & \gamma = 1.4, & c_p = 875 \text{ J/kg}^\circ\text{K}, \\ \sigma = 40 \text{ mhos/m}, & v_1 = 500 \text{ m/sec}, & T(0) = 3000^\circ\text{K}, \\ p(0) = 4 \times 10^5 \text{ N/m}^2, & \rho(0) = 0.534 \text{ kg/m}^3, & M^2(0) = 0.238, \\ w(0) = 0.2 \text{ m}, & d(0) = 0.1 \text{ m}, & l = 0.95 \text{ m}, \\ K = \frac{1}{2}, & V = 150 \text{ V}, & B_2(0) = 3 \text{ Wb/m}^2. \end{array}$$

First we use (13.2.51) to calculate the constant C_1 :

$$C_1 = 0.0322/\text{m}.$$

Then the expressions for the variables follow from (13.2.52) to (13.2.57):

$$\begin{aligned}\frac{T(x_1)}{T(0)} &= (1 - 0.0322x_1), & \frac{p(x_1)}{p(0)} &= (1 - 0.0322x_1)^7, \\ \frac{\rho(x_1)}{\rho(0)} &= (1 - 0.0322x_1)^6, & \frac{A(x_1)}{A(0)} &= \frac{1}{(1 - 0.0322x_1)^6}, \\ \frac{d(x_1)}{d(0)} = \frac{w(x_1)}{w(0)} &= \frac{1}{(1 - 0.0322x_1)^3}, & M^2(x_1) &= \frac{0.238}{(1 - 0.0322x_1)}.\end{aligned}$$

These variations with x_1 are plotted in Fig. 13.2.7. Compare the curves in this figure with those in Fig. 13.2.3 to learn how the slight variation of the channel area can reduce the changes in properties along the channel. Because the Mach number has changed so slightly over the length of the channel, the channel can be made much longer without reaching Mach one. This was not the case for the constant-area channel.

Further comparisons can be made in the constant-area channel with both compressible and incompressible fluids. Assuming the same inlet dimensions and properties for each of the three cases, we list several quantities in Table 13.2.2. Note that the constant-area generator with incompressible fluid produces the same power as the variable-area generator but with a larger pressure drop, and that the constant-area generator with compressible fluid produces the most power. This is due to the acceleration of the gas down the channel, as indicated by Fig. 13.2.3. This small increase in power occurs at the expense of a large increase in pressure and temperature drops

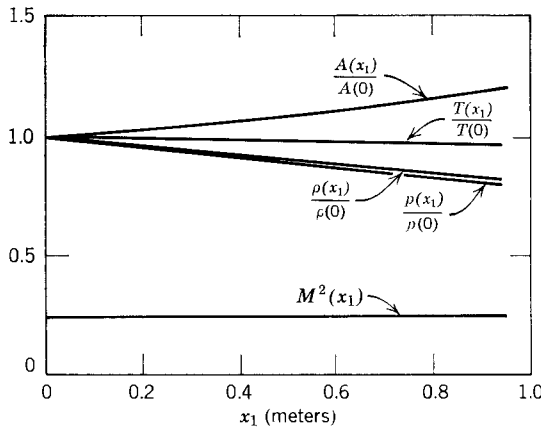


Fig. 13.2.7 Variation of properties along a variable-area channel designed to have constant velocity and constant loading factor while acting as a generator.

Table 13.2.2 Properties of MHD Generators

	Constant-Area Incompressible Fluid	Constant-Area Compressible Fluid	Variable-Area Constant Velocity Constant Loading Factor
Terminal voltage (volts)	150	150	150
Terminal current (amperes)	2,850	4,100	2,850
Power output (watts)	427,000	615,000	427,000
Pressure drop (newtons per square meter)	0.95×10^5	2.11×10^5	0.80×10^5
Temperature drop (degrees Kelvin)	...	420	93

over the variable-area generator. Although it is beyond the scope of this book, it is worthwhile to remark that this increase in power output from the constant-area channel results in the generation of considerable entropy which makes the energy in the exhaust fluid less available than with the variable-area channel.

In our analysis of the variable area channel we defined a set of constraints that allowed the complete solution of the differential equations in closed form. Several other sets of constraints allow direct integration of the equations. For still others numerical integration is necessary for solution.

It must be recognized that when a set of constraints is selected and closed-form solutions are obtained the design of a generator is fixed. In our example this means we specify the dimensions $[d(x_1)]$, and $w(x_1)]$. Now, if we wish to operate this channel with a different set of inlet conditions, magnetic flux density, and/or applied voltage, we can no longer, in general, determine how the properties vary along the channel by literal integration. Instead, we must integrate numerically. Thus, if we wanted to fix the inlet properties to the channel we designed in our example and to find the output current and power as a function of load resistance for the range from open-circuit to short circuit, our solution in closed form would represent only one point on the curve. The remainder of the points would have to be found by numerical integration.

The preceding analysis of a variable-area MHD machine with a compressible working fluid is the basic technique in the study of electromechanical coupling in conduction-type MHD generators. Several types of machine have been built or proposed.* A cutaway drawing of one machine is shown in Fig. 13.2.8 and a photograph in Fig. 13.2.9.

* T. R. Brogan, "MHD Power Generation," *IEEE Spectrum*, 1, 58-65 (February 1964).

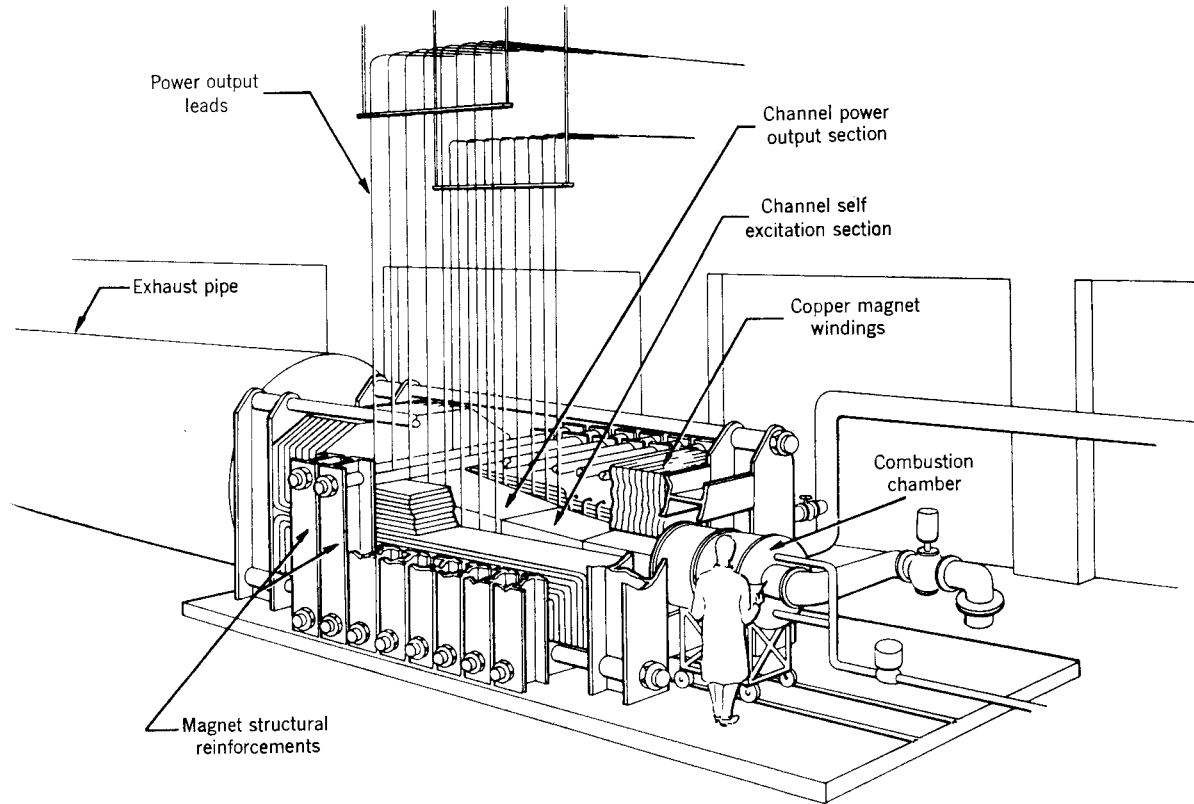


Fig. 13.2.8 Cutaway drawing of Avco Mark V rocket-driven, self-excited MHD power generator. Oxygen and fuel are burned in the combustion chamber to create a 5000°F electrically conducting gas which flows through the channel, where it interacts with the magnetic field to generate power. The magnet coil is excited by part of the generator output. For a gross power output of 31.3 MW, 7.7 MW are used to energize the field coils. (Courtesy of Avco-Everett Research Laboratory, a division of Avco Corporation.)

Courtesy of Textron Corporation. Used with permission.

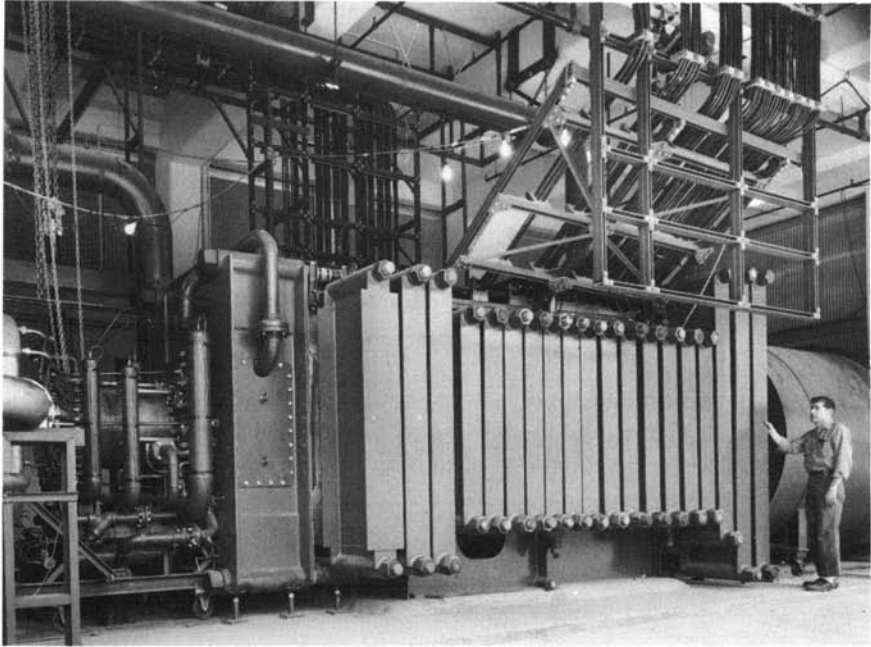


Fig. 13.2.9 Photograph of Avco Mark V generator described in Fig. 13.2.8. (Courtesy of Avco-Everett Research Laboratory, a division of Avco Corporation.)

Courtesy of Textron Corporation. Used with permission.

13.2.3 Coupling with Propagating Disturbances

Recall from Section 12.2.3 that in the analysis of Alfvén waves propagating through an incompressible fluid of high electrical conductivity the fluid motion was entirely transverse. Even though the assumption of incompressibility was made, it was not necessary for the type of fluid motion described. Thus Alfvén waves are also found in an inviscid gas of high electrical conductivity.

Because a gas is compressible, it will also transmit longitudinal (acoustic) waves that are very much like the longitudinal elastic waves analyzed in Chapter 11. The propagation of acoustic waves in a gas can be affected by bulk electromechanical coupling when the gas has high electrical conductivity and is immersed in a transverse magnetic field. These modified disturbances are called magnetoacoustic waves. The same phenomena also occur in liquids because liquids are slightly compressible. The effect of bulk electromechanical coupling on acoustic waves in a liquid, however, is much less pronounced than in a gas. Consequently, we use our mathematical model of a gas to describe acoustic waves first and then to describe magnetoacoustic waves.

13.2.3a Acoustic Waves

As already stated, we shall study longitudinal disturbances, and thus we assume the rectangular channel in Fig. 13.2.10, which has rigid walls perpendicular to the x_2 - and x_3 -axes and infinite length in the x_1 -direction. At $x_1 = 0$ a close-fitting piston, perpendicular to the x_1 -direction, can be driven in the x_1 -direction by a mechanical source. The channel is filled with a gas, with gas constant R and specific heat capacity at constant volume c_v , that can be represented as ideal. With this arrangement, the piston will drive disturbances that are uniform across the channel and that will propagate along the channel. The infinite length in the x_1 -direction precludes reflections of the disturbance.

It is clear from the configuration of Fig. 13.2.10 that with disturbances driven by the piston uniformly in an x_2 - x_3 -plane there will be no variation of properties with x_2 or x_3 and there will only be an x_1 -component of velocity v_1 . Thus we can write the equations of motion for the gas in one-space-dimensional forms:

conservation of mass (13.1.1)

$$\frac{D_1 \rho}{Dt} = -\rho \frac{\partial v_1}{\partial x_1}, \quad (13.2.64)$$

where now

$$\frac{D_1}{Dt} = \left(\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x_1} \right), \quad (13.2.65)$$

conservation of momentum (13.1.17)

$$\rho \frac{D_1 v_1}{Dt} = \frac{\partial p}{\partial x_1}, \quad (13.2.66)$$

conservation of energy (13.1.29)

$$\rho \frac{D_1}{Dt} (u + \tfrac{1}{2} v_1^2) = - \frac{\partial}{\partial x_1} (p v_1), \quad (13.2.67)$$

and the equations of state (13.1.10) and (13.1.14)

$$p = \rho R T, \quad du = c_v dT. \quad (13.2.68)$$

Before proceeding to analyze the propagation of disturbances, it will be useful to simplify the equations somewhat. First, we use the equations of state to eliminate u and then T from the conservation of energy.

$$\rho \frac{c_v}{R} \frac{D_1}{Dt} \left(\frac{p}{\rho} \right) + \rho v_1 \frac{D_1 v_1}{Dt} = - \frac{\partial}{\partial x_1} (p v_1). \quad (13.2.69)$$

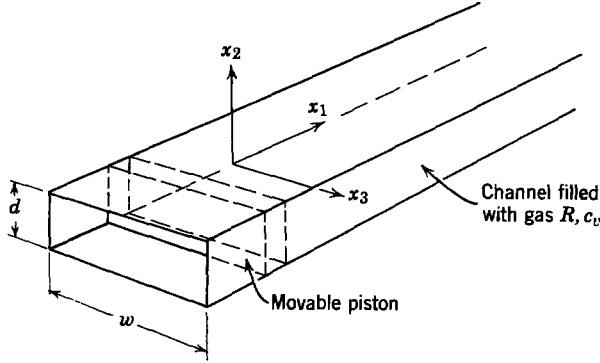


Fig. 13.2.10 Configuration for studying propagation of longitudinal (acoustic) disturbances in a gas.

Next, the conservation of momentum equation simplifies this expression to

$$\rho \frac{c_v}{R} \frac{D_1}{Dt} \left(\frac{p}{\rho} \right) = -p \frac{\partial v_1}{\partial x_1}. \quad (13.2.70)$$

Finally, the conservation of mass equation eliminates the space derivative of v_1 and the derivative on the left is expanded to obtain

$$\frac{D_1 p}{Dt} = \frac{\gamma p}{\rho} \frac{D_1 \rho}{Dt}. \quad (13.2.71)$$

An equation of the same form can be obtained for three-dimensional variations of properties.

Equations 13.2.64, 13.2.66, and 13.2.71 are sufficient to describe the propagation of disturbances through the gas; these equations, however, are nonlinear. For the remainder of this section, we assume that the disturbances involve small perturbations from an equilibrium condition such that the equations can be linearized. Thus we represent the three relevant variables in terms of equilibrium quantities (subscript o) and perturbation quantities (primed)

$$p = p_o + p', \quad (13.2.72a)$$

$$\rho = \rho_o + \rho', \quad (13.2.72b)$$

$$v_1 = v'_1. \quad (13.2.72c)$$

At equilibrium the gas is at rest; thus the equilibrium value of v_1 is zero.

Substitution of (13.2.72a-c) into (13.2.64), (13.2.66), and (13.2.71) and

retention of only linear terms in the perturbation quantities yield

$$\frac{\partial \rho'}{\partial t} = -\rho_o \frac{\partial v_1'}{\partial x_1}, \quad (13.2.73)$$

$$\rho_o \frac{\partial v_1'}{\partial t} = -\frac{\partial p'}{\partial x_1}, \quad (13.2.74)$$

$$p' = \frac{\gamma p_o}{\rho_o} \rho'. \quad (13.2.75)$$

In obtaining (13.2.75), the linearized version of (13.2.71) has been integrated and the constant of integration set to zero because both perturbation quantities are zero at equilibrium.

Elimination of p' and ρ' from (13.2.73) to (13.2.75) yields a single equation with v_1' as the unknown:

$$\frac{\partial^2 v_1'}{\partial t^2} = \frac{\gamma p_o}{\rho_o} \frac{\partial^2 v_1'}{\partial x_1^2}. \quad (13.2.76)$$

This is a wave equation (see Section 11.4.1) that describes longitudinal (acoustic) waves that propagate with a sound speed given by*

$$a_s = \left(\frac{\gamma p_o}{\rho_o} \right)^{1/2}. \quad (13.2.77)$$

Refer now to Fig. 13.2.10. We specify that the piston be driven with small amplitude oscillations such that the velocity of the gas at $x_1 = 0$ is constrained to be

$$v_1'(0, t) = V_m \cos \omega t. \quad (13.2.78)$$

Because the channel is infinitely long in the x_1 -direction, disturbances will propagate only in the positive x_1 -direction (there are no reflected waves). Thus the velocity of the gas at any point along the channel for steady-state conditions is

$$v_1'(x_1, t) = V_m \cos \left(\omega t - \frac{\omega}{a_s} x_1 \right). \quad (13.2.79)$$

That this is a solution of (13.2.76) which satisfies the boundary condition of (13.2.78) can be verified by direct substitution.

We can now use (13.2.79) in (13.2.73) to find the perturbation density

$$\rho'(x_1, t) = \rho_o \frac{V_m}{a_s} \cos \left(\omega t - \frac{\omega}{a_s} x_1 \right). \quad (13.2.80)$$

* This is the same speed as that given by (13.2.11).

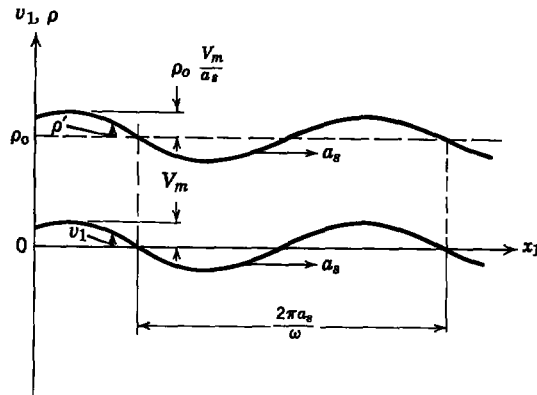


Fig. 13.2.11 Density and velocity variations in a sound wave of frequency ω propagating in the positive x_1 -direction.

Sketches of the variation of density and velocity as functions of space at a given instant of time are shown in Fig. 13.2.11. Note that the velocity and density perturbations are in phase and that the whole pattern propagates in the positive x_1 -direction with the acoustic speed a_s .

It is clear from the nature of the wave equation (13.2.76) that sound waves propagate in our assumed perfect medium without dispersion. Thus all the techniques and conclusions of Section 9.1.1 apply equally well to sound waves.

It is also worthwhile at this point to comment that no heat conduction term appears in the conservation of energy equation. This is the model that best describes sound waves from the audio-frequency range up to the megacycle per second range.

In modeling the slight compressibility of liquids to describe mechanical behavior during moderate changes in pressure the temperature is immaterial. Consequently, the conservation of energy equation and the thermal equation of state are dropped, and the mechanical equation of state is conventionally written as*

$$\frac{d\rho}{\rho} = \kappa dp, \quad (13.2.81)$$

where κ is the compressibility. For small perturbations about an equilibrium with the definitions of (13.2.72a,b) (13.2.81) can be linearized and integrated to obtain

$$p' = \frac{1}{\kappa \rho_0} \rho'. \quad (13.2.82)$$

* See, for example, H. B. Callen, *Thermodynamics*, Wiley, New York, 1960, pp. 344–349.

If this expression is used in place of (13.2.75) with (13.2.73) and (13.2.74), it will be found that a wave equation like that of (13.2.76) will result and $(\gamma p_o/\rho_o)$ will be replaced by $(1/\kappa\rho_o)$. Thus for a liquid with density ρ_o and compressibility κ the acoustic speed is

$$a_s = \frac{1}{\sqrt{\kappa\rho_o}}. \quad (13.2.83)$$

With this modification all the results already obtained for acoustic waves in inviscid gases hold equally well for acoustic waves in inviscid liquids.

In this mathematical development we used a lossless fluid model with the mathematical result that a plane disturbance propagates with no attenuation. In all real fluids viscosity (mechanical loss) dissipates energy and damps disturbances. In most practical problems, however, the damping is slight and can be treated mathematically as a perturbation of the lossless analysis, much like the process used to introduce electrical losses in transmission lines.† Although the problem of viscous damping of acoustic waves is not analyzed in this book, the concept and mathematical model of viscosity is introduced in Chapter 14, and it is a straightforward process to include viscous terms as perturbations on the lossless analysis and evaluate viscous damping of acoustic waves.

13.2.3b Magnetoacoustic Waves

Now that we have described the physical nature and mathematical characterization of ordinary acoustic waves, we add bulk electromechanical coupling to see how acoustic waves are modified to magnetoacoustic waves. The physical system to be used is the rectangular channel of Fig. 13.2.10, with electric and magnetic modifications, as illustrated in Fig. 13.2.12. The channel is fitted with pole pieces and an excitation winding which produce, at equilibrium, a flux density that is uniform throughout the channel and has only an x_2 -component:

$$\mathbf{B} = \mathbf{i}_2 B_o. \quad (13.2.84)$$

The walls of the channel that are perpendicular to the x_3 -axis are made of highly conducting electrodes. The movable piston is also made of highly conducting material.

Because of the high conductivity of the gas, the electrodes, and the piston and because of the presence of an applied magnetic field, the electromagnetic part of this system is represented by a quasi-static, magnetic field system.

† See, for example, R. B. Adler, L. J. Chu, and R. M. Fano, *Electromagnetic Energy Transmission and Radiation*, Wiley, New York, 1960, Chapter 5, p. 179.

First, the momentum and energy equations (13.2.66) and (13.2.67) must be modified to include coupling terms, thus:

conservation of momentum (13.1.17) is

$$\rho \frac{D_1 v_1}{Dt} = - \frac{\partial p}{\partial x_1} - J_3(B_o + B'_2), \quad (13.2.86)$$

and conservation of energy (13.1.29) is

$$\rho \frac{D_1}{Dt} (u + \frac{1}{2} v_1^2) = - \frac{\partial}{\partial x_1} (p v_1) + J_3 E_3. \quad (13.2.87)$$

Next, recognizing that the equilibrium flux density B_o is not a function of time or space, the relevant electromagnetic equations are:

Ampere's law (1.1.1)*

$$\frac{1}{\mu_0} \frac{\partial B'_2}{\partial x_1} = J_3, \quad (13.2.88)$$

Faraday's law (1.1.5)*

$$\frac{\partial E_3}{\partial x_1} = \frac{\partial B'_2}{\partial t}, \quad (13.2.89)$$

and Ohm's law $\mathbf{J}' = \sigma \mathbf{E}'$ written as†

$$J_3 = \sigma [E_3 + v_1(B_o + B'_2)]. \quad (13.2.90)$$

Note that $\nabla \cdot \mathbf{B} = 0$ is automatically satisfied by the functional form of $\mathbf{B}$ that results in this problem.

The equations necessary for describing magnetoacoustic disturbances are (13.2.86) to (13.2.90), plus the conservation of mass (13.2.64) and the equations of state (13.2.68). As in the case of acoustic waves, these equations are nonlinear; thus we assume perturbations small enough to allow us to linearize the equations of motion. Again we represent the relevant variables in terms of equilibrium quantities (subscript o) and perturbation quantities (primed).

$$p = p_o + p', \quad (13.2.91a)$$

$$\rho = \rho_o + \rho', \quad (13.2.91b)$$

$$T = T_o + T', \quad (13.2.91c)$$

$$v_1 = v'_1, \quad (13.2.91d)$$

$$B_2 = B_o + B'_2, \quad (13.2.91e)$$

$$J_3 = J'_3, \quad (13.2.91f)$$

$$E_3 = E'_3. \quad (13.2.91g)$$

* Table 1.2, Appendix G.

† See Table 6.1, Appendix G.

Note that velocity, current density, and electric field intensity have zero equilibrium values.

First, linearization of Ohm's law (13.2.90) in the limit where $\sigma \rightarrow \infty$ gives

$$E_3 = -v_1 B_o. \quad (13.2.92)$$

Substitution of this result in (13.2.89) yields

$$\frac{1}{B_o} \frac{\partial B'_2}{\partial t} = -\frac{\partial v'_1}{\partial x_1}. \quad (13.2.93)$$

Linearization of (13.2.64) (conservation of mass) and division of the result by ρ_o yields

$$\frac{1}{\rho_o} \frac{\partial \rho'}{\partial t} = -\frac{\partial v'_1}{\partial x_1}. \quad (13.2.94)$$

Subtraction of (13.2.94) from (13.2.93) and integration with respect to time (recognizing that for equilibrium conditions all perturbation quantities go to zero) yields

$$\frac{B'_2}{B_o} = \frac{\rho'}{\rho_o}. \quad (13.2.95)$$

This shows that perturbations in flux density follow perturbations in mass density. This is formal mathematical acknowledgment that for $\sigma \rightarrow \infty$ the time constant for diffusion of magnetic flux lines through the gas goes to infinity and the flux lines are essentially frozen into the material.

It can be verified by following a process similar to that for (13.2.69) to (13.2.71) for small-signal linearized equations that (13.2.71) still holds for perturbation quantities:

$$\frac{D_1 p'}{Dt} = \frac{\gamma p_o}{\rho_o} \frac{D_1 \rho'}{Dt}. \quad (13.2.96)$$

Integration of this expression and use of (13.2.77) to define acoustic speed a_s yield

$$p' = a_s^2 \rho'. \quad (13.2.97)$$

Linearization of the conservation of momentum (13.2.86) yields

$$\rho_o \frac{\partial v'_1}{\partial t} = -\frac{\partial p'}{\partial x_1} - \frac{B_o}{\mu_o} \frac{\partial B'_2}{\partial x_1}. \quad (13.2.98)$$

In writing this equation, we have used (13.2.88) to eliminate J'_3 .

The use of (13.2.97) to eliminate p' from (13.2.98) and the use of (13.2.95) to eliminate B'_2 yield

$$\rho_o \frac{\partial v'_1}{\partial t} = \left(a_s^2 + \frac{B_o^2}{\mu_o \rho_o} \right) \frac{\partial \rho'}{\partial x_1}. \quad (13.2.99)$$

The use of this expression and the linearized conservation of mass (13.2.94) to eliminate ρ' yields the single equation for v_1' :

$$\frac{\partial^2 v_1'}{\partial t^2} = \left(a_s^2 + \frac{B_o^2}{\mu_0 \rho_o} \right) \frac{\partial^2 v_1'}{\partial x_1^2}. \quad (13.2.100)$$

Comparison of this result with (13.2.76) for ordinary acoustic waves shows that (13.2.100) describes longitudinal waves that propagate without dispersion with a propagation speed a given by

$$a = \left(a_s^2 + \frac{B_o^2}{\mu_0 \rho_o} \right)^{1/2}. \quad (13.2.101)$$

These waves are called magnetoacoustic waves and a is the magnetoacoustic velocity because the propagation speed is given by (13.2.101) as a combination of the acoustic velocity a_s and another velocity $\sqrt{B_o^2/\mu_0 \rho_o}$, which depends on magnetic flux density. This other velocity is numerically equal to the Alfvén velocity a_b obtained for transverse electromechanical waves and defined in (12.2.88).

Provided we replace a_s with a , as defined in (13.2.101), all the comments made about acoustic waves in the preceding section hold true for magnetoacoustic waves. Because of the bulk electromechanical coupling, it will be instructive to study the physical makeup of a magnetoacoustic wave. To provide a basis for comparison with ordinary acoustic waves we assume the same driving function we used for the acoustic wave example, namely, that the piston at $x_1 = 0$ is driven with small amplitude at angular frequency ω such that the gas velocity at $x_1 = 0$ is

$$v_1'(0, t) = V_m \cos \omega t. \quad (13.2.102)$$

The gas velocity at any point in the gas is then

$$v_1'(x_1, t) = V_m \cos \left(\omega t - \frac{\omega}{a} x_1 \right). \quad (13.2.103)$$

This can be verified as the solution by seeing that the boundary condition (13.2.102) and the differential equation (13.2.100) are both satisfied. In addition, the infinite length in the x_1 -direction results in no reflected waves traveling in the negative x_1 -direction.

We now use the conservation of mass (13.2.94) and (13.2.95) to write

$$\frac{B_2'(x_1, t)}{B_o} = \frac{\rho'(x_1, t)}{\rho_o} = \frac{V_m}{a} \cos \left(\omega t - \frac{\omega}{a} x_1 \right). \quad (13.2.104)$$

Finally, we use (13.2.88) to evaluate J_3 :

$$J_3(x_1, t) = \frac{B_o \omega V_m}{\mu_0 a^2} \sin \left(\omega t - \frac{\omega}{a} x_1 \right). \quad (13.2.105)$$

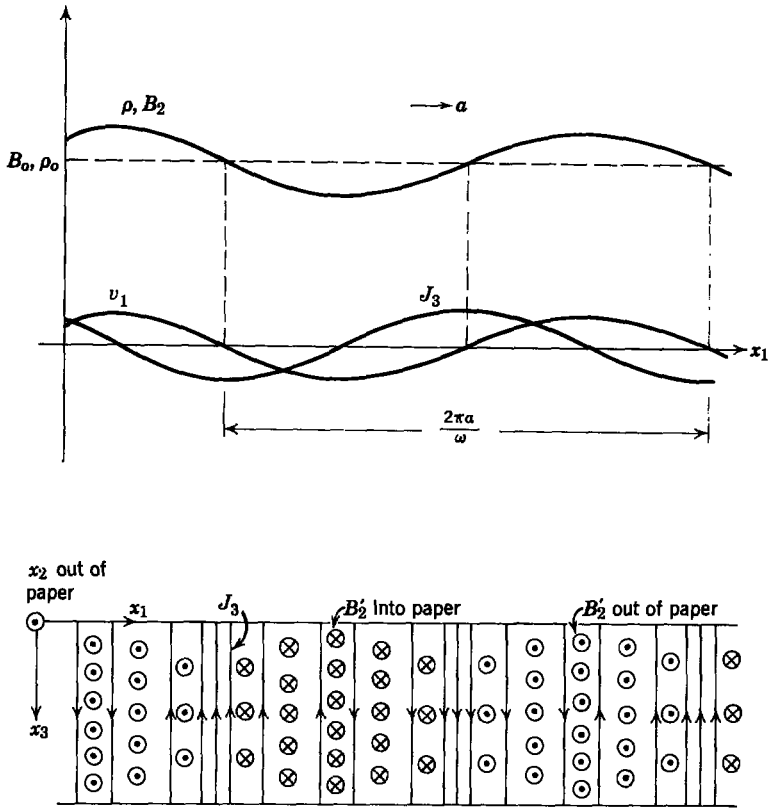


Fig. 13.2.13 Gas and electromagnetic variables in a magnetoacoustic wave of frequency ω propagating in the positive x_1 -direction.

The variables described by (13.2.103) to (13.2.105) are illustrated for one instant of time in Fig. 13.2.13. As time passes, this pattern propagates with speed a in the positive x_1 -direction. In describing J_3 the density of lines indicates the intensity of the current density, and for B'_2 the density of the circles indicates the strength of the flux density. We already know that B'_2 is excited by J_3 . This can be verified by the right-hand rule or by (13.2.88). Also, as indicated by (13.2.95), the perturbation flux density and mass density are linearly related. Thus, when the gas is compressed, magnetic flux lines are compressed. This compression of flux lines induces a current density J_3 , which interacts with the equilibrium flux density to produce a force that resists the compression. This makes the gas essentially less compressible, raises the effective continuum "spring constant," and makes the propagation velocity greater than the ordinary acoustic velocity.

It is clear from the pattern of current density in Fig. 13.2.13 why the highly conducting electrodes are necessary to close the current paths and maintain the one-dimensional nature of the problem.

In the example the waves were driven mechanically by a piston; they could have been driven equally well by local perturbations in flux density or current density. Furthermore, these waves can interact with an electric circuit that couples either to the flux density or to the current density. Thus magnetoacoustic waves provide the opportunity for continuum electromechanical coupling between a channel of highly conducting gas and an electric circuit.*

Although viscosity provides the loss mechanism that ultimately damps ordinary acoustic waves, magnetoacoustic waves are damped both by viscosity and by electrical losses that result from current flow in the presence of finite conductivity. In virtually all cases in which magnetoacoustic waves can be excited experimentally electrical losses predominate as the damping mechanism, and it is the limited electrical conductivity of gases that restricts the possibilities for practical utilization of magnetoacoustic waves for electro-mechanical coupling. This limitation is explored extensively in the literature.†

To illustrate the kinds of conditions necessary for the propagation of magnetoacoustic waves, we select conditions in which the waves have been excited and detected‡:

$$\begin{aligned} \text{Helium gas,} \quad \rho_o &= 0.0016 \text{ kg/m}^3, \\ B_o &= 0.32 \text{ Wb/m}^2, \quad T_o = 15,000^\circ\text{K}, \\ R &= 2080 \text{ J/kg}^\circ\text{K}, \quad p_o = 0.5 \times 10^5 \text{ N/m}^2 \left(\frac{1}{2} \text{ atm}\right). \\ \gamma &= 1.67, \end{aligned}$$

The extremely high temperature is necessary to achieve high enough conductivity that will allow magnetoacoustic wave propagation without excessive damping. Needless to say, this was a pulsed experiment. From the data given the sound velocity is

$$a_s = \left(\frac{\gamma p_o}{\rho_o} \right)^{1/2} = 7240 \text{ m/sec.}$$

The Alfvén velocity is

$$a_b = \left(\frac{B_o^2}{\mu_o \rho_o} \right)^{1/2} = 7150 \text{ m/sec.}$$

* H. A. Haus, "Alternating Current Generation with Moving Conducting Fluids," *J. Appl. Phys.*, **33**, 2161 (June 1962).

† G. L. Wilson and H. H. Woodson, "Excitation and Detection of Magnetoacoustic Waves in a Rotating Plasma Accelerator," *AIAA*, Vol. 5, No. 9, Sept. 1967, pp. 1633-1641.

‡ Wilson and Woodson, *loc. cit.*

The magnetoacoustic velocity is

$$a = \sqrt{a_s^2 + a_b^2} = 10,200 \text{ m/sec.}$$

It is clear from these numerical values that in a gas a moderate flux density will yield a magnetoacoustic velocity that is considerably greater than the ordinary acoustic velocity; thus the electromechanical coupling in the wave is easily made strong.

Magnetoacoustic waves can also be excited in conducting liquids such as liquid metals; however, because of the high density of liquids it is difficult to obtain an Alfvén velocity large enough to affect appreciably the propagation velocity of longitudinal disturbances. It is easy to show that the propagation velocity of magnetoacoustic waves in conducting liquids is still given by

$$a = \sqrt{a_s^2 + a_b^2},$$

where a_s is the sound velocity given by (13.2.83) and a_b is the Alfvén velocity given by (12.2.88).

To determine how much the propagation velocity of a longitudinal disturbance can be affected in a conducting liquid by an applied magnetic field consider mercury for which the sound velocity and density are

$$a_s = 1410 \text{ m/sec,}$$

$$\rho_o = 13,600 \text{ kg/m}^3.$$

The flux density necessary to give an Alfvén velocity that is 10 per cent of the sound velocity is

$$B_o = 18.5 \text{ Wb/m}^2.$$

This flux density (185,000 gauss) is obtainable at present only in large, high-field research magnets and it is a factor of 10 higher than obtainable with conventional iron-core electromagnets. A less dense liquid metal like sodium or potassium would require less flux density. For obtainable fields, however, the effect of a magnetic field is still small. Conducting gases, on the other hand, have low enough densities that the Alfvén velocity can be greater than the sound velocity at moderate flux densities, as we illustrated earlier.

In general, the propagation of disturbances in conducting fluids immersed in magnetic fields involves complex combinations of ordinary acoustic waves (longitudinal waves) and Alfvén waves (transverse waves) both propagating along magnetic field lines, and magnetoacoustic waves (longitudinal waves) propagating normal to magnetic field lines. These separate component waves couple through electromagnetic and gas variables

and are all damped by loss mechanisms. Thus the analysis of a disturbance, in general, is quite complex. Nonetheless, many phenomena can be understood in terms of the simple component waves we have studied separately.

13.3 DISCUSSION

In this chapter we have gone one step further in the analysis of electro-mechanical interactions between electrical systems and conducting fluids by using a compressible fluid model. The effects of compressibility on the basic conduction-type MHD machines were shown. Compressible fluids were shown to propagate longitudinal (acoustic) waves, and under appropriate conditions (long enough magnetic diffusion time) these waves can be modified significantly by the presence of a transverse magnetic field. Although the phenomena described and the techniques used in their analyses have important engineering applications, they were also intended to be indicative of the techniques available for the study of still other types of electromechanical interactions with fluids.

In Chapter 14 we introduce viscosity, another fluid-mechanical effect. We limit the discussion to incompressible fluids to highlight the principal effects of viscosity in MHD systems.

PROBLEMS

13.1. A static compressible fluid is subject to a gravitational force per unit volume $-\rho g$ (Fig. 13P.1). Under the assumption that the fluid has a constant temperature T_0 and that the fluid is a perfect gas so that $p = \rho RT$ find the distribution of density $\rho(x_1)$. The density at $x_1 = 0$ is ρ_0 .

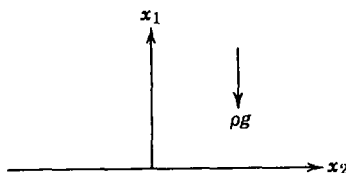


Fig. 13P.1

13.2. The MHD generator illustrated in Fig. 13P.2 uses a gas with constant specific heat capacities c_p and c_v and constant scalar conductivity σ . The dimensions are defined in the figure and it is assumed that the inlet values of all quantities are known. The loading factor K is to be held constant and the magnetic flux density is adjusted to satisfy the relation $B^2(z) A(z) = B_i^2 A_i$. For the constraint that the pressure be constant along the channel determine how the velocity v varies as a function of z .

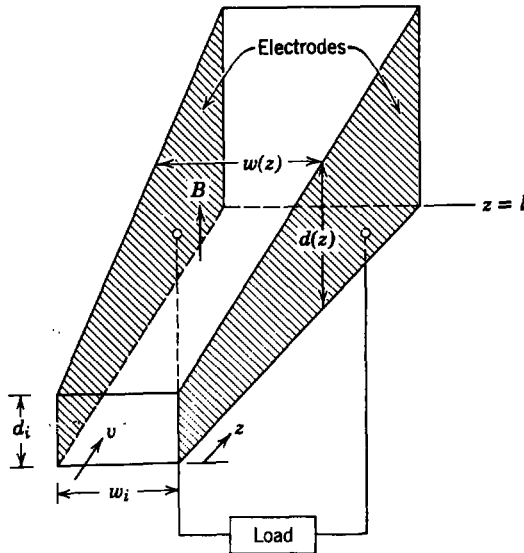


Fig. 13P.2

13.3. The dc MHD generator of Fig. 13P.3 has constant width w between the electrodes, and the magnetic flux density B_0 is constant along the length of the channel. The gas is assumed ideal with constant specific heats and with constant electrical conductivity. The inlet quantities ρ_i , p_i , T_i , and v_i are assumed known. The electrodes are short circuited together.

- Find the channel depth $d(z)$ necessary to maintain the temperature constant along the channel.
- Find the mass density variation along the channel for the channel geometry found in part (a).

13.4. For the MHD generator of Fig. 13P.4 assume that all inlet quantities are known (p_i , ρ_i , T_i , d_i , w_i , A_i) and that the working gas has constant, scalar conductivity σ and ratio of specific heats γ . We apply a magnetic flux density B_0 , which is constant, over the length of the channel. We now specify that we wish to design a channel such that the loading factor K will be constant along the length of the channel.

- For maintaining constant Mach number M along the channel find the following functions of z that satisfy the given conditions:

$$\frac{A(z)}{A_i}, \frac{d(z)}{d_i}, \frac{w(z)}{w_i}, \frac{v(z)}{v_i}, \frac{p(z)}{p_i}, \frac{T(z)}{T_i}.$$

- Repeat part (a) for maintaining constant velocity v along the channel.
- Assume that the given data is

$$\begin{aligned} \gamma &= 1.4, & \sigma &= 40 \text{ mhos/m}, & R &= 240 \text{ J/kg}^\circ\text{K}, \\ p_i &= 5 \times 10^5 \text{ N/m}^2, & \rho_i &= 0.7 \text{ kg/m}^3, & T_i &= 3000^\circ\text{K}, \\ v_i &= 700 \text{ m/sec}, & w_i &= 0.4 \text{ m} & d_i &= 0.2 \text{ m}. \\ K &= 0.5, & B_0 &= 4 \text{ Wb/m}^2 \end{aligned}$$

For the generators of parts (a) and (b) find the length of generator necessary to reduce the total enthalpy per unit mass ($c_p T + \frac{1}{2} v^2$) by 10%.

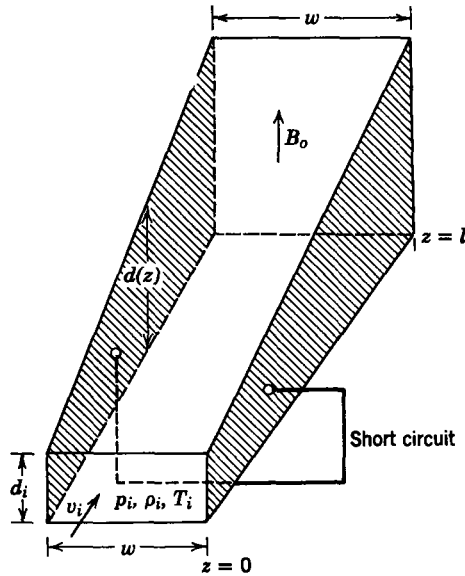


Fig. 13P.3

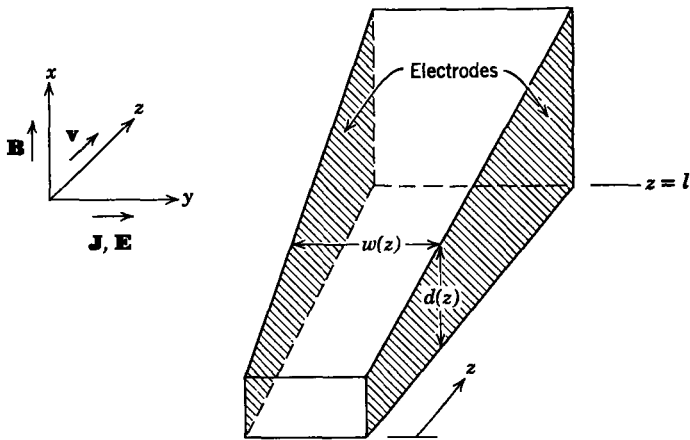


Fig. 13P.4

13.5. An MHD generator (Fig. 13P.4) uses an ideal gas with a constant ratio of specific heats $\gamma = 1.4$ and a constant, scalar, electrical conductivity $\sigma = 50$ mhos/m. At the inlet to the MHD generator channel the parameters and variables are adjusted to have the values

$$\begin{aligned} B_i &= 4 \text{ Wb/m}^2, & v_i &= 700 \text{ m/sec}, \\ \rho_i &= 0.7 \text{ kg/m}^3, & M_i^2 &= 0.5, \\ p_i &= 5 \times 10^5 \text{ N/m}^2, & d_i &= 0.5 \text{ m} \\ w_i &= 1 \text{ m}, \end{aligned}$$

The aspect ratio of the channel is to remain constant so that the electric and magnetic fields will vary as

$$\frac{B(z)}{B_i} = \frac{E(z)}{E_i} = \left[\frac{w(z)}{w_i} \right]^{-1} = \left[\frac{d(z)}{d_i} \right]^{-1} = \left[\frac{A(z)}{A_i} \right]^{-1/4},$$

where A is the channel cross sectional area. For the conditions specified and for a loading factor $K = 0.5$ complete the following:

- Find the area as a function of z necessary to keep the gas velocity constant.
- For the channel of part (a) specify the length l necessary to reduce the gas temperature by 20 percent; that is, find l such that $T(l)/T_i = 0.8$.
- Calculate and plot curves of $p(z)$, $\rho(z)$, $T(z)$ and $M^2(z)$ over the length of the channel.
- Calculate the total electrical power drawn from this generator under the conditions given.

13.6. An ion propulsion device is represented schematically in Fig. 13P.6. The lateral dimensions are much larger than the separation of the accelerator electrodes so that fringing fields can be neglected. Ions, each having a charge q and mass m are injected with negligible initial velocity at $x = 0$. The system operates in the steady state in the space-charge-limited mode, in which case the solution for the electric field between the electrodes is

$$E = i_x \frac{4}{3} \frac{V_o}{L^{3/4}} x^{1/4},$$

and the solution for the charge density between the electrodes is

$$\rho_e = \frac{4}{9} \frac{\epsilon_0 V_o}{L^{1/2} x^{3/2}}$$

The charge on each ion does not change during the acceleration process; that is,

$$\frac{\rho_e}{\rho_m} = \frac{q}{m},$$

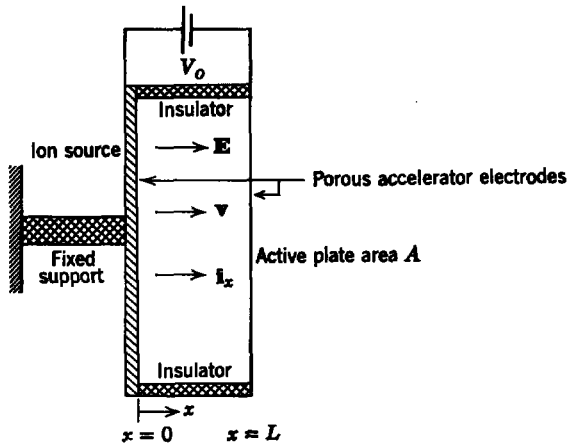


Fig. 13P.6

where ρ_m is mass density. The pressure in the ion gas is negligibly small. For steady-state operation complete the following:

- Find the velocity of the ion gas as a function of x between the electrodes.
- Find the magnitude and direction of the force that must be supplied by the fixed support to keep the accelerator system at rest.

13.7. In Fig. 13P.7 a liquid is placed between rigid walls and a movable piston. The liquid has a speed of sound a . The system is uniform in the x_2 - and x_3 -directions ($\partial/\partial x_2 = \partial/\partial x_3 = 0$). Assuming that the piston is moved by a velocity source $V = V_o \cos \omega t$ at $x_1 = -L$,

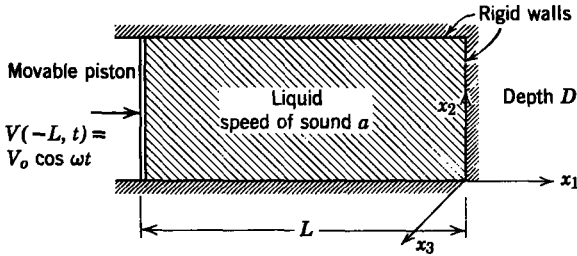


Fig. 13P.7

complete the following:

- Find the pressure in the liquid at $x_1 = -L$. Take the equilibrium pressure inside and outside the liquid as p_o .
- At what frequencies will there be resonances in the pressure, $p(-L, t)$?

13.8. A perfectly conducting compressible inviscid fluid fills the region $0 < x_2 < d$ as shown in Fig. 13P.8. When the fluid is static, it is permeated by a magnetic field $\mathbf{H} = H_o \mathbf{i}_3$. The fluid is set in motion by a piston at $x_1 = -L$ having velocity as shown, and is constrained at $x_1 = 0$ by a piston having mass M .

- Derive a differential equation (one equation in one unknown) for the velocity $v_1(x_1, t)$. Use a linearized analysis.
- Find the velocity of the piston, which in equilibrium is at $x_1 = 0$, under the conditions shown in Fig. 13P.8.

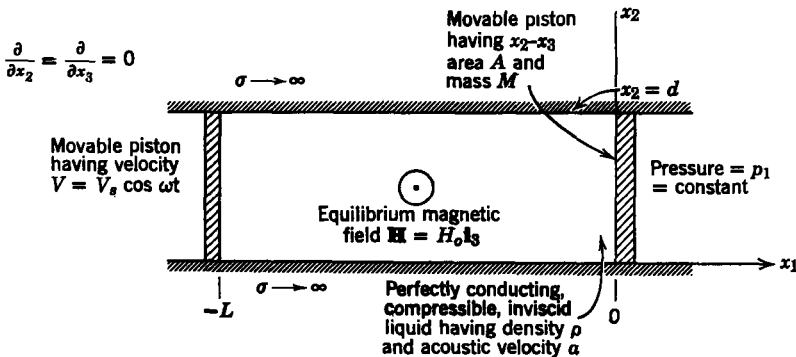


Fig. 13P.8

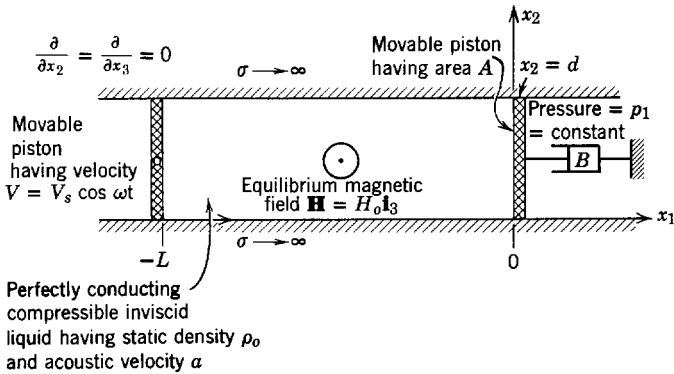


Fig. 13P.9

13.9. The region $0 < x_2 < d$ in Fig. 13P.9 is filled with a perfectly conducting inviscid compressible fluid. When the fluid is static, it is permeated by a magnetic field $\mathbf{H} = H_0 \mathbf{i}_3$. The fluid is excited by a piston at $x_1 = -L$ with velocity $v(t)$, as shown, and constrained at $x_1 = 0$ by a movable massless insulating piston connected to a dashpot.

- Derive a differential equation (one equation in one unknown) for the velocity $v_1(x_1, t)$. Use a linearized analysis.
- The system is to be used as a delay line in which the object is to delay a signal by a fixed time without distorting the signal. If the signal is to be transmitted delayed and undistorted to $x_1 = 0$, the backward traveling wave must be eliminated by choosing a specific value for the dashpot coefficient B . Find this value in terms of the appropriate constants of the system.

13.10. The system shown in Fig. 13P.10 consists of a cylinder of cross-sectional area A_c containing a liquid of compressibility κ and equilibrium density ρ_l at equilibrium pressure p_0 . The fluid is constrained at $x = 0$ by the closed end of the cylinder and at the other end by a rigid piston of mass M and thickness Δ . The equilibrium length of the liquid in the x -direction is L_1 . The left face of the piston is connected to a thin elastic rod of cross-sectional area A_s , modulus of elasticity E , and equilibrium density ρ_s . The equilibrium length of the elastic rod is L_2 . The left end of the elastic rod is driven by a stress source $T_0 + T_s(t)$, where $T_s(t) \ll T_0$. For small-signal dynamic operation around an equilibrium point the general solutions are for the elastic rod: $-(L_1 + L_2 + \Delta) < x < -(L_1 + \Delta)$, $T(x, t)$ and $v_e(x, t)$; for the fluid $-L_1 < x < 0$, $p(x, t)$ and $v_l(x, t)$. Assume that except for constants to be determined by boundary conditions these solutions are known. Set up the equations that describe all the boundary conditions necessary for specializing constants in the general solutions. *Note.* You are not required to solve these equations.

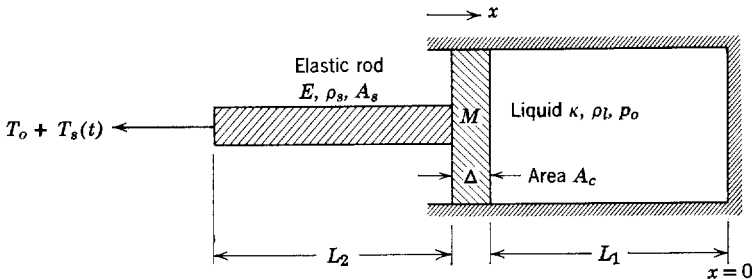


Fig. 13P.10

13.11. A slightly compressible inviscid fluid flows with a steady velocity V_0 in the x -direction (Fig. 13P.11). The motions are in the x -direction and only a function of (x, t) . The velocity of sound in the fluid is a_s .

- Find the dispersion equation for small disturbances in the form of $p = \text{Re} [\hat{p}e^{j(\omega t - kx)}]$.
- Under what condition will the phases of both waves propagate in the positive x -direction?

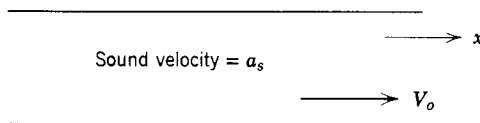


Fig. 13P.11

13.12. A static, inviscid fluid of conductivity σ is immersed in a uniform magnetic field H_0 (Fig. 13P.12). In the limit in which $\sigma \rightarrow \infty$, it is possible for magnetoacoustic plane waves to propagate along the x -axis. In this problem investigate the consequences of having a finite conductivity σ .

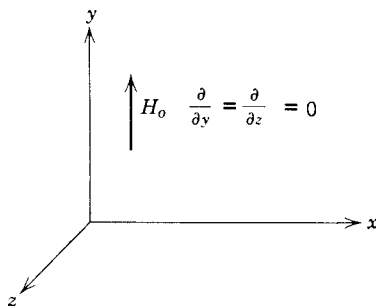


Fig. 13P.12

- Write the linearized equations of motion for perturbations that are compressional (along the x -axis) and depend only on (x, t) . (You need not combine these equations.)
- Consider solutions of the form $v_x = \text{Re} [\hat{v}e^{j(\omega t - kx)}]$ and find the dispersion relation between ω and k . Show that in the limit $\sigma \rightarrow \infty$ the lossless dispersion equation is retained.
- Show that when σ is small there are two pairs of waves, each pair consisting of a forward and backward traveling wave. What would you call these waves?
- Consider the case in which σ is very large but finite (slight losses) and in which the excitation is sinusoidal (ω is real). Find an approximate expression for the rate at which waves decay in space. *Hint.* Write the dispersion equation in the form

$$k^2 = \frac{\omega^2}{a^2} - j \frac{f(\omega, k)}{\sigma},$$

where a is the phase velocity of magnetoacoustic waves. When σ is large but finite, the second term can be approximated by making $k^2 \approx (\omega^2/a^2)$. (Why?)

Chapter 14

ELECTROMECHANICAL COUPLING WITH VISCOUS FLUIDS

14.0 INTRODUCTION

In Chapters 12 and 13 mathematical descriptions of lossless fluids are presented in a study of several basic types of electromechanical interaction with fluids. In the introductory section (12.0) of Chapter 12, we indicated that viscosity can have some marked effects on electromechanical interactions, especially when the system involves the flow of a fluid near a solid boundary. In this chapter the earlier fluid models are generalized to include the effects of fluid mechanical losses (viscosity), and the generalized models are used to study the effects of viscosity on some electromechanical interactions.

In our considerations of viscosity we limit our attention to incompressible fluids. The viscous effects we shall study also occur in compressible fluids, and the incompressible model we shall use is often employed to estimate the effects of viscosity on the flow of compressible fluids. Nonetheless, our model does not include mechanical losses due to longitudinal (dilatational) distortion of the fluid, and thus it is inadequate for a study of the effects of viscosity on longitudinal disturbances such as acoustic and magnetoacoustic waves. At the appropriate point in the development we indicate how the necessary extension can be made.

Our model for viscous fluids is restricted to *Newtonian fluids* whose stress-strain rate relations are linear. This model is a good representation of most fluids of interest in electromechanical interactions. It is analogous to the model of a linear resistance in electric circuits. Non-Newtonian fluids require suitable nonlinear models.*

* R. R. Long, *Mechanics of Solids and Fluids*, Prentice-Hall, Englewood Cliffs, N.J., 1961, pp. 69–72.

14.1 VISCOUS FLUIDS

In Section 12.1.4 the mechanical interaction force between adjacent fluid particles in an inviscid fluid is represented by the hydrostatic pressure p . When a viscous fluid is at rest, the mechanical interaction of adjacent particles is still described by a hydrostatic pressure. With the fluid in motion, it is subjected to pressure forces just as an inviscid fluid is, but, in addition, there is a force due to friction between adjacent particles that are in relative motion. The effects of this friction force are accounted for mathematically by defining a fluid *viscosity*.

14.1.1 Mathematical Description of Viscosity

A simple, one-dimensional example helps to introduce a derivation of the stress-tensor (hence force density) that represents the fluid friction.† Figure 14.1.1 shows a viscous fluid contained between parallel plates. We can imagine such a situation in which the system has large dimensions in the x_1 - and x_3 -directions. The plates are set into steady relative motion by externally applied forces. The viscous fluid near the upper plate tends to move with it. Similarly, the fluid at $x_2 = 0$ tends to move with the lower plate. Forces have to be applied to the two plates to maintain relative motion and, when steady-state conditions exist, there must be equal and opposite forces acting on the two plates, as indicated in Fig. 14.1.1. If each plate has the area A , there is a shear stress (force per unit area), given by

$$T_{12} = \frac{f}{A}. \quad (14.1.1)$$

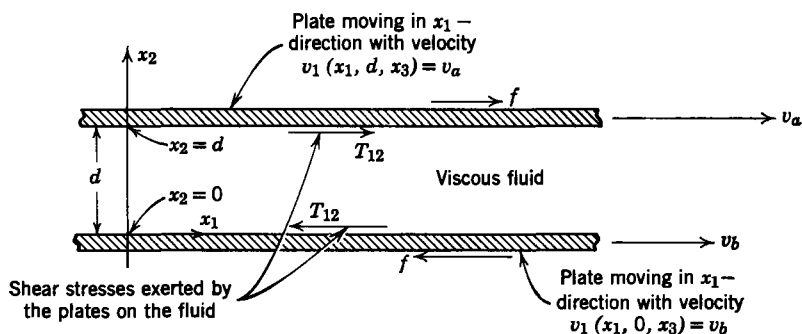


Fig. 14.1.1 Simple example of shear flow.

† This example is considered in more detail in Section 14.1.3.

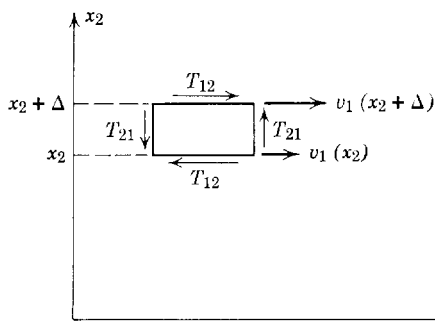


Fig. 14.1.2 Equilibrium for a small element of fluid of thickness Δ .

Experiments show that for a variety of fluids the shear stress T_{12} applied to the fluid by the plates is directly proportional to the difference of the plate velocities and inversely proportional to the plate spacing d :

$$T_{12} = \mu \left(\frac{v_a - v_b}{d} \right). \quad (14.1.2)$$

The constant of proportionality μ is defined as the *coefficient of viscosity*. This constant μ describes a Newtonian fluid. A non-Newtonian fluid does not exhibit the linear relation between velocity difference and shear stress.

An experiment of this kind leads us to postulate that for a certain type of fluid flow an element of fluid of infinitesimal thickness, shown in Fig. 14.1.2, is held in equilibrium by the shear stress T_{12} , where

$$T_{12} = \lim_{\Delta \rightarrow 0} \mu \left[\frac{v_1(x_2 + \Delta) - v_1(x_2)}{\Delta} \right] = \mu \frac{\partial v_1}{\partial x_2}. \quad (14.1.3)$$

Moreover, because the torque on the infinitesimal element must be zero,

$$T_{21} = T_{12}. \quad (14.1.4)$$

From (8.1.10)* we conclude that the force density due to viscosity in our simple one-dimensional problem is

$$F_1 = \mu \frac{\partial^2 v_1}{\partial x_2^2}. \quad (14.1.5)$$

Because the motion is steady and the pressure is uniform, this force density F_1 must be zero. Two boundary conditions are then required to integrate (14.1.5). Our intuition tells us that the fluid moves with (sticks to) a contiguous boundary; for example, in Fig. 14.1.1 the constants are determined by the conditions that

$$\begin{aligned} v_1(d) &= v_a, \\ v_1(0) &= v_b, \end{aligned}$$

* See Table 8.1, Appendix G.

and it follows that

$$v_1 = \frac{(v_a - v_b)}{d} x_2 + v_b.$$

Of course, the force density and stress tensor deduced are of limited validity. The steps outlined, however, serve to indicate the approach that is now used to establish a stress tensor of more general validity. The similarity of this example and the one-dimensional elasticity problem of Section 9.1* suggest that we approach the problem of deriving the viscous stresses by the same techniques that were used to find the elastic stress tensor. Equation 14.1.3 emphasizes the basic difference between fluid and elastic media. Rather than a linear relation between stress and strain, as we found in dealing with elastic media [see (11.2.32)], we now find a linear relation between stress and *rate of strain*. Hence we can establish the viscous stress tensor in two steps: first, we relate the velocity to the rate of strain, which is defined in a way analogous to that in Section 11.2.1 for strain, and, second, we introduce the empirical relationship between stress and the rate of strain. We can then find the force density from the stress tensor by simply taking the divergence of the stress tensor (8.2.7)†.

14.1.1a The Strain-Rate Tensor

The strain-rate tensor, like the strain tensor, is defined by geometrical considerations. It is defined in such a way that its components represent those types of fluid flow that would be expected to give rise to a viscous stress. Because the relation between stress and strain rate (as found in the laboratory) is linear, it is possible to superimpose various types of deformation rates to describe an arbitrary deformation rate. Our development now parallels that used in connection with the strain tensor which described elastic media (Section 11.2.1).

A one-dimensional flow, such as that used to introduce this section, is shown in Fig. 14.1.3a. If we now consider the flow in the neighborhood of point *A*, the velocity field may be divided into a pure translation, as shown in Fig. 14.1.3b, and the flow of Fig. 14.1.3c. A pure translation cannot give rise to a viscous stress, for the particles that interact with those at *A* from above or below or to the right or left are moving at the same velocity as those at point *A*.

Now, in turn, we can divide the remaining flow field into two parts, as shown in Fig. 14.1.4. There is now a part that represents a rigid-body rotation about point *A* and a part that we call a *shear flow*. Viscous stresses would not be expected to arise from rigid-body rotation any more than they would from rigid-body translation. Hence a flow with the character of that shown in Fig. 14.1.4c must be related to the viscous stress.

* See Table 9.2, Appendix G.

† Appendix G.

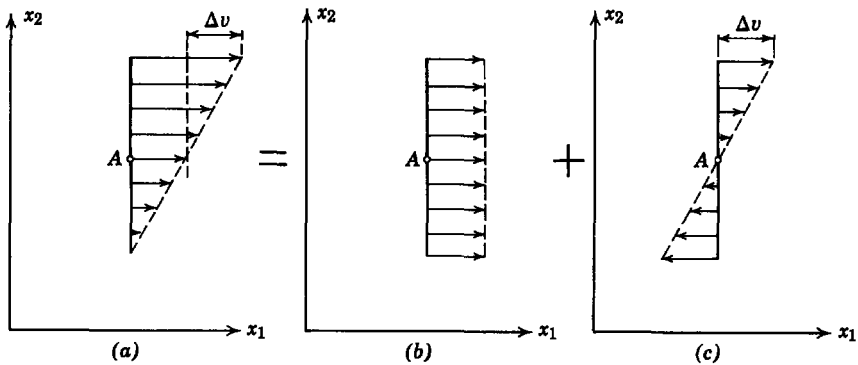


Fig. 14.1.3 Decomposition of flow into rigid-body translation of point A and flow with respect to that point.

It is worthwhile to recall the attributes of a flow field $\mathbf{v}(x_1, x_2, x_3, t)$ that correspond to rotation. Clearly, if fluid is rotating about point A , the line integral of $\mathbf{v}$ along a contour C that encloses point A is some finite number, say Γ .

$$\oint_C \mathbf{v} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{v}) \cdot \mathbf{n} \, da = \Gamma. \quad (14.1.6)$$

We have used Stokes's theorem to transform the line integral to an integral over the surface enclosed by the contour C . From (14.1.6) it is apparent that the magnitude of the rotation about a point is proportional to $\nabla \times \mathbf{v}$. For the simple flow field shown in Figs. 14.1.3 and 14.1.4 there is only an i_3 -component of the curl, and that is

$$(\nabla \times \mathbf{v})_3 = \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right). \quad (14.1.7)$$

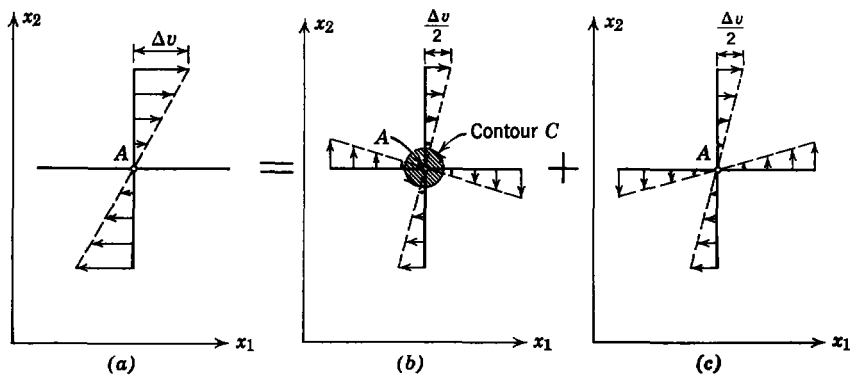


Fig. 14.1.4 Decomposition of flow into rotation and shear components: (a) total flow with respect to point A ; (b) rigid-body rotation; (c) shear flow,

In our one-dimensional example we found that the stress T_{12} is proportional to $\partial v_1/\partial x_2$ [see (14.1.3)]. We can however, write this derivative as

$$\frac{\partial v_1}{\partial x_2} = \frac{1}{2} \left(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} \right) + \frac{1}{2} \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right), \quad (14.1.8)$$

and we see that it includes the rotation of (14.1.7). Equation 14.1.8 is the analytical representation of Fig. 14.1.4. Our point is that it is reasonable to define as the component of the *shear rate* $\dot{e}_{12}$

$$\dot{e}_{12} = \frac{1}{2} \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right), \quad (14.1.9)$$

since it represents the only part of the deformation rate that is not (locally) a rigid-body translation or rotation. From the symmetry of the x_1 - and x_2 -coordinates it follows that $\dot{e}_{12} = \dot{e}_{21}$. Although our remarks have been made for flow fields in the x_1 - x_2 -plane, they apply equally well with other combinations of the coordinates. Hence we define

$$\dot{e}_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right). \quad (14.1.10)$$

Although so far we have discussed the situation in which $i \neq j$ in (14.1.10), those components given by $i = j$ also represent a rate of deformation that was not present in our simple example. If $\nabla \cdot \mathbf{v} \neq 0$, it is possible for the fluid to execute a motion of the kind illustrated in Fig. 14.1.5b. There the fluid in the region of the point A is either expanding or contracting. We can characterize this dilatational motion by the three terms of the divergence, recognized as $\dot{e}_{ii}$ when $i = j$. The mechanism by which a dilatational motion can produce a viscous stress is not defined by the simple experiment discussed in the introduction to this section. It is therefore not surprising that we find it necessary to define (and measure by some other means than a shear flow) a second coefficient of viscosity. It is important to recognize that the strain-rate, *defined* by (14.1.10), unlike the strain *approximated* by (11.2.10), involves no approximations about magnitudes of motion.

We can arrive at our definition of the strain-rate tensor in a more precise fashion by considering the relative velocity of fluid at two adjacent points in the flow field. For this purpose we define two points in a cartesian coordinate system which are at $\mathbf{r}$ and $\mathbf{r} + \Delta \mathbf{r}$, where

$$\mathbf{r} = \mathbf{i}_1 x_1 + \mathbf{i}_2 x_2 + \mathbf{i}_3 x_3 = \mathbf{i}_i x_i,$$

$$\Delta \mathbf{r} = \mathbf{i}_1 \Delta x_1 + \mathbf{i}_2 \Delta x_2 + \mathbf{i}_3 \Delta x_3 = \mathbf{i}_i \Delta x_i.$$

These coordinate vectors are shown in Fig. 14.1.6.

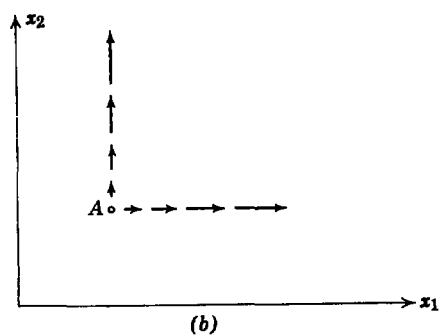
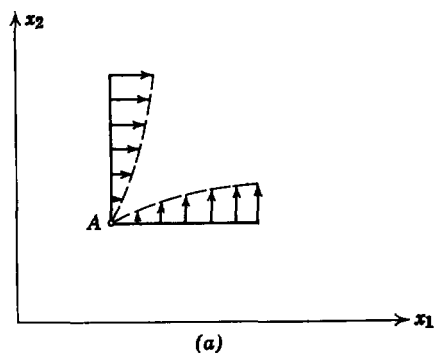


Fig. 14.1.5 Illustration of shear and dilatation at point A : (a) shear; (b) dilatation.

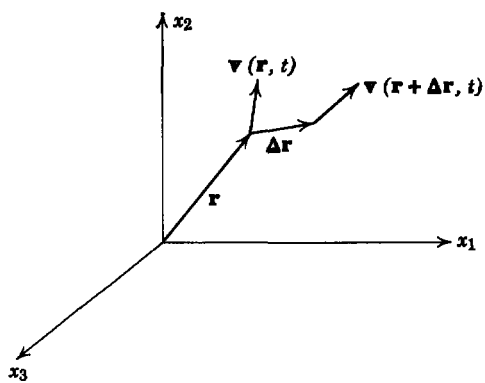


Fig. 14.1.6 System for defining rate of strain.

To find the velocity of the particle at $\mathbf{r} + \Delta\mathbf{r}$ with respect to the particle at $\mathbf{r}$ we expand the velocity in a Taylor series about the coordinate $\mathbf{r}$ and (because we are interested only in the region near $\mathbf{r}$) keep only linear terms. For the i th component we obtain

$$\Delta v_i = v_i(\mathbf{r} + \Delta\mathbf{r}, t) - v_i(\mathbf{r}, t) = \frac{\partial v_i}{\partial x_j} \Delta x_j. \quad (14.1.11)$$

We add and subtract $\frac{1}{2}(\partial v_j/\partial x_i) \Delta x_j$ on the right side of this expression to obtain

$$\Delta v_i = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) \Delta x_j + \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \Delta x_j. \quad (14.1.12)$$

The first term on the right represents rigid-body rotation with no rate of deformation, whereas the second term on the right represents the flow field left after translation and rotation have been subtracted out. (We removed the translation when we considered the *difference* between velocities at neighboring points.) The second coefficient of Δx_j in (14.1.12) is the strain rate (14.1.10). We have shown that, given the components of $\dot{e}_{ij}$ at the point $\mathbf{r}$, we can specify (except for the rotation) the difference in velocity between that point and a neighboring point an infinitesimal distance $\Delta\mathbf{r}$ away.

The strain-rate tensor is related to the velocity vector in exactly the same mathematical way that the strain is related to the displacement vector. Hence a proof that $\dot{e}_{ij}$ is, in fact, a tensor could begin with our knowledge that the velocity $\mathbf{v}$ is a vector and would follow identically the steps given in Section 11.2.1b, which prove that the strain is a tensor.

14.1.1b Stress-Strain-Rate Relations

At the beginning of this section we discussed a simple experiment that provided the relationship between the rate of shear strain and the shear stress,

$$T_{ij} = 2\mu\dot{e}_{ij}, \quad i \neq j. \quad (14.1.13)$$

It is possible to make a simple argument that this relation remains correct in the presence of a normal stress. Figure 14.1.7 shows a hypothetical situation in which we imagine that a normal stress T_{11} results in a strain rate $\dot{e}_{12}$. If we rotate the coordinate axes as shown in this figure, the normal stress remains unchanged in magnitude and direction but the resulting strain rate has reversed its sign. It is concluded that the rates of shear strain must be independent of the normal stresses. (This is a formal statement that, given a T_{11} , symmetry requires that the resulting rate of shear strain can be neither positive nor negative, hence must be zero.)

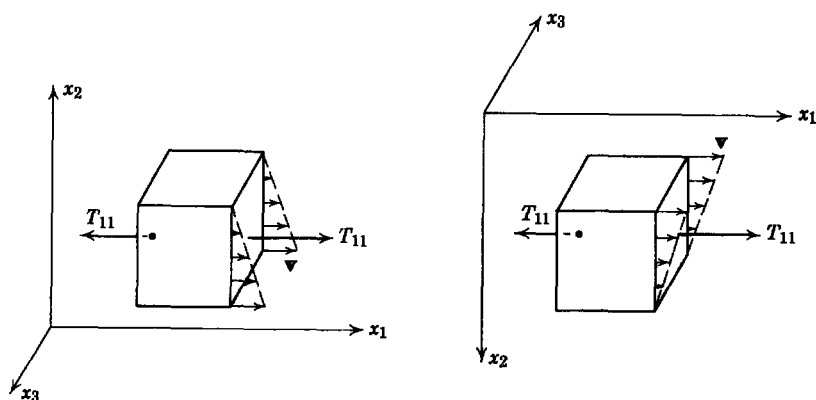


Fig. 14.1.7 Hypothetical situation in which a normal stress results in a shear strain rate.

Similarly, it is found that the dilatational strain rates depend only on the normal stresses. If k_1 and k_2 are experimentally determined constants, then

$$\begin{aligned}\dot{\epsilon}_{11} &= k_1 T_{11} - k_2 (T_{22} + T_{33}), \\ \dot{\epsilon}_{22} &= k_1 T_{22} - k_2 (T_{11} + T_{33}), \\ \dot{\epsilon}_{33} &= k_1 T_{33} - k_2 (T_{22} + T_{11}).\end{aligned}\tag{14.1.14}$$

A dilatational strain-rate $\dot{\epsilon}_{11}$ is shown in Fig. 14.1.8. Intuitively, we expect this type of flow to be caused by the stresses T_{11} , as shown in Fig. 14.1.8a,

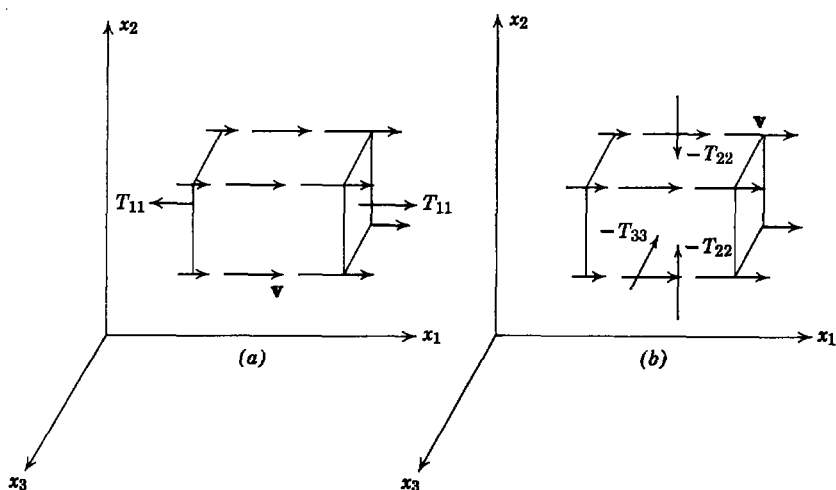


Fig. 14.1.8 Example of a dilatational strain rate showing two possible normal stresses.

or by normal stresses T_{22} and T_{33} , probably reversed in sign and not necessarily with the same magnitude as T_{11} . From symmetry we expect T_{22} and T_{33} to have an equivalent effect on the strain rate $\dot{\epsilon}_{11}$, so that there are only two constants in each of the relations (14.1.14). Because the fluid is isotropic, the equations must have the same form for strain-rate in each of the axis directions.

Again, a simple argument shows that shear stresses should not appear in (14.1.14). Suppose that a shear stress T_{12} resulted in a normal strain rate $\dot{\epsilon}_{11}$, as shown in Fig. 14.1.9. A change of coordinates generated by rotating the original system about a 45° axis in the x_1 - x_2 plane results in a fluid element subject to the same shear stress but displaying a strain rate $\dot{\epsilon}_{22}$ (as viewed in the first coordinate system) rather than $\dot{\epsilon}_{11}$. Symmetry and the isotropy of the fluid require that the dilatational strain rates depend only on the normal stresses.

A simple example is now used to illustrate that the experimentally determined constants k_1 and k_2 in (14.1.14) are not independent of the coefficient μ in (14.1.13).

Example 14.1.1. A fluid is subject to a stress $T_{12} = T_{21} = T_o$ in the x_1, x_2, x_3 -coordinate system. Show that the stress and strain rate in a coordinate system x'_1, x'_2, x'_3 defined by $x'_i = a_{ij}x_j$, where

$$a_{ij} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (a)$$

have only normal terms and find the relation between k_1, k_2 , and μ that must therefore exist.

Equation 14.1.13 shows that

$$\dot{\epsilon}_{12} = \dot{\epsilon}_{21} = \frac{T_o}{2\mu}, \quad \text{all other components} = 0 \quad (b)$$

Because the stress and strain rate are tensors, we can find them as expressed in the x'_1, x'_2, x'_3 -coordinate system by using the transformations

$$\begin{aligned} T'_{ij} &= a_{ik}a_{jl}T_{kl}, \\ \dot{\epsilon}'_{ij} &= a_{ik}a_{jl}\dot{\epsilon}_{kl}. \end{aligned} \quad (c)$$

Hence we find that

$$T'_{ij} = \begin{bmatrix} T_o & 0 & 0 \\ 0 & -T_o & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \dot{\epsilon}'_{ij} = \begin{bmatrix} \frac{T_o}{2\mu} & 0 & 0 \\ 0 & -\frac{T_o}{2\mu} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (d)$$

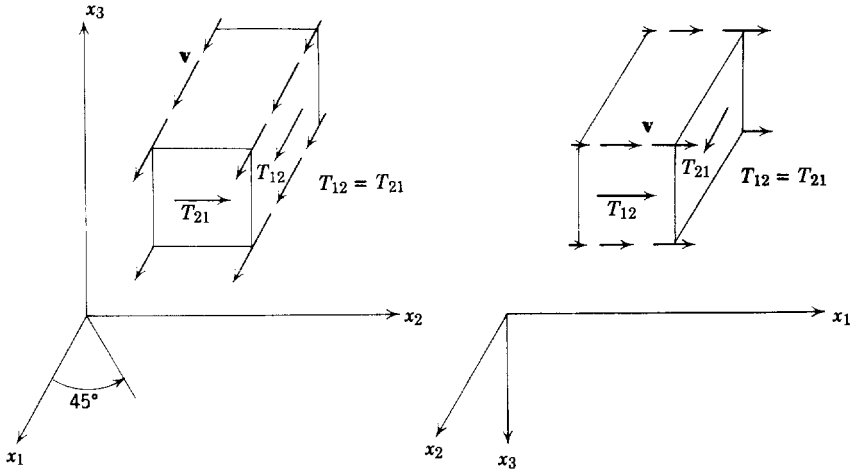


Fig. 14.1.9 Hypothetical situation in which a shear stress results in a dilatational strain rate.

It follows from these equations and (14.1.14) that

$$\dot{\epsilon}'_{11} = \frac{T_o}{2\mu} = k_1 T_o + k_2 T_o \quad \text{or} \quad k_1 + k_2 = \frac{1}{2\mu}, \quad (\text{e})$$

which is the desired relationship between μ , k_1 , and k_2 .

14.1.1c The Equations of Fluid Dynamics as Modified by Viscosity

Equations 14.1.14 allow us to find the stress components T_{11} , T_{22} , and T_{33} in terms of the strain-rate components $\dot{\epsilon}_{11}$, $\dot{\epsilon}_{22}$, and $\dot{\epsilon}_{33}$. This result and (14.1.13) provide a concise expression for the viscous stress in terms of the strain rates:

$$T_{ij} = 2\mu(\dot{\epsilon}_{ij}) + \alpha\delta_{ij}\dot{\epsilon}_{kk}. \quad (14.1.15)$$

We have used (e) in Example 14.1.1 to define the experimentally determined constant α in terms of the constants k_1 and k_2 :

$$\alpha = \frac{k_2}{(k_1 + k_2)(k_1 - 2k_2)}. \quad (14.1.16)$$

The numbers μ and α are physical constants that characterize the viscous properties of a fluid. The constant α , however, will not be found in the literature but rather a constant, which is a linear combination of α and μ , is defined as the *second coefficient of viscosity* η .*

$$\eta = \alpha + \frac{2}{3}\mu. \quad (14.1.17)$$

* See, for example, K. F. Herzfeld and T. A. Litovitz, *Absorption and Dispersion of Ultrasonic Waves*, Academic, New York, 1959, pp. 353–361.

Reference to (14.1.10) shows that

$$\dot{\epsilon}_{kk} = \frac{\partial v_k}{\partial x_k} = \nabla \cdot \mathbf{v}; \quad (14.1.18)$$

thus the second coefficient of viscosity affects dilatational motion and has a damping effect on longitudinal disturbances (acoustic waves).

As stated earlier, we shall restrict our treatment of viscosity to examples with incompressible fluid models for which $\nabla \cdot \mathbf{v} = 0$. Thus (14.1.15) is simplified to the form

$$T_{ij} = 2\mu \dot{\epsilon}_{ij}. \quad (14.1.19)$$

Equation 14.1.19 is that part of the stress on an element of incompressible fluid created by viscous effects. The total stress of mechanical origin must include the hydrostatic pressure p [see (12.1.34)] and may be written as

$$T_{ij}^m = -\delta_{ij}p + \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right). \quad (14.1.20)$$

We now use this stress tensor to write the conservation of momentum for an incompressible fluid in a form that includes the viscous force density [see (12.1.14), (12.1.15), and (12.1.19)]:

$$\rho \frac{Dv_i}{Dt} = F_i^e + \rho g_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_j} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right). \quad (14.1.21)$$

The order of partial differentiation is immaterial; thus

$$\frac{\partial}{\partial x_j} \left(\frac{\partial v_j}{\partial x_i} \right) = \frac{\partial}{\partial x_i} \left(\frac{\partial v_j}{\partial x_j} \right) = \frac{\partial}{\partial x_i} (\nabla \cdot \mathbf{v}),$$

which is zero for an incompressible fluid. Consequently, (14.1.21) assumes the simpler form

$$\rho \frac{Dv_i}{Dt} = F_i^e + \rho g_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_j} \left(\frac{\partial v_i}{\partial x_j} \right). \quad (14.1.22)$$

In vector form this equation is

$$\rho \frac{D\mathbf{v}}{Dt} = \mathbf{F}^e + \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{v}. \quad (14.1.23)$$

Equation 14.1.22 or 14.1.23 is called the *Navier-Stokes equation* of fluid mechanics; for an incompressible fluid this force equation is used with a conservation of mass equation to describe the fluid dynamics. Of course, boundary conditions and the force density of electric origin $\mathbf{F}^e$ must be known.

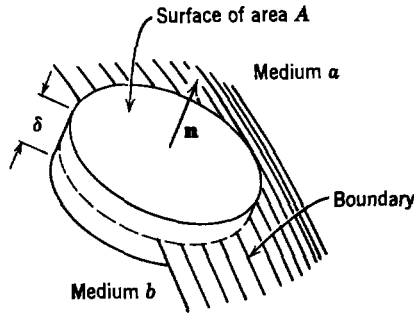


Fig. 14.1.10 Geometry for deriving boundary conditions.

14.1.2 Boundary Conditions

We specify that a boundary exists between media a and b , as illustrated in Fig. 14.1.10. The unit vector $\mathbf{n}$, which is normal to the boundary, is positive from medium b to medium a and has components n_1 , n_2 , and n_3 in the cartesian coordinate system x_1 , x_2 , and x_3 . Subscripts a and b are used to denote parameters and variables in the two media.

In order for the boundary to separate the two media, the velocities at the boundary must satisfy the relation

$$\mathbf{n} \cdot \mathbf{v}_a = \mathbf{n} \cdot \mathbf{v}_b = v_n. \quad (14.1.24)$$

This merely states that the normal component of velocity must be continuous at the boundary. Furthermore, the fluid particles at the boundary must have the same normal component of velocity as the boundary; otherwise the media are interdiffusing or they are moving apart. The result of (14.1.24) can be derived formally from the conservation of mass.

We now define a right circular cylindrical volume V with end surfaces of area A and height δ , as illustrated in Fig. 14.1.10. The volume V is assumed to be small enough that in its vicinity the boundary is essentially plane. The volume is oriented so that the ends of area A are parallel to the boundary and the boundary intersects the volume as illustrated in Fig. 14.1.10. We make the further restriction that δ be so small that the lateral area of the cylindrical surface will be much smaller than the area A .

We use the conservation of momentum as expressed by (12.1.21)

$$\rho \frac{Dv_i}{Dt} = \frac{\partial T_{ij}}{\partial x_j} \quad (14.1.25)$$

to integrate throughout the volume V :

$$\int_V \rho \frac{Dv_i}{Dt} dV = \int_V \frac{\partial T_{ij}}{\partial x_j} dV = \oint_S T_{ij} n_j da. \quad (14.1.26)$$

When we let the volume V go to zero, the restriction of finite mass density and finite acceleration makes the left side zero. The result is

$$0 = (T_{ij}^a - T_{ij}^b)n_j. \quad (14.1.27)$$

This expression states that the traction $(T_{ij}n_j)$ must be continuous at a surface.

Another type of boundary condition that must be specified is the condition that holds for a viscous fluid in contact with a solid surface. This situation is depicted in two dimensions in Fig. 14.1.11. We select our coordinate system so that the normal $\mathbf{n}$ to the surface is in the x_2 -direction and the velocity of the fluid parallel to the boundary is in the x_1 -direction. We assume no externally applied body forces (no electromagnetic or gravity forces). The condition in (14.1.24) indicates that the fluid adjacent to the boundary can have no normal component

$$v_2 = 0. \quad (14.1.28)$$

The condition in (14.1.27) indicates that the traction at the surface must be continuous:

$$\tau_1^b = \mu \frac{\partial v_1}{\partial x_2}, \quad (14.1.29a)$$

$$\tau_2^b = -p, \quad (14.1.29b)$$

$$\tau_3^b = 0. \quad (14.1.29c)$$

Equation 14.1.29b gives the normal component of traction on the solid surface due to the fluid, and it is just the negative of the pressure. Equation 14.1.29a gives the tangential component of traction due to the fluid. Thus, if we know how v_1 varies with x_2 , we can calculate the viscous stress transmitted to the solid surface.

Although (14.1.29a) expresses the derivative of the velocity at the boundary, we also need to know the velocity at the boundary. We specify that the relative tangential velocity at the boundary is zero,

$$(\mathbf{v}_a - \mathbf{v}_b) \times \mathbf{n} = 0, \quad (14.1.30)$$

where $\mathbf{v}_a$ is the velocity of the fluid and $\mathbf{v}_b$ is the velocity of the boundary. The physical reasoning to justify this condition is as follows: a viscous fluid

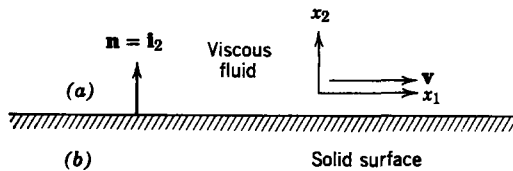


Fig. 14.1.11 A boundary between a viscous fluid and a solid surface.

exhibits friction when it flows along a solid surface, although this surface friction may have a different constant than the bulk coefficient of viscosity. If there were any tangential slippage at the boundary, it would represent an impulse in $(\partial v_1/\partial x_2)$ at the boundary, and with a finite friction coefficient it would require an infinite surface shear force. We conclude that there can be no slippage between a viscous fluid and a solid boundary.

14.1.3 Fluid-Mechanical Examples

Now that we have established the momentum equation and boundary conditions that describe incompressible viscous fluid flow it is appropriate to study some fluid flow examples without electromechanical coupling to establish ideas concerning the principal effects of viscosity that will show up later in electromechanical examples.

We consider again the example introduced in Section 14.1.1. The system is shown in Fig. 14.1.12. A viscous fluid is constrained between two parallel rigid plates separated by a distance d . The lower plate is fixed and the upper plate is made to move in the x_1 -direction with constant velocity v_o . The fluid can be considered homogeneous and incompressible, $\nabla \cdot \mathbf{v} = 0$, and has a coefficient of viscosity μ . The hydrostatic pressure is constrained to be constant throughout the fluid, the system is in the steady state, and the plates are large enough in the x_1 - and x_3 -directions that we can neglect edge effects. Neglect the force of gravity. We want to find all components of velocity between the plates and the tangential traction applied to the plates by the fluid.

From (14.1.23), the momentum equation is:

$$\rho(\mathbf{v} \cdot \nabla)\mathbf{v} = \mu \nabla^2 \mathbf{v}.$$

The upper plate is driven in the x_1 -direction so we expect that $v_3 = 0$. Furthermore, the large dimensions in the x_1 and x_3 directions allow us to assume

$$\partial/\partial x_1 = \partial/\partial x_3 = 0.$$

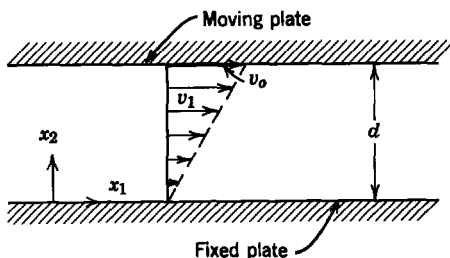


Fig. 14.1.12 A simple example of shear flow.

The use of these assumptions with $\nabla \cdot \mathbf{v} = 0$ and the boundary condition at $x_2 = \pm d$ yield

$$v_2 = 0.$$

The momentum equation thus reduces to

$$0 = \frac{\partial^2 v_1}{\partial x_2^2}.$$

Integration of this equation twice yields:

$$v_1 = C_1 x_2 + C_2.$$

The requirement of continuous tangential velocity at a boundary (14.1.30) gives

$$\text{at } x_2 = 0, \quad v_1 = 0, \quad C_2 = 0,$$

$$\text{at } x_2 = d, \quad v_1 = v_o, \quad C_1 = \frac{v_o}{d}.$$

Thus

$$v_1 = v_o \frac{x_2}{d}.$$

This profile, referred to as *plane Couette flow*, is sketched in Fig. 14.1.12.

The use of (14.1.27) yields for the tangential component of traction

$$\text{at } x_2 = 0, \quad \tau_1 = \mu \frac{v_o}{d},$$

$$\text{at } x_2 = d, \quad \tau_1 = -\mu \frac{v_o}{d}.$$

It is apparent that the viscous traction tends to oppose the relative motion of the two plates as we intuitively expected.

The type of flow described in this example is used in a device specifically designed to measure the coefficient of viscosity. It consists of two concentric cylinders with a small annular space between them. One cylinder is fixed and the other can be rotated about its axis. The fluid is introduced into the space between the cylinders, and the torque required to rotate one cylinder at a particular speed is measured. This torque is used with the known lever arm and the expression for surface traction to calculate the viscosity.

As a second example, consider the pressure-driven, steady flow of an incompressible viscous fluid between two fixed parallel plates separated by a distance $2d$, as illustrated in Fig. 14.1.13. The lateral extent of the plates

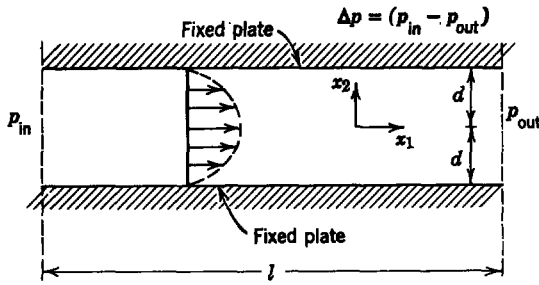


Fig. 14.1.13 Two-dimensional viscous flow.

is large enough compared with $2d$ that we can neglect edge effects. We orient our coordinate axes x_1 and x_2 so that there is no flow or variation of velocity along x_3 . We assume that there is only an x_1 -component of velocity and show that this satisfies the differential equations and the boundary conditions.

In this case our assumptions are

$$v_2 = v_3 = 0, \quad \frac{\partial}{\partial x_3} = 0.$$

With these assumptions $\nabla \cdot \mathbf{v} = 0$ reduces to

$$\frac{\partial v_1}{\partial x_1} = 0, \quad (14.1.31)$$

and we can write the x_1 -component of (14.1.22) as

$$0 = -\frac{\partial p}{\partial x_1} + \mu \frac{\partial^2 v_1}{\partial x_2^2}. \quad (14.1.32)$$

When we take the partial derivative of this expression with respect to x_1 and use (14.1.31), we find

$$\frac{\partial^2 p}{\partial x_1^2} = 0, \quad (14.1.33)$$

which shows that $\partial p / \partial x_1$ is constant. Setting

$$\frac{\partial p}{\partial x_1} = -\frac{\Delta p}{l}, \quad (14.1.34)$$

where Δp is the pressure difference impressed across the length l in the x_1 -direction (the pressure decreases with x_1 so that the flow is in the positive x_1 -direction), we can write (14.1.32) as

$$\mu \frac{\partial^2 v_1}{\partial x_2^2} = -\frac{\Delta p}{l}. \quad (14.1.35)$$

Integrating this expression twice with respect to x_2 yields

$$v_1 = -\frac{\Delta p}{2\mu l} x_2^2 + C_1 x_2 + C_2. \quad (14.1.36)$$

We first use the symmetry condition that $dv_1/dx_2 = 0$ at $x_2 = 0$ to set $C_1 = 0$. We then use the boundary conditions that $v_1 = 0$ at $x_2 = \pm d$ to obtain

$$C_2 = \frac{\Delta p d^2}{2\mu l}.$$

The final result, referred to as *plane Poiseuille flow*, is

$$v_1 = \frac{\Delta p}{2\mu l} (d^2 - x_2^2). \quad (14.1.37)$$

The parabolic velocity profile indicated by (14.1.37) is sketched in Fig. 14.1.13. It can be shown by direct substitution that this solution also satisfies the x_2 - and x_3 -components of (14.1.22).

14.2 ELECTROMECHANICAL COUPLING WITH VISCOUS FLUIDS

To illustrate in simple contexts how viscosity affects electromechanical coupling and vice versa we reconsider the two examples of the preceding section with electromechanical coupling added. In both cases we apply magnetic fields and assume that the fluids have high enough electrical conductivity that a quasi-static magnetic field system is the appropriate electromagnetic model.

14.2.1 Electromechanical Coupling with Shear Flow

The system to be analyzed is shown in Fig. 14.2.1*b*. It consists of two parallel, highly conducting plates, separated by an incompressible conducting liquid with conductivity σ and coefficient of viscosity μ . The plates and liquid are both nonmagnetic. The lower plate is fixed and the upper plate is moving with velocity v_o in the x_1 -direction. A uniform flux density B_o is applied (by a system not shown) in the x_3 -direction and a uniform current density J_o is injected in the x_2 -direction. The system is operating in the steady state ($\partial/\partial t = 0$). This is a representation of a liquid metal brush like those used in the acyclic generator of Fig. 6.4.13. (See Fig. 14.2.1*a*.) Our use of a plane geometry is a simplification based on the fact that the thickness d of the fluid is very small compared with the radius of curvature. Our analysis allows assessment of the effect of B_o on the characteristics of the brush, primarily the voltage drop of the brush and the losses, both electrical and mechanical, that heat the fluid.

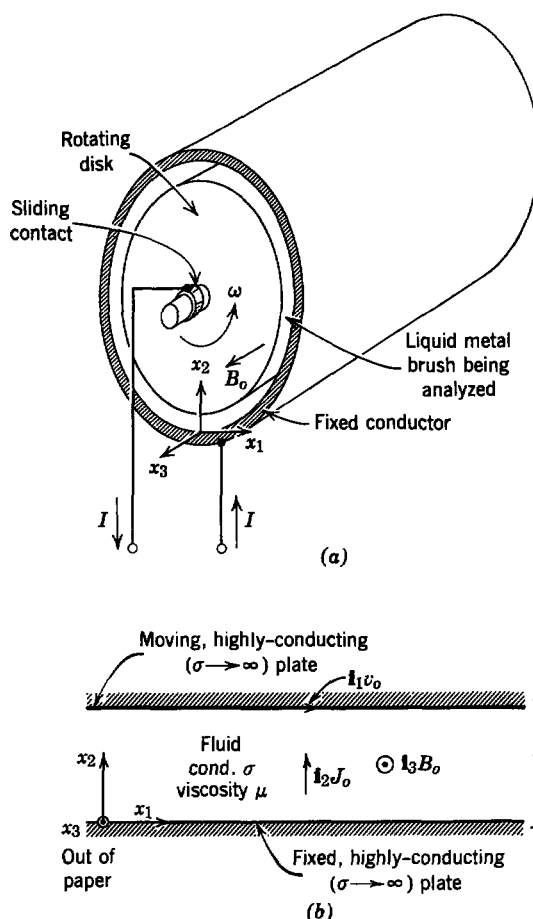


Fig. 14.2.1 Configuration for analyzing the effect of electromechanical coupling on shear flow. (a) physical system, (b) mathematical approximation.

Recognizing that this planar model is a representation of an annular system that closes on itself in the x_1 -direction and that the system is symmetrical, we conclude that no properties can vary with x_1 ; thus

$$\frac{\partial}{\partial x_1} = 0.$$

Furthermore, the system is large enough in the x_3 -direction that we can neglect end effects. The current paths external to the fluid close in such a way that the field induced by current density J_0 according to Ampère's law is only in the x_1 -direction. With these assumptions about the system

we can specify the relevant variables as

$$\begin{aligned} \mathbf{v} &= \mathbf{i}_1 v_1(x_2), & \mathbf{J} &= \mathbf{i}_2 J_o \\ \mathbf{E} &= \mathbf{i}_2 E_2(x_2), & \mathbf{B} &= \mathbf{i}_3 B_o + \mathbf{i}_1 B_1(x_2). \end{aligned} \quad (14.2.1)$$

These explicit functional dependences will be shown to satisfy the relevant differential equations and boundary conditions as the analysis proceeds.

With the variables defined above and the restrictions cited earlier, the x_1 -component of the momentum equation (14.1.22) is

$$0 = J_o B_o + \mu \frac{\partial^2 v_1}{\partial x_2^2}. \quad (14.2.2)$$

The quantities J_o , B_o , and μ are constants and v_1 varies only with x_2 ; consequently, (14.2.2) is integrated twice to obtain

$$v_1 = -\frac{J_o B_o}{2\mu} x_2^2 + C_1 x_2 + C_2, \quad (14.2.3)$$

where C_1 and C_2 are constants of integration to be determined by boundary conditions. The boundary condition of (14.1.30) which requires no slippage yields

$$\begin{aligned} \text{at } x_2 &= 0, & v_1 &= 0, \\ \text{at } x_2 &= d, & v_1 &= v_o. \end{aligned}$$

The use of these two conditions to evaluate C_1 and C_2 in (14.2.3) leads to the resulting velocity

$$v_1 = v_o \frac{x_2}{d} + \frac{J_o B_o d^2}{2\mu} \frac{x_2}{d} \left(1 - \frac{x_2}{d}\right). \quad (14.2.4)$$

The first term is just the linear variation obtained in Fig. 14.1.12 with no electromechanical coupling. The second term is a parabolic profile that results from the electromechanical coupling. The second term can be positive or negative, depending on the sign of the product $J_o B_o$. The two terms in (14.2.4) are sketched with the composite profile for two conditions in Fig. 14.2.2. The $\mathbf{J} \times \mathbf{B}$ force density in the x_1 -direction is uniform; consequently, the parabolic profile is expected from the results of the second example of Section 14.1.3, which was driven by a constant force density due to a pressure gradient (see Fig. 14.1.13). It is clear from Fig. 14.2.2 that the magnetic force can have a marked effect on the profile, even reversing the velocity in Fig. 14.2.2b. The profiles indicate more shear strain rate with the magnetic force than without; thus, as we shall see subsequently, we can expect increased fluid mechanical losses.

At this point it is well to note that there is an x_3 component of $\mathbf{J} \times \mathbf{B}$ ($J_o B_1$). Because B_1 is excited by J_o it depends only on x_3 , thus the x_3 component

The use of this expression, with (14.2.4) for v_1 , in (14.2.5) and evaluation of the integral lead to

$$V = \frac{J_o}{\sigma} d + B_o v_o \frac{d}{2} + \frac{J_o B_o^2 d^3}{12\mu}. \quad (14.2.7)$$

The first term is just the voltage that results with J_o in the fluid at rest. The second term can be interpreted as the speed voltage generated by the linear velocity variation of simple shear flow (see Fig. 14.2.2). The last term is the speed voltage generated by the parabolic velocity profile of Fig. 14.2.2.

The electrical power input p_e per unit area in an x_1 - x_3 -plane is found by taking the product

$$p_e = J_o V = \frac{J_o^2}{\sigma} d + J_o B_o v_o \frac{d}{2} + \frac{J_o^2 B_o^2 d^3}{12\mu}. \quad (14.2.8)$$

The first and third terms are always positive, but the second term can be negative, in which case the brush can act as an MHD generator and produce net electrical power. This is not a practical source of power, but it does indicate the nature of possible electrical characteristics of liquid metal brushes in the presence of a magnetic field.

The traction (force per unit area) that must be applied to the upper plate to maintain the steady motion is found from (14.1.29a) to be

$$\tau_1 = \mu \left. \frac{\partial v_1}{\partial x_2} \right|_{x_2=d}. \quad (14.2.9)$$

The use of (14.2.4) in this expression yields

$$\tau_1 = \mu \frac{v_o}{d} - \frac{J_o B_o d}{2}. \quad (14.2.10)$$

The second term is negative because, as indicated in Fig. 14.2.2a, the $\mathbf{J} \times \mathbf{B}$ force drives the fluid faster and tries to accelerate the upper plate, thus requiring a negative component of traction to maintain steady velocity.

The mechanical input power p_m per unit area is found as

$$p_m = \tau_1 v_o = \mu \frac{v_o^2}{d} - \frac{J_o B_o v_o d}{2}. \quad (14.2.11)$$

Once again the brush may actually produce net mechanical power.

To find the total power per unit area p_t put into the brush electrically and mechanically we add (14.2.8) and (14.2.11) to obtain the result.

$$p_t = p_e + p_m = \frac{J_o^2}{\sigma} d + \mu \frac{v_o^2}{d} + \frac{J_o^2 B_o^2 d^3}{12\mu}. \quad (14.2.12)$$

The total power input is always positive, even though (14.2.8) and (14.2.11) indicate that there may be net electrical output or net mechanical output.

In (14.2.12) the first term is simply the Joule loss associated with current in the conducting fluid; the second term is the viscous loss that would result from simple shear flow; and the third term results from the electromechanical coupling.

It should be clear from the form of (14.2.12) that for a system in which the quantities σ , μ , J_o , v_o , and B_o are known there is a brush thickness that minimizes brush losses. This optimum brush thickness can be found by a straightforward application of differential calculus. We shall not carry out this process here. Rather, we shall assume liquid metal brush parameters typical of configurations used in practice and compare the brush properties with those of carbon brushes. At the same time we shall assess the effects of electromechanical coupling on the liquid metal brush.

The parameters assumed for the liquid metal brush are given in Table 14.2.1. Substitution of these quantities into (14.2.7) and (14.2.12) yields

$$V = 0.015 + 0.15B_o + 83.3B_o^2 \text{ V},$$

$$p_t = 2.26 \times 10^4 + 1.25 \times 10^8 B_o^2 \text{ W/m}^2.$$

Table 14.2.1 Parameters Assumed for Liquid Metal Brush

Material	Mercury
Conductivity	$\sigma = 10^8 \text{ mhos/m}$
Viscosity	$\mu = 1.5 \times 10^{-3} \text{ kg/m-sec}$
Current density	$J_o = 1.5 \times 10^6 \text{ A/m}^2$
Velocity	$v_o = 30 \text{ m/sec}$
Brush thickness	$d = 10^{-2} \text{ m}$

These two quantities are plotted on logarithmic scales as functions of flux density B_o in Fig. 14.2.3. The characteristics of a metal-graphite brush of the type normally used for slip rings* are also shown. The metal-graphite brush operates at only *one tenth* the current density of the liquid metal brush; thus for the same total current the solid brush will require 10 times as much contact area. At very low flux density the liquid metal brush has a voltage drop roughly two orders of magnitude lower than that of a solid brush, and the power loss per unit area is three orders of magnitude lower. When the flux density gets to the range of 0.1 to 1 Wb/m² the liquid metal brush performance deteriorates rapidly as flux density increases. The curves of Fig. 14.2.3 demonstrate clearly the superiority of liquid metal brushes with respect to contact drop and losses, but they also demonstrate that magnetic fields can degrade the liquid metal brush performance markedly.

In this example we have demonstrated how electromechanical coupling

* The characteristics of the metal-graphite brush were taken from *Standard Handbook for Electrical Engineers*, A. E. Knowlton, ed., McGraw-Hill, New York, 9th ed., 1957, Sections 4-234 and 8-120.

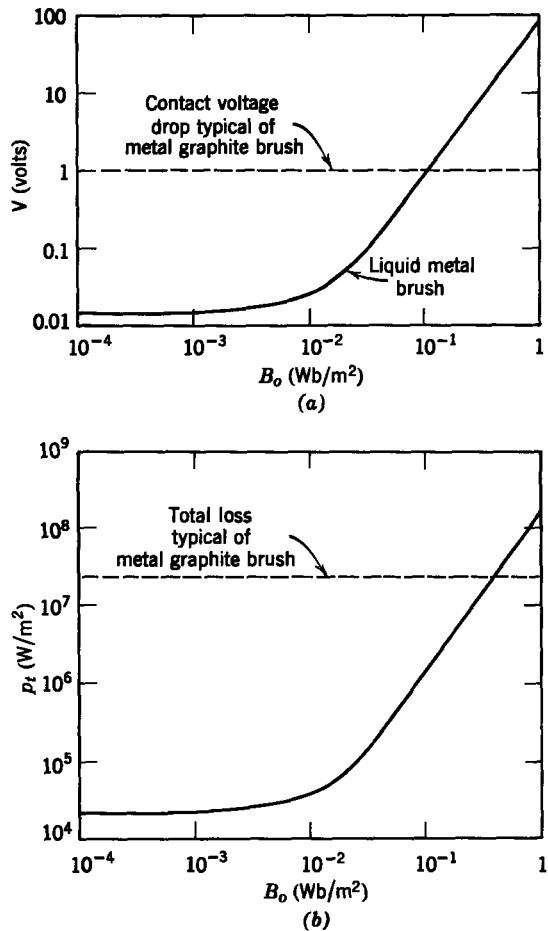


Fig. 14.2.3 Voltage and brush loss as functions of applied flux density for liquid metal brushes compared with metal graphite brushes: (a) brush voltage; (b) total brush loss.

can have a considerable effect on the behavior of a fluid mechanical system in a configuration of practical significance. In the next section we treat another configuration in which electromechanical coupling can have significant effects.

14.2.2 Electromechanical Coupling with Pressure-Driven Flow (Hartmann Flow)

In this section we consider the effects of viscosity on fluid flow in a rectangular channel. The electrical conductivity of the fluid is high enough to justify a magnetic field system model and the fluid is subjected to transverse magnetic field and current. Thus this example allows us to assess the effects

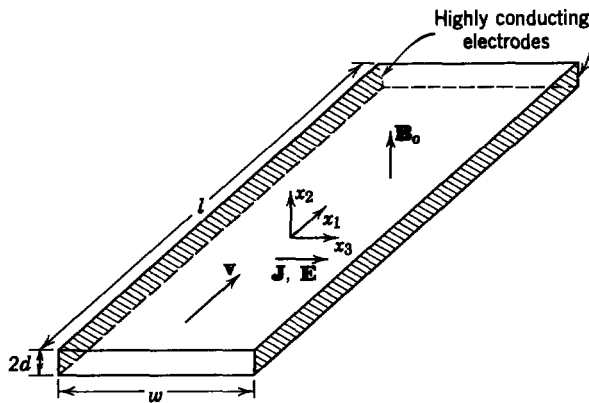


Fig. 14.2.4 Configuration for studying Hartmann flow.

of viscosity on the conduction-type MHD machines considered in Chapters 12 and 13 and to determine the conditions under which the inviscid fluid models used there are accurate.

The flow pattern to be analyzed is conventionally called *Hartmann flow* and will be studied in the configuration of Fig. 14.2.4. The channel has a length l , a width w , and a depth $2d$. The aspect ratio of the channel is quite large

$$\frac{w}{2d} \gg 1,$$

and we are interested in the flow properties near the center of the channel. Thus we can neglect variations with x_3 . The sides of the channel that lie in the x_1 - x_2 planes are highly conducting electrodes and can be connected to an external circuit with the result that there can be net current flow across any x_1 - x_2 plane.

The fluid is assumed to be incompressible with coefficient of viscosity μ and electrical conductivity σ . There is an applied flux density $\mathbf{B}_0$,

$$\mathbf{B}_0 = i_2 B_0, \quad (14.2.13)$$

and we assume that the magnetic flux density due to current flow in the fluid is negligible compared to B_0 (low magnetic Reynolds number). We also assume steady flow.

We neglect end effects, and thus we are considering the two-dimensional problem in Fig. 14.2.5. The symmetry of the simplified problem allows us to assume the following forms for the variables:

$$\mathbf{v} = i_1 v_1(x_2), \quad (14.2.14a)$$

$$\mathbf{E} = i_3 E_3(x_2), \quad (14.2.14b)$$

$$\mathbf{J} = i_3 J_3(x_2). \quad (14.2.14c)$$

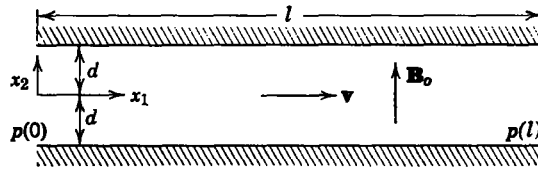


Fig. 14.2.5 Two-dimensional configuration for Hartmann flow.

All of these variables are functions of x_2 only, as indicated. It should be clear at this point that our example is simply the one shown in Fig. 14.1.13 with the addition of a magnetic force density. Thus we expect our result to be similar to and interpretable in terms of the results given in Fig. 14.1.13. From Faraday's law (1.1.5)* we obtain

$$\frac{dE_3}{dx_2} = 0, \quad (14.2.15)$$

from which we obtain

$$E_3 = \text{constant}. \quad (14.2.16)$$

The x_3 -component of Ohm's law, (12.2.18) is

$$J_3 = \sigma(E_3 + v_1 B_0). \quad (14.2.17)$$

The x_1 -component of the momentum equation (14.1.22) is

$$0 = -\frac{\partial p}{\partial x_1} + \mu \frac{d^2 v_1}{dx_2^2} - J_3 B_0. \quad (14.2.18)$$

Recognizing that J_3 and v_1 are not functions of x_1 , we differentiate (14.2.18) with respect to x_1 to obtain

$$\frac{\partial^2 p}{\partial x_1^2} = 0, \quad (14.2.19)$$

which shows that

$$\frac{\partial p}{\partial x_1} = \text{constant}. \quad (14.2.20)$$

Denoting the pressure drop over the length of the channel as Δp ,

$$\Delta p = p(0) - p(l), \quad (14.2.21)$$

we can write (14.2.20)

$$\frac{\partial p}{\partial x_1} = -\frac{\Delta p}{l}. \quad (14.2.22)$$

Compare these results with (14.1.33) and (14.1.34). We assume that Δp is

* Table 1.2, Appendix G.

maintained constant by external means; thus it represents an independent input to the system.

We now substitute (14.2.17) and (14.2.22) in (14.2.18) and rearrange the resulting expression to obtain

$$\frac{d^2 v_1}{dx_2^2} - \frac{\sigma B_o^2}{\mu} v_1 = -\frac{\Delta p}{\mu l} + \frac{\sigma B_o E_3}{\mu}. \quad (14.2.23)$$

Equation 14.2.16 indicates that E_3 is constant; thus we can solve this linear differential equation with constant coefficients to obtain, in general,

$$v_1 = C_1 \sinh M \frac{x_2}{d} + C_2 \cosh M \frac{x_2}{d} + \frac{\Delta p}{\sigma B_o^2 l} - \frac{E_3}{B_o}, \quad (14.2.24)$$

where we have defined the *Hartmann number* M as

$$M = B_o d \left(\frac{\sigma}{\mu} \right)^{1/2}. \quad (14.2.25)$$

We must now apply the boundary condition that at

$$x_2 = \pm d, \quad v_1 = 0. \quad (14.2.26)$$

This imposes the requirement that v_1 be an even function of x_2 ; thus

$$C_1 = 0, \quad (14.2.27)$$

and the constant C_2 is then given by

$$C_2 = \frac{-1}{\cosh M} \left(\frac{\Delta p}{\sigma B_o^2 l} - \frac{E_3}{B_o} \right). \quad (14.2.28)$$

We now use (14.2.27) and (14.2.28) to write (14.2.24) as

$$v_1 = \left(\frac{\Delta p}{\sigma B_o^2 l} - \frac{E_3}{B_o} \right) \left[1 - \frac{\cosh M(x_2/d)}{\cosh M} \right]. \quad (14.2.29)$$

To complete the solution for the velocity profile we must specify the value of the electric field E_3 . We can fix E_3 by the application of a terminal voltage as we did in Section 12.2.1a. Alternatively, we can fix E_3 by constraining the total current that can flow across an x_1 - x_2 plane. To complete the present example we assume that the external terminals are open-circuited; the result is no net current across an x_1 - x_2 plane. Mathematically, this requirement can be written as

$$\int_{-d}^d J_3 dx_2 = 0. \quad (14.2.30)$$

We use (14.2.17) with this expression to obtain

$$E_3 = -\frac{B_o}{2d} \int_{-d}^d v_1 dx_2. \quad (14.2.31)$$

We now use (14.2.29) in (14.2.31), evaluate the integral, and solve for E_3 to obtain

$$E_3 = -\frac{\Delta p}{\sigma l B_o} \left(\frac{M \cosh M - \sinh M}{\sinh M} \right). \quad (14.2.32)$$

Substitution of (14.2.32) into (14.2.29) and simplification yield

$$v_1 = \frac{\Delta p}{\sigma B_o^2 l} \left\{ \frac{M [\cosh M - \cosh M(x_2/d)]}{\sinh M} \right\}. \quad (14.2.33)$$

It will be easy to interpret our results if we assume that the mechanical properties are fixed and vary the applied magnetic field to vary M . For this purpose it is more useful to write (14.2.33) in the form

$$v_1 = \frac{\Delta p d^2}{\mu l} \left[\frac{\cosh M - \cosh M(x_2/d)}{M \sinh M} \right]. \quad (14.2.34)$$

As a check on this expression, it is easily verified that the limit taken as $M \rightarrow 0$ yields the same velocity profile as in the second example of Section 14.1.3 (see Fig. 14.1.13). This is as it should be, for when $M \rightarrow 0$ in (14.2.34) this profile is achieved by eliminating electrical and retaining only viscous effects.

To interpret the meaning of the profile shape of (14.2.34) we assume a system of fixed dimensions, with a fluid of fixed properties and constrain the pressure drop Δp to be fixed. We now vary the applied flux density B_o and ask how the shape of the profile changes. Such changes are indicated in Fig. 14.2.6, in which the profiles are plotted for three values of M . Note that the profile for $M = 0$ is the same as that plotted in Fig. 14.1.13.

To interpret the results of Fig. 14.2.6 we observe first that in the absence of electromagnetic forces the velocity profile is parabolic. This is the profile given with viscous effects alone. When electromagnetic forces are present, any local variation of velocity will generate circulating currents that interact with the applied magnetic field to reduce the local velocity variations. Thus the presence of the applied magnetic field tends to make the velocity uniform. When the applied magnetic field is large (M large), the velocity is uniform near the center of the channel and varies appreciably in the vicinity of the walls where the velocity must go to zero. Whether the velocity profile is flat or parabolic depends on the value of the Hartmann number M given by (14.2.25) as

$$M = B_o d \left(\frac{\sigma}{\mu} \right)^{1/2}.$$

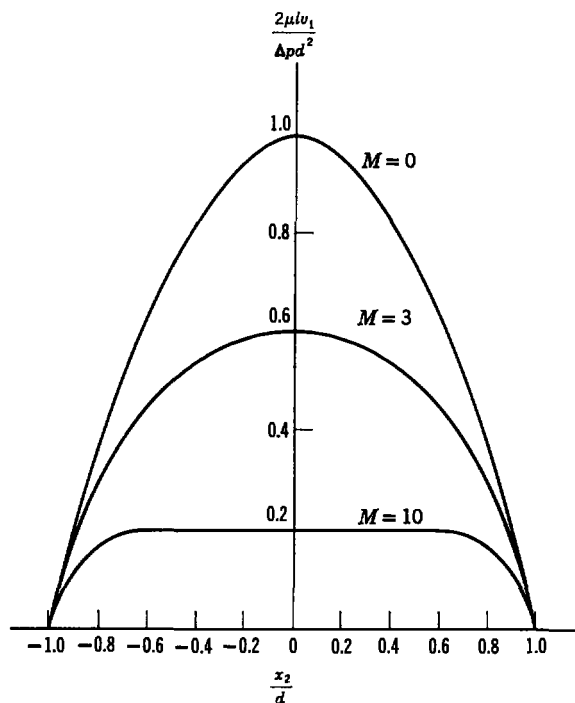


Fig. 14.2.6 Variation of velocity profile with Hartmann number.

For low values of M viscous forces predominate and the profile tends to be parabolic. For high values of M electromagnetic forces predominate and the velocity profile tends to be flat. Thus the Hartmann number is interpreted as a measure of the relative magnitude of electromagnetic and viscous forces. It is clear that for high values of M the model of uniform flow velocity used earlier in Section 12.2.1*a* is valid over most of the channel width. We consider this situation subsequently with more precision.

It is evident from the curves of Fig. 14.2.6 that with a fixed pressure drop the volume flow rate of the fluid is reduced by the presence of an applied magnetic field. If we designate the volume flow rate by $\dot{V}_{oi}$, it is given by

$$\dot{V}_{oi} = w \int_{-d}^d v_1 dx_2 \text{ m}^3/\text{sec}. \quad (14.2.35)$$

We now use (14.2.34) in this expression, perform the integration, and simplify to obtain

$$\dot{V}_{oi} = \frac{2\Delta p d^3 w}{3\mu l} \left[\frac{3}{M^2} (M \coth M - 1) \right]. \quad (14.2.36)$$

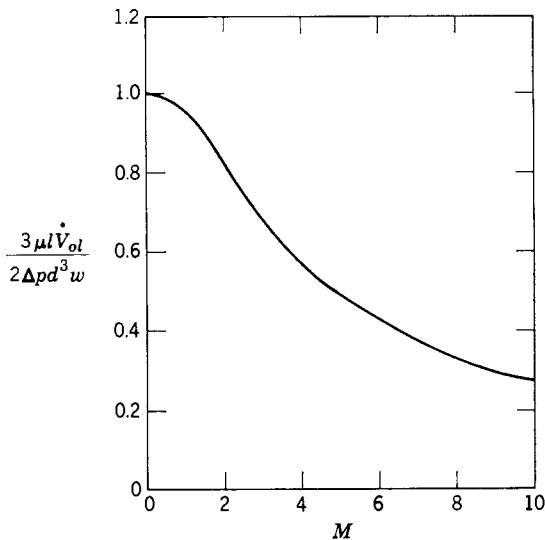


Fig. 14.2.7 Variation of volume flow rate with Hartmann number.

In the limit as $M \rightarrow 0$ the volume flow rate becomes

$$(\dot{V}_{ol})_{M=0} = \frac{2\Delta p d^3 w}{3\mu l},$$

which is the volume flow rate in the presence of viscous forces only. The expression of (14.2.36), normalized to the value for $M \rightarrow 0$,

$$\frac{3\mu l \dot{V}_{ol}}{2\Delta p d^3 w} = \frac{3}{M^2} (M \coth M - 1), \quad (14.2.37)$$

is plotted as a function of M in Fig. 14.2.7. It is evident from the curve of Fig. 14.2.7 that the presence of a magnetic field that yields an appreciable Hartmann number will markedly reduce the volume flow rate for a given pressure drop.

To ascertain the order of magnitude of Hartmann number that can be obtained with real conducting fluids consider liquid mercury which has

$$\sigma = 10^6 \text{ mhos/m}, \quad \mu = 1.5 \times 10^{-3} \text{ kg/m-sec}$$

If we consider a system with a channel depth of 2 cm,

$$d = 10^{-2} \text{ m},$$

and an applied flux density of

$$B_o = 1 \text{ Wb/m}^2 = 10,000 \text{ gauss},$$

the resulting Hartmann number is

$$M = 260.$$

Thus the flow of mercury under these conditions is very strongly affected by electromagnetic forces. The same is true of other liquid metals.

Alternatively, a seeded combustion gas would be used in the variable-area channel considered in Section 13.2.2. This system would have approximately the following constants,

$$\begin{aligned}\sigma &= 40 \text{ mhos/m}, & \mu &= 10^{-5} \text{ kg/m-sec}, \\ B_o &= 4 \text{ Wb/m}^2, & d &= 10^{-1} \text{ m},\end{aligned}$$

and a Hartmann number

$$M = 800.$$

Thus, even with an ionized gas with its very small conductivity, this large Hartmann number indicates that magnetic forces predominate over viscous forces.

To be more precise about how viscosity affects a conduction-type MHD machine we remove the constraint of no net current and operate the system in Fig. 14.2.4 as we did the MHD machine in Section 12.2.1a. In the present system the channel depth is $2d$, whereas in Fig. 12.2.3 the channel depth is d . Redefining the quantities for the inviscid fluid model to account for this difference, we have [see (12.2.21), (12.2.22), and (12.2.24)] for the inviscid model

$$\text{internal resistance,} \quad R_i = \frac{w}{2\sigma l d}, \quad (14.2.38)$$

$$\text{voltage equation,} \quad IR_i = v_o B_o w - V, \quad (14.2.39)$$

$$\text{pressure drop,} \quad \Delta p = \frac{IB}{2d}, \quad (14.2.40)$$

where I and V are defined in Fig. 14.2.4 and v_o is the fluid velocity for an inviscid fluid model.

We now remove the constraint of no net current (14.2.30) and instead specify that the electric field intensity be given in terms of the terminal voltage as

$$E_3 = -\frac{V}{w}. \quad (14.2.41)$$

From Ohm's law (14.2.17) the current density becomes

$$J_3 = \sigma \left(-\frac{V}{w} + v_1 B_o \right) \quad (14.2.42)$$

and the terminal current is

$$I = \int_{-d}^d J_3 l \, dx_2. \quad (14.2.43)$$

Substitution of (14.2.42) into (14.2.43) yields

$$I = -\frac{V}{R_i} + \sigma l B_o \int_{-d}^d v_1 \, dx_2. \quad (14.2.44)$$

When the space average velocity is defined

$$\langle v_1 \rangle = \frac{1}{2d} \int_{-d}^d v_1 \, dx_2, \quad (14.2.45)$$

we can write (14.2.44) in the form

$$IR_i = -V + B_o w \langle v_1 \rangle. \quad (14.2.46)$$

This expression is the same as (14.2.39) for the inviscid model except that v_o for the inviscid case is replaced by average velocity in the viscous case.

The most important aspect of viscosity in an MHD machine is how much of the pressure gradient goes into viscous losses and how much is balanced by the magnetic force density. To answer this question we evaluate the average velocity by using (14.2.29) in (14.2.45) to obtain

$$\langle v_1 \rangle = \left(\frac{\Delta p}{\sigma B_o^2 l} + \frac{V}{B_o w} \right) \left(1 - \frac{\tanh M}{M} \right). \quad (14.2.47)$$

The use of this result in (14.2.46) and solution for Δp yield

$$\Delta p = \frac{B_o I / 2d + (B_o \sigma l V / w) (\tanh M / M)}{1 - \tanh M / M}. \quad (14.2.48)$$

Noting that the first term in the numerator is the pressure drop in an inviscid machine, as given by (14.2.40), we write the ratio

$$\frac{\Delta p}{\Delta p_i} = \frac{1 + (V / IR_i) (\tanh M / M)}{1 - \frac{\tanh M}{M}}, \quad (14.2.49)$$

where Δp_i is the ideal pressure drop given by (14.2.40). This ratio is plotted as a function of Hartmann number M in Fig. 14.2.8 with

$$\frac{V}{IR_i} = 1,$$

which is the condition for maximum electrical power extraction from the flowing fluid. Note that the Hartmann number axis is a logarithmic scale.

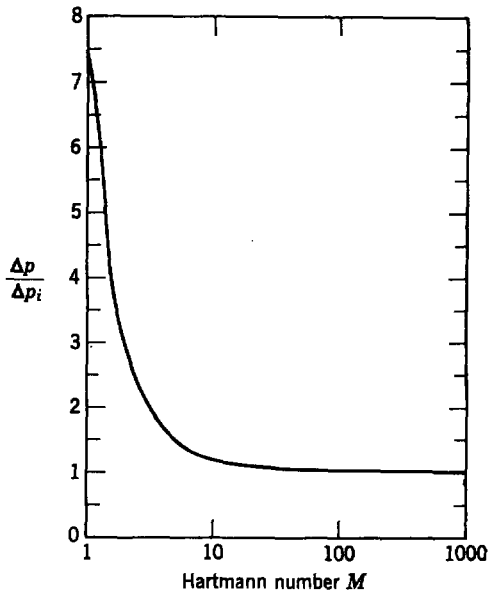


Fig. 14.2.8 Ratio of actual pressure drop and ideal pressure drop for an MHD generator loaded to extract maximum electric power.

It is clear that the error is large for a small Hartmann number with a pressure drop 22 per cent more than the ideal for $M = 10$. At $M = 100$ the error is 2 per cent, and at $M = 1000$ the error is 0.2 per cent. This shows that for a Hartmann number of the order of 100 or larger the inviscid fluid model used to analyze MHD machines in Chapters 12 and 13 is quite accurate with respect to the neglect of the effects of viscosity.

14.3 DISCUSSION

In this chapter we have added the effects of viscosity to the mathematical description of incompressible fluids. In the process we have indicated how this description is modified for compressible fluids. We have analyzed and discussed two applications of the equations to steady flow problems in which viscous effects and electromechanical coupling compete; and one or the other can predominate, depending on the relative values of the critical parameters.

It is a straightforward process to extend the techniques of this chapter to dynamic situations; for example, we can use the description of viscosity in this chapter to study viscous damping of the Alfvén waves defined and analyzed in a lossless system in Section 12.2.3.

PROBLEMS

14.1. Rework the example of Sec. 14.2.1 with the applied flux density in the x_2 direction. Assume that no current can flow in the x_3 direction. In particular obtain expressions for velocity profile (like Eq. 14.2.4), voltage (like Eq. 14.2.7), traction (like Eq. 14.2.10), and total power per unit area (like Eq. 14.2.12). Make plots of voltage and loss per unit area for the constants of Table 14.2.1 and compare the results with those plotted in Fig. 14.2.3.

14.2. A viscous liquid flows through a circular pipe, as shown in Fig. 14P.2. At the inlet the pressure is uniform and equal to p_1 , and at the outlet it is still uniform, but p_2 . The volume rate of flow is Qm^3/sec . Under the assumption that the flow is axisymmetric and steady and that the velocity is low enough that the fluid can be considered incompressible, find the velocity profile $v_z(r)$. *Hint.* Look for solutions where $\mathbf{v} = v_z(r)\mathbf{i}_z$ and $p = p(z)$.

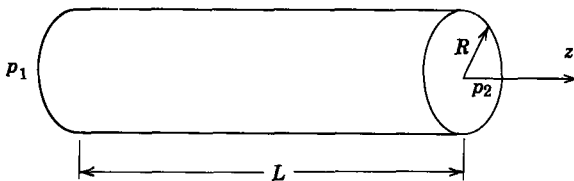


Fig. 14P.2

14.3. The channel shown in Fig. 14P.3 contains a viscous fluid of conductivity σ moving in the x_1 -direction. You are to analyze this problem using the Hartmann flow solutions (Section 14.2.2). The highly conducting electrodes are connected by a load resistance R .

- Given the pressure drop from inlet to outlet, the dimensions of the system, field B_o , and conductivity σ , what is the power dissipated in the resistance?
- What value of R should be used to dissipate the largest possible power in the load?
- If the fluid is mercury, $B_o = 20,000$ gauss, $d = 1$ cm, $l = 1$ m, and $w = 10$ cm, what is the Hartmann number? What is the value of the optimum resistance found in part (b)?

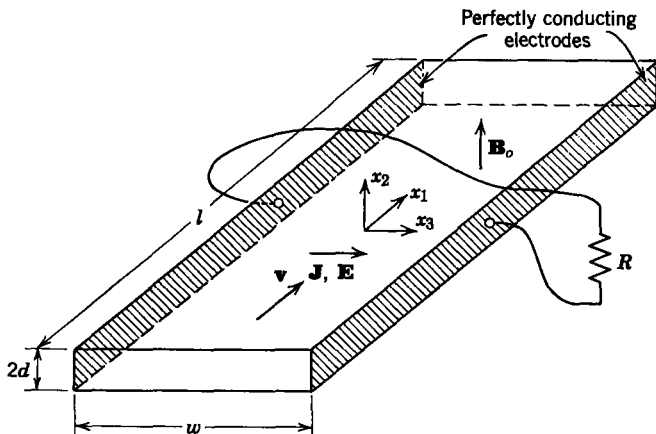


Fig. 14P.3

Appendix F

GLOSSARY OF COMMONLY USED SYMBOLS

Section references indicate where symbols of a given significance are introduced; grouped symbols are accompanied by their respective references. The absence of a section reference indicates that a symbol has been applied for a variety of purposes. Nomenclature used in examples is not included.

Symbol	Meaning	Section
A	cross-sectional area	
A_i	coefficient in differential equation	5.1.1
(A_n^+, A_n^-)	complex amplitudes of components of n th mode	9.2.1
A_w	cross-sectional area of armature conductor	6.4.1
a	spacing of pole faces in magnetic circuit	8.5.1
$a, (a_c, a_d)$	phase velocity of acoustic related waves	13.2.1, 11.4.1
a_b	Alfvén velocity	12.2.3
(a, b, c)	Lagrangian coordinates	11.1
a_i	constant coefficient in differential equation	5.1.1
a_p	instantaneous acceleration of point p fixed in material	2.2.1c
B, B_r, B_θ	damping constant for linear, angular and square law dampers	2.2.1b, 4.1.1, 5.2.2
B, B_i, B_0	magnetic flux density	1.1.1a, 8.1, 6.4.2
B_i	induced flux density	7.0
$(B_r, B_{ra}, B_{rb}, B_{rm})$	radial components of air-gap flux densities	4.1.4
$[B_{rf}, (B_{rf})_{av}]$	radial flux density due to field current	6.4.1
b	width of pole faces in magnetic circuit	8.5
b	half thickness of thin beam	11.4.2b
C	contour of integration	1.1.2a
$C, (C_a, C_b), C_o$	capacitance	2.1.2, 7.2.1a, 5.2.1
C	coefficient in boundary condition	9.1.1
C	the curl of the displacement	11.4
(C^+, C^-)	designation of characteristic lines	9.1.1

Symbol	Meaning	Section
c_p	specific heat capacity at constant pressure	13.1.2
c_v	specific heat capacity at constant volume	13.1.2
D	electric displacement	1.1.1a
d	length	
da	elemental area	1.1.2a
$d\mathbf{f}_n$	total elemental force on material in rigid body	2.2.1c
$d\mathbf{l}$	elemental line segment	1.1.2a
$d\mathbf{T}_n$	torque on elemental volume of material	2.2.1c
dV	elemental volume	1.1.2b
E	constant of motion	5.2.1
E	Young's modulus or the modulus of elasticity	9.1
E, E_o	electric field intensity	1.1.1a, 5.1.2d
E_f	magnitude of armature voltage generated by field current in a synchronous machine	4.1.6a
E_t	induced electric field intensity	7.0
e_{11}, e_{ij}	strain tensor	9.1, 11.2
$\dot{e}_{ij}$	strain-rate tensor	14.1.1a
F	magnetomotive force (mmf)	13.2.2
$\mathbf{F}$	force density	1.1.1a
$\hat{F}$	complex amplitude of $f(t)$	5.1.1
F_0	amplitude of sinusoidal driving force	9.1.3
f	equilibrium tension of string	9.2
f	driving function	5.1.1
$f, \mathbf{f}, f^o, f^s, f_j, f_i, f_1$	force	2.2.1, 2.2.1c, 3.1, 5.1.2a, 3.1.2b, 8.1, 9.1
f	arbitrary scalar function	6.1
f'	scalar function in moving coordinate system	6.1
f	three-dimensional surface	6.2
$\int$	integration constant	11.4.2a
G	a constant	5.1.2c
G	shear modulus of elasticity	11.2.2
G	speed coefficient	6.4.1
G	conductance	3.1
g	air-gap length	5.2.1
$g, \mathbf{g}$	acceleration of gravity	5.1.2c, 12.1.3
(H, H_x, H_y, H_z)	magnetic field intensity	1.1.1a
h	specific enthalpy	13.1.2
$I, I, (I_r, I_\theta), I_f$	electrical current	10.4.3, 12.2.1a, 4.1.2, 6.4.1
$(i, i_1, i_2, \dots, i_n), (i_{ar}, i_{as}, i_{br}, i_{bs}), i_a, (i_a, i_b, i_c), (i_f, i_d), (i_r, i_\theta)$	electrical current	2.1, 4.1.3, 6.4.1, 4.1.7, 6.4.1, 4.1

Symbol	Meaning	Section
$\mathbf{i}_n$	unit vector perpendicular to area of integration	6.2.1
$\mathbf{i}_s$	unit vector normal to surface of integration	6.2.1
$(\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z), (\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$	unit vectors in coordinate directions	2.2.1c
J, J_f	current density	7.0, 1.1.1a
$J, J_r, (J_x, J_y, J_z)$	moment of inertia	5.1.2b, 4.1.1, 2.2.1c
J_{zz}, J_{yz}	products of inertia	2.2.1c
j	$\sqrt{-1}$	4.1.6a
K	loading factor	13.2.2
K, K_f	surface current density	7.0, 1.1.1a
K	linear or torsional spring constant	2.2.1a
K_i	induced surface current density	7.0
$k, k_o, (k_r, k_i)$	wavenumber	7.1.3, 10.1.3, 10.0
k	summation index	2.1.1
k	maximum coefficient of coupling	4.1.6b
k_n	n th eigenvalue	9.2
$(L, L_1, L_2), (L_a, L_f),$ $L_m, (L_0, L_2),$ $(L_r, L_s, L_{sr}), L_{ss}$	inductance	2.1.1, 6.4.1, 2.1.1, 4.2.1, 4.1.1, 4.2.4
L	length of incremental line segment	6.2.1
l	value of relative displacement for which spring force is zero	2.2.1a
l, l_w, l_y	length	
M	Hartmann number	14.2.2
M	mass of one mole of gas in kilograms	13.1.2
M	Mach number	13.2.1
M	mass	2.2.1c
M	number of mechanical terminal pairs	2.1.1
M, M_s	mutual inductance	4.1.1, 4.2.4
$\mathbf{M}$	magnetization density	1.1.1a
m	mass/unit length of string	9.2
N	number of electrical terminal pairs	2.1.1
N	number of turns	5.2.2
n	number density of ions	12.3.1
n	integer	7.1.1
$\mathbf{n}$	unit normal vector	1.1.2
$\mathbf{P}$	polarization density	1.1.1a
P	power	12.2.1a
p	number of pole pairs in a machine	4.1.8
p	power per unit area	14.2.1
P	pressure	5.1.2d and 12.1.4
P_o, P_o, P_m, P_r	power	4.1.6a, 4.1.6b, 4.1.2, 4.1.6b
Q	electric charge	7.2.1a
q, q_i, q_k	electric charge	1.1.3 and 2.1.2, 8.1, 2.1.2
R, R_t, R_o	radius	

Symbol	Meaning	Section
$R, R_a, R_b, R_f, R_r, R_s$	resistance	
(R, R_g)	gas constant	13.1.2
R_s	electric Reynolds number	7.0
R_m	magnetic Reynolds number	7.0
r	radial coordinate	
$\mathbf{r}$	position vector of material	2.2.1c
$\mathbf{r}'$	position vector in moving reference frame	6.1
$\mathbf{r}_m$	center of mass of rigid body	2.2.1c
S	reciprocal modulus of elasticity	11.5.2c
S	surface of integration	1.1.2a
S	normalized frequency	7.2.4
S	membrane tension	9.2
S_s	transverse force/unit length acting on string	9.2
s	complex frequency	5.1.1
(s, s_{mT})	slip	4.1.6b
s_i	i th root of characteristic equation, a	5.1.1
	natural frequency	
T	period of oscillation	5.2.1
T	temperature	13.1.2
$T, T, T^e, T_{em}, T_m,$ T_0, T_1	torque	2.2.1c, 5.1.2b, 3.1.1, 4.1.6b, 4.1.1, 6.4.1, 6.4.1
$\mathbf{T}$	surface force	8.4
T_{ij}^m	mechanical stress tensor	13.1.2
T_{mn}	the component of the stress-tensor in the m th-direction on a cartesian surface with a normal vector in the n th-direction	8.1
T_{or}	constant of coulomb damping	4.1.1
T_0	initial stress distribution on thin rod	9.1.1
T	longitudinal stress on a thin rod	9.1.1
T_z	transverse force per unit area on membrane	9.2
T_2	transverse force per unit area acting on thin beam	11.4.2b
t	time	1.1.1
t'	time measured in moving reference frame	6.1
U	gravitational potential	12.1.3
U	longitudinal steady velocity of string or membrane	10.2
u	internal energy per unit mass	13.1.1
u	surface coordinate	11.3
$u_0(x - x_0)$	unit impulse at $x = x_0$	9.2.1
u	transverse deflection of wire in x -direction	10.4.3
$u_{-1}(t)$	unit step occurring at $t = 0$	5.1.2b
V, V_m	velocity	7.0, 13.2.3
V	volume	1.1.2
V, V_a, V_f, V_o, V_s	voltage	
V	potential energy	5.2.1

Symbol	Meaning	Section
$v, \mathbf{v}$	velocity	
$(v, v_1, \dots, v_k)$	voltage	2.1.1
$v', (v_a, v_b, v_c),$ v_f, v_{0c}, v_t	voltage	
v_n	velocity of surface in normal direction	6.2.1
v_o	initial velocity distribution on thin rod	9.1.1
v_p	phase velocity	9.1.1 and 10.2
$\mathbf{v}^r$	relative velocity of inertial reference frames	6.1
v_s	$\sqrt{f/m}$ for a string under tension f and having mass/unit length m	10.1.1
v	longitudinal material velocity on thin rod	9.1.1
v	transverse deflection of wire in y -direction	10.4.3
(W_e, W_m)	energy stored in electromechanical coupling	3.1.1
(W'_e, W'_m, W')	coenergy stored in electromechanical coupling	3.1.2b
W''	hybrid energy function	5.2.1
w	width	5.2.2
w	energy density	11.5.2c
w'	coenergy density	8.5
X	equilibrium position	5.1.2a
$(x, x_1, x_2, \dots, x_k)$	displacement of mechanical node	2.1.1
x	dependent variable	5.1.1
x_p	particular solution of differential equation	5.1.1
$(x_1, x_2, x_3), (x, y, z)$	cartesian coordinates	8.1, 6.1
(x', y', z')	cartesian coordinates of moving frame	6.1
(α, β)	constants along C^+ and C^- characteristics, respectively	9.1.1
(α, β)	see (10.2.20) or (10.2.27)	
α	transverse wavenumber	11.4.3
(α, β)	angles used to define shear strain	11.2
(α, β)	constant angles	4.1.6b
α	space decay parameter	7.1.4
α	damping constant	5.1.2b
α	equilibrium angle of torsional spring	2.2.1a
γ	ratio of specific heats	13.1.2
γ	piezoelectric constant	11.5.2c
$\gamma, \gamma_0, \gamma'$	angular position	
$\Delta_d(t)$	slope excitation of string	10.2.1b
Δ_0	amplitude of sinusoidal slope excitation	10.2.1b
$\Delta \mathbf{r}$	distance between unstressed material points	11.2.1a
$\Delta \mathbf{s}$	distance between stressed positions of material points	11.2.1a
$\delta()$	incremental change in ()	8.5
$\delta, \delta_1, \delta_0$	displacement of elastic material	11.1, 9.1, 11.4.2a
δ	thickness of incremental volume element	6.2.1
δ	torque angle	4.1.6a

Symbol	Meaning	Section
δ_{ij}	Kronecker delta	8.1
(δ_+, δ_-)	wave components traveling in the $\pm x$ -directions	9.1.1
ϵ	linear permittivity	1.1.1b
ϵ_0	permittivity of free space	1.1.1a
η	efficiency of an induction motor	4.1.6b
η	second coefficient of viscosity	14.1.1c
$\theta, \theta_i, \theta_m$	angular displacement	2.1.1, 3.1.1, 5.2.1
θ	power factor angle; phase angle between current and voltage	4.1.6a
θ	equilibrium angle	5.2.1
$\dot{\theta}$	angular velocity of armature	6.4.1
θ_m	maximum angular deflection	5.2.1
$(\lambda, \lambda_1, \lambda_2, \dots, \lambda_k)$	magnetic flux linkage	2.1.1, 6.4.1, 4.1.7,
λ_a		4.1.3, 4.1
$(\lambda_a, \lambda_b, \lambda_c)$		
$(\lambda_{as}, \lambda_{bs}, \lambda_{cs})$		
(λ_r, λ_s)		
λ	Lamé constant for elastic material	11.2.3
λ	wavelength	7.1.4
μ	linear permeability	1.1.1a
$\mu, (\mu_+, \mu_-)$	mobility	12.3.1, 1.1.1b
μ	coefficient of viscosity	14.1.1
μ_d	coefficient of dynamic friction	2.2.1b
μ_0	permeability of free space	1.1.1a
μ_s	coefficient of static friction	2.2.1b
ν	Poisson's ratio for elastic material	11.2.2
ν	damping frequency	10.1.4
(ξ, ξ)	continuum displacement	8.5
ξ_0	initial deflection of string	9.2
ξ_d	amplitude of sinusoidal driving deflection	9.2
$(\xi_n(x), \xi_n(x))$	n th eigenfunctions	9.2.1b
(ξ_+, ξ_-)	amplitudes of forward and backward traveling waves	9.2
$\dot{\xi}_0(x)$	initial velocity of string	9.2
ρ	mass density	2.2.1c
ρ_f	free charge density	1.1.1a
ρ_s	surface mass density	11.3
Σ	surface of discontinuity	6.2
σ	conductivity	1.1.1a
σ_f	free surface charge density	1.1.1a
σ_m	surface mass density of membrane	9.2
σ_0	surface charge density	7.2.3
σ_s	surface conductivity	1.1.1a
σ_u	surface charge density	7.2.3
τ	surface traction	8.2.1
τ, τ_d	diffusion time constant	7.1.1, 7.1.2a
τ	relaxation time	7.2.1a

Symbol	Meaning	Section
τ_e	electrical time constant	5.2.2
τ_m	time for air gap to close	5.2.2
τ_o	time constant	5.1.3
τ_t	traversal time	7.1.2a
ϕ	electric potential	7.2
ϕ	magnetic flux	2.1.1
ϕ	cylindrical coordinate	2.1.1
ϕ	potential for H when $J_f = 0$	8.5.2
ϕ	flow potential	12.2
χ_e	electric susceptibility	1.1.1b
χ_m	magnetic susceptibility	1.1.1a
ψ	the divergence of the material displacement	11.4
ψ	angle defined in Fig. 6.4.2	6.4.1
ψ	angular position in the air gap measured from stator winding (a) magnetic axis	4.1.4
ψ	electromagnetic force potential	12.2
ψ	angular deflection of wire	10.4.3
Ω	equilibrium rotational speed	5.1.2b
Ω	rotation vector in elastic material	11.2.1a
Ω_n	real part of eigenfrequency (10.1.47)	10.1.4
$\omega, (\omega_r, \omega_s)$	radian frequency of electrical excitation	4.1.6a, 4.1.2
ω	natural angular frequency (Im s)	5.1.2b
ω, ω_m	angular velocity	2.2.1c, 4.1.2
ω_c	cutoff frequency for evanescent waves	10.1.2
ω_d	driving frequency	9.2
ω_n	n th eigenfrequency	9.2
ω_o	natural angular frequency	5.1.3
(ω_r, ω_i)	real and imaginary parts of ω	10.0
∇	nabla	6.1
∇_Σ	surface divergence	6.2.1

1.8. B/c*

Appendix G

SUMMARY OF PARTS I AND II AND USEFUL THEOREMS

IDENTITIES

$$\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{B} \times \mathbf{C},$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi,$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B},$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B},$$

$$\nabla(\phi\psi) = \phi \nabla\psi + \psi \nabla\phi,$$

$$\nabla \cdot (\psi\mathbf{A}) = \mathbf{A} \cdot \nabla\psi + \psi \nabla \cdot \mathbf{A},$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B},$$

$$\nabla \cdot \nabla\phi = \nabla^2\phi,$$

$$\nabla \cdot \nabla \times \mathbf{A} = 0,$$

$$\nabla \times \nabla\phi = 0,$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A},$$

$$(\nabla \times \mathbf{A}) \times \mathbf{A} = (\mathbf{A} \cdot \nabla)\mathbf{A} - \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A}),$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

$$\nabla \times (\phi\mathbf{A}) = \nabla\phi \times \mathbf{A} + \phi \nabla \times \mathbf{A},$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}.$$

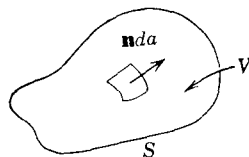
THEOREMS

$$\int_a^b \nabla \phi \cdot d\mathbf{l} = \phi_b - \phi_a.$$



Divergence theorem

$$\oint_S \mathbf{A} \cdot \mathbf{n} \, da = \int_V \nabla \cdot \mathbf{A} \, dV$$



Stokes's theorem

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} \, da$$

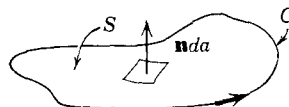
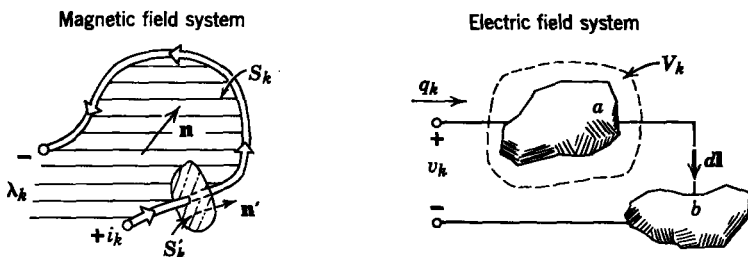


Table 1.2 Summary of Quasi-Static Electromagnetic Equations

	Differential Equations		Integral Equations	
Magnetic field system	$\nabla \times \mathbf{H} = \mathbf{J}_f$	(1.1.1)	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot \mathbf{n} \, da$	(1.1.20)
	$\nabla \cdot \mathbf{B} = 0$	(1.1.2)	$\oint_S \mathbf{B} \cdot \mathbf{n} \, da = 0$	(1.1.21)
	$\nabla \cdot \mathbf{J}_f = 0$	(1.1.3)	$\oint_S \mathbf{J}_f \cdot \mathbf{n} \, da = 0$	(1.1.22)
	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	(1.1.5)	$\oint_C \mathbf{E}' \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da$	(1.1.23)
			where $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$	
Electric field system	$\nabla \times \mathbf{E} = 0$	(1.1.11)	$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	(1.1.24)
	$\nabla \cdot \mathbf{D} = \rho_f$	(1.1.12)	$\oint_S \mathbf{D} \cdot \mathbf{n} \, da = \int_V \rho_f \, dV$	(1.1.25)
	$\nabla \cdot \mathbf{J}_f = -\frac{\partial \rho_f}{\partial t}$	(1.1.14)	$\oint_S \mathbf{J}_f' \cdot \mathbf{n} \, da = -\frac{d}{dt} \int_V \rho_f \, dV$	(1.1.26)
	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$	(1.1.15)	$\oint_C \mathbf{H}' \cdot d\mathbf{l} = \int_S \mathbf{J}_f' \cdot \mathbf{n} \, da + \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{n} \, da$	(1.1.27)
			where $\mathbf{J}_f' = \mathbf{J}_f - \rho_f \mathbf{v}$	
			$\mathbf{H}' = \mathbf{H} - \mathbf{v} \times \mathbf{D}$	

Table 2.1 Summary of Terminal Variables and Terminal Relations



Definition of Terminal Variables

Flux

$$\lambda_k = \int_{S_k} \mathbf{B} \cdot \mathbf{n} \, da$$

Charge

$$q_k = \int_{V_k} \rho_f \, dV$$

Current

$$i_k = \int_{S'_k} \mathbf{J}_f \cdot \mathbf{n}' \, da$$

Voltage

$$v_k = \int_a^b \mathbf{E} \cdot d\mathbf{l}$$

Terminal Conditions

$$v_k = \frac{d\lambda_k}{dt}$$

$$i_k = \frac{dq_k}{dt}$$

$$\lambda_k = \lambda_k(i_1 \cdots i_N; \text{geometry})$$

$$q_k = q_k(v_1 \cdots v_N; \text{geometry})$$

$$i_k = i_k(\lambda_1 \cdots \lambda_N; \text{geometry})$$

$$v_k = v_k(q_1 \cdots q_N; \text{geometry})$$

Table 3.1 Energy Relations for an Electromechanical Coupling Network with N Electrical and M Mechanical Terminal Pairs*

Magnetic Field Systems		Electric Field Systems	
Conservation of Energy			
$dW_m = \sum_{j=1}^N i_j d\lambda_j - \sum_{j=1}^M f_j^e dx_j$	(a)	$dW_e = \sum_{j=1}^N v_j dq_j - \sum_{j=1}^M f_j^e dx_j$	(b)
$dW'_m = \sum_{j=1}^N \lambda_j di_j + \sum_{j=1}^M f_j^e dx_j$	(c)	$dW'_e = \sum_{j=1}^N q_j dv_j + \sum_{j=1}^M f_j^e dx_j$	(d)
Forces of Electric Origin, $j = 1, \dots, M$			
$f_j^e = - \frac{\partial W_m(\lambda_1, \dots, \lambda_N; x_1, \dots, x_M)}{\partial x_j}$	(e)	$f_j^e = - \frac{\partial W_e(q_1, \dots, q_N; x_1, \dots, x_M)}{\partial x_j}$	(f)
$f_j^e = \frac{\partial W'_m(i_1, \dots, i_N; x_1, \dots, x_M)}{\partial x_j}$	(g)	$f_j^e = \frac{\partial W'_e(v_1, \dots, v_N; x_1, \dots, x_M)}{\partial x_j}$	(h)
Relation of Energy to Coenergy			
$W_m + W'_m = \sum_{j=1}^N \lambda_j i_j$	(i)	$W_e + W'_e = \sum_{j=1}^N v_j q_j$	(j)
Energy and Coenergy from Electrical Terminal Relations			
$W_m = \sum_{j=1}^N \int_0^{\lambda_j} i_j(\lambda_1, \dots, \lambda_{j-1}, \lambda'_j, 0, \dots, 0; x_1, \dots, x_M) d\lambda'_j$	(k)	$W_e = \sum_{j=1}^N \int_0^{q_j} v_j(q_1, \dots, q_{j-1}, q'_j, 0, \dots, 0; x_1, \dots, x_M) dq'_j$	(l)
$W'_m = \sum_{j=1}^N \int_0^{i_j} \lambda_j(i_1, \dots, i_{j-1}, i'_j, 0, \dots, 0; x_1, \dots, x_M) di'_j$	(m)	$W'_e = \sum_{j=1}^N \int_0^{v_j} q_j(v_1, \dots, v_{j-1}, v'_j, 0, \dots, 0; x_1, \dots, x_M) dv'_j$	(n)

* The mechanical variables f_j and x_j can be regarded as the j th force and displacement or the j th torque T_j and angular displacement θ_j .

Table 6.1 Differential Equations, Transformations, and Boundary Conditions for Quasi-static Electromagnetic Systems with Moving Media

	Differential Equations		Transformations		Boundary Conditions	
Magnetic field systems	$\nabla \times \mathbf{H} = \mathbf{J}_f$	(1.1.1)	$\mathbf{H}' = \mathbf{H}$	(6.1.35)	$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K}_f$	(6.2.14)
	$\nabla \cdot \mathbf{B} = 0$	(1.1.2)	$\mathbf{B}' = \mathbf{B}$	(6.1.37)	$\mathbf{n} \cdot (\mathbf{B}^a - \mathbf{B}^b) = 0$	(6.2.7)
	$\nabla \cdot \mathbf{J}_f = 0$	(1.1.3)	$\mathbf{J}'_f = \mathbf{J}_f$	(6.1.36)	$\mathbf{n} \cdot (\mathbf{J}_f^a - \mathbf{J}_f^b) + \nabla_\Sigma \cdot \mathbf{K}_f = 0$	(6.2.9)
	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	(1.1.5)	$\mathbf{E}' = \mathbf{E} + \mathbf{v}^r \times \mathbf{B}$	(6.1.38)	$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = v_n(\mathbf{B}^a - \mathbf{B}^b)$	(6.2.22)
	$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$	(1.1.4)	$\mathbf{M}' = \mathbf{M}$	(6.1.39)		
Electric field systems	$\nabla \times \mathbf{E} = 0$	(1.1.11)	$\mathbf{E}' = \mathbf{E}$	(6.1.54)	$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = 0$	(6.2.31)
	$\nabla \cdot \mathbf{D} = \rho_f$	(1.1.12)	$\mathbf{D}' = \mathbf{D}$	(6.1.55)	$\mathbf{n} \cdot (\mathbf{D}^a - \mathbf{D}^b) = \sigma_f$	(6.2.33)
			$\rho'_f = \rho_f$	(6.1.56)		
	$\nabla \cdot \mathbf{J}_f = -\frac{\partial \rho_f}{\partial t}$	(1.1.14)	$\mathbf{J}'_f = \mathbf{J}_f - \rho_f \mathbf{v}^r$	(6.1.58)	$\mathbf{n} \cdot (\mathbf{J}_f^a - \mathbf{J}_f^b) + \nabla_\Sigma \cdot \mathbf{K}_f = v_n(\rho_f^a - \rho_f^b) - \frac{\partial \sigma_f}{\partial t}$	(6.2.36)
	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$	(1.1.15)	$\mathbf{H}' = \mathbf{H} - \mathbf{v}^r \times \mathbf{D}$	(6.1.57)	$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K}_f + v_n \mathbf{n} \times [\mathbf{n} \times (\mathbf{D}^a - \mathbf{D}^b)]$	(6.2.38)
	$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$	(1.1.13)	$\mathbf{P}' = \mathbf{P}$	(6.1.59)		

From Chapter 8; The Stress Tensor and Related Tensor Concepts

In what follows we assume a right-hand cartesian coordinate system x_1, x_2, x_3 . The component of a vector in the direction of an axis carries the subscript of that axis. When we write F_m we mean the m th component of the vector F , where m can be 1, 2, or 3. When the index is repeated in a single term, it implies summation over the three values of the index

$$\frac{\partial H_n}{\partial x_n} = \frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial x_2} + \frac{\partial H_3}{\partial x_3} = \nabla \cdot \mathbf{H}$$

and

$$H_n \frac{\partial}{\partial x_n} = H_1 \frac{\partial}{\partial x_1} + H_2 \frac{\partial}{\partial x_2} + H_3 \frac{\partial}{\partial x_3} = \mathbf{H} \cdot \nabla.$$

This illustrates the *summation convention*. On the other hand, $\partial H_m / \partial x_n$ represents any one of the nine possible derivatives of components of $\mathbf{H}$ with respect to coordinates. We define the *Kronecker delta* δ_{mn} which has the values

$$\delta_{mn} = \begin{cases} 1, & \text{when } m = n, \\ 0, & \text{when } m \neq n. \end{cases} \quad (8.1.7)$$

The component T_{mn} of the stress tensor can be physically interpreted as the m th component of the traction (force per unit area) applied to a surface with a normal vector in the n -direction.

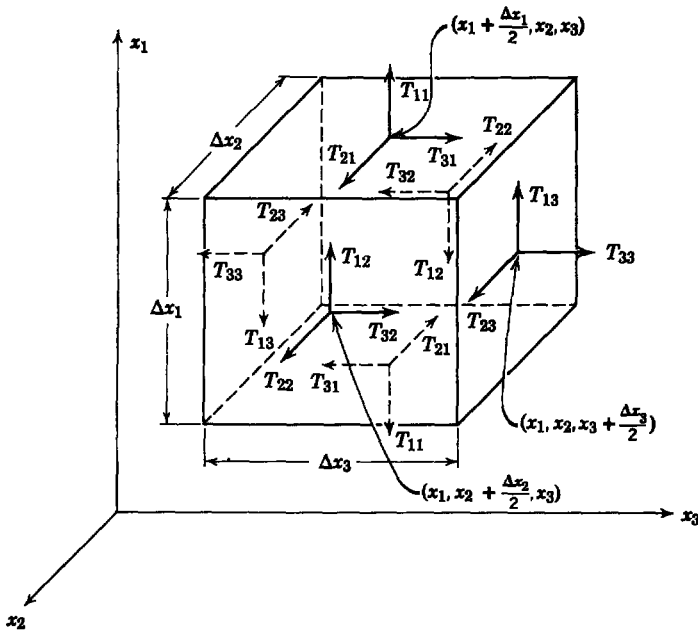


Fig. 8.2.2 Rectangular volume with center at (x_1, x_2, x_3) showing the surfaces and directions of the stresses T_{mn} .

The x_1 -component of the total force applied to the material within the volume of Fig. 8.2.2 is

$$\begin{aligned}
 f_1 = & T_{11} \left(x_1 + \frac{\Delta x_1}{2}, x_2, x_3 \right) \Delta x_2 \Delta x_3 - T_{11} \left(x_1 - \frac{\Delta x_1}{2}, x_2, x_3 \right) \Delta x_2 \Delta x_3 \\
 & + T_{12} \left(x_1, x_2 + \frac{\Delta x_2}{2}, x_3 \right) \Delta x_1 \Delta x_3 - T_{12} \left(x_1, x_2 - \frac{\Delta x_2}{2}, x_3 \right) \Delta x_1 \Delta x_3 \\
 & + T_{13} \left(x_1, x_2, x_3 + \frac{\Delta x_3}{2} \right) \Delta x_1 \Delta x_2 - T_{13} \left(x_1, x_2, x_3 - \frac{\Delta x_3}{2} \right) \Delta x_1 \Delta x_2.
 \end{aligned} \tag{8.2.3}$$

Here we have evaluated the components of the stress tensor at the centers of the surfaces on which they act; for example, the stress component T_{11} acting on the top surface is evaluated at a point having the same x_2 - and x_3 -coordinates as the center of the volume but an x_1 coordinate $\Delta x_1/2$ above the center.

The dimensions of the volume have already been specified as quite small. In fact, we are interested in the limit as the dimensions go to zero. Consequently, each component of the stress tensor is expanded in a Taylor series about the value at the volume center with only linear terms in each series retained to write (8.2.3) as

$$\begin{aligned}
 f_1 = & \left(T_{11} + \frac{\Delta x_1}{2} \frac{\partial T_{11}}{\partial x_1} - T_{11} + \frac{\Delta x_1}{2} \frac{\partial T_{11}}{\partial x_1} \right) \Delta x_2 \Delta x_3 \\
 & + \left(T_{12} + \frac{\Delta x_2}{2} \frac{\partial T_{12}}{\partial x_2} - T_{12} + \frac{\Delta x_2}{2} \frac{\partial T_{12}}{\partial x_2} \right) \Delta x_1 \Delta x_3 \\
 & + \left(T_{13} + \frac{\Delta x_3}{2} \frac{\partial T_{13}}{\partial x_3} - T_{13} + \frac{\Delta x_3}{2} \frac{\partial T_{13}}{\partial x_3} \right) \Delta x_1 \Delta x_2
 \end{aligned}$$

or

$$f_1 = \left(\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} \right) \Delta x_1 \Delta x_2 \Delta x_3. \tag{8.2.4}$$

All terms in this expression are to be evaluated at the center of the volume (x_1, x_2, x_3) . We have thus verified our physical intuition that space-varying stress tensor components are necessary to obtain a net force.

From (8.2.4) we can obtain the x_1 -component of the force density $\mathbf{F}$ at the point (x_1, x_2, x_3) by writing

$$F_1 = \lim_{\Delta x_1, \Delta x_2, \Delta x_3 \rightarrow 0} \frac{f_1}{\Delta x_1 \Delta x_2 \Delta x_3} = \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3}. \tag{8.2.5}$$

The limiting process makes the expansion of (8.2.4) exact. The summation convention is used to write (8.2.5) as

$$F_1 = \frac{\partial T_{1n}}{\partial x_n}. \quad (8.2.6)$$

A similar process for the other two components of the force and force density yields the general result that the m th component of the force density at a point is

$$F_m = \frac{\partial T_{mn}}{\partial x_n}. \quad (8.2.7)$$

Now suppose we wish to find the m th component of the total force $\mathbf{f}$ on material contained within the volume V . We can find it by performing the volume integration:

$$f_m = \int_V F_m dV = \int_V \frac{\partial T_{mn}}{\partial x_n} dV. \quad (8.1.13)$$

When we define the components of a vector $\mathbf{A}$ as

$$A_1 = T_{m1}, \quad A_2 = T_{m2}, \quad A_3 = T_{m3}, \quad (8.1.14)$$

we can write (8.1.13) as

$$f_m = \int_V \frac{\partial A_n}{\partial x_n} dV = \int_V (\nabla \cdot \mathbf{A}) dV. \quad (8.1.15)$$

We now use the divergence theorem to change the volume integral to a surface integral,

$$f_m = \oint_S \mathbf{A} \cdot \mathbf{n} da = \oint_S A_n n_n da, \quad (8.1.16)$$

where n_n is the n th component of the outward-directed unit vector $\mathbf{n}$ normal to the surface S and the surface S encloses the volume V . Substitution from (8.1.14) back into this expression yields

$$f_m = \oint_S T_{mn} n_n da. \quad (8.1.17)$$

where $T_{mn} n_n$ is the m th component of the surface traction $\boldsymbol{\tau}$.

The traction $\boldsymbol{\tau}$ is a vector. The components of this vector depend on the coordinate system in which $\boldsymbol{\tau}$ is expressed; for example, the vector might be directed in one of the coordinate directions (x_1, x_2, x_3) , in which case there would be only one nonzero component of $\boldsymbol{\tau}$. In a second coordinate system (x'_1, x'_2, x'_3) , this same vector might have components in all of the coordinate directions. Analyzing a vector into orthogonal components along the coordinate axes is a familiar process. The components in a cartesian coordinate system (x'_1, x'_2, x'_3) are related to those in the cartesian coordinate system (x_1, x_2, x_3) by the three equations

$$\tau'_p = a_{pr} \tau_r, \quad (8.2.10)$$

where a_{pr} is the cosine of the angle between the x'_p -axis and the x_r -axis.

Similarly, the components of the stress tensor transform according to the equation

$$T'_{pq} = a_{pr}a_{qs}T_{rs}. \quad (8.2.17)$$

This relation provides the rule for finding the components of the stress in the primed coordinates, given the components in the unprimed coordinates. It serves the same purpose in dealing with tensors that (8.2.10) serves in dealing with vectors.

Equation 8.2.10 is the transformation of a vector τ from an unprimed to a primed coordinate system. There is, in general, nothing to distinguish the two coordinate systems. We could just as well define a transformation from the primed to the unprimed coordinates by

$$\tau_s = b_{sp}\tau'_p, \quad (8.2.18)$$

where b_{sp} is the cosine of the angle between the x_s -axis and the x'_p -axis. But b_{sp} , from the definition following (8.2.10), is then also

$$b_{sp} \equiv a_{ps}; \quad (8.2.19)$$

that is, the transformation which reverses the transformation (8.2.10) is

$$\tau_s = a_{ps}\tau'_p. \quad (8.2.20)$$

Now we can establish an important property of the direction cosines a_{ps} by transforming the vector τ to an arbitrary primed coordinate system and then transforming the components τ'_m back to the unprimed system in which they must be the same as those we started with. Equation 8.2.10 provides the first transformation, whereas (8.2.20) provides the second; that is, we substitute (8.2.10) into (8.2.20) to obtain

$$\tau_s = a_{ps}a_{pr}\tau_r. \quad (8.2.21)$$

Remember that we are required to sum on both p and r ; for example, consider the case in which $s = 1$:

$$\begin{aligned} \tau_1 &= (a_{11}a_{11} + a_{21}a_{21} + a_{31}a_{31})\tau_1 \\ &\quad + (a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32})\tau_2 \\ &\quad + (a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33})\tau_3. \end{aligned} \quad (8.2.22)$$

This relation must hold in general. We have not specified either a_{ps} or τ_m . Hence the second two bracketed quantities must vanish and the first must be unity. We can express this fact much more concisely by stating that in general

$$a_{ps}a_{pr} = \delta_{sr} \quad (8.2.23)$$

Table 8.1 Electromagnetic Force Densities, Stress Tensors, and Surface Force Densities for Quasi-static Magnetic and Electric Field Systems*

Description	Force Density $\mathbf{F}$	Stress Tensor T_{mn} $F_m = \frac{\partial T_{mn}}{\partial x_n}$ (8.1.10)	Surface Force Density* $T_m = [T_{mn}]n_n$ (8.4.2)
Force on media carrying free current density $\mathbf{J}_f$, μ constant	$\mathbf{J}_f \times \mathbf{B}$ (8.1.3)	$T_{mn} = \mu H_m H_n - \delta_{mn} \frac{1}{2} \mu H_k H_k$ (8.1.11)	$\mathbf{T} = \mathbf{K}_f \times \mu \langle \mathbf{H} \rangle$ $\mathbf{K}_f = \mathbf{n} \times [\mathbf{H}]$ (8.4.3)
Force on media supporting free charge density ρ_f , ϵ constant	$\rho_f \mathbf{E}$ (8.3.3)	$T_{mn} = \epsilon E_m E_n - \delta_{mn} \frac{1}{2} \epsilon E_k E_k$ (8.3.10)	$\mathbf{T} = \sigma_f \langle \mathbf{E} \rangle$ $\sigma_f = \mathbf{n} \cdot [\epsilon \mathbf{E}]$ (8.4.8)
Force on free current plus magnetization force in which $\mathbf{B} = \mu \mathbf{H}$ both before and after media are deformed	$\mathbf{J}_f \times \mathbf{B} - \frac{1}{2} \mathbf{H} \cdot \mathbf{H} \nabla \mu$ $+ \frac{1}{2} \nabla \left(\mathbf{H} \cdot \mathbf{H} \rho \frac{\partial \mu}{\partial \rho} \right)$ (8.5.38)	$T_{mn} = \mu H_m H_n$ $- \frac{1}{2} \delta_{mn} \left(\mu - \rho \frac{\partial \mu}{\partial \rho} \right) H_k H_k$ (8.5.41)	
Force on free charge plus polarization force in which $\mathbf{D} = \epsilon \mathbf{E}$ both before and after media are deformed	$\rho_f \mathbf{E} - \frac{1}{2} \mathbf{E} \cdot \mathbf{E} \nabla \epsilon$ $+ \frac{1}{2} \nabla \left(\mathbf{E} \cdot \mathbf{E} \rho \frac{\partial \epsilon}{\partial \rho} \right)$ (8.5.45)	$T_{mn} = \epsilon E_m E_n$ $- \frac{1}{2} \delta_{mn} \left(\epsilon - \rho \frac{\partial \epsilon}{\partial \rho} \right) E_k E_k$ (8.5.46)	

$$* \langle \mathbf{A} \rangle \equiv \frac{\mathbf{A}^a + \mathbf{A}^b}{2}$$

$$[\mathbf{A}] \equiv \mathbf{A}^a - \mathbf{A}^b$$

Table 9.1 Modulus of Elasticity E and Density ρ for Representative Materials*

Material	E -units of 10^{11} N/m ²	ρ -units of 10^3 kg/m ³	v_p -units† of m/sec
Aluminum (pure and alloy)	0.68–0.79	2.66–2.89	5100
Brass (60–70% Cu, 40–30% Zn)	1.0–1.1	8.36–8.51	3500
Copper	1.17–1.24	8.95–8.98	3700
Iron, cast (2.7–3.6% C)	0.89–1.45	6.96–7.35	4000
Steel (carbon and low alloy)	1.93–2.20	7.73–7.87	5100
Stainless steel (18% Cr, 8% Ni)	1.93–2.06	7.65–7.93	5100
Titanium (pure and alloy)	1.06–1.14	4.52	4900
Glass	0.49–0.79	2.38–3.88	4500
Methyl methacrylate	0.024–0.034	1.16	1600
Polyethylene	$1.38\text{--}3.8 \times 10^{-3}$	0.915	530
Rubber	$0.79\text{--}4.1 \times 10^{-5}$	0.99–1.245	46

* See S. H. Crandall, and N. C. Dahl, *An Introduction to the Mechanics of Solids*, McGraw-Hill, New York, 1959, for a list of references for these constants and a list of these constants in English units.

† Computed from average values of E and ρ .

**Table 9.2 Summary of One-Dimensional Mechanical Continua
Introduced in Chapter 9**

Thin Elastic Rod
$\rho \frac{\partial^2 \delta}{\partial t^2} = E \frac{\partial^2 \delta}{\partial x^2} + F_x$ $T = E \frac{\partial \delta}{\partial x}$ δ—longitudinal (x) displacement T—normal stress ρ—mass density E—modulus of elasticity F_x—longitudinal body force density
Wire or "String"
$m \frac{\partial^2 \xi}{\partial t^2} = f \frac{\partial^2 \xi}{\partial x^2} + S_z$ ξ—transverse displacement m—mass/unit length f—tension (constant force) S_z—transverse force/unit length
Membrane
$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right) + T_z$ ξ—transverse displacement σ_m—surface mass density S—tension in y- and z-directions (constant force per unit length) T_z—z-directed force per unit area

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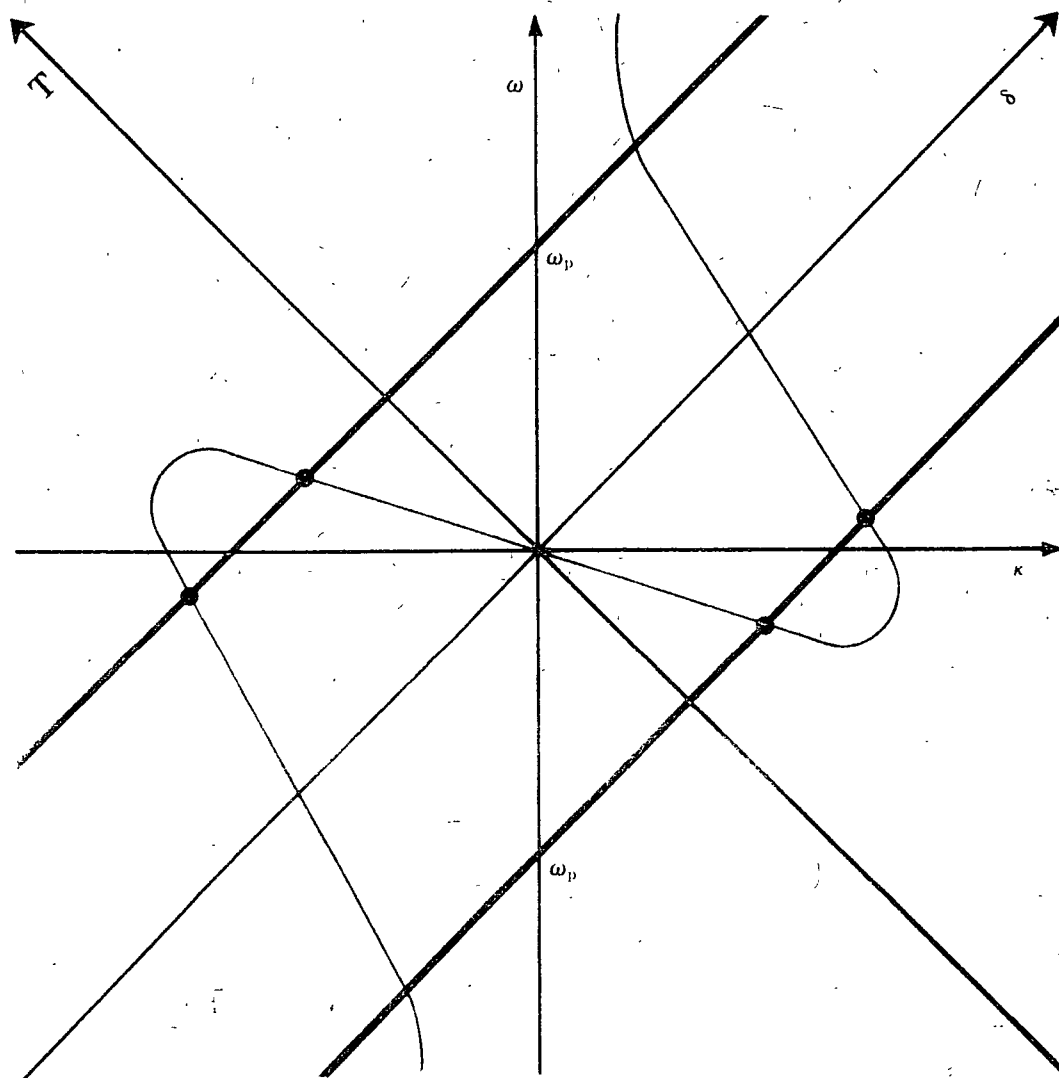
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SOLUTIONS MANUAL FOR

ELECTROMECHANICAL DYNAMICS

PART I: Discrete Systems

HERBERT H. WOODSON JAMES R. MELCHER



JOHN WILEY & SONS, INC. NEW YORK • LONDON • SYDNEY

Mark Zeln

SOLUTIONS MANUAL FOR

ELECTROMECHANICAL

DYNAMICS

Part I: Discrete Systems

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PREFACE TO: SOLUTIONS MANUAL FOR
ELECTROMECHANICAL DYNAMICS, PART I:
DISCRETE SYSTEMS

This manual presents in an informal format solutions to the problems found at the ends of chapters in Part I of the book, Electromechanical Dynamics. It is intended as an aid for instructors, and in special circumstances for use by students. We have included a sufficient amount of explanatory material that solutions, together with problem statements, are in themselves a teaching aid. They are substantially as found in our records for the course 6.06 as taught at M.I.T. over a period of several years.

Typically, the solutions were originally written up by graduate student tutors whose responsibility it was to conduct one-hour tutorials once a week with students in pairs. These tutorials focused on the homework, with the problem solutions reproduced and given to the students upon receipt of their own homework solutions.

It is difficult to give proper credit to all of those who contributed to these solutions, because the individuals involved range over teaching assistants, instructors, and faculty who have taught the material over a period of more than four years. However, a significant contribution was made by D.S. Guttman who took major responsibility for the solutions in Chapter 6.

The manuscript was typed by Mrs. Barbara Morton, whose patience and expertise were invaluable.

H.H. Woodson
J.R. Melcher

Cambridge, Massachusetts
July, 1968

PROBLEM 2.1

LUMPED ELECTROMECHANICAL ELEMENTS

Using the second Maxwell equation we write that the flux of B into the movable slab equals the flux of B out of the movable slab

$$\mu_o H_1 LD = \mu_o H_2 aD + \mu_o H_3 bD$$

or

$$H_1 L = H_2 a + H_3 b \quad (c)$$

Note that in determining the relative strengths of H_1, H_2 and H_3 in this last equation we have let $(a-x) \approx a$, $(b-y) \approx b$ to simplify the solution. This means that we are assuming that

$$x/a \ll 1, y/b \ll 1 \quad (d)$$

Solving for H_1 using (a), (b), and (c)

$$H_1 = \frac{nI(y/a + x/b)}{(c-b-y)(y/a + x/b) + L(y/a \cdot x/b)}$$

The flux of B through the n turns of the coil is then

$$\begin{aligned} (x,y,i) &= nB_1 LD = n\mu_o H_1 LD \\ &= \frac{\mu_o n^2 (y/a + x/b) LD i}{(c-b-y)(y/a+x/b) + L(y/a \cdot x/b)} \end{aligned}$$

Because we have assumed that the air gaps are short compared to their cross-sectional dimensions we must have

$$\frac{(c-b-y)}{L} \ll 1, y/a \ll 1 \text{ and } x/b \ll 1$$

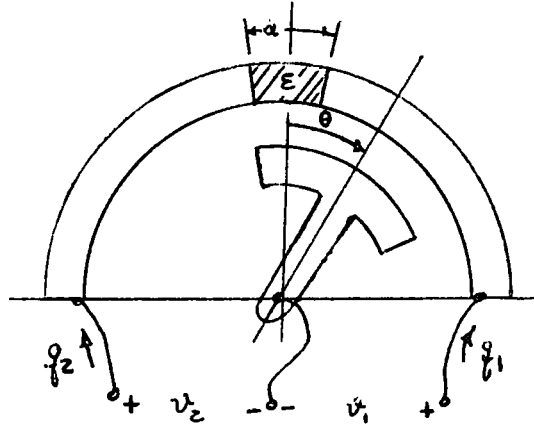
in addition to the constraints of (d) for our expression for λ to be valid. If we assume that $a > L > c > b > (c-b)$ as shown in the diagram, these conditions become

$$x \ll b$$

$$y \ll b$$

LUMPED ELECTROMECHANICAL ELEMENTS

PROBLEM 2.2



Because the charge is linearly related to the applied voltages we

know that $q_1(V_1, V_2, \theta) = q_1(V_1, 0, \theta) + q_1(0, V_2, \theta)$

$$q_1(V_1, 0, \theta) = \frac{\epsilon V_1}{R\alpha} w l + \epsilon_0 \frac{V_1}{g} \left(\frac{\pi}{4} + \theta\right) R l$$

$$q_1(0, V_2, \theta) = -\frac{\epsilon V_2}{R\alpha} w l$$

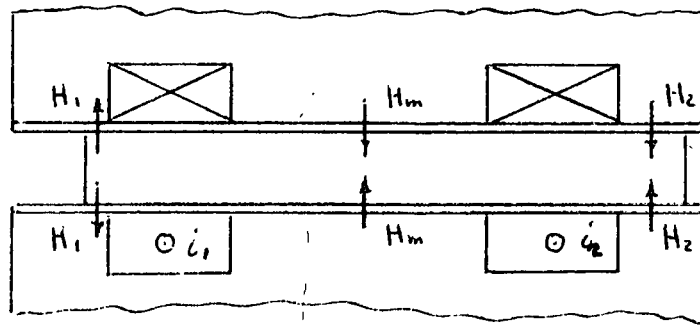
Hence

$$q_1(V_1, V_2, \theta) = V_1 l \left(\frac{\epsilon w}{R\alpha} + \frac{\epsilon_0 (\pi/4 + \theta) R}{g} \right) - V_2 l \frac{\epsilon w}{R\alpha}$$

$$q_2(V_1, V_2, \theta) = -V_1 l \frac{\epsilon w}{R\alpha} + V_2 l \left(\frac{\epsilon w}{R} + \frac{\epsilon_0 (\pi/4 - \theta) R}{g} \right)$$

PROBLEM 2.3

The device has cylindrical symmetry so that we assume that the fields in the gaps are essentially radial and denoted as shown in the figure.



Ampere's law can be

integrated around each of the current loops to obtain the relations

LUMPED ELECTROMECHANICAL ELEMENTS

PROBLEM 2.3 (Continued)

$$gH_1 + gH_m = NI_1 \quad (a)$$

$$gH_2 - gH_m = NI_2 \quad (b)$$

In addition, the net flux into the plunger must be zero, and so

$$\mu_o(d-x)2\pi rH_1 - 2d(2\pi r)\mu_o H_m - (d+x)(2\pi r)\mu_o H_2 \quad (c)$$

These three equations can be solved for any one of the intensities. In particular we are interested in H_1 and H_2 , because the terminal fluxes can be written simply in terms of these quantities. For example, the flux linking the (1) winding is N times the flux through the air gap to the left

$$\lambda_1 = \mu_o N(d-x)(2\pi r)H_1 \quad (d)$$

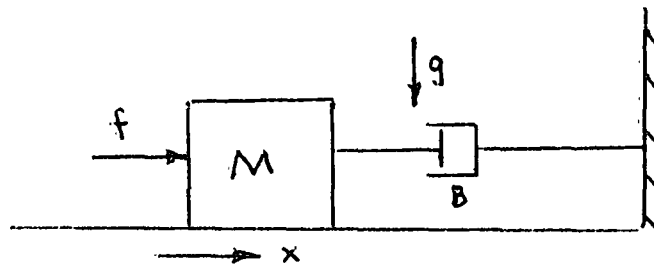
Similarly, to the right,

$$\lambda_2 = \mu_o N(d+x)(2\pi r)H_2 \quad (e)$$

Now, if we use the values of H_1 and H_2 found from (a) - (c), we obtain the terminal relations of Prob. 2.3 with

$$L_o = \frac{\mu_o \pi r N^2 d}{2g}$$

PROBLEM 2.4



LUMPED ELECTROMECHANICAL ELEMENTS

PROBLEM 2.4 (Continued)

Part a

$$\sum_1 f_1 = Ma = M \frac{dx^2}{dt^2}$$

$$f_{\text{DAMPER}} = -B \frac{dx}{dt} ; f_{\text{coul}} = -\mu_d Mg \frac{dx_1}{dt} \frac{dx_1}{|dx_1|}$$

$$M \frac{d^2x}{dt^2} = f(t) - B \frac{dx}{dt} + f_{\text{coul}}$$

or

$$M \frac{d^2x}{dt^2} + B \frac{dx}{dt} = f(t) - \mu_d Mg \frac{dx_1}{dt} \frac{dx_1}{|dx_1|}$$

Part b

First we recognize that the block will move so that $\frac{dx_1}{dt} > 0$, hence

$$f_{\text{coul}} = -\mu_d Mg \frac{dx_1}{dt} > 0$$

Then for $t > 0$

$$M \frac{d^2x}{dt^2} + B \frac{dx}{dt} = -\mu_d Mg$$

which has a solution

$$x(t) = -\frac{\mu_d Mg}{B} t + c_1 + c_2 e^{-(B/M)t}$$

Equating singularities at $t = 0$

$$M \frac{d^2x}{dt^2}(0) = I_o \mu_o(t) \quad \text{or} \quad \frac{d^2x}{dt^2}(0) = \frac{I_o}{M} \mu_o(t)$$

Then since $x(0^-) = \frac{dx}{dt}(0^-) = \frac{d^2x}{dt^2}(0^-) = 0$

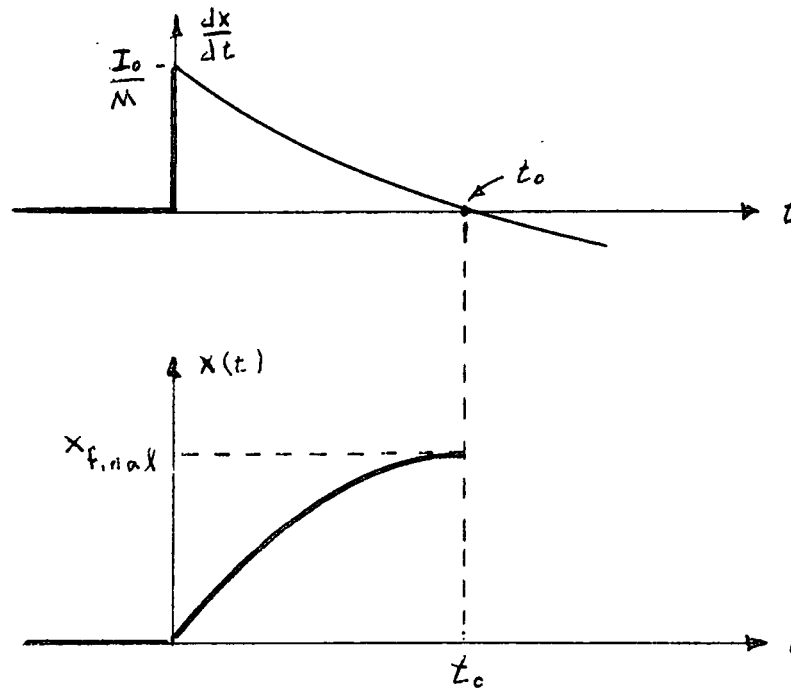
LUMPED ELECTROMECHANICAL ELEMENTS

PROBLEM 2.4 (Continued)

$$\frac{dx}{dt}(0^+) = \frac{I_o}{M} ; x(0^+) = 0$$

$$\text{Hence } x(t) = u_{-1}(t) \left[-\frac{\mu_d M g}{B} t + \left(\frac{I_o}{B} + \mu_d g \left(\frac{M}{B} \right)^2 \right) (1 - e^{-(B/M)t}) \right]$$

Actually, this solution will only hold until t_o , where $\frac{dx}{dt}(t_o) = 0$, at which point the mass will stop.



PROBLEM 2.5

Part a

Equation of motion

$$M \frac{d^2 x}{dt^2} + B \frac{dx}{dt} = f(t)$$

$$(1) f(t) = I_o u_o(t)$$

$$x(t) = u_{-1}(t) \frac{I_o}{B} (1 - e^{-(B/M)t})$$

LUMPED ELECTROMECHANICAL ELEMENTS

PROBLEM 2.5 (Continued)

as shown in Prob. 2.4 with $\mu_d = 0$.

$$(2) \quad f(t) = F_o u_{-1}(t)$$

Integrating the answer in (1)

$$x(t) = \frac{F_o}{B} \left[t + \frac{M}{B} (e^{-(B/M)t} - 1) \right] u_{-1}(t)$$

Part b

Consider the node connecting the damper and the spring; there must be no net force on this node or it will suffer infinite acceleration.

$$-B \frac{dx}{dt} + K(y-x) = 0$$

or

$$B/K \frac{dx}{dt} + x = y(t)$$

$$1. \quad \text{Let } y(t) = Au_o(t)$$

$$\frac{B}{K} \frac{dx}{dt} + x = 0 \quad t > 0$$

$$x(t) = C e^{-K/Bt} \quad t > 0$$

But at $t = 0$

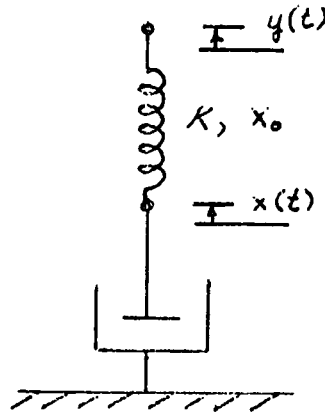
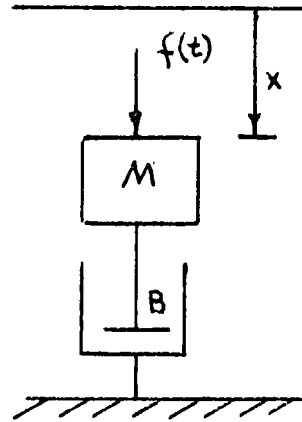
$$\frac{B}{K} \frac{dx}{dt}(0) = Au_o(0)$$

Now since $x(t)$ and $\frac{dx}{dt}(t)$ are zero for $t < 0$

$$x(0^+) = \frac{AK}{B} = C$$

$$x(t) = u_{-1}(t) \frac{AK}{B} e^{-(K/B)t} \quad \text{all } t$$

$$2. \quad \text{Let } y(t) = Au_{-1}(t)$$



LUMPED ELECTROMECHANICAL ELEMENTS

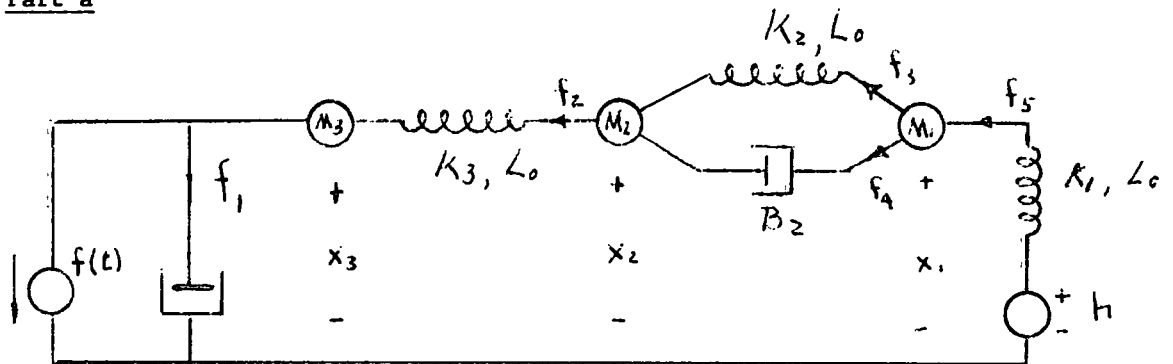
PROBLEM 2.5 (Continued)

Integrating the answer in (1)

$$x(t) = u_{-1}(t) Y_0 (1 - e^{-(K/B)t}) \quad \text{all } t$$

PROBLEM 2.6

Part a



$$f_1 = B_3 \frac{dx_3}{dt}; \quad f_2 = K_3(x_2 - x_3 - L_0)$$

$$f_3 = K_2(x_1 - x_2 - L_0); \quad f_4 = B_2 \frac{d}{dt}(x_1 - x_2)$$

$$f_5 = K_1(h - x_1 - L_0)$$

Part b

Summing forces at the nodes and using Newton's law

$$K_1(h - x_1 - L_0) = K_2(x_1 - x_2 - L_0) + B_2 \frac{d(x_1 - x_2)}{dt} + M_1 \frac{d^2 x_1}{dt^2}$$

$$K_2(x_1 - x_2 - L_0) + B_2 \frac{d(x_1 - x_2)}{dt} = K_3(x_2 - x_3 - L_0) + M_2 \frac{d^2 x_2}{dt^2}$$

$$K_3(x_2 - x_3 - L_0) = f(t) + B_3 \frac{dx_3}{dt} + M_3 \frac{d^2 x_3}{dt^2}$$

LUMPED ELECTROMECHANICAL ELEMENTS

PROBLEM 2.6 (Continued)

Let's solve these equations for the special case

$$M_1 = M_2 = M_3 = B_2 = B_3 = L_0 = 0$$

Now nothing is left except three springs pulled by force $f(t)$. The three equations are now

$$K_1(h-x_1) = K_2(x_1-x_2) \quad (a)$$

$$K_2(x_1-x_2) = K_3(x_2-x_3) \quad (b)$$

$$K_3(x_2-x_3) = f(t) \quad (c)$$

We write the equation of geometric constraint

$$x_3 + (x_2-x_3) + (x_1-x_2) + (h-x_1) - h = 0$$

or

$$(h-x_3) = (x_2-x_3) + (x_1-x_2) + (h-x_1) \quad (d)$$

which is really a useful identity rather than a new independent equation.

Substituting in (a) and (b) into (d)

$$\begin{aligned} (h-x_3) &= \frac{K_3(x_2-x_3)}{K_3} + \frac{K_3(x_2-x_3)}{K_2} + \frac{K_3(x_2-x_3)}{K_1} \\ &= K_3(x_2-x_3) \left(\frac{1}{K_3} + \frac{1}{K_2} + \frac{1}{K_1} \right) \end{aligned}$$

which can be plugged into (c)

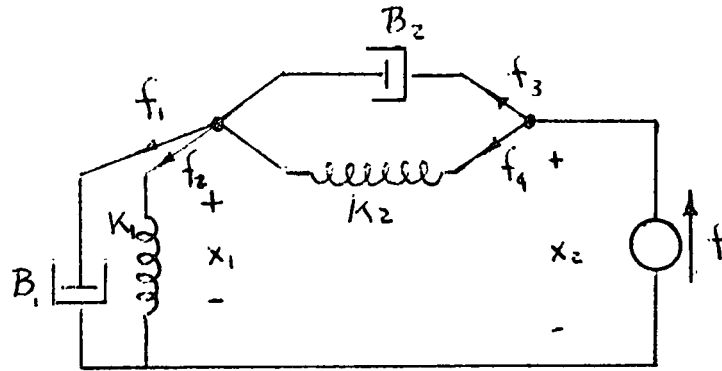
$$\left(\frac{1}{K_3} + \frac{1}{K_2} + \frac{1}{K_1} \right)^{-1} (h-x_3) = f(t)$$

which tells us that three springs in series act like a spring with

$$K' = \left(\frac{1}{K_3} + \frac{1}{K_2} + \frac{1}{K_1} \right)^{-1}$$

LUMPED ELECTROMECHANICAL ELEMENTS

PROBLEM 2.7



$$f_1 = B_1 \frac{dx_1}{dt} \quad f_2 = K_1 x_1$$

$$f_3 = B_2 \frac{d(x_2 - x_1)}{dt} \quad f_4 = K_2 (x_2 - x_1)$$

Node equations:

$$\text{Node 1} \quad B_1 \frac{dx_1}{dt} + K_1 x_1 = B_2 \frac{d(x_2 - x_1)}{dt} + K_2 (x_2 - x_1)$$

$$\text{Node 2} \quad B_2 \frac{d(x_2 - x_1)}{dt} + K_2 (x_2 - x_1) = f$$

To find natural frequencies let \$f = 0\$

$$B_1 \frac{dx_1}{dt} + K_1 x_1 = 0 \quad \text{Let } x_1 = e^{st}$$

$$B_1 s + K_1 = 0 \quad s_1 = -K_1/B_1$$

$$B_2 \frac{d(x_2 - x_1)}{dt} + K_2 (x_2 - x_1) = 0 \quad \text{Let } (x_2 - x_1) = e^{st}$$

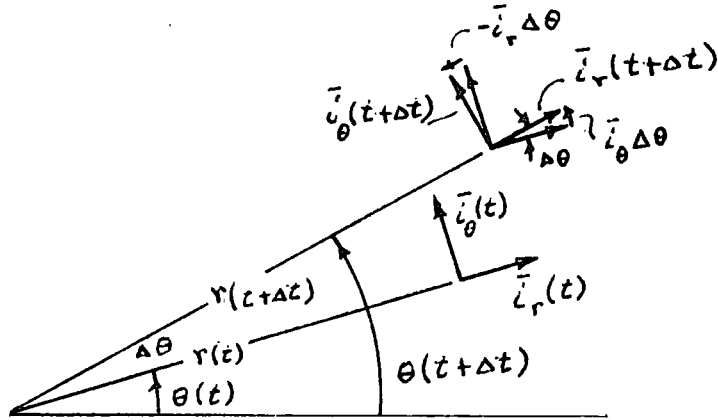
$$B_2 s + K_2 = 0 \quad s_2 = -K_2/B_2$$

The general solution when \$f = 0\$ is then

$$x_1 = c_1 e^{-(K_1/B_1)t}$$

$$x_2 = (x_2 - x_1) + x_1 = c_1 e^{-(K_1/B_1)t} + c_2 e^{-(K_2/B_2)t}$$

PROBLEM 2.8



From the diagram, the change in $\bar{i}_r$ in the time Δt is $\bar{i}_\theta \Delta \theta$. Hence

$$\frac{d\bar{i}_r}{dt} = \lim_{\Delta t \rightarrow 0} \bar{i}_\theta \frac{\Delta \theta}{\Delta t} = \bar{i}_\theta \frac{d\theta}{dt} \quad (a)$$

Similarly,

$$\frac{d\bar{i}_\theta}{dt} = \lim_{\Delta t \rightarrow 0} -\bar{i}_r \frac{\Delta \theta}{\Delta t} = -\bar{i}_r \frac{d\theta}{dt} \quad (b)$$

Then, the product rule of differentiation on $\bar{v}$ gives

$$\frac{d\bar{v}}{dt} = \frac{d\bar{i}_r}{dt} \frac{dr}{dt} + \bar{i}_r \frac{d^2 r}{dt^2} + \frac{d\bar{i}_\theta}{dt} \left(r \frac{d\theta}{dt} \right) + \bar{i}_\theta \frac{d}{dt} \left(r \frac{d\theta}{dt} \right) \quad (c)$$

and the required acceleration follows by combining these equations.

PROBLEM 3.1

This problem is a simple extension of that considered in Sec. 3.2, having the purpose of emphasizing how the geometric dependence of the electrical force depends intimately on the electrical constraints.

Part a

The system is electrically linear. Hence, $W'_m = \frac{1}{2} L i^2$ and the force f that must be applied to the plunger is

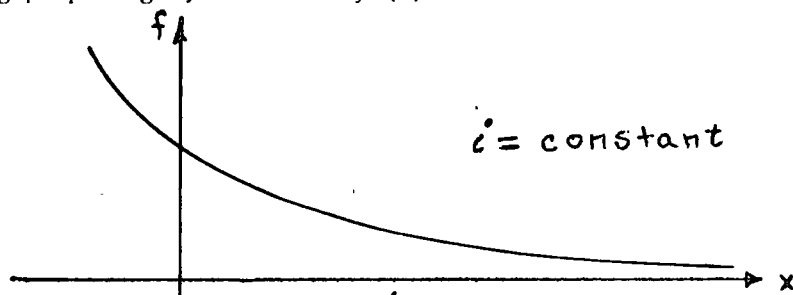
$$f = -f^e = \frac{1}{2a} \frac{L_0 i^2}{\left(1 + \frac{x}{a}\right)^2} \quad (a)$$

The terminal equation can be used to write this force in terms of λ

$$f = -f^e = \lambda^2 / 2aL_0 \quad (b)$$

Part b

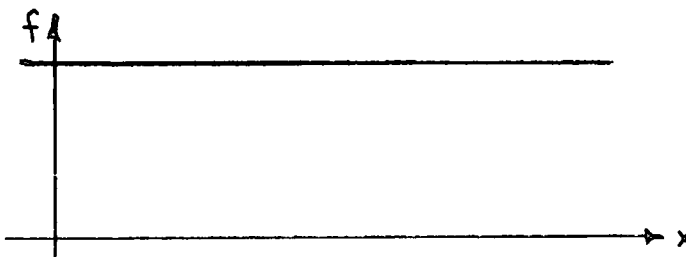
With the current constant, the force decreases rapidly as a function of the plunger gap spacing x , as shown by (a) and the sketch below



With the current constant, the drop in $\int \vec{H} \cdot d\vec{\ell}$ across the gap increases with x , and hence the field in the gap is reduced by increasing x .

Part c

By contrast with part b, at constant λ , the force is independent of x



PROBLEM 3.1 (Continued)

With this constraint, the field in the gap must remain constant, independent of the position x .

PROBLEM 3.2

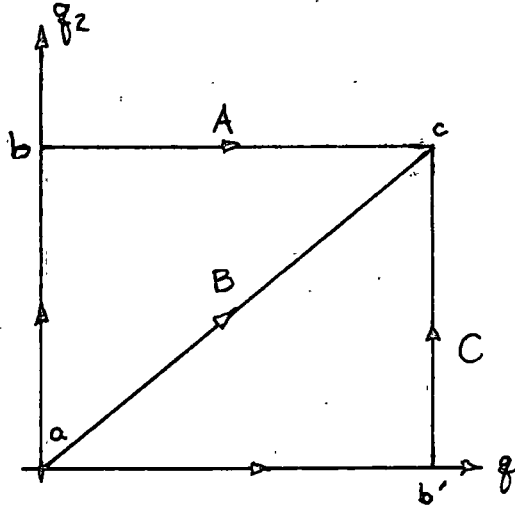
Part a

The terminal relations are

$$\begin{aligned} v_1 &= S_{11}q_1 + S_{12}q_2 \\ v_2 &= S_{21}q_1 + S_{22}q_2 \end{aligned} \quad (a)$$

Energy input can result only through the electrical terminal pairs, because the mechanical terminal pairs are constrained to constant position. Thus,

$$W_e = \int v_1 dq_1 + v_2 dq_2 \quad (b)$$



First carry out this line integral along the contour A: from $a \rightarrow b$, $q_1 = 0$, while from $b \rightarrow c$, $dq_2 = 0$. Hence,

$$W_e = \int_0^{Q_2} v_2(0, q_2) dq_2 + \int_0^{Q_1} v_1(q_1, Q_2) dq_1 \quad (c)$$

and using (a),

$$W_e = \int_0^{Q_2} S_{22}q_2 dq_2 + \int_0^{Q_1} (S_{11}q_1 + S_{12}Q_2) dq_1 \quad (d)$$

and for path A,

$$W_e = \frac{1}{2} S_{22}Q_2^2 + S_{12}Q_1Q_2 + S_{11}Q_1^2 \quad (e)$$

If instead of path A, we use C, the roles of q_1 and q_2 are simply reversed. Mathematically this means $1 \rightarrow 2$ and $2 \rightarrow 1$ in the above. Hence, for path C

$$W_e = \frac{1}{2} S_{11}Q_1^2 + S_{21}Q_2Q_1 + S_{22}Q_2^2 \quad (f)$$

To use path B in carrying out the integration of (b), we relate q_2 and q_1

PROBLEM 3.2 (Continued)

$$q_2 = \frac{Q_2}{Q_1} q_1 \quad (g)$$

Then, (a) becomes,

$$v_1 = [s_{11} + \frac{s_{12}Q_2}{Q_1}]q_1; \quad v_2 = [s_{21} + \frac{s_{22}Q_2}{Q_1}]q_1 \quad (h)$$

and, from (b), where dq_2 and $Q_2 dq_1/Q_1$

$$w_e = \int_0^{Q_1} [s_{11} + \frac{s_{12}Q_2}{Q_1}]q_1 dq_1 + \int_0^{Q_1} [s_{21} + \frac{s_{22}Q_2}{Q_1}]q_1 \frac{Q_2}{Q_1} dq_1 \quad (i)$$

or

$$w_e = \frac{1}{2} s_{11} Q_1^2 + \frac{1}{2} s_{12} Q_2 Q_1 + \frac{1}{2} s_{21} Q_1 Q_2 + \frac{1}{2} s_{22} Q_2^2 \quad (j)$$

Part b

The integrations along paths A, B and C are the same only if $s_{21} = s_{12}$ as can be seen by comparing (e), (f) and (j).

Part c

Conservation of energy requires

$$dw_e(q_1, q_2) = v_1 dq_1 + v_2 dq_2 = \frac{\partial w_e}{\partial q_1} dq_1 + \frac{\partial w_e}{\partial q_2} dq_2 \quad (k)$$

Since q_1 and q_2 are independent variables

$$v_1 = \frac{\partial w_e}{\partial q_1}; \quad v_2 = \frac{\partial w_e}{\partial q_2} \quad (l)$$

Taking cross derivatives of these two expressions and combining gives

$$\frac{\partial v_1}{\partial q_2} = \frac{\partial v_2}{\partial q_1} \quad (m)$$

or, from (a), $s_{12} = s_{21}$.

PROBLEM 3.3

The electric field intensity between the plates is

$$E = v/a \quad (a)$$

LUMPED-PARAMETER ELECTROMECHANICS

PROBLEM 3.3 (Continued)

Hence, the surface charge adjacent to the free space region on the upper plate is

$$\sigma_f = \epsilon_o v/a \quad (b)$$

while that next to the nonlinear dielectric slab is

$$\sigma_f = \alpha \frac{v^3}{a} + \epsilon_o \frac{v}{a} \quad (c)$$

It follows that the total charge on the upper plate is

$$q = \frac{dx \epsilon_o v}{a} + d(l-x) \left[\frac{\alpha v^3}{a} + \frac{\epsilon_o v}{a} \right] \quad (d)$$

The electric co-energy is

$$W'_e = \int q dv = \frac{d \epsilon_o v^2}{2a} + \frac{d(l-x) \alpha v^4}{4a^3} \quad (e)$$

Then, the force of electrical origin is

$$f^e = \frac{\partial W'_e}{\partial x} = - \frac{d \alpha v^4}{4a^3} \quad (f)$$

PROBLEM 3.4

Part a

The magnetic field intensity in the gap must first be related to the excitation current. From Ampere's law,

$$Ni = dH_d + xH_x \quad (a)$$

where the fields H_d and H_x are directed counterclockwise around the magnetic circuit when they are positive. These fields are further related because the magnetic flux into the movable member must equal that out of it

$$\mu_o w b H_d = \mu_o w a H_x \quad (b)$$

From these two expressions

$$H_x = Ni / \left(\frac{da}{b} + x \right) \quad (c)$$

LUMPED-PARAMETER ELECTROMECHANICS

PROBLEM 3.4 (Continued)

The flux linked by the electrical terminals is $\lambda = N\mu_o a_w H_x$ which in view of (c) is

$$\lambda = Li; L = \frac{N^2 \mu_o a_w}{(\frac{da}{b} + x)} \quad (d)$$

Part b

The system is electrically linear. Hence, $W_m = \frac{1}{2} \lambda^2 / L$ (See Sec. 3.1.2b) and from (d),

$$W_m = \frac{1}{2} \lambda^2 \frac{(\frac{da}{b} + x)}{N^2 \mu_o a_w} \quad (e)$$

Part c

From conservation of energy $f^e = -\partial W_m / \partial x$, $W_m = W_m(\lambda, x)$. Hence,

$$f^e = -\frac{1}{2} \lambda^2 / (N^2 \mu_o a_w) \quad (f)$$

Part d

In view of (d) the current node equation can be written as (remember that the terminal voltage is $d\lambda/dt$)

$$I(t) = \frac{1}{R} \frac{d\lambda}{dt} + \frac{\lambda(\frac{da}{b} + x)}{N^2 \mu_o a_w} \quad (g)$$

Part e

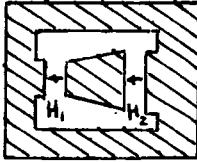
The inertial force due to the mass M must be equal to two other forces, one due to gravity and the other f^e . Hence,

$$M \frac{d^2 x}{dt^2} = Mg - \frac{1}{2} \frac{\lambda^2}{N^2 \mu_o a_w} \quad (h)$$

(g) and (h) are the required equations of motion, where (λ, x) are the dependent variables.

PROBLEM 3.5

Part a



From Ampere's Law

$$H_1(a+x) + H_2(a-x) = N_1 i_1 + N_2 i_2$$

Because $\oint_S \vec{B} \cdot \vec{n} da = 0$

$$\mu_o H_1 A_1 = \mu_o H_2 A_2$$

solving for H_1

$$H_1 = \frac{N_1 i_1 + N_2 i_2}{a(1 + \frac{A_1}{A_2}) + x(1 - \frac{A_1}{A_2})}$$

Now the flux ϕ in each air gap must be the same because

$$\phi = \mu_o H_1 A_1 = \mu_o H_2 A_2$$

and the flux linkages are determined to be $\lambda_1 = N_1 \phi$ and $\lambda_2 = N_2 \phi$. Using these ideas

$$\lambda_1 = N_1^2 L(x) i_1 + N_1 N_2 L(x) i_2$$

$$\lambda_2 = N_2 N_1 L(x) i_1 + N_2^2 L(x) i_2$$

where
$$L(x) = \frac{\mu_o A_1}{a(1 + \frac{A_1}{A_2}) + x(1 - \frac{A_1}{A_2})}$$

Part b

From part a the system is electrically linear, hence

$$W'_m = L(x) [\frac{1}{2} N_1^2 i_1^2 + N_1 N_2 i_1 i_2 + \frac{1}{2} N_2^2 i_2^2]$$

where
$$L(x) = \frac{\mu_o A_1}{a(1 + \frac{A_1}{A_2}) + x(1 - \frac{A_1}{A_2})}$$

PROBLEM 3.6

Part a

Conservation of energy requires that

$$dW = i d\lambda - f^e dx \quad (a)$$

In addition,

$$dW = \frac{\partial W}{\partial \lambda} d\lambda + \frac{\partial W}{\partial x} dx \quad (b)$$

so that

$$i = \frac{\partial W}{\partial \lambda} ; f^e = - \frac{\partial W}{\partial x} \quad (c)$$

Now if we take cross-derivatives of these last relations and combine,

$$\frac{\partial i}{\partial x} = - \frac{\partial f^e}{\partial \lambda} \quad (d)$$

This condition of reciprocity between the electrical and mechanical terminal pairs must be satisfied if the system is to be conservative. For the given terminal relations,

$$\frac{\partial i}{\partial x} = - \frac{I_o}{a} \left[\frac{\lambda}{\lambda_o} + \left(\frac{\lambda}{\lambda_o} \right)^3 \right] / \left(1 + \frac{x}{a} \right)^2 \quad (e)$$

$$- \frac{\partial f^e}{\partial \lambda} = - \frac{I_o}{a} \left[\frac{\lambda}{\lambda_o} + \frac{\lambda}{\lambda_o^3} \right] / \left(1 + \frac{x}{a} \right)^2 \quad (f)$$

and the system is conservative.

Part b

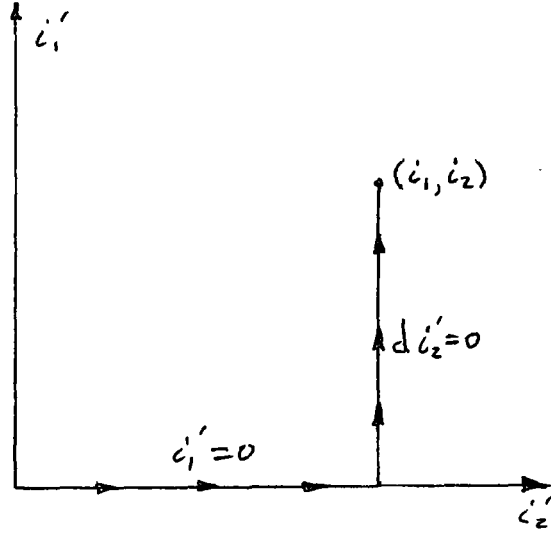
The stored energy is

$$W_e = \int i d\lambda = \frac{I_o}{[1 + \frac{x}{a}]} \left[\frac{1}{2} \frac{\lambda^2}{\lambda_o} + \frac{1}{4} \frac{\lambda^4}{\lambda_o^3} \right] \quad (g)$$

PROBLEM 3.7

To find the co-energy from the electrical terminal relations alone, we must assume that in the absence of electrical excitations there is no force of electrical origin. Then, the system can be assembled mechanically, with the currents constrained to zero, and there will be no contribution of co-energy in the process (see

Sec. 3.1.1). The co-energy input through the electrical terminal pairs with the mechanical system held fixed is



$$W'_m = \int \lambda_1 di_1 + \lambda_2 di_2 \quad (a)$$

For the path shown in the (i_1, i_2) plane of the figure, this becomes

$$W'_m = \int_0^{i_2} \lambda_2(0, i_2') di_2' + \int_0^{i_1} \lambda_1(i_1', i_2) di_1' \quad (b)$$

and in view of the given terminal relations, the required co-energy is

$$W'_m = \frac{c}{4} x_2 i_2^4 + b x_1 x_2 i_2 i_1 + \frac{a}{4} x_1 i_1^4 \quad (c)$$

PROBLEM 3.8

Steps (a) and (b) establish the flux in the rotor winding.

$$\lambda_2 = I_o L_m \quad (a)$$

With the current constrained on the stator coil, as in step (c), the current i_1 is known, and since the flux λ_2 is also known, we can use the second terminal equations to solve for the current in the rotor winding as a function of the angular position

$$i_2 = \frac{L_m}{L_2} [I_o - I(t) \cos \theta] \quad (b)$$

This is the electrical equation of motion for the system. To complete the picture, the torque equation must be found. From the terminal relations, the co-energy is

PROBLEM 3.8 (Continued)

$$W'_m = \int \lambda_1 di_1 + \lambda_2 di_2 = \frac{1}{2} I_1 i_1^2 + i_1 i_2 L_m \cos \theta + \frac{1}{2} i_2^2 L_2 \quad (c)$$

and hence, the electrical torque is

$$T^e = \frac{\partial W'_m}{\partial \theta} = -i_1 i_2 L_m \sin \theta \quad (d)$$

Now, we use this expression in the torque equation, with i_2 given by (b) and $i_1 = I(t)$

$$\frac{J d^2 \theta}{dt^2} = - \frac{I I_m^2}{L_2} (I_0 - I(t) \cos \theta) \sin \theta \quad (e)$$

This is the required equation of motion. Note that we did not substitute i_2 from (b) into the co-energy expression and then take the derivative with respect to θ . This gives the wrong answer because we have assumed in using the basic energy method to find the torque that i_1, i_2 and θ are thermodynamically independent variables.

PROBLEM 3.9

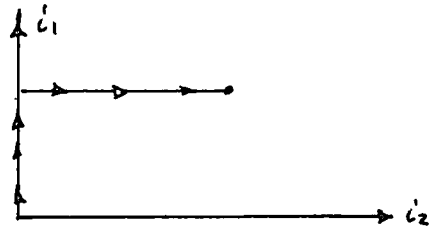
Part a

From the terminal relations, the electrical co-energy is (Table 3.1.1)

$$W'_m = \int \lambda_1 di_1 + \lambda_2 di_2 \quad (a)$$

or

$$W'_m = \frac{1}{4} a x_1^2 i_1^4 + b x_2^2 x_1 i_1 i_2 + \frac{1}{4} c x_2^2 i_2^4 \quad (b)$$



Part b

It follows that the required forces are

$$f_1^e = \frac{\partial W'_m}{\partial x_1} = \frac{1}{2} a x_1 i_1^4 + b x_2^2 i_1 i_2 \quad (c)$$

$$f_2^e = \frac{\partial W'_m}{\partial x_2} = 2 b x_2 x_1 i_1 i_2 + \frac{1}{2} c x_2 i_2^4 \quad (d)$$

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PROBLEM 3.9 (Continued)

Part c

There are four equations of motion in the dependent variables i_1, i_2, x_1 and x_2 : two of these are the electrical voltage equations, which in view of the terminal equations for the λ 's, are

$$-i_1 R_1 = \frac{d}{dt}(ax_1^2 i_1^3 + bx_2^2 x_1 i_2) \quad (e)$$

$$v_2(t) - i_2 R_2 = \frac{d}{dt}(bx_2^2 x_1 i_1 + cx_2^2 i_2^3) \quad (f)$$

and two are the mechanical force equations

$$0 = \frac{1}{2} ax_1 i_1^4 + bx_2^2 i_1 i_2 - Kx_1 \quad (g)$$

$$0 = 2bx_2 x_1 i_1 i_2 + \frac{1}{2} cx_2 i_2^4 - B \frac{dx_2}{dt} \quad (h)$$

PROBLEM 3.10

Part a

Because the terminal relations are expressed as functions of the current and x , it is most appropriate to use the co-energy to find the force. Hence,

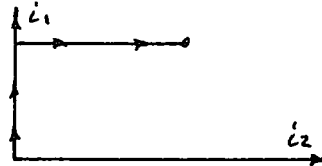
$$W'_m = \int \lambda_1 di_1 + \lambda_2 di_2 \quad (a)$$

which becomes,

$$W'_m = \frac{1}{2} L_o i_1^2 + \frac{1}{2} A i_1^2 i_2^2 x + \frac{1}{2} L_o i_2^2 \quad (b)$$

From this it follows that the force is,

$$f^e = \frac{1}{2} A i_1^2 i_2^2 \quad (c)$$



Part b

The currents i_1 and i_2 and x will be used as the dependent variables. Then, the voltage equations for the two electrical circuits can be written, using the electrical terminal equations, as

$$e_1(t) = i_1 R_1 + \frac{d}{dt}(L_o i_1 + A i_1 i_2^2 x) \quad (d)$$

$$e_2(t) = i_2 R_2 + \frac{d}{dt}(A i_1^2 i_2 x + L_o i_2) \quad (e)$$

PROBLEM 3.10 (Continued)

The equation for mechanical equilibrium of the mass M is the third equation of motion

$$M \frac{d^2 x}{dt^2} = -K(x-x_0) + \frac{1}{2} A i_1^2 i_2^2 \quad (f)$$

PROBLEM 3.11

Part a

The electrical torques are simply found by taking the appropriate derivatives of the co-energy (see Table 3.1.1)

$$T_1^e = \frac{\partial W'_m}{\partial \theta} = -M \sin \theta \cos \psi i_1 i_2 \quad (a)$$

$$T_2^e = \frac{\partial W'_m}{\partial \psi} = -M \cos \theta \sin \psi i_1 i_2 \quad (b)$$

Part b

The only torques acting on the rotors are due to the fields. In view of the above expressions the mechanical equations of motion, written using θ, ψ, i_1 and i_2 as dependent variables, are

$$J \frac{d^2 \theta}{dt^2} = -M \sin \theta \cos \psi i_1 i_2 \quad (c)$$

$$J \frac{d^2 \psi}{dt^2} = -M \cos \theta \sin \psi i_1 i_2 \quad (d)$$

Remember that the terminal voltages are the time rates of change of the respective fluxes. Hence, we can make use of the terminal equations to write the current node equations for each of the circuits as

$$I_1(t) = C \frac{d^2}{dt^2} (L_1 i_1 + M i_2 \cos \theta \cos \psi) + i_1 \quad (e)$$

$$I_2(t) = G \frac{d}{dt} (M i_1 \cos \theta \cos \psi + L_2 i_2) + i_2 \quad (f)$$

Thus, we have four equations, two mechanical and two electrical, which involve the dependent variables θ, ψ, i_1 and i_2 and the known driving functions I_1 and I_2 .

LUMPED PARAMETER-ELECTROMECHANICS

PROBLEM 3.12

We can approach this problem in two ways. First from conservation of energy,

$$dW'_m = \lambda_1 di_1 + \lambda_2 di_2 + \lambda_3 di_3 \quad (a)$$

and

$$dW'_m = \frac{\partial W'_m}{\partial i_1} di_1 + \frac{\partial W'_m}{\partial i_2} di_2 + \frac{\partial W'_m}{\partial i_3} di_3 \quad (b)$$

Hence,

$$\lambda_1 = \frac{\partial W'_m}{\partial i_1}; \lambda_2 = \frac{\partial W'_m}{\partial i_2}; \lambda_3 = \frac{\partial W'_m}{\partial i_3} \quad (c)$$

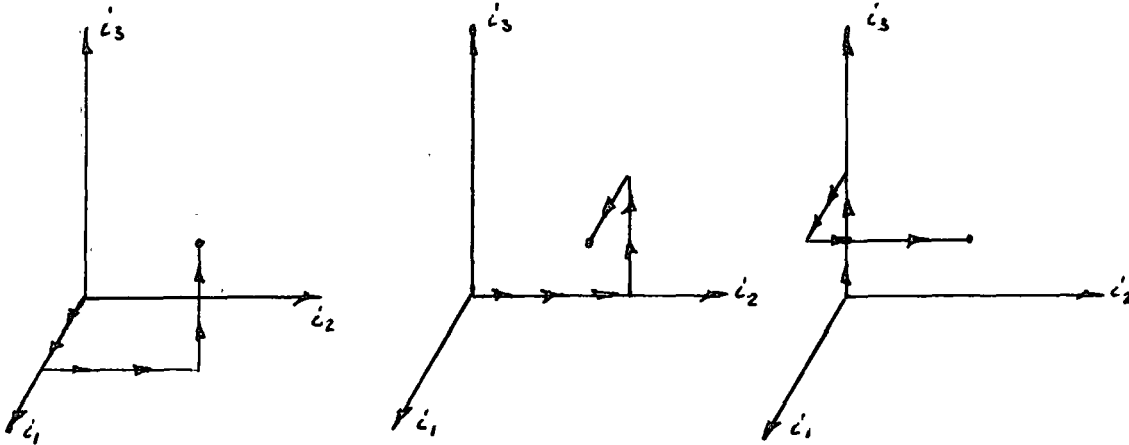
Taking combinations of cross-derivatives, this gives

$$\frac{\partial \lambda_1}{\partial i_2} = \frac{\partial \lambda_2}{\partial i_1}; \frac{\partial \lambda_2}{\partial i_3} = \frac{\partial \lambda_3}{\partial i_2}; \frac{\partial \lambda_3}{\partial i_1} = \frac{\partial \lambda_1}{\partial i_3} \quad (d)$$

or

$$L_{12} = L_{21}; L_{23} = L_{32}; L_{31} = L_{13} \quad (e)$$

Another way to show the same thing is to carry out the integrations along the three different paths shown



Since

$$W'_m = \int \lambda_1 di_1 + \lambda_2 di_2 + \lambda_3 di_3 \quad (f)$$

PROBLEM 3.12 (Continued)

these paths of integration lead to differing results. For path (a), we have

$$W_m = \frac{1}{2} L_{11} i_1^2 + L_{21} i_1 i_2 + \frac{1}{2} L_{22} i_2^2 + L_{31} i_1 i_3 + L_{32} i_2 i_3 + \frac{1}{2} L_{33} i_3^2 \quad (g)$$

while for path (b)

$$W_m = \frac{1}{2} L_{22} i_2^2 + L_{32} i_2 i_3 + \frac{1}{2} L_{33} i_3^2 + \frac{1}{2} L_{11} i_1^2 + L_{12} i_2 i_1 + L_{13} i_3 i_1 \quad (h)$$

and path (c)

$$W_m = \frac{1}{2} L_{33} i_3^2 + \frac{1}{2} L_{11} i_1^2 + L_{13} i_3 i_1 + L_{21} i_1 i_2 + \frac{1}{2} L_{22} i_2^2 + L_{23} i_3 i_2 \quad (i)$$

These equations will be identical only if (e) holds.

PROBLEM 3.13

Part a

When $\theta = 0$, there is no overlap between the stator and rotor plates, as compared to complete overlap when $\theta = \pi/2$. Because the total exposed area between one pair of stator and rotor plates is $\pi R^2/2$, at an angle θ the area is

$$A = \frac{\pi R^2}{2} \frac{\theta}{(\pi/2)} = \theta R^2 \quad (a)$$

There are $2N-1$ pairs of such surfaces, and hence the total capacitance is

$$C = (2N-1)\theta R^2 \epsilon_0 / g \quad (b)$$

The required terminal relation is then $q = Cv$.

Part b

The system is electrically linear. Hence, $W'_e = \frac{1}{2} Cv^2$ and

$$T^e = \frac{\partial W'_e}{\partial \theta} = \frac{(2N-1)R^2 \epsilon_0 v^2}{2g} \quad (c)$$

Part c

There are three torques acting on the shaft, one due to the torsional spring, the second from viscous damping and the third the electrical torque.

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PROBLEM 3.13 (Continued)

$$J \frac{d^2\theta}{dt^2} = -K(\theta-\alpha) - B \frac{d\theta}{dt} + \frac{1}{2} \frac{v_o^2 (2N-1) R^2 \epsilon_o}{g} \quad (d)$$

Part d

The voltage circuit equation, in view of the electrical terminal equation is simply

$$V_o(t) = R \frac{d}{dt} \left[\frac{(2N-1) R^2 \epsilon_o v_o}{g} \right] + v \quad (e)$$

Part e

When the rotor is in static equilibrium, the derivatives in (d) vanish and we can solve for $\theta-\alpha$,

$$\theta-\alpha = \frac{v_o^2 (2N-1) R^2 \epsilon_o}{2gK} \quad (f)$$

This equation would comprise a theoretical calibration for the voltmeter if effects of fringing fields could be ignored. In practice, the plates are shaped so as to somewhat offset the square law dependence of the deflections.

PROBLEM 3.14

Part a

Fringing fields are ignored near the ends of the metal coaxial cylinders. In the region between the cylinders, the electric field has the form $\vec{E} = A \vec{1}_r / r$, where r is the radial distance from the axis and A is a constant determined by the voltage. This solution is both divergence and curl free, and hence satisfies the basic electric field equations (See Table 1.2) everywhere between the cylinders. The boundary conditions on the surfaces of the dielectric slab are also satisfied because there is no normal electric field at a dielectric interface and the tangential electric fields are continuous. To determine the constant A , note that

$$\int_a^b E_r dr = -v = A \ln\left(\frac{b}{a}\right); A = -v / \ln\left(\frac{b}{a}\right) \quad (a)$$

The surface charge on the inner surface of the outer cylinder in the regions adjacent to free space is then

PROBLEM 3.14 (Continued)

$$\sigma_f = \frac{v\epsilon_o}{\ln(\frac{b}{a})b} \quad (b)$$

while that adjacent to regions occupied by the dielectric is

$$\sigma_f = \frac{v\epsilon}{\ln(\frac{b}{a})b} \quad (c)$$

It follows that the total charge on the outer cylinder is

$$q = v \frac{\pi}{\ln(\frac{b}{a})} [L(\epsilon_o + \epsilon) - x(\epsilon - \epsilon_o)] \quad (d)$$

Part b

Conservation of power requires

$$v \frac{dq}{dt} = \frac{dW}{dt} + f_e \frac{dx}{dt} \quad (e)$$

Parts c and d

It follows from integration of (c) that

$$W_e = \frac{1}{2} \frac{q^2}{C} \quad \text{or} \quad W'_e = \frac{1}{2} C v^2 \quad (f)$$

where

$$C = \frac{\pi}{\ln(\frac{b}{a})} [L(\epsilon_o + \epsilon) - x(\epsilon - \epsilon_o)]$$

Part e

The force of electrical origin is therefore

$$f^e = \frac{\partial W'_e}{\partial x} = -\frac{1}{2} v^2 \frac{\pi}{\ln(\frac{b}{a})} (\epsilon - \epsilon_o) \quad (g)$$

Part f

The electrical constraints of the system have been left unspecified. The mechanical equation of motion, in terms of the terminal voltage v , is

$$M \frac{d^2 x}{dt^2} = -K(x-l) - \frac{1}{2} \frac{v^2 \pi}{\ln(\frac{b}{a})} (\epsilon - \epsilon_o) \quad (h)$$

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PROBLEM 3.14 (Continued)

Part g

In static equilibrium, the inertial term makes no contribution, and (h) can be simply solved for the equilibrium position x .

$$x = \ell - \frac{1}{2} \frac{V_o^2 \pi (\epsilon - \epsilon_o)}{K \ln(\frac{b}{a})} \quad (i)$$

PROBLEM 3.15

Part a

Call r the radial distance from the origin O . Then, the field in the gap to the right is, (from Ampere's law integrated across the gaps at a radius r

$$H_{\theta}^r = Ni / (\beta - \alpha - \theta) r \quad (\text{directed to the right}) \quad (a)$$

and to the left

$$H_{\theta}^l = Ni / (\beta - \alpha + \theta) r \quad (\text{directed to the left}) \quad (b)$$

These fields satisfy the conditions that $\nabla \times \vec{H} = 0$ and $\nabla \cdot \vec{B} = 0$ in the gaps. The flux is computed by integrating the flux density over the two gaps and multiplying by N .

$$\lambda = \mu_o DN \int_a^b (H_{\theta}^l + H_{\theta}^r) dr \quad (c)$$

which, in view of (a) and (b) becomes,

$$\lambda = Li, \quad L = \mu_o DN^2 \ln(\frac{b}{a}) \left[\frac{1}{\beta - \alpha + \theta} + \frac{1}{\beta - \alpha - \theta} \right] \quad (d)$$

Part b

The system is electrically linear, and hence the co-energy is simply (See Sec. 3.1.2b)

$$W'_m = \frac{1}{2} Li^2 \quad (e)$$

Part c

The torque follows from (e) as

PROBLEM 3.15 (Continued)

$$T^e = -\frac{1}{2} \mu_o D N^2 \ln\left(\frac{b}{a}\right) \left[\frac{1}{(\beta-\alpha+\theta)^2} - \frac{1}{(\beta-\alpha-\theta)^2} \right] i^2 \quad (f)$$

Part d

The torque equation is then

$$J \frac{d^2 \theta}{dt^2} = -K\theta + T^e \quad (g)$$

Part e

This equation is satisfied if $\theta=0$, and hence it is possible for the wedge to be in static equilibrium at this position.

PROBLEM 3.16

We ignore fringing fields. Then the electric field is completely between the center plate and the outer plates, where it has the value $E = v/b$. The constraints on the electrical terminals further require that $v = V_o - Ax$.

The surface charge on the outer plates is $\epsilon_o v/b$ and hence the total charge q on these plates is

$$q = 2(a-x) \frac{d\epsilon_o}{b} v \quad (a)$$

It follows that the co-energy is

$$W_e' = (a-x) \frac{d\epsilon_o}{b} v^2 \quad (b)$$

and the electrical force is

$$f^e = \frac{\partial W_e'}{\partial x} = -\frac{d\epsilon_o}{b} v^2 \quad (c)$$

Finally, we use the electrical circuit conditions to write

$$f^e = -\frac{d\epsilon_o}{b} (V_o - Ax)^2 \quad (d)$$

The major point to be made in this situation is this. One might substitute the voltage, as it depends on x , into (b) before taking the derivative. This clearly gives an answer not in agreement with (d). We have assumed in writing (c) that the variables (v, x) remain thermodynamically independent until after the force has been found. Of course, in the actual situation, external constraints

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PROBLEM 3.16 (Continued)

relate these variables, but these constraints can only be introduced with care in the energy functions. To be safe they should not be introduced until after the force has been found.

PROBLEM 3.17

Part a

The magnetic field intensities in the gaps can be found by using Ampere's law integrated around closed contours passing through the gaps. These give

$$H_g = N(i_1 + i_2)/g \quad (a)$$

$$H_1 = Ni_1/d \quad (b)$$

$$H_2 = Ni_2/d \quad (c)$$

In the magnetic material, the flux densities are

$$B_1 = \frac{\alpha N^3 i_1^3}{d^3} + \frac{\mu_o Ni_1}{d} \quad (d)$$

$$B_2 = \frac{\alpha N^3 i_2^3}{d^3} + \frac{\mu_o Ni_2}{d} \quad (e)$$

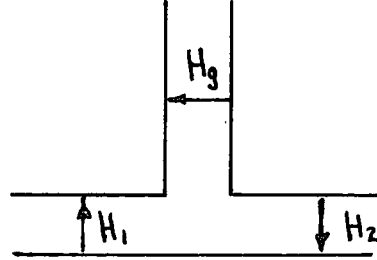
The flux linking the individual coils can now be computed as simply the flux through the appropriate gaps. For example, the flux λ_1 is

$$\lambda_1 = ND[\ell \mu_o H_g + x \mu_o H_1 + (\ell - x) B_1] \quad (f)$$

which upon substitution from the above equations becomes the first terminal relation. The second is obtained in a similar manner.

Part b

The co-energy is found by integrating, first on i_1 with $i_2 = 0$ and then on i_2 with i_1 fixed at its final value. Hence,



PROBLEM 3.17 (Continued)

$$\begin{aligned}
 W_m' &= \int \lambda_1 di_1 + \lambda_2 di_2 \\
 &= \frac{1}{2} L_o (1 + \frac{d}{g}) i_1^2 + \frac{1}{4} L_o \beta (1 - \frac{x}{\ell}) i_1^4 + L_o \frac{d}{g} i_1 i_2 \\
 &\quad + \frac{1}{4} L_o \beta (\frac{x}{\ell}) i_2^4 + \frac{1}{2} L_o (1 + \frac{d}{g}) i_2^2
 \end{aligned} \tag{g}$$

Part c

The force of electrical origin follows from the co-energy functions as,

$$f^e = - \frac{1}{4} L_o \frac{\beta}{\ell} i_1^4 + \frac{1}{4} \frac{L_o \beta}{\ell} i_2^4 \tag{h}$$

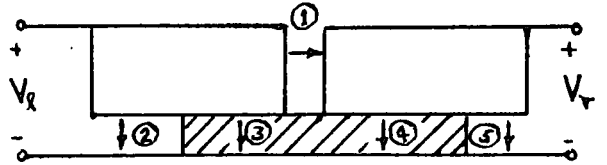
PROBLEM 3.18

Part a

Assuming simple uniform E fields in the gaps

$$E_1 = (V_\ell - V_r)/g; E_2 = V_\ell/d = E_3$$

$$E_4 = E_5 = V_r/d$$



These fields leave surface charge densities on the top electrodes

$$\sigma_1 = \epsilon_o (V_\ell - V_r)/g, \sigma_2 = \epsilon_o V_\ell/d$$

$$\sigma_3 = [\alpha (V_\ell/d)^2 + \epsilon_o] (V_\ell/d)$$

$$\sigma_4 = [\alpha (V_r/d)^2 + \epsilon_o] (V_r/d)$$

$$\sigma_5 = \epsilon_o (V_r/d)$$

These surface charge densities cause net charges on the electrodes of

$$q_\ell = \frac{\epsilon_o w b}{g} (V_\ell - V_r) + \frac{\epsilon_o w L V_\ell}{d} + \alpha w (L-x) \left(\frac{V_\ell}{d}\right)^3$$

$$q_r = \frac{\epsilon_o w b}{g} (V_r - V_\ell) + \frac{\epsilon_o w L}{d} V_r + \alpha w (x-g) \left(\frac{V_r}{d}\right)^3$$

LUMPED-PARAMETER ELECTROMECHANICS

PROBLEM 3.18 (Continued)

Part b

$$\begin{aligned}
 W' &= \int_0^{q_1} q_\ell dV_\ell + \int_0^{q_2} q_r dV_r \\
 &\quad q_2=0 \quad q_1=q_1 \\
 &= \epsilon_o w \left(\frac{b}{g} + \frac{L}{d} \right) \frac{V_\ell^2}{2} + \frac{\alpha w (L-x) d}{4} \left(\frac{V_\ell}{d} \right)^4 \\
 &\quad + \epsilon_o w \left(\frac{b}{g} + \frac{L}{d} \right) \frac{V_r^2}{2} - \frac{\epsilon_o w b}{g} V_\ell V_r + \frac{\alpha w (x-g) d}{4} \left(\frac{V_r}{d} \right)^4 \\
 f_e &= \frac{\partial W'}{\partial x} = \frac{\alpha w d}{4} \left[\left(\frac{V_r}{d} \right)^4 - \left(\frac{V_\ell}{d} \right)^4 \right] \quad (\text{pulled to side with more voltage})
 \end{aligned}$$

PROBLEM 3.19

Part a

The rotating plate forms a simple capacitor plate with respect to the other two curved plates. There is no mutual capacitance if the fringing fields are ignored. For example, the terminal relations over the first half cycle of the rotor are

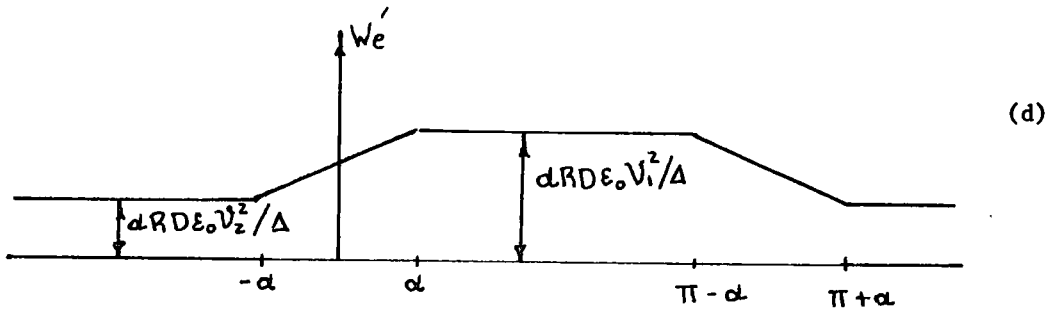
$$-\alpha < \theta < \alpha; \quad q_1 = \frac{(\alpha + \theta) R D \epsilon_o V_1}{\Delta}; \quad q_2 = \frac{(\alpha - \theta) R D \epsilon_o V_2}{\Delta} \quad (a)$$

$$\alpha < \theta < \pi - \alpha; \quad q_1 = \frac{2\alpha R D \epsilon_o V_1}{\Delta}; \quad q_2 = 0 \quad (b)$$

So that the co-energy can be simply written as the sum of the capacitances for the two outer electrodes relative to the rotor.

$$W'_e = \frac{1}{2} C_1 V_1^2 + \frac{1}{2} C_2 V_2^2 \quad (c)$$

The dependence of this quantity on θ is as shown below

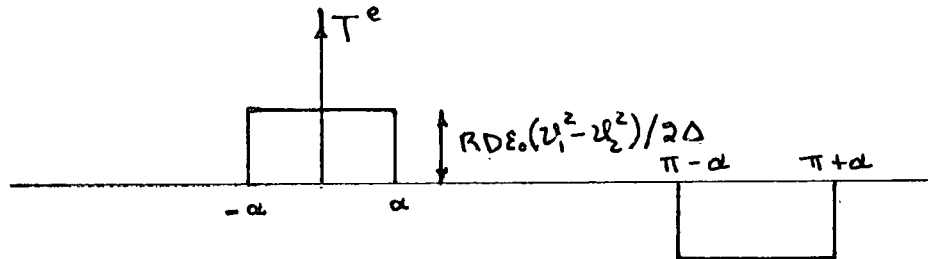


LUMPED-PARAMETER ELECTROMECHANICS

PROBLEM 3.19 (Continued)

Part b

The torque is the spatial derivative of the above function



Part c

The torque equation is then

$$J \frac{d^2 \theta}{dt^2} = T^e$$

where T^e is graphically as above.

PROBLEM 3.20

Part a

The electric energy is

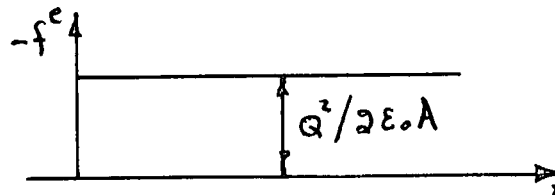
$$W_e = \frac{1}{2} q^2 / C \quad (a)$$

where

$$C = \epsilon A / d (1 + \frac{\epsilon x}{\epsilon_0 d}) \quad (b)$$

It follows that the force on the upper plate due to the electric field is,

$$-f = f^e = - \frac{\partial W_e}{\partial x} = - \frac{1}{2} \frac{Q^2}{\epsilon_0 A}$$



So long as the charge on the plate is constant, so also is the force.

LUMPED-PARAMETER ELECTROMECHANICS

PROBLEM 3.20 (Continued)

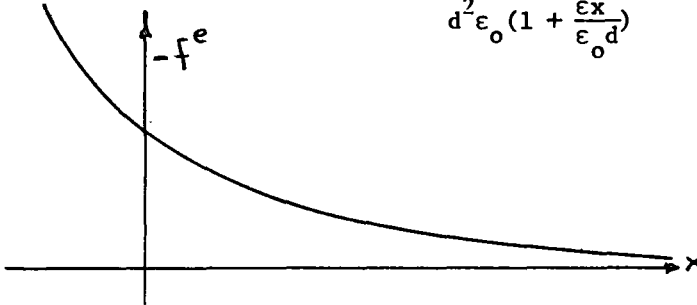
Part b

The electric co-energy is

$$W'_e = \frac{1}{2} C v^2 \quad (c)$$

and hence the force, in terms of the voltage is

$$-f = f^e = \frac{\partial W'_e}{\partial x} = \frac{1}{2} \frac{v^2 \epsilon^2 A}{d^2 \epsilon_o (1 + \frac{\epsilon x}{\epsilon_o d})^2} \quad (d)$$



The energy converted to mechanical form is $\int f^e dx$. The contribution to this integral from $d \rightarrow c$ and $b \rightarrow a$ in the figure is zero. Hence,

$$\begin{aligned} \text{Energy converted to mechanical form} &= \int_{\epsilon_o d/\epsilon}^{2\epsilon_o d/\epsilon} f^e(2Q_o, x) dx \\ &+ \int_{2\epsilon_o d/\epsilon}^{\epsilon_o d/\epsilon} f^e(Q_o, x) dx = -\frac{3}{2} \frac{dQ_o^2}{A\epsilon} \quad (e) \end{aligned}$$

That is, the energy $3dQ_o^2/2A\epsilon$ is converted from mechanical to electrical form.

PROBLEM 3.21

Part a

The magnetic energy stored in the coupling is

$$W_m = \frac{1}{2} \lambda^2 / L \quad (a)$$

where $L = L_o / (1 + \frac{x}{a})$

Hence, in terms of λ , the force of electrical origin is

$$-f = f^e = -\frac{\partial W_m}{\partial x} = -\lambda^2 / 2aL_o \quad (b)$$

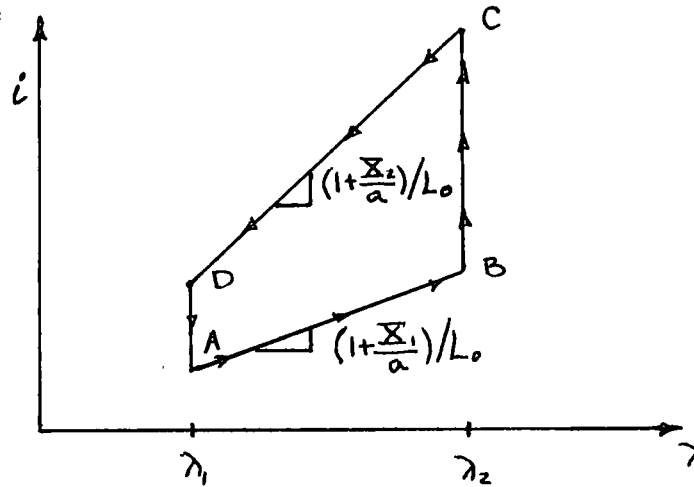
PROBLEM 3.21 (Continued)

Part b

According to the terminal equation, i depends on (λ, x) according to

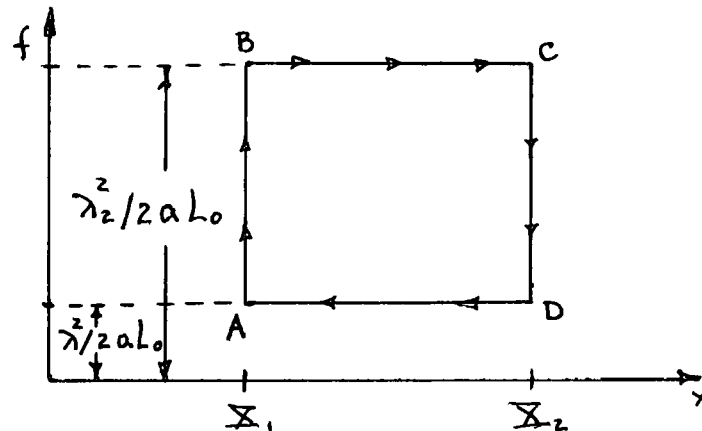
$$i = \frac{\lambda}{L_0} \left(1 + \frac{x}{a}\right) \quad (c)$$

Thus, the process represented in the λ - x plane has the corresponding path in the i - λ plane



Path c

At the same time, the force traverses a loop in the f - x plane which, from (b) is,



LUMPED-PARAMETER ELECTROMECHANICS

PROBLEM 3.21 (Continued)

Part d

The energy converted per cycle to mechanical form is $\int f^e dx$. Hence,

$$\text{Energy converted to mechanical form} = \int_B^C f^e dx + \int_D^A f^e dx \quad (d)$$

$$= -(\lambda_2^2 - \lambda_1^2)(X_2 - X_1)/2aL_0 \quad (e)$$

That is, the energy converted to electrical form per cycle is

$(\lambda_2^2 - \lambda_1^2)(X_2 - X_1)/2aL_0$. (Note that the energy stored in the coupling, summed around the closed path, is zero because the coupling is conservative.)

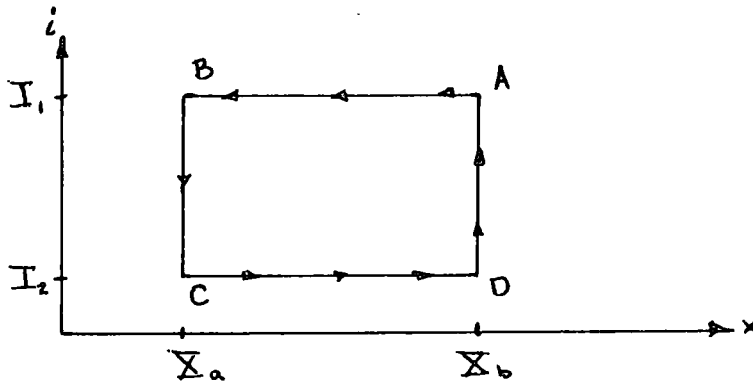
PROBLEM 3.22

Part a

The plates are pushed apart by the fields. Therefore energy is converted from mechanical form to either electrical form or energy storage in the coupling as the plate is moved from X_b to X_a . To make the net conversion from mechanical to electrical form, we therefore make the current the largest during this phase of the cycle or, $I_1 > I_2$.

Part b

With the currents related as in part a, the cycle appears in the i - x plane as shown



LUMPED-PARAMETER ELECTROMECHANICS

PROBLEM 3.22 (Continued)

Quantitatively, the magnetic field intensity into the paper is $H = I/D$ so that $\lambda = \mu_0 I x h / D$. Hence,

$$W'_m = \frac{1}{2} \left(\frac{\mu_0 x h}{D} \right) I^2 \quad (a)$$

and

$$f^e = \frac{\partial W'_m}{\partial x} = \frac{1}{2} \left(\frac{\mu_0 h}{D} \right) I^2 \quad (b)$$

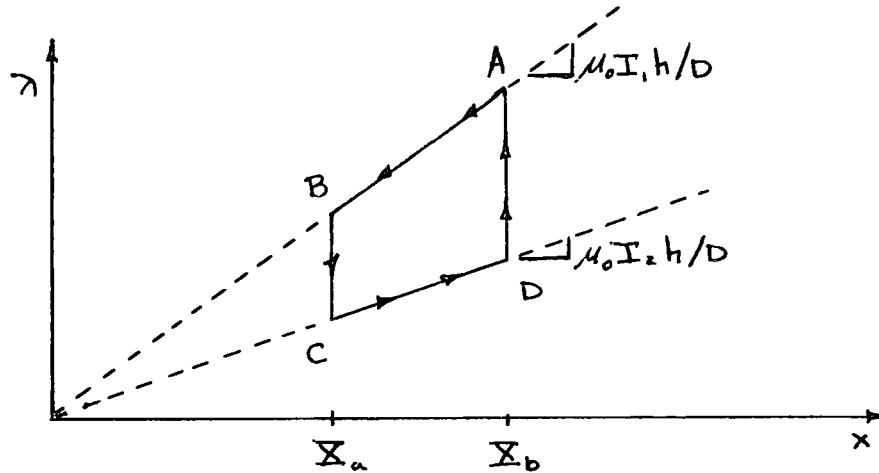
Because the cycle is closed, there is no net energy stored in the coupling, and the energy converted to electrical form is simply that put in in mechanical form:

$$\text{Mechanical to electrical energy per cycle} = - \int_A^B f^e dx - \int_C^D f^e dx \quad (c)$$

$$= \frac{1}{2} \frac{\mu_0 h}{D} (x_b - x_a) (I_1^2 - I_2^2) \quad (d)$$

Part c

From the terminal equation and the defined cycle conditions, the cycle in the λ - x plane can be pictured as



The energy converted to electrical form on each of the legs is

LUMPED-PARAMETER ELECTROMECHANICS

PROBLEM 3.22 (Continued)

$$(A \rightarrow B) \quad - \int I_1 d\lambda = - \int_{\mu_o I_1 X_b h/D}^{\mu_o I_1 X_a h/D} I_1 d\lambda = - \frac{\mu_o I_1^2 h}{D} (X_a - X_b) \quad (e)$$

$$(B \rightarrow C) \quad - \int i d\lambda = - \int_{\mu_o I_1 X_a h/D}^{\mu_o I_2 X_a h/D} \frac{\lambda D d\lambda}{\mu_o X_a h} = - \frac{\mu_o X_a h}{D} (I_2^2 - I_1^2) \quad (f)$$

$$(C \rightarrow D) \quad - \int i_2 d\lambda = \frac{\mu_o I_2^2 h}{D} (X_a - X_b) \quad (g)$$

$$(D \rightarrow A) \quad - \int i d\lambda = \frac{\mu_o X_b h}{2D} (I_2^2 - I_1^2) \quad (h)$$

The sum of these is equal to (c). Note however that the mechanical energy input on each leg is not necessarily converted to electrical form, but can be stored in the coupling.

PROBLEM 4.1
Part a

With stator current acting alone the situation is as depicted at the right. Recognizing by symmetry that $H_{rs}(\psi+\pi) = -H_{rs}(\psi)$ we use the contour shown and Ampere's law to get

$$2H_{rs}(\psi)g = \int_{\psi}^{\psi+\pi} \left[\frac{N_s i_s}{2(R+g)} \sin \psi' \right] (R+g) d\psi' = N_s i_s \cos \psi$$

from which

$$H_{rs}(\psi) = \frac{N_s i_s \cos \psi}{2g}$$

and

$$B_{rs}(\psi) = \frac{\mu_0 N_s i_s \cos \psi}{2g}$$

Part b

Following the same procedure for rotor excitation alone we obtain

$$B_{rr}(\psi) = \frac{\mu_0 N_r i_r \cos(\psi-\theta)}{2g}$$

Note that this result is obtained from part (a) by making the replacements

$$N_s \rightarrow N_r$$

$$i_s \rightarrow i_r$$

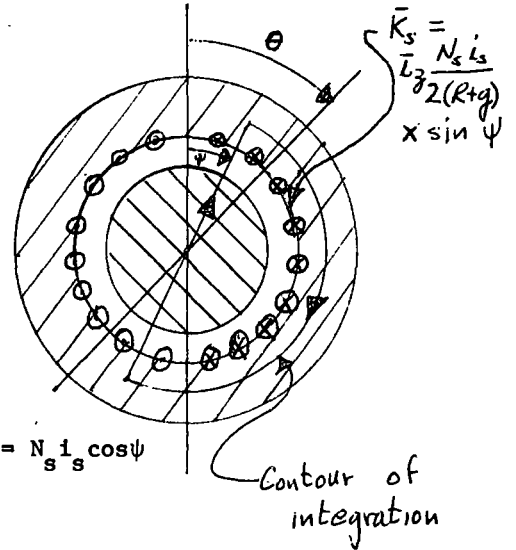
$$\psi \rightarrow (\psi-\theta)$$

Part c

The flux density varies around the periphery and the windings are distributed, thus a double integration is required to find inductances, whether they are found from stored energy or from flux linkages. We will use flux linkages.

The total radial flux density is

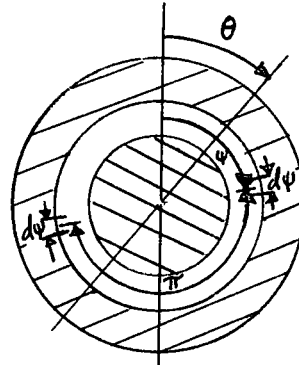
$$B_r = B_{rs} + B_{rr} = \frac{\mu_0}{2g} [N_s i_s \cos \psi + N_r i_r \cos(\psi-\theta)]$$



ROTATING MACHINES

PROBLEM 4.1 (Continued)

Taking first the elemental coil on the stator having sides of angular span $d\psi$ at positions ψ and $\psi+\pi$ as illustrated. This coil links an amount of flux



$$d\lambda_s = \underbrace{\left[\frac{N_s}{2(R+g)} \sin\psi \right] (R+g) d\psi}_{\text{number of turns in elemental coil}} \underbrace{\left[- \int_{\psi}^{\psi+\pi} B_r(\psi') (R+g) \ell d\psi' \right]}_{\text{flux linking one turn of elemental coil}}$$

$$d\lambda_s = - \frac{\mu_0 N_s (R+g) \ell}{4g} \sin\psi d\psi \int_{\psi}^{\psi+\pi} [N_s i_s \cos\psi' + N_r i_r \cos(\psi' - \theta)] d\psi'$$

$$d\lambda_s = \frac{\mu_0 N_s (R+g) \ell}{2g} \sin\psi [N_s i_s \sin\psi + N_r i_r \sin(\psi - \theta)] d\psi$$

To find the total flux linkage with the stator coil we add up all of the contributions

$$\lambda_s = \frac{\mu_0 N_s (R+g) \ell}{2g} \int_0^\pi \sin\psi [N_s i_s \sin\psi + N_r i_r \sin(\psi - \theta)] d\psi$$

$$\lambda_s = \frac{\mu_0 N_s (R+g) \ell}{2g} \left[\frac{\pi}{2} N_s i_s + \frac{\pi}{2} N_r i_r \cos\theta \right]$$

This can be written as

$$\lambda_s = L_s i_s + M i_r \cos\theta$$

where

$$L_s = \frac{\pi \mu_0 N_s^2 R \ell}{4g}$$

$$M = \frac{\pi \mu_0 N_s N_r R \ell}{4g}$$

and we have written $R+g \approx R$ because $g \ll R$.

When a similar process is carried out for the rotor winding, it yields

$$\lambda_r = L_r i_r + M i_s \cos\theta$$

where

$$L_r = \frac{\pi \mu_0 N_r^2 R \ell}{4g}$$

and M is the same as calculated before.

PROBLEM 4.2

Part a

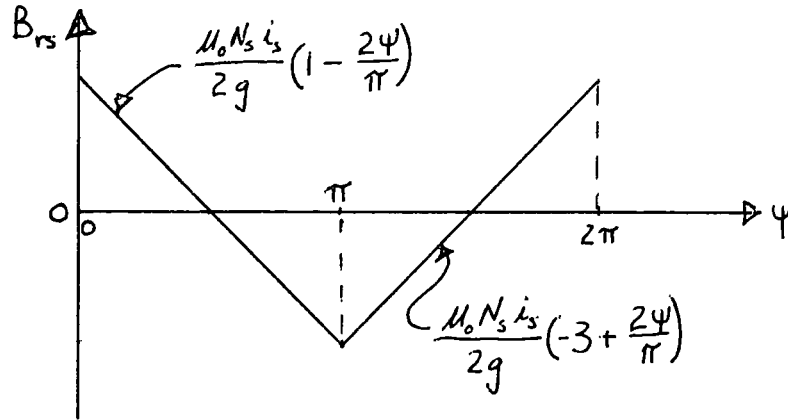
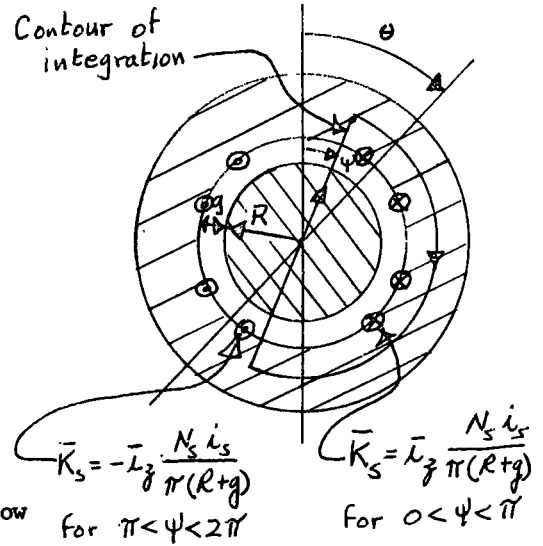
Application of Ampere's law
with the contour shown and use of the
symmetry condition

$$H_{rs}(\psi + \pi) = -H_{rs}(\psi) \text{ yields}$$

$$2H_{rs}(\psi)g = N_s i_s \left(1 - \frac{2\psi}{\pi}\right); \text{ for } 0 < \psi < \pi$$

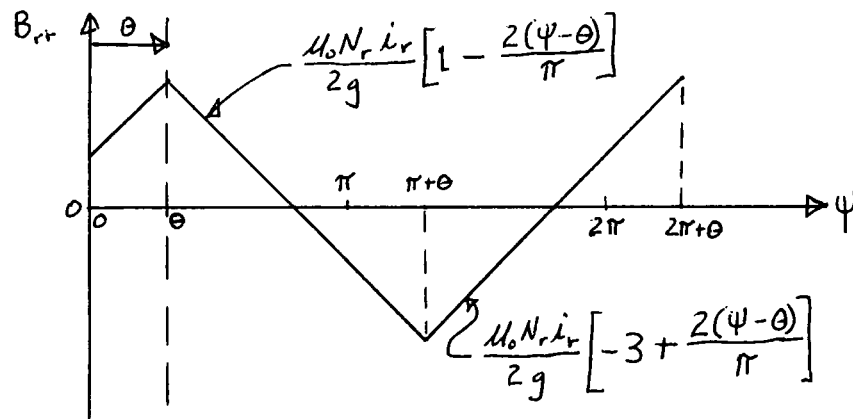
$$2H_{rs}(\psi)g = N_s i_s \left(-3 + \frac{2\psi}{\pi}\right); \text{ for } \pi < \psi < 2\pi$$

The resulting flux density is sketched below



Part b

The same process applied to excitation of the rotor winding yields



PROBLEM 4.2 (Continued)

Part c

For calculating inductances it will be helpful to have both flux densities and turn densities in terms of Fourier series. The turn density on the stator is expressible as

$$n_s = \frac{4N_s}{\pi^2(R+g)} \sum_{\text{nodd}} \frac{1}{n} \sin n\psi$$

and the turn density on the rotor is

$$n_r = \frac{4N_r}{\pi^2 R} \sum_{\text{nodd}} \frac{1}{n} \sin(\psi-\theta)$$

and the flux densities are expressible as

$$B_{rs} = \sum_{\text{nodd}} \frac{4\mu_o N_s i_s}{\pi^2 g n^2} \cos n\psi$$

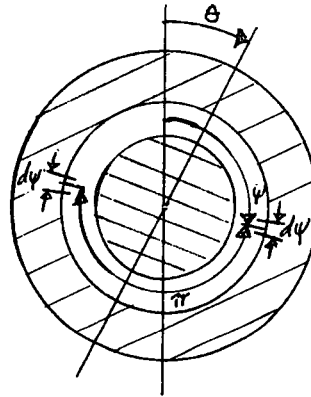
$$B_{rr} = \sum_{\text{nodd}} \frac{4\mu_o N_r i_r}{\pi^2 g n^2} \cos n(\psi-\theta)$$

The total radial flux density is

$$B_r = B_{rs} + B_{rr}$$

First calculating stator flux linkages, we first consider the elemental coil having sides $d\psi$ long and π radians apart

$$d\lambda_s = \underbrace{n_s(R+g)d\psi}_{\text{number of turns}} \underbrace{\left[-\int_{\psi}^{\psi+\pi} B_r(\psi')(R+g)l d\psi' \right]}_{\text{flux linking one turn of elemental coil}}$$



Substitution of series for B_r yields

$$d\lambda_s = n_s(R+g)^2 l d\psi \left[\sum_{\text{nodd}} \frac{8\mu_o N_s i_s}{\pi^2 g n^3} \sin n\psi + \sum_{\text{nodd}} \frac{8\mu_o N_r i_r}{\pi^2 g n^3} \sin n(\psi-\theta) \right]$$

The total flux linkage with the stator coil is

$$\lambda_s = \frac{32\mu_o N_s (R+g)l}{\pi^4 g} \int_0^\pi \left[\sum_{\text{nodd}} \frac{1}{n} \sin n\psi \right] \left[\sum_{\text{nodd}} \frac{N_s i_s}{n^3} \sin n\psi + \sum_{\text{nodd}} \frac{N_r i_r}{n^3} \sin n(\psi-\theta) \right] d\psi$$

ROTATING MACHINES

PROBLEM 4.2 (Continued)

Recognition that

$$\int_0^\pi \sin n\psi \sin m(\psi-\theta) d\psi = 0 \text{ when } m \neq n$$

simplifies the work in finding the solution

$$\lambda_s = \frac{32\mu_o N_s (R+g)\ell}{\pi^4 g} \sum_{\text{nodd}} \left(\frac{\pi N_s i_s}{2n^4} + \frac{\pi N_r i_r}{2n^4} \cos n\theta \right)$$

This can be written in the form

$$\lambda_s = L_s i_s + \sum_{\text{nodd}} M_n \cos n\theta i_r$$

where

$$L_s = \frac{16\mu_o N_s^2 R\ell}{\pi^3 g} \sum_{\text{nodd}} \frac{1}{n^4}$$

$$M_n = \frac{16\mu_o N_s N_r R\ell}{\pi^3 g n^4}$$

In these expressions we have used the fact that $g \ll R$ to write $R+g \approx R$.

A similar process with the rotor winding yields

$$\lambda_r = L_r i_r + \sum_{\text{nodd}} M_n \cos n\theta i_s$$

where

$$L_r = \frac{16\mu_o N_r^2 R\ell}{\pi^3 g} \sum_{\text{nodd}} \frac{1}{n^4}$$

and M_n is as given above.

PROBLEM 4.3

With reference to the solution of Prob. 4.2, if the stator winding is sinusoidally distributed, λ_s becomes

$$\lambda_s = \frac{32\mu_o N_s (R+g)\ell}{\pi^4 g} \int_0^\pi \sin\psi \left[N_s i_s \sin\psi + \sum_{\text{nodd}} \frac{N_r i_r}{n^3} \sin n(\psi-\theta) \right] d\psi$$

Because $\int_0^\pi \sin\psi \sin n(\psi-\theta) d\psi = 0$ when $n \neq 1$

$$\lambda_s = \frac{32\mu_o N_s (R+g)\ell}{\pi^4 g} \int_0^\pi \sin\psi \left[N_s i_s \sin\psi + N_r i_r \sin(\psi-\theta) \right] d\psi$$

and the mutual inductance will contain no harmonic terms.

Similarly, if the rotor winding is sinusoidally distributed,

ROTATING MACHINES

PROBLEM 4.3 (Continued)

$$\lambda_s = \frac{32\mu_o N_s (R+g)\ell}{\pi^4 g} \int_0^\pi \left[\sum_{n \text{ odd}} \frac{1}{n} \sin n\psi \right] \left[\sum_{n \text{ odd}} \frac{N_s i_s}{n^3} \sin n\psi + N_r i_r \sin(\psi-\theta) \right] d\psi$$

Using the orthogonality condition

$$\int_0^\pi \sin n\psi \sin(\psi-\theta) d\psi = 0 \text{ when } n \neq 1$$

$$\lambda_s = \frac{32\mu_o N_s (R+g)\ell}{\pi^4 g} \int_0^\pi \left[\sum_{n \text{ odd}} \frac{N_s i_s}{4n} \sin^2 n\psi + N_r i_r \sin\psi \sin(\psi-\theta) \right] d\psi$$

and the mutual inductance once again contains only a space fundamental term.

PROBLEM 4.4

Part a

The open-circuit stator voltage is

$$v_s = \frac{d\lambda_s}{dt} = \frac{d}{dt} \left[I \sum_{n \text{ odd}} \frac{M_o}{4} \cos n\omega t \right]$$

$$v_s(t) = - \sum_{n \text{ odd}} \frac{\omega M_o I}{3n} \sin n\omega t$$

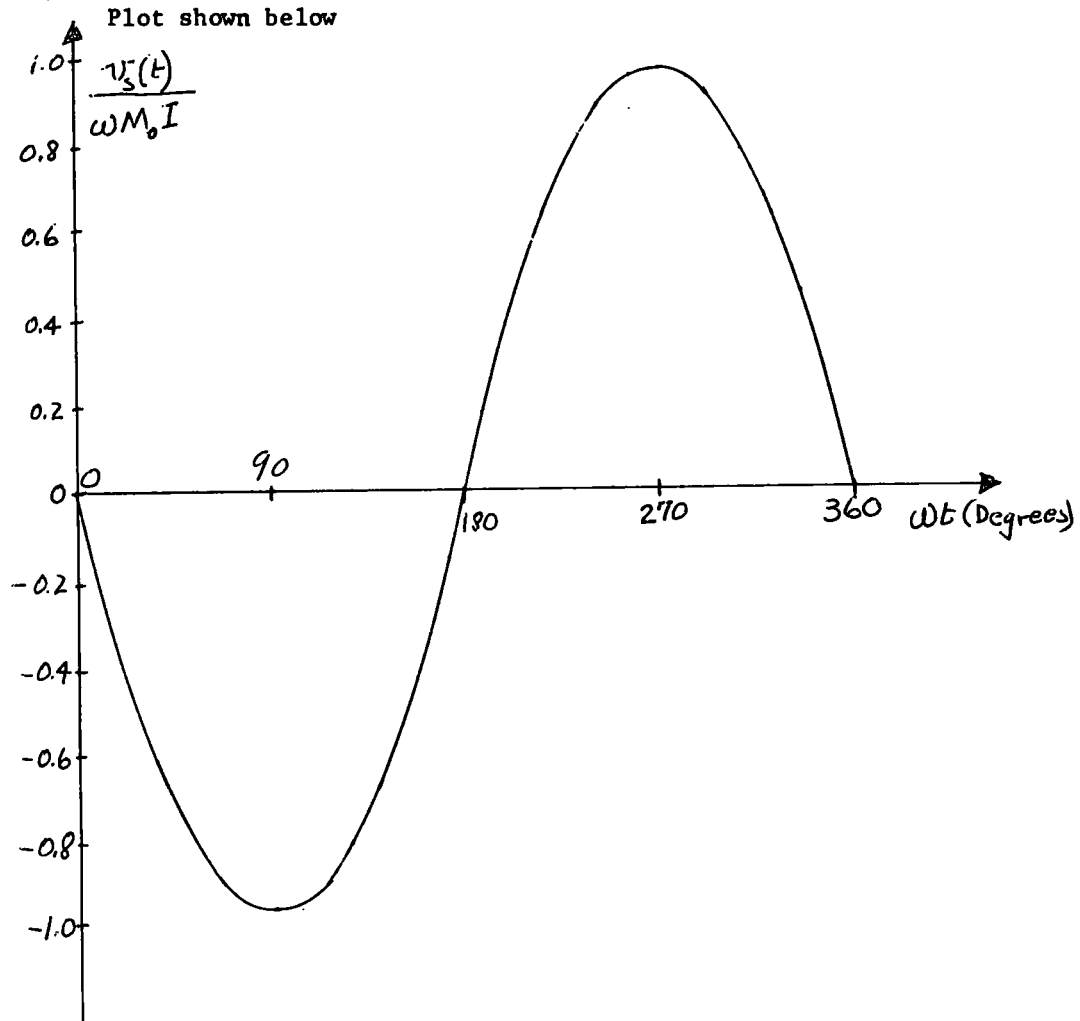
Part b

$$\frac{V_{s3}}{V_{s1}} = \frac{1}{3} ; \quad \frac{V_{s3}}{V_{s1}} = \frac{1}{27} \approx 4 \text{ percent}$$

This indicates that uniform turn density does not yield unreasonably high values of harmonics.

PROBLEM 4.4 (Continued)

Part c



PROBLEM 4.5

Given electrical terminal relations are

$$\lambda_s = L_s i_s + M i_r \cos \theta$$

$$\lambda_r = M i_s \cos \theta + L_r i_r$$

System is conservative so energy or coenergy is independent of path. Select currents and θ as independent variables and use coenergy (see Table 3.1). Assemble system first mechanically, then electrically so torque is not needed in calculation of coenergy. Selecting one of many possible paths of integration for i_s and i_r we have

ROTATING MACHINES

PROBLEM 4.5 (Continued)

$$W'_m(i_s, i_r, \theta) = \int_0^{i_s} \lambda_s(i'_s, 0, \theta) di'_s + \int_0^{i_r} \lambda_r(i_s, i'_r, \theta) di'_r$$

$$W'_m(i_s, i_r, \theta) = \frac{1}{2} L_s i_s^2 + M i_r i_s \cos \theta + \frac{1}{2} L_r i_r^2$$

$$T^e = \frac{\partial W'_m(i_s, i_r, \theta)}{\partial \theta} = -M i_r i_s \sin \theta$$

PROBLEM 4.6

The conditions existing at the time the rotor winding terminals are short-circuited lead to the constant rotor winding flux linkages

$$\lambda_r = M I_o$$

This constraint leads to a relation between i_r and $i_s = i(t)$

$$M I_o = M i_s \cos \theta + L_r i_r$$

$$i_r = \frac{M}{L_r} [I_o - i(t) \cos \theta]$$

The torque equation (4.1.8) is valid for any terminal constraint, thus

$$T^e = -M i_r i_s \sin \theta = -\frac{M^2}{L_r} i(t) [I_o - i(t) \cos \theta] \sin \theta$$

The equation of motion for the shaft is then

$$J \frac{d^2 \theta}{dt^2} = -\frac{M^2}{L_r} i(t) [I_o - i(t) \cos \theta] \sin \theta$$

PROBLEM 4.7

Part a

Coenergy is

$$W'_m(i_s, i_r, \theta) = \frac{1}{2} L_s i_s^2 + \frac{1}{2} L_r i_r^2 + L_{sr}(\theta) i_s i_r$$

$$T^e = \frac{\partial W'_m(i_s, i_r, \theta)}{\partial \theta} = i_s i_r \frac{dL_{sr}(\theta)}{d\theta}$$

$$T^e = -i_s i_r [M_1 \sin \theta + 3M_3 \sin 3\theta]$$

Part b

With the given constraints

$$T^e = -I_s I_r \sin \omega_s t \sin \omega_r t [M_1 \sin(\omega_m t + \gamma) + 3M_3 \sin 3(\omega_m t + \gamma)]$$

ROTATING MACHINES

PROBLEM 4.7 (Continued)

Repeated application of trigonometric identities leads to:

$$T^e = -\frac{M_1 I_s I_r}{4} \left\{ \sin[(\omega_m + \omega_s - \omega_r)t + \gamma] + \sin[(\omega_m - \omega_s + \omega_r)t + \gamma] \right. \\ \left. - \sin[(\omega_m + \omega_s + \omega_r)t + \gamma] - \sin[(\omega_m - \omega_s - \omega_r)t + \gamma] \right\} \\ - \frac{3M_3 I_s I_r}{4} \left\{ \sin[(3\omega_m + \omega_s - \omega_r)t + 3\gamma] + \sin[(3\omega_m - \omega_s + \omega_r)t + 3\gamma] \right. \\ \left. - \sin[(3\omega_m + \omega_s + \omega_r)t + 3\gamma] - \sin[(3\omega_m - \omega_s - \omega_r)t + 3\gamma] \right\}$$

To have a time-average torque, one of the coefficients of time must equal zero. This leads to the eight possible mechanical speeds

$$\omega_m = \pm \omega_s \pm \omega_r \text{ and } \omega_m = \pm \frac{\omega_s \pm \omega_r}{3}$$

For

$$\omega_m = \pm(\omega_s - \omega_r) \\ T_{avg}^e = -\frac{M_1 I_s I_r}{4} \sin \gamma$$

For

$$\omega_m = \pm(\omega_s + \omega_r) \\ T_{avg}^e = \frac{M_1 I_s I_r}{4} \sin \gamma$$

For

$$\omega_m = \pm \frac{(\omega_s - \omega_r)}{3} \\ T^e = -\frac{3M_3 I_s I_r}{4} \sin 3\gamma$$

For

$$\omega_m = \pm \frac{(\omega_s + \omega_r)}{3} \\ T_{avg}^e = \frac{3M_3 I_s I_r}{4} \sin 3\gamma$$

PROBLEM 4.8

From 4.1.8 and the given constraints the instantaneous torque is

$$T^e = -I_r M \sin \omega_r t \cos(\omega_m t + \gamma) (I_{s1} \sin \omega_s t + I_{s3} \sin 3\omega_s t)$$

Repeated use of trigonometric identities leads to:

ROTATING MACHINES

PROBLEM 4.8 (Continued)

$$T^e = - \frac{I_r I_{s1}^M}{4} \left\{ \begin{aligned} &\cos[(\omega_r + \omega_m - \omega_s)t + \gamma] - \cos[(\omega_r + \omega_m + \omega_s)t + \gamma] \\ &+ \cos[(\omega_r - \omega_m - \omega_s)t - \gamma] - \cos[(\omega_r - \omega_m + \omega_s)t - \gamma] \end{aligned} \right\}$$

$$- \frac{I_r I_{s3}^M}{4} \left\{ \begin{aligned} &\cos[(\omega_r + \omega_m - 3\omega_s)t + \gamma] - \cos[(\omega_r + \omega_m + 3\omega_s)t + \gamma] \\ &+ \cos[(\omega_r - \omega_m - 3\omega_s)t - \gamma] - \cos[(\omega_r - \omega_m + 3\omega_s)t - \gamma] \end{aligned} \right\}$$

For a time-average torque one of the coefficients of t must be zero. This leads to eight values of ω_m :

$$\omega_m = \pm \omega_r \pm \omega_s \quad \text{and} \quad \omega_m = \pm \omega_r \pm 3\omega_s$$

For

$$\omega_m = \pm(\omega_r - \omega_s)$$

$$T_{\text{avg}}^e = - \frac{I_r I_{s1}^M}{4} \cos \gamma$$

For

$$\omega_m = \pm(\omega_r + \omega_s)$$

$$T_{\text{avg}}^e = \frac{I_r I_{s1}^M}{4} \cos \gamma$$

For

$$\omega_m = \pm(\omega_r - 3\omega_s)$$

$$T_{\text{avg}}^e = - \frac{I_r I_{s3}^M}{4} \cos \gamma$$

For

$$\omega_m = \pm(\omega_r + 3\omega_s)$$

$$T_{\text{avg}}^e = \frac{I_r I_{s3}^M}{4} \cos \gamma$$

PROBLEM 4.9

Electrical terminal relations are 4.1.19-4.1.22. For conservative system, coenergy is independent of path and if we bring system to its final mechanical configuration before exciting it electrically there is no contribution to the coenergy from the torque term. Thus, of the many possible paths of integration we choose one

ROTATING MACHINES

PROBLEM 4.9 (Continued)

$$\begin{aligned} W'_m(i_{as}, i_{bs}, i_{ar}, i_{br}, \theta) = & \int_0^{i_{as}} \lambda_{as}(i'_{as}, 0, 0, 0, \theta) di'_{as} \\ & + \int_0^{i_{bs}} \lambda_{bs}(i_{as}, i'_{bs}, 0, 0, \theta) di'_{bs} \\ & + \int_0^{i_{ar}} \lambda_{ar}(i_{as}, i_{bs}, i'_{ar}, 0, \theta) di'_{ar} \\ & + \int_0^{i_{br}} \lambda_{br}(i_{as}, i_{bs}, i_{ar}, i'_{br}, \theta) di'_{br} \end{aligned}$$

The use of 4.1.19-4.1.22 in this expression yields

$$\begin{aligned} W'_m = & \int_0^{i_{as}} L_s i'_{as} di'_{as} + \int_0^{i_{bs}} L_s i'_{bs} di'_{bs} \\ & + \int_0^{i_{ar}} (L_r i'_{ar} + M_{as} \cos \theta + M_{bs} \sin \theta) di'_{ar} \\ & + \int_0^{i_{br}} (L_r i'_{br} - M_{as} \sin \theta + M_{bs} \cos \theta) di'_{br} \end{aligned}$$

Evaluation of these integrals yields

$$\begin{aligned} W'_m = & \frac{1}{2} L_s i_{as}^2 + \frac{1}{2} L_s i_{bs}^2 + \frac{1}{2} L_r i_{ar}^2 + \frac{1}{2} L_r i_{br}^2 \\ & + M_{as} i_{ar} \cos \theta + M_{bs} i_{ar} \sin \theta \\ & - M_{as} i_{br} \sin \theta + M_{bs} i_{br} \cos \theta \end{aligned}$$

The torque of electric origin is then (see Table 3.1)

$$\begin{aligned} T^e = & \frac{\partial W'_m(i_{as}, i_{bs}, i_{ar}, i_{br}, \theta)}{\partial \theta} \\ T^e = & -M[i_{as} i_{ar} \sin \theta - i_{bs} i_{ar} \cos \theta + i_{as} i_{br} \cos \theta + i_{bs} i_{br} \sin \theta] \end{aligned}$$

PROBLEM 4.10

Part a

Substitution of currents into given expressions for flux density

$$\begin{aligned} B_r = & B_{ra} + B_{rb} \\ B_r = & \frac{\mu_0 N}{2g} [I_a \cos \omega t \cos \psi + I_b \sin \omega t \sin \psi] \end{aligned}$$

ROTATING MACHINES

PROBLEM 4.10 (Continued)

Part b

Application of trigonometric identities and simplification yield.

$$B_r = \frac{\mu_o N}{2g} \left[\frac{I_a}{2} \cos(\omega t - \psi) + \frac{I_a}{2} \cos(\omega t + \psi) \right]$$

$$+ \frac{I_b}{2} \cos(\omega t - \psi) - \frac{I_b}{2} \cos(\omega t + \psi)]$$

$$B_r = \frac{\mu_o N}{4g} [(I_a + I_b) \cos(\omega t - \psi) + (I_a - I_b) \cos(\omega t + \psi)]$$

The forward wave is

$$B_{rf} = \frac{\mu_o N(I_a + I_b)}{4g} \cos(\omega t - \psi)$$

For constant phase on the forward wave

$$\omega t - \psi = \text{constant}$$

$$\omega_f = \frac{d\psi}{dt} = \omega$$

The backward wave is

$$B_{rb} = \frac{\mu_o N(I_a - I_b)}{4g} \cos(\omega t + \psi)$$

For

$$\omega t + \psi = \text{constant}$$

$$\omega_b = \frac{d\psi}{dt} = -\omega$$

Part c

The ratio of amplitudes is

$$\frac{B_{rbm}}{B_{rfm}} = \frac{I_a - I_b}{I_a + I_b}$$

$$\frac{B_{rbm}}{B_{rfm}} \rightarrow 0 \quad \text{as} \quad I_a \rightarrow I_b$$

Part d

When $I_b = -I_a$

$$B_{rf} = 0$$

This has simply reversed the phase sequence.

ROTATING MACHINES

PROBLEM 4.11

Part a

$$B_r = B_{ra} + B_{rb}$$

$$B_r = \frac{\mu_o NI}{2g} [\cos \omega t \cos \psi + \sin(\omega t + \beta) \sin \psi]$$

Part b

Using trigonometric identities

$$B_r = \frac{\mu_o NI}{2g} [\cos \omega t \cos \psi + \cos \beta \sin \omega t \sin \psi + \sin \beta \cos \omega t \sin \psi]$$

$$B_r = \frac{\mu_o NI}{2g} \left[\frac{1}{2} \cos(\omega t - \psi) + \frac{1}{2} \cos(\omega t + \psi) + \frac{\cos \beta}{2} \cos(\omega t - \psi) - \frac{\cos \beta}{2} \cos(\omega t + \beta) + \frac{\sin \beta}{2} \sin(\omega t + \psi) - \frac{\sin \beta}{2} \sin(\omega t - \psi) \right]$$

$$B_r = \frac{\mu_o NI}{4g} [(1 + \cos \beta) \cos(\omega t - \psi) - \sin \beta \sin(\omega t - \psi) + (1 - \cos \beta) \cos(\omega t + \psi) + \sin \beta \sin(\omega t + \psi)]$$

Forward wave is

$$B_{rf} = \frac{\mu_o NI}{4g} [(1 + \cos \beta) \cos(\omega t - \psi) - \sin \beta \sin(\omega t - \psi)]$$

For constant phase

$$\omega t - \psi = \text{constant}$$

and

$$\omega_f = \frac{d\psi}{dt} = \omega$$

Backward wave is

$$B_{rb} = \frac{\mu_o NI}{4g} [(1 - \cos \beta) \cos(\omega t + \psi) + \sin \beta \sin(\omega t + \psi)]$$

For constant phase

$$\omega t + \psi = \text{constant}$$

and

$$\omega_b = \frac{d\psi}{dt} = -\omega$$

ROTATING MACHINES

PROBLEM 4.11 (Continued)

Part c

The ratio of amplitudes is

$$\frac{B_{\text{rbm}}}{B_{\text{rfm}}} = \frac{\sqrt{(1-\cos\beta)^2 + \sin^2\beta}}{\sqrt{(1+\cos\beta)^2 + \sin^2\beta}} = \sqrt{\frac{1-\cos\beta}{1+\cos\beta}}$$

$$\text{as } \beta \rightarrow 0, \frac{B_{\text{rbm}}}{B_{\text{rfm}}} \rightarrow 0..$$

Part d

The forward wave amplitude will go to zero when $\beta = \pi$. The phase sequence has been reversed by reversing the phase of the current in the b-winding.

PROBLEM 4.12

Equation 4.1.53 is

$$p_e = v_{as} i_{as} + v_{bs} i_{bs}$$

For steady state balanced conditions we can write

$$i_{as} = I \cos \omega t; \quad i_{bs} = I \sin \omega t$$

$$v_{as} = V \cos(\omega t + \phi); \quad v_{bs} = V \sin(\omega t + \phi)$$

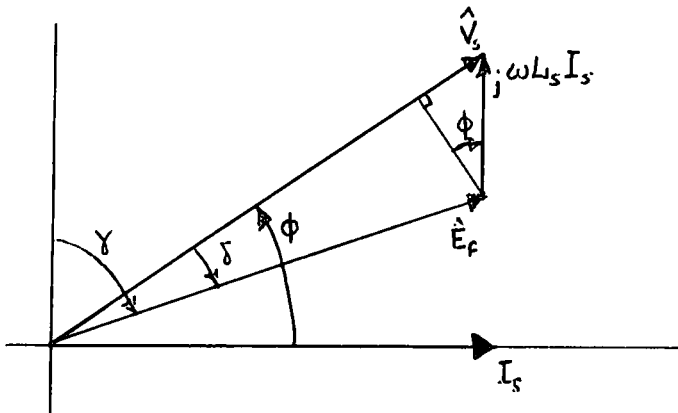
then

$$p_e = VI[\cos\omega t \cos(\omega t + \phi) + \sin\omega t \sin(\omega t + \phi)]$$

Using trigonometric identities

$$P_e = VI \cos \phi$$

Referring to Fig. 4.1.13(b) we have the vector diagram



PROBLEM 4.12 (Continued)

From this figure it is clear that

$$\omega L_s I_s \cos \phi = -E_f \sin \delta$$

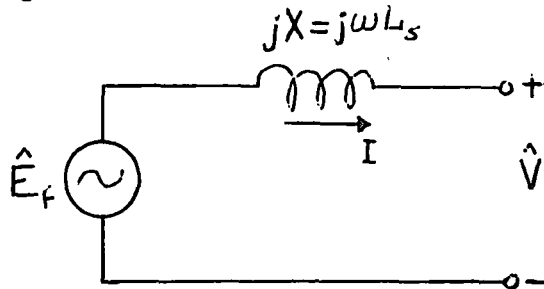
(remember that $\delta < 0$)

$$\text{Then } p_e = - \frac{VE_f}{\omega L_s} \sin \delta$$

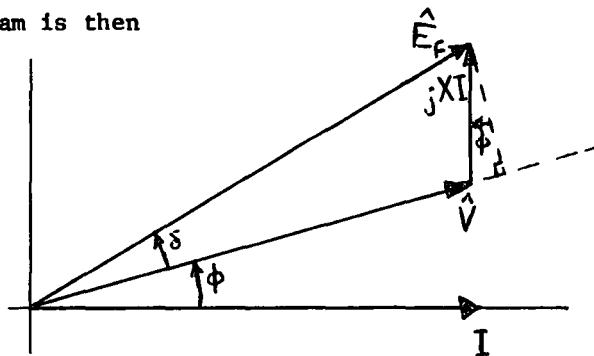
which was to be shown.

PROBLEM 4.13

For the generator we adopt the notation for one phase of the armature circuit (see Fig. 4.1.12 with current convention reversed)



The vector diagram is then



From the vector diagram

$$XI \sin \phi = E_f \cos \delta - V$$

$$XI \cos \phi = E_f \sin \delta$$

Also, the mechanical power input is

$$P = \frac{E_f V}{X} \sin \delta$$

Eliminating ϕ and δ from these equations and solving for I yields

ROTATING MACHINES

PROBLEM 4.13 (Continued)

$$I = \frac{V}{X} \sqrt{\left(\frac{E_f}{V}\right)^2 - 2 \sqrt{\left(\frac{E_f}{V}\right)^2 - \left(\frac{PX}{V^2}\right)^2} + 1}$$

Normalizing as indicated in the problem statement we define

I_o = rated armature current

I_{fo} = field current to give rated voltage
on open circuit.

P_o = rated power

$$\frac{I}{I_o} = \frac{V}{I_o X} \sqrt{\left(\frac{I_f}{I_{fo}}\right)^2 + 1 - 2 \sqrt{\left(\frac{I_f}{I_{fo}}\right)^2 - \left(\frac{P}{P_o}\right)^2 \left(\frac{P_o X}{V^2}\right)^2}}$$

Injecting given numbers and being careful about rms and peak quantities we have

$$\frac{I}{I_o} = 0.431 \sqrt{\left(\frac{I_f}{I_{fo}}\right)^2 + 1 - 2 \sqrt{\left(\frac{I_f}{I_{fo}}\right)^2 - 3.92 \left(\frac{P}{P_o}\right)^2}}$$

$$I_{fo} = 2,030 \text{ amps}$$

and

$$\left(\frac{I_f}{I_{fo \text{ max}}}\right) = 3.00$$

The condition that $\delta = \frac{\pi}{2}$ is

$$E_f = \frac{PX}{V}$$

$$\left(\frac{I_f}{I_{fo \text{ min}}}\right) = \frac{PX}{\omega M I_{fo} V} = \frac{PX}{V^2} = 1.98 \frac{P}{P_o}$$

For unity p.f., $\cos \phi = 1$, $\sin \phi = 0$

$$E_f \cos \delta = V \text{ and } E_f \sin \delta = IX$$

eliminating δ we have

$$\frac{I}{I_o} = \frac{V}{XI_o} \sqrt{\left(\frac{E_f}{V}\right)^2 - 1}$$

$$\frac{I}{I_o} = 0.431 \sqrt{\left(\frac{I_f}{I_{fo}}\right)^2 - 1}$$

ROTATING MACHINES

PROBLEM 4.13 (Continued)

for 0.85 p.f.

$$E_f \sin \delta = 0.85 IX$$

$$E_f \cos \delta - V = \sqrt{1 - (0.85)^2} IX$$

eliminating δ , solving for I , and normalizing yields

$$\frac{I}{I_o} = 0.431 \left| [-0.527 \pm \sqrt{(\frac{I_f}{I_{fo}})^2 - 0.722}] \right|$$

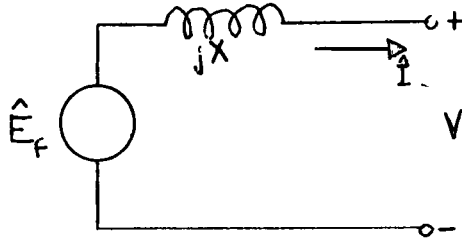
This is double-valued and the magnitude of the bracketed term is used.

The required curves are shown on the next page.

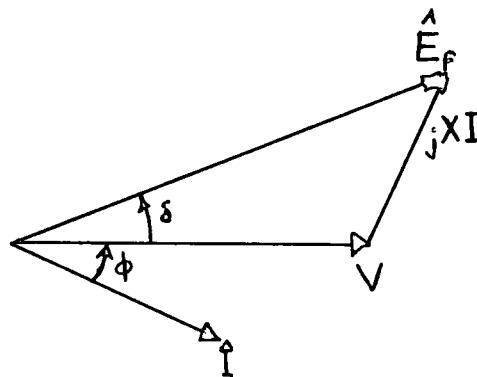
PROBLEM 4.14

The armature current limit is defined by a circle of radius VI_o , where I_o is the amplitude of rated armature current.

To find the effect of the field current limit we must express the complex power in terms of field current. Defining quantities in terms of this circuit

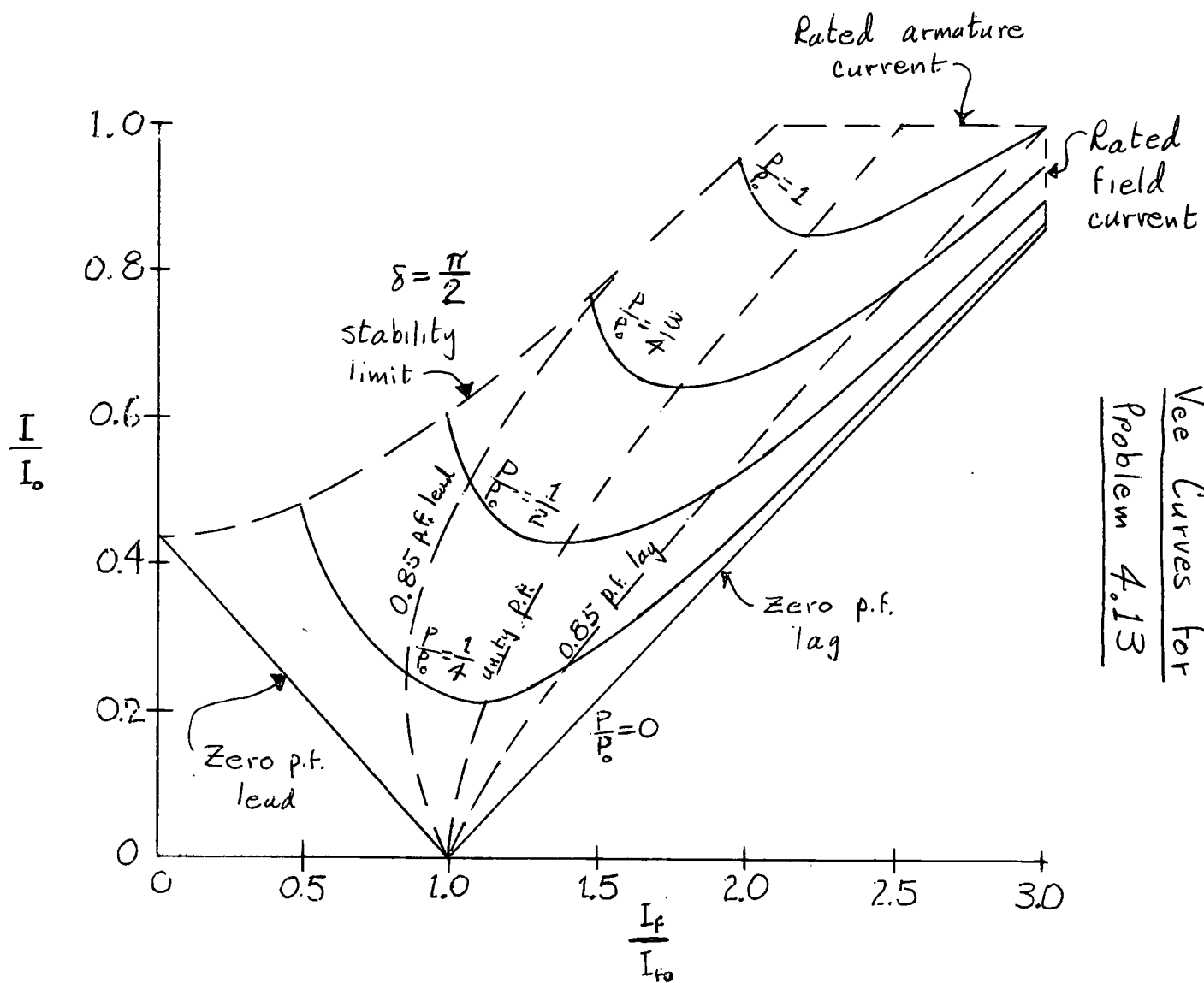


The vector diagram is



$$\hat{I} = \frac{\hat{E}_f - V}{jX}$$

$$P + jQ = \hat{V}\hat{I}^* = \frac{\hat{V}\hat{E}_f^* - V^2}{-jX} = j \frac{\hat{V}\hat{E}_f e^{-j\delta}}{X} - j \frac{V^2}{X}$$



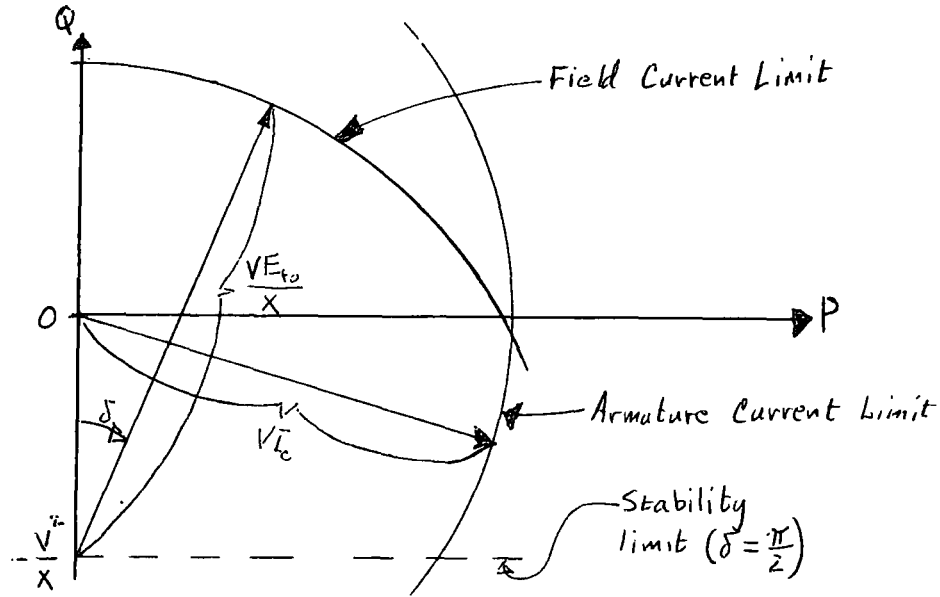
Vee Curves for
Problem 4.13

PROBLEM 4.14 (Continued)

If we denote the voltage for maximum field current as E_{fo} , this expression becomes

$$P + jQ = -j \frac{V^2}{X} + \frac{VE_{fo}}{X} \sin \delta + j \frac{VE_{fo}}{X} \cos \delta$$

On a $P+jQ$ plane this trajectory is as sketched below



The stability limit ($\delta = \frac{\pi}{2}$) is also shown in the sketch, along with the armature current limit.

The capability curve for the generator of Prob. 4.13 is shown on the next page.

P and Q are normalized to 724 MVA.

PROBLEM 4.15

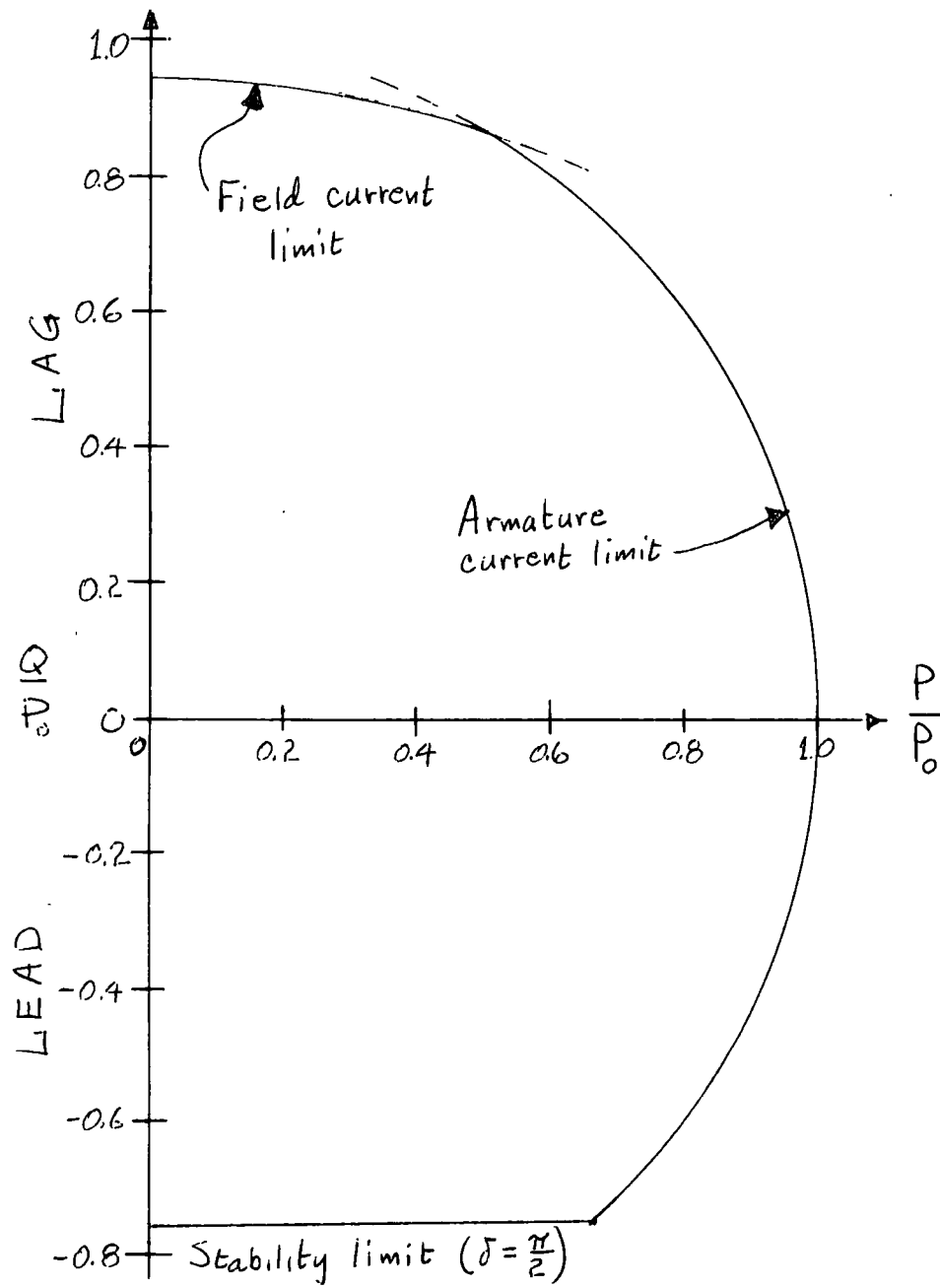
The steady state deflection ψ of the rotatable frame is found by setting sum of torques to zero

$$T^e + T_S = 0 = T^e - K\psi \quad (1)$$

where T^e is electromagnetic torque. This equation is solved for ψ .

Torque T^e is found from

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Capability curve of Problem 4.14

PROBLEM 4.15 (Continued)

$$T^e = \frac{\partial W'_m(i_1, i_2, i_3, \phi, \psi)}{\partial \psi}$$

and the magnetic coenergy for this electrically linear system is

$$W'_m = \frac{1}{2} L i_1^2 + \frac{1}{2} L i_2^2 + \frac{1}{2} L i_3^2 + M i_1 i_3 \cos(\phi - \psi) + M i_2 i_3 \sin(\phi - \psi)$$

from which

$$T^e = M i_1 i_3 \sin(\phi - \psi) - M i_2 i_3 \cos(\phi - \psi)$$

For constant shaft speed ω , the shaft position is

$$\phi = \omega t.$$

Then, with $i_3 = I_o$ as given

$$\frac{d\lambda_1}{dt} = -\omega M I_o \sin(\omega t - \psi) + L \frac{di_1}{dt} = -i_1 R$$

and

$$\frac{d\lambda_2}{dt} = \omega M I_o \cos(\omega t - \psi) + L \frac{di_2}{dt} = -i_2 R$$

Using the given assumptions that

$$\left| L \frac{di_1}{dt} \right| \ll \left| R i_1 \right| \quad \text{and} \quad \left| L \frac{di_2}{dt} \right| \ll \left| R i_2 \right|$$

we have

$$i_1 = \frac{\omega M I_o}{R} \sin(\omega t - \psi)$$

$$i_2 = -\frac{\omega M I_o}{R} \cos(\omega t - \psi)$$

and the torque T^e is

$$T^e = M I_o \left(\frac{\omega M I_o}{R} \right) [\sin^2(\omega t - \psi) + \cos^2(\omega t - \psi)]$$

Hence, from (1)

$$\psi = \frac{(M I_o)^2}{K R} \omega$$

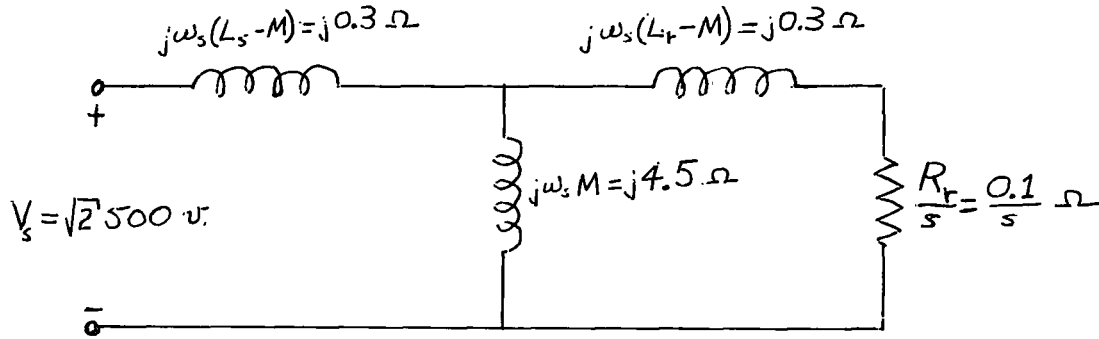
which shows that pointer displacement ψ is a linear function of shaft speed ω which is in turn proportional to car speed.

Suppose we had not neglected the voltage drops due to self inductance. Would the final result still be the same?

ROTATING MACHINES

PROBLEM 4.16

The equivalent circuit with parameter values as given is



From (4.1.82) the torque is

$$T^e = \frac{\left(\frac{k^2}{\omega_s}\right) \left(\frac{L_r}{L_s}\right) \left(\frac{R_r}{s}\right) V_s^2}{[\omega_s(1-k^2)L_r]^2 + (R_r/s)^2}$$

where $k^2 = \frac{M^2}{L_r L_s}$ and $s = \frac{\omega_s - \omega_m}{\omega_s}$

Solution of (4.1.81) for I_s yields

$$I_s = \sqrt{\frac{\frac{R_r^2}{s^2} + (\omega_s L_r)^2}{\frac{R_r^2}{s^2} + [\omega_s L_r(1-k^2)]^2}} \left(\frac{V_s}{\omega_s L_s}\right)$$

volt-ampere input is simply (for two phases)

$$(VA)_{in} = V_s I_s$$

The electrical input power can be calculated in a variety of ways, the simplest being to recognize that in the equivalent circuit the power dissipated in R_r/s (for two phases) is just ω_s times the electromagnetic torque, hence

$$P_{in} = T^e \omega_s$$

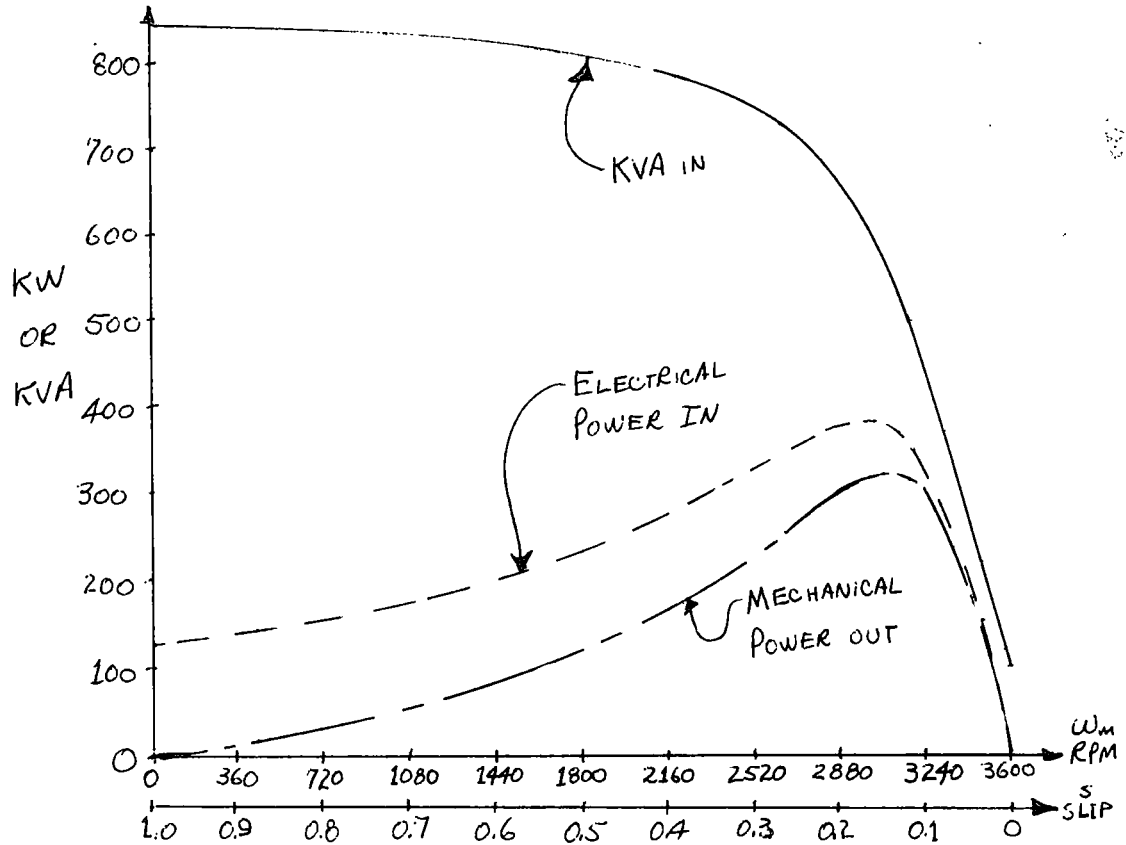
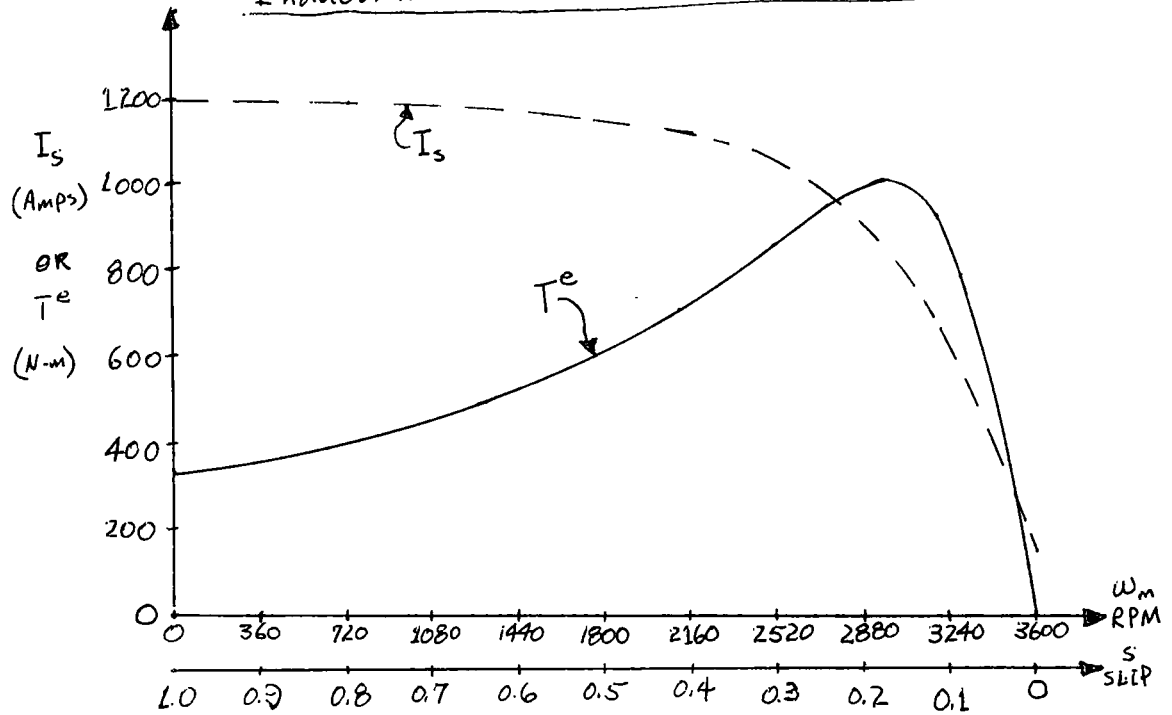
Finally, the mechanical power output is

$$P_{mech} = T^e \omega_m$$

These five quantities are shown plotted in the attached graphs. Numerical constants used in the computations are

ROTATING MACHINES

Induction Machine Curves for Problem 4.16



ROTATING MACHINES

PROBLEM 4.16 (Continued)

$$\omega_s L_s = \omega_s L_r = \omega_s M + 0.3 = 4.8\Omega$$

$$k^2 = \left(\frac{4.5}{4.8}\right)^2 = 0.878$$

$$T^e = \frac{\frac{117}{s}}{0.342 + \frac{0.01}{s^2}} \text{ newton-meters}$$

$$I_s = \sqrt{\frac{23.0 + \frac{0.01}{s^2}}{0.342 + \frac{0.01}{s^2}}} \quad 147 \text{ amps } \rho K.$$

$$s_{mT} = 0.188$$

PROBLEM 4.17

Part a

For ease in calculation it is useful to write the mechanical speed as

$$\omega_m = (1-s)\omega_s$$

and the fan characteristic as

$$T_m = -B\omega_s^3(1-s)^3$$

With $\omega_s = 120\pi \text{ rad/sec}$

$$B\omega_s^3 = 400 \text{ newton-meters}$$

The results of Prob. 4.16 for torque yields

$$400(1-s)^3 = \frac{\frac{117}{s}}{0.342 + \frac{0.01}{s^2}}$$

Solution of this equation by cut-and-try for s yields:

$$s = 0.032$$

$$\text{Then } P_{\text{mech}} = (400)(1-s)^3\omega_m = (400)(\omega_s)(1-s)^4$$

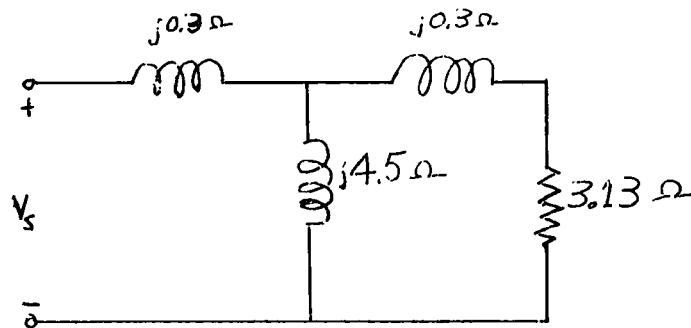
$$P_{\text{mech}} = 133 \text{ kilowatts into fan}$$

$$P_{\text{input}} = \frac{P_{\text{mech}}}{1-s} = 138 \text{ kilowatts}$$

Circuit seen by electrical source is

ROTATING MACHINES

PROBLEM 4.17 (Continued)



Input impedance is

$$Z_{in} = j0.3 + \frac{(j4.5)(3.13 + j0.3)}{3.13 + j4.8} = \frac{-2.79 + j15.0}{3.13 + j4.8}$$

$$\angle Z_{in} = 100.6^\circ - 56.8^\circ = 43.8^\circ$$

Hence,

$$\text{p.f.} = \cos \angle Z_{in} = 0.72 \text{ lagging}$$

Part b

Electromagnetic torque scales as the square of the terminal voltage, thus

$$T^e = \frac{\frac{117}{s}}{0.342 + \frac{0.01}{s^2}} \left(\frac{V_s}{V_{so}} \right)^2$$

where $V_{so} = \sqrt{2}$ 500 volts peak. The slip for any terminal voltage is now found from

$$400(1-s)^3 = \frac{\frac{117}{s}}{0.342 + \frac{0.01}{s^2}} \left(\frac{V_s}{V_{so}} \right)^2$$

The mechanical power into the fan is

$$P_{mech} = 400 \omega_s (1-s)^4$$

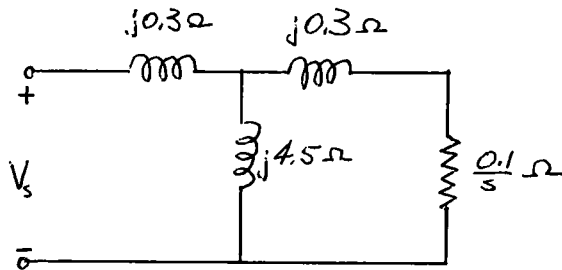
electrical power input is

$$P_{in} = \frac{P_{mech}}{1-s}$$

ROTATING MACHINES

PROBLEM 4.17 (Continued)

and the power factor is found as the cosine of the angle of the input impedance of the circuit



These quantities are plotted as required on the attached graph.

PROBLEM 4.18

Part a

The solution to Prob. 4.1 can be used to find the flux densities here. For the stator a-winding, the solution of Prob. 4.1 applies directly, thus, the radial component of flux density due to current in stator winding a is

$$B_{ra}(\psi) = \frac{\mu_0 N_s i_a}{2g} \cos \psi$$

Windings b and c on the stator are identical with the a winding except for the indicated angular displacements, thus,

$$B_{rb}(\psi) = \frac{\mu_0 N_s i_b}{2g} \cos(\psi - \frac{2\pi}{3})$$

$$B_{rc}(\psi) = \frac{\mu_0 N_s i_c}{2g} \cos(\psi - \frac{4\pi}{3})$$

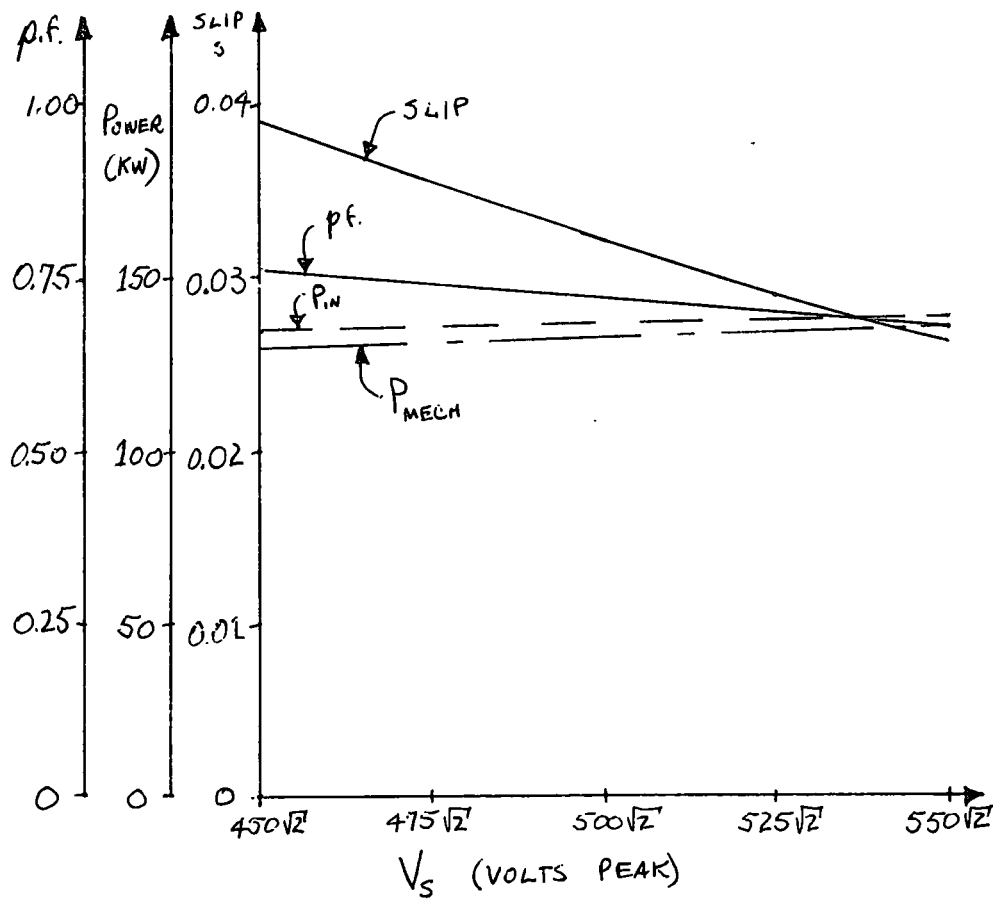
The solution in Prob. 4.1 for the flux density due to rotor winding current applies directly here, thus,

$$B_{rr}(\psi) = \frac{\mu_0 N_r i_r}{2g} \cos(\psi - \theta)$$

Part b

The method of part (c) of Prob. 4.1 can be used and the results of that analysis applied directly by replacing rotor quantities by stator b-winding quantities and θ by $2\pi/3$. The resulting mutual inductance is (assuming $g \ll R$)

ROTATING MACHINES



Induction Machine Curves for Problem 4.17

ROTATING MACHINES

PROBLEM 4.18 (Continued)

$$L_{ab} = \frac{\pi \mu_o N_s^2 R \ell}{2g} \cos \frac{2\pi}{3}$$

$$L_{ab} = -\frac{\mu_o N_s^2 R \ell}{4g} = -\frac{L_s}{2}$$

where L_s is the self inductance of one stator winding alone. Note that $L_{ac} = L_{ab}$ because of relative geometry.

Part c

The λ - i relations are thus

$$\lambda_a = L_s i_a - \frac{L_s}{2} i_b - \frac{L_s}{2} i_c + M \cos \theta i_r$$

$$\lambda_b = -\frac{L_s}{2} i_a + L_s i_b - \frac{L_s}{2} i_c + M \cos(\theta - \frac{2\pi}{3}) i_r$$

$$\lambda_c = -\frac{L_s}{2} i_a - \frac{L_s}{2} i_b + L_s i_c + M \cos(\theta - \frac{4\pi}{3}) i_r$$

$$\lambda_r = M \cos \theta i_a + M \cos(\theta - \frac{2\pi}{3}) i_b$$

$$+ M \cos(\theta - \frac{4\pi}{3}) i_c + L_r i_r$$

where from Prob. 4.1,

$$L_s = \frac{\pi \mu_o N_s^2 R \ell}{2g}$$

$$M = \frac{\pi \mu_o N_s N_r R \ell}{2g}$$

$$L_r = \frac{\pi \mu_o N_r^2 R \ell}{2g}$$

Part d

The torque of electric origin is found most easily by using magnetic coenergy which for this electrically linear system is

$$W'_m(i_a, i_b, i_c, i_r, \theta) = \frac{1}{2} L_s (i_a^2 + i_b^2 + i_c^2)$$

$$+ \frac{1}{2} L_s (i_a i_b + i_a i_c + i_b i_c) + M \cos \theta i_r i_a$$

$$+ M \cos(\theta - \frac{2\pi}{3}) i_r i_b + M \cos(\theta - \frac{4\pi}{3}) i_r i_c$$

The torque of electric origin is

PROBLEM 4.18 (Continued)

$$T^e = \frac{\partial W'_m(i_a, i_b, i_c, i_r, \theta)}{\partial \theta}$$

$$T^e = -Mi_r [i_a \sin \theta + i_b \sin(\theta - \frac{2\pi}{3}) + i_c \sin(\theta - \frac{4\pi}{3})]$$

PROBLEM 4.19

Part a

Superimposing the three component stator flux densities from Part a of Prob. 4.18, we have

$$B_{rs} = \frac{\mu_o N_s}{2g} [i_a \cos \psi + i_b \cos(\psi - \frac{2\pi}{3}) + i_c \cos(\psi - \frac{4\pi}{3})]$$

Substituting the given currents

$$B_{rs} = \frac{\mu_o N_s}{2g} [I_a \cos \omega t \cos \psi + I_b \cos(\omega t - \frac{2\pi}{3}) \cos(\psi - \frac{2\pi}{3}) + I_c \cos(\omega t - \frac{4\pi}{3}) \cos(\psi - \frac{4\pi}{3})]$$

Using trigonometric identities and simplifying yields

$$B_{rs} = \frac{\mu_o N_s}{2g} \left[\left(\frac{I_a + I_b + I_c}{2} \right) \cos(\omega t - \psi) + \frac{1}{2} (I_a + I_b \cos \frac{4\pi}{3} + I_c \cos \frac{2\pi}{3}) \cos(\omega t + \psi) + \frac{1}{2} (I_b \sin \frac{4\pi}{3} + I_c \sin \frac{2\pi}{3}) \sin(\omega t + \psi) \right]$$

Positive traveling wave has point of constant phase defined by

$$\omega t - \psi = \text{constant}$$

from which

$$\frac{d\psi}{dt} = \omega$$

This is positive traveling wave with amplitude

$$B_{rfm} = \frac{\mu_o N_s}{4g} (I_a + I_b + I_c)$$

Negative traveling wave has point of constant phase

$$\omega t + \psi = \text{constant}$$

from which

$$\frac{d\psi}{dt} = -\omega$$

This defines negative traveling wave with amplitude

ROTATING MACHINES

PROBLEM 4.19 (Continued)

$$B_{rbm} = \frac{\mu_0 N_s}{4g} \sqrt{\left(I_a - \frac{I_b}{2} - \frac{I_c}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2} I_b + \frac{\sqrt{3}}{2} I_c\right)^2}$$

Part b

When three phase currents are balanced

$$I_a = I_b = I_c$$

and $B_{rbm} = 0$ leaving only a forward (positive) traveling wave.

PROBLEM 4.20

Part a

Total radial flux density due to stator excitation is

$$B_{rs} = \frac{\mu_0 N}{2g} (i_a \cos 2\psi + i_b \sin 2\psi)$$

Substituting given values for currents

$$B_{rs} = \frac{\mu_0 N}{2g} (I_a \cos \omega t \cos 2\psi + I_b \sin \omega t \sin 2\psi)$$

Part b

$$B_{rs} = \frac{\mu_0 N}{2g} \left| \left(\frac{I_a + I_b}{2}\right) \cos(\omega t - 2\psi) + \left(\frac{I_a - I_b}{2}\right) \cos(\omega t + 2\psi) \right|$$

The forward (positive-traveling) component has constant phase defined by

$$\omega t - 2\psi = \text{constant}$$

from which

$$\frac{d\psi}{dt} = \frac{\omega}{2}$$

The backward (negative-traveling) component has constant phase defined by

$$\omega t + 2\psi = \text{constant}$$

from which

$$\frac{d\psi}{dt} = -\frac{\omega}{2}$$

Part c

From part b, when $I_a = I_b$, $I_a - I_b = 0$ and the backward-wave amplitude goes to zero. When $I_b = -I_a$, $I_a + I_b = 0$ and the forward-wave amplitude goes to zero.

ROTATING MACHINES

PROBLEM 4.21

Referring to the solution for Prob. 4.20,

Part a

$$B_{rs} = \frac{\mu_o N}{2g} (i_a \cos p\psi + i_b \sin p\psi)$$

$$B_{rs} = \frac{\mu_o N}{2g} (I_a \cos \omega t \cos p\psi + I_b \sin \omega t \sin p\psi)$$

Part b

Using trigonometric identities yields

$$B_{rs} = \frac{\mu_o N}{2g} \left[\left(\frac{I_a + I_b}{2} \right) \cos(\omega t - p\psi) + \left(\frac{I_a - I_b}{2} \right) \cos(\omega t + p\psi) \right]$$

Forward wave has constant phase

$$\omega t - p\psi = \text{constant}$$

from which

$$\frac{d\psi}{dt} = \frac{\omega}{p}$$

Backward wave has constant phase

$$\omega t + p\psi = \text{constant}$$

from which

$$\frac{d\psi}{dt} = -\frac{\omega}{p}$$

Part c

From part b, when $I_b = I_a$, $I_a - I_b = 0$, and backward-wave amplitude goes to zero. When $I_b = -I_a$, $I_a + I_b = 0$, and forward-wave amplitude goes to zero.

PROBLEM 4.22

This is an electrically linear system, so the magnetic coenergy is

$$W'_m(i_s, i_r, \theta) = \frac{1}{2}(L_o + L_2 \cos 2\theta) i_s^2 + \frac{1}{2} L_r i_r^2 + M i_r i_s \cos \theta$$

Then the torque is

$$T^e = \frac{\partial W'_m(i_s, i_r, \theta)}{\partial \theta} = -M i_r i_s \sin \theta - L_2 i_s^2 \sin 2\theta$$

PROBLEM 4.23

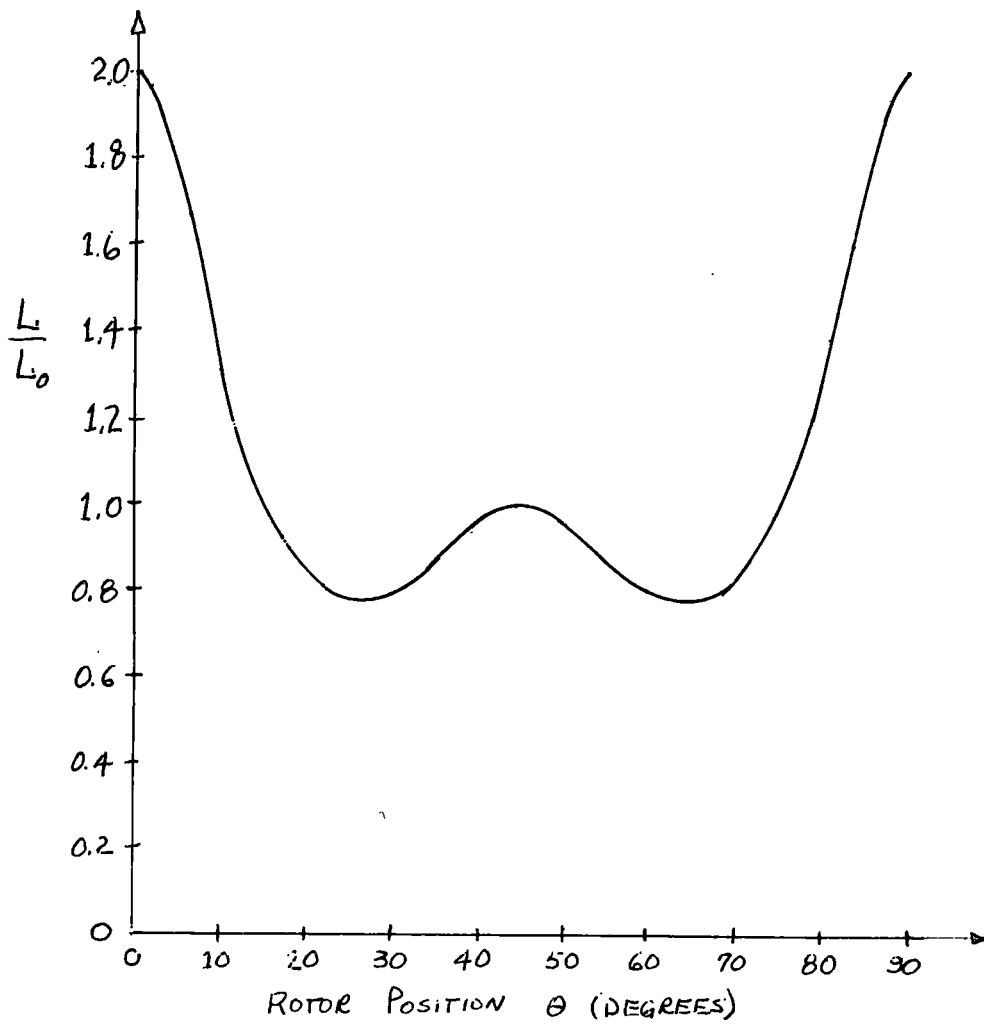
Part a

$$L = \frac{L_o}{(1 - 0.25 \cos 4\theta - 0.25 \cos 8\theta)}$$

ROTATING MACHINES

PROBLEM 4.23 (Continued)

The variation of this inductance with θ is shown plotted below.



PROBLEM 4.23 (Continued)

From this plot and the configuration of Fig. 4P.23, it is evident that minimum reluctance and maximum inductance occur when $\theta = 0, \pi/2, \pi, \dots, \frac{n}{2}\pi, \dots$. The inductance is symmetrical about $\theta = 0, \frac{\pi}{2}, \dots$ and about $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \dots, \frac{\pi}{4} + \frac{n\pi}{2}, \dots$ as it should be. Minimum inductance occurs on both sides of $\theta = \frac{\pi}{4}$ which ought to be maximum reluctance.

The general trend of the inductance is correct for the geometry of Fig. 4P.23 but the equation would probably be a better representation if the sign of the 8θ term were reversed.

Part b

For this electrically linear system, the magnetic stored energy is

$$W_m(\lambda, \theta) = \frac{1}{2} \frac{\lambda^2}{L}$$

$$W_m(\lambda, \theta) = \frac{\lambda^2(1 - 0.25 \cos 4\theta - 0.25 \cos 8\theta)}{2L_o}$$

The torque is then

$$T^e = - \frac{\partial W_m(\lambda, \theta)}{\partial \theta}$$

$$T^e = - \frac{\lambda^2}{2L_o} (\sin 4\theta + 2 \sin 8\theta)$$

Part c

With $\lambda = \Lambda_o \cos \omega t$ and $\theta = \Omega t + \delta$

$$T^e = - \frac{\Lambda_o^2 \cos^2 \omega t}{2L_o} [\sin(4\Omega t + 4\delta) + 2 \sin(8\Omega t + 8\delta)]$$

Repeated use of trig identities yields for the instantaneous converted power

$$\Omega T^e = - \frac{\Omega \Lambda_o^2}{4L_o} [\sin(4\Omega t + 4\delta) + 2 \sin(8\Omega t + 8\delta)$$

$$+ \frac{1}{2} \sin(2\omega t + 4\Omega t + 4\delta) + \frac{1}{2} \sin(4\Omega t - 2\omega t + 4\delta)$$

$$+ \sin(2\omega t + 8\Omega t + 8\delta) + \sin(8\Omega t - 2\omega t + 8\delta)]$$

This can only have a non-zero average value when $\Omega \neq 0$ and a coefficient of t in one argument is zero. This gives 4 conditions

$$\Omega = \pm \frac{\omega}{2}, \pm \frac{\omega}{4}$$

When $\Omega = \pm \frac{\omega}{2}$

$$[\Omega T^e]_{\text{avg}} = - \frac{\Omega \Lambda_o^2}{8L_o} \sin 4\delta$$

ROTATING MACHINES

PROBLEM 4.23 (Continued)

and when $\Omega = \pm \frac{\omega}{4}$

$$[\Omega T^e]_{\text{avg}} = -\frac{\Omega \Lambda_o^2}{4L_o} \sin 8\delta$$

PROBLEM 4.24

It will be helpful to express the given ratings in alternative ways.

Rated output power = 6000 HP = 4480 KW at 0.8 p.f. this is

$$\frac{4480}{0.8} = 5600 \text{ KVA total}$$

or

2800 KVA per phase

The rated phase current is then

$$I_s = \frac{2800 \times 10^3}{3 \times 10^3} = 933 \text{ amps rms} = 1320 \text{ amps pk.}$$

Given:

Direct axis reactance $\omega(L_o + L_2) = 4.0 \text{ ohms}$

Quadrature axis reactance $\omega(L_o - L_2) = 2.2 \text{ ohms}$

$\omega L_o = 3.1 \text{ ohms}$

$\omega L_2 = 0.9 \text{ ohms}$

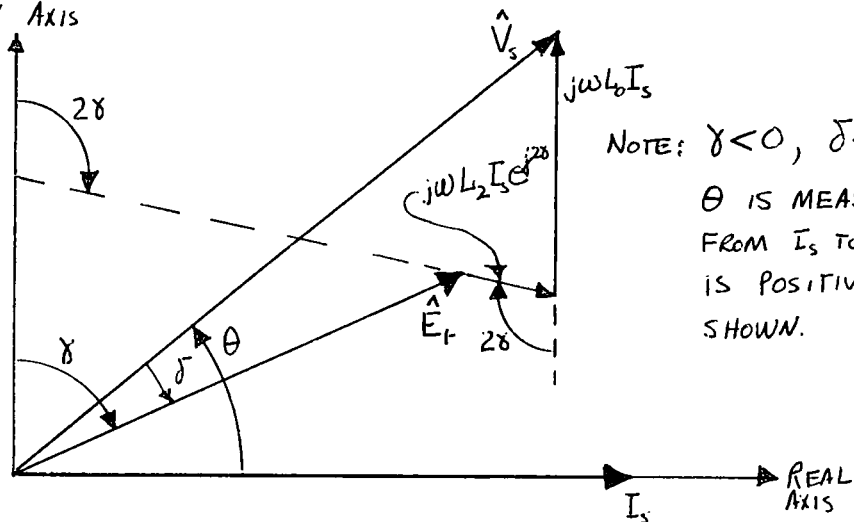
The number of poles is not given in the problem statement. We assume 2 poles.

Part a

Rated field current can be found in several ways, all involving cut-and-try procedures. Our method will be based on a vector diagram like that of

Fig. 4.2.5(a), thus

IMAGINARY AXIS



NOTE: $\gamma < 0$, $\delta < 0$, AND θ IS MEASURED FROM I_s TO V_s AND IS POSITIVE AS SHOWN.

ROTATING MACHINES

PROBLEM 4.24 (Continued)

Evaluating the horizontal and vertical components of $\hat{V}_s$ we have (remember that $\gamma < 0$)

$$V_s \cos \theta = E_f \cos\left(\frac{\pi}{2} + \gamma\right) + \omega L_2 I_s \cos\left(\frac{\pi}{2} + 2\gamma\right)$$

$$V_s \sin \theta = E_f \sin\left(\frac{\pi}{2} + \gamma\right) + \omega L_2 I_s \sin\left(\frac{\pi}{2} + 2\gamma\right) + \omega L_o I_s$$

Using trigonometric identities we rewrite these as

$$V_s \cos \theta = -E_f \sin \gamma - \omega L_2 I_s \sin 2\gamma$$

$$V_s \sin \theta = E_f \cos \gamma + \omega L_2 I_s \cos 2\gamma + \omega L_o I_s$$

Next, it will be convenient to normalize these equations to V_s ,

$$\cos \theta = -e_f \sin \gamma - \frac{\omega L_2 I_s}{V_s} \sin 2\gamma$$

$$\sin \theta = e_f \cos \gamma + \frac{\omega L_2 I_s}{V_s} \cos 2\gamma + \frac{\omega L_o I_s}{V_s}$$

where

$$e_f = \frac{E_f}{V_s}$$

Solution of these two equations for e_f yields

$$e_f = \frac{\sin \theta - \frac{\omega L_2 I_s}{V_s} \cos 2\gamma - \frac{\omega L_o I_s}{V_s}}{\cos \gamma}$$

$$e_f = \frac{-\cos \theta - \frac{\omega L_2 I_s}{V_s} \sin 2\gamma}{\sin \gamma}$$

For rated conditions as given the constants are:

$$\cos \theta = \text{p.f.} = 0.8$$

$$\sin \theta = -0.6 \text{ (negative sign for leading p.f.)}$$

$$\frac{\omega L_2 I_s}{V_s} = 0.280; \quad \frac{\omega L_o I_s}{V_s} = 0.964$$

Solution by trial and error for a value of γ that satisfies both equations simultaneously yields

$$\gamma = -148^\circ$$

ROTATING MACHINES

PROBLEM 4.24 (Continued)

and the resulting value for e_f is

$$e_f = 1.99$$

yielding for the rated field current

$$I_r = \frac{V_s e_f}{\omega M} = 24.1 \text{ amps.}$$

where V_s is in volts peak.

Part b

The V-curves can be calculated in several ways. Our choice here is to first relate power converted to terminal voltage and field generated voltage by multiplying (4.2.46) by ω , thus

$$P = \omega T^e = - \frac{E_f V_s}{X_d} \sin \delta - \frac{(X_d - X_q) V_s^2}{2 X_d X_q} \sin 2\delta$$

where

$$X_d = \omega(L_o + L_2)$$

$$X_q = \omega(L_o - L_2)$$

We normalize this expression with respect to V_s^2/X_d , then

$$\frac{PX_d}{V_s^2} = - e_f \sin \delta - \frac{(X_d - X_q)}{2 X_q} \sin 2\delta$$

Pull-out torque occurs when the derivative of this power with respect to δ goes to zero. Thus pull-out torque angle is defined by

$$\frac{\partial}{\partial \delta} \left(\frac{PX_d}{V_s^2} \right) = -e_f \cos \delta - \frac{(X_d - X_q)}{X_q} \cos 2\delta = 0$$

The use of (4.2.44) and (4.2.45) then yield the armature (stator) current amplitude as

$$I_s = \sqrt{\left(\frac{V_s}{X_q} \sin \delta \right)^2 + \left(\frac{V_s}{X_d} \cos \delta - \frac{E_f}{X_d} \right)^2}$$

A more useful form is

$$I_s = \frac{V_s}{X_d} \sqrt{\left(\frac{X_d}{X_q} \sin \delta \right)^2 + (\cos \delta - e_f)^2}$$

The computation procedure used here was to fix the power and assume values of δ over a range going from either rated armature current or rated field current to pull-out. For each value of δ , the necessary value of e_f is calculated

ROTATING MACHINES

PROBLEM 4.24 (Continued)

from the expression for power as

$$e_f = \frac{\frac{PX_d}{V_s^2} + \frac{(X_d - X_q)}{2X_q} \sin 2\delta}{-\sin \delta}$$

and then the armature current magnitude is calculated from

$$I_s = \frac{V_s}{X_d} \sqrt{\left(\frac{X_d}{X_q} \sin \delta\right)^2 + (\cos \delta - e_f)^2}$$

For zero load power, $\gamma = 0$ and $\delta = 0$ and, from the vector diagram given earlier, the armature current amplitude is

$$I_s = \frac{|V_s - E_f|}{\omega(L_o + L_2)}$$

with pull-out still defined as before. The required V-curves are shown in the following graph. Note that pull-out conditions are never reached because range of operation is limited by rated field current and rated armature current.

PROBLEM 4.25

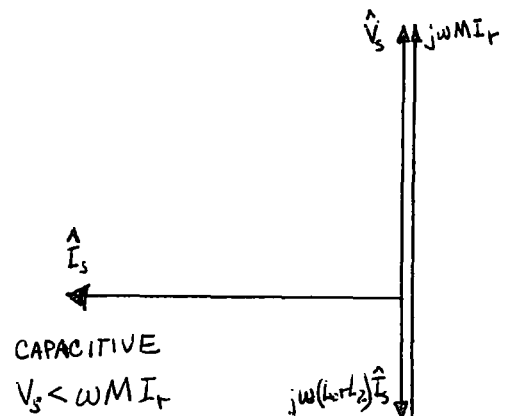
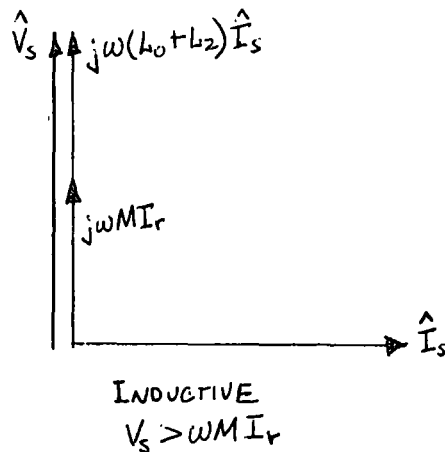
Equation (4.2.41) is (assuming arbitrary phase for I_s)

$$\hat{V}_s = j\omega L_o \hat{I}_s + j\omega L_2 \hat{I}_s e^{j2\gamma} + j\omega M I_r e^{j2\gamma}$$

With $\gamma = 0$ as specified

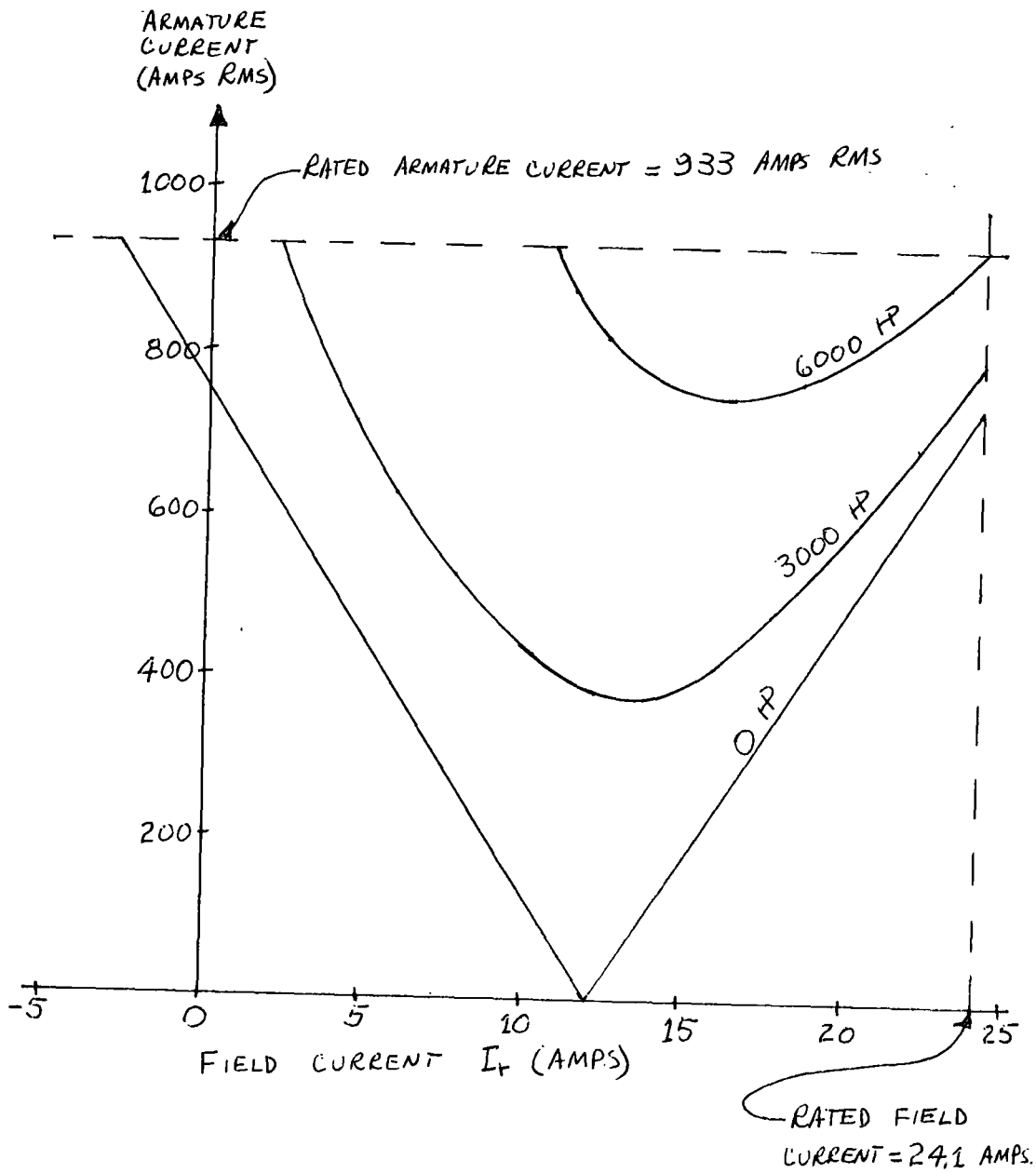
$$\hat{V}_s = j\omega(L_o + L_2) \hat{I}_s + j\omega M I_r$$

The two vector diagrams required are



ROTATING MACHINES

V-CURVES FOR PROBLEM 4,24



PROBLEM 4.26

Part a

From Fig. 4P.26(a)

$$\frac{\hat{V}}{\hat{V}_s} = \frac{\frac{1}{Y} e^{j\phi}}{jX_s + \frac{1}{Y} e^{j\phi}}$$

from which the ratio of the magnitudes is

$$\frac{|\hat{V}|}{|\hat{V}_s|} = \frac{\frac{1}{Y}}{\sqrt{\left(\frac{1}{Y} \cos\phi\right)^2 + \left(\frac{1}{Y} \sin\phi + X_s\right)^2}}$$

 For the values $Y = 0.01$ mho, $X_s = 10$ ohms

$$\frac{|\hat{V}|}{|\hat{V}_s|} = \frac{100}{\sqrt{(100 \cos\phi)^2 + (100 \sin\phi + 10)^2}}$$

 Then, for $\phi = 0$

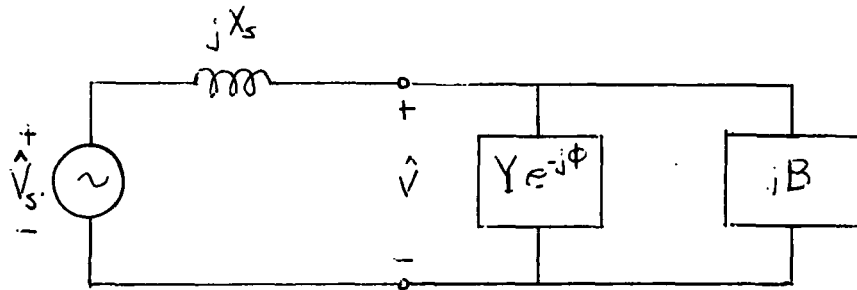
$$\frac{|\hat{V}|}{|\hat{V}_s|} = \frac{100}{\sqrt{10,000 + 100}} = 0.995$$

 and, for $\phi = 45^\circ$

$$\frac{|\hat{V}|}{|\hat{V}_s|} = \frac{100}{\sqrt{\left(\frac{100}{\sqrt{2}}\right)^2 + \left(\frac{100}{\sqrt{2}} + 10\right)^2}} = 0.932$$

Part b

It is instructive to represent the synchronous condenser as a susceptance jB , then when B is positive the synchronous condenser appears capacitive. Now the circuit is



ROTATING MACHINES

PROBLEM 4.26 (Continued)

Now the voltage ratio is

$$\frac{\hat{V}}{\hat{V}_s} = \frac{1}{Y e^{-j\phi} + jB} = \frac{1}{\frac{1}{Y e^{-j\phi} + jB} + jX_s} = \frac{1}{1 + jX_s Y e^{-j\phi} - BX_s}$$

$$\frac{\hat{V}}{\hat{V}_s} = \frac{1}{1 - BX_s + X_s Y \sin\phi + jX_s Y \cos\phi}$$

Then

$$\frac{|\hat{V}|}{|\hat{V}_s|} = \frac{1}{\sqrt{(1 - BX_s + X_s Y \sin\phi)^2 + (X_s Y \cos\phi)^2}}$$

For $\phi = 0$

$$\frac{|\hat{V}|}{|\hat{V}_s|} = \frac{1}{\sqrt{(1 - BX_s)^2 + (X_s Y)^2}}$$

If this is to be unity

$$(1 - BX_s)^2 + (X_s Y)^2 = 1$$

$$1 - BX_s = \sqrt{1 - (X_s Y)^2}$$

$$B = \frac{1 - \sqrt{1 - (X_s Y)^2}}{X_s}$$

for the constants given

$$B = \frac{1 - \sqrt{1 - 0.01}}{10} = \frac{0.005}{10} = 0.0005 \text{ mho}$$

Volt-amperes required from synchronous condenser

$$(VA)_{sc} = |\hat{V}|^2 B = (2)(10^{10})(5)(10^{-4}) = 10,000 \text{ KVA}$$

Real power supplied to load

$$P_L = |\hat{V}|^2 Y \cos \phi = |\hat{V}|^2 Y \text{ for } \phi = 0$$

Then

$$\frac{(VA)_{sc}}{P_L} = \frac{B}{Y} = \frac{0.0005}{0.01} = 0.05$$

For $\phi = 0$ the synchronous condenser needs to supply reactive volt amperes equal to 5 percent of the load power to regulate the voltage perfectly.

ROTATING MACHINES

PROBLEM 4.26 (Continued)

For $\phi = 45^\circ$

$$\frac{|\hat{V}|}{|\hat{V}_s|} = \frac{1}{\sqrt{\left(1 - BX_s + \frac{X_s Y}{\sqrt{2}}\right)^2 + \left(\frac{X_s Y}{\sqrt{2}}\right)^2}}$$

In order for this to be unity

$$\begin{aligned} \left(1 - BX_s + \frac{X_s Y}{\sqrt{2}}\right)^2 + \left(\frac{X_s Y}{\sqrt{2}}\right)^2 &= 1 \\ B &= \frac{1 + \frac{X_s Y}{\sqrt{2}} - \sqrt{1 - \left(\frac{X_s Y}{\sqrt{2}}\right)^2}}{X_s} \end{aligned}$$

For the constants given

$$B = \frac{1 + 0.0707 - \sqrt{1 - 0.005}}{10} = 0.00732 \text{ mho}$$

Volt-amperes required from synchronous condenser

$$(VA)_{sc} = |\hat{V}|^2 B = (2)(10^{10})(7.32)(10^{-3}) = 146,400 \text{ KVA}$$

Real power supplied to load

$$P_L = |\hat{V}|^2 Y \cos \phi = \frac{|\hat{V}|^2 Y}{\sqrt{2}} \text{ for } \phi = 45^\circ$$

Then

$$\frac{(VA)_{sc}}{P_L} = \frac{B\sqrt{2}}{Y} = \frac{(\sqrt{2})(0.00732)}{0.01} = 1.04$$

Thus for a load having power factor of 0.707 lagging a synchronous condenser needs to supply reactive volt-amperes equal to 1.04 times the power supplied to the load to regulate the voltage perfectly.

These results, of course, depend on the internal impedance of the source. That given is typical of large power systems.

PROBLEM 4.27

Part a

This part of this problem is very much like part a of Prob. 4.24. Using results from that problem we define

ROTATING MACHINES

PROBLEM 4.27 (Continued)

$$e_f = \frac{E_f}{V_s} = \frac{\omega M I_r}{V_s}$$

where V_s is in volts peak. Then

$$e_f = \frac{\sin \theta - \frac{\omega L_2 I_s}{V_s} \cos 2\gamma - \frac{\omega L_o I_s}{V_s}}{\cos \gamma}$$

$$e_f = \frac{-\cos \theta - \frac{\omega L_2 I_s}{V_s} \sin 2\gamma}{\sin \gamma}$$

From the constants given

$$\cos \theta = 1.0; \quad \sin \theta = 0$$

$$\omega L_o = 2.5 \text{ ohms} \quad \omega L_2 = 0.5 \text{ ohm}$$

Rated power

$$P_L = 1000 \text{ HP} = 746 \text{ KW}$$

Armature current at rated load is

$$I_s = \frac{746,000}{\sqrt{2} \cdot 1000} = 527 \text{ amps peak} = 373 \text{ amps RMS}$$

Then

$$\frac{\omega L_2 I_s}{V_s} = 0.186; \quad \frac{\omega L_o I_s}{V_s} = 0.932$$

Using the constants

$$e_f = \frac{-0.186 \cos 2\gamma - 0.932}{\cos \gamma}$$

$$e_f = \frac{-1 - 0.186 \sin 2\gamma}{\sin \gamma}$$

The use of trial-and-error to find a value of γ that satisfies these two equations simultaneously yields

$$\gamma = -127^\circ \text{ and } e_f = 1.48$$

Using the given constants we obtain

$$I_r = \frac{e_f V_s}{\omega M} = \frac{(1.48)(\sqrt{2})(1000)}{150} = 14.0 \text{ amps}$$

For L_f/R_f very large compared to a half period of the supply voltage the field

ROTATING MACHINES

PROBLEM 4.27 (Continued)

current will essentially be equal to the peak of the supply voltage divided by the field current; thus, the required value of R_f is

$$R_f = \frac{V_s}{I_r} = \frac{\sqrt{2} (1000)}{14.0} \approx 100 \text{ ohms}$$

Part b

We can use (4.2.46) multiplied by the rotational speed ω to write the output power as

$$P_L = \omega T^e = - \frac{E_f V_s}{X_d} \sin \delta - \frac{(X_d - X_q) V_s^2}{2X_d X_q} \sin 2\delta$$

where

$$X_d = \omega(L_o + L_2) = \text{direct axis reactance}$$

$$X_q = \omega(L_o - L_2) = \text{quadrature axis reactance}$$

With the full-wave rectifier supplying the field winding we can express

$$E_f = \omega M I_r = \frac{\omega M}{R_f} V_s$$

Then

$$P_L = - \frac{\omega M V_s^2}{R_f X_d} \sin \delta - \frac{(X_d - X_q) V_s^2}{2X_d X_q} \sin 2\delta$$

Factoring out V_s^2 yields

$$P_L = V_s^2 \left[- \frac{\omega M}{R_f X_d} \sin \delta - \frac{(X_d - X_q)}{2X_d X_q} \sin 2\delta \right]$$

Substitution of given constants yields

$$746 \times 10^3 = V_s^2 [-0.500 \sin \delta - 0.083 \sin 2\delta]$$

To find the required curve it is easiest to assume δ and calculate the required V_s , the range of δ being limited by pull-out which occurs when

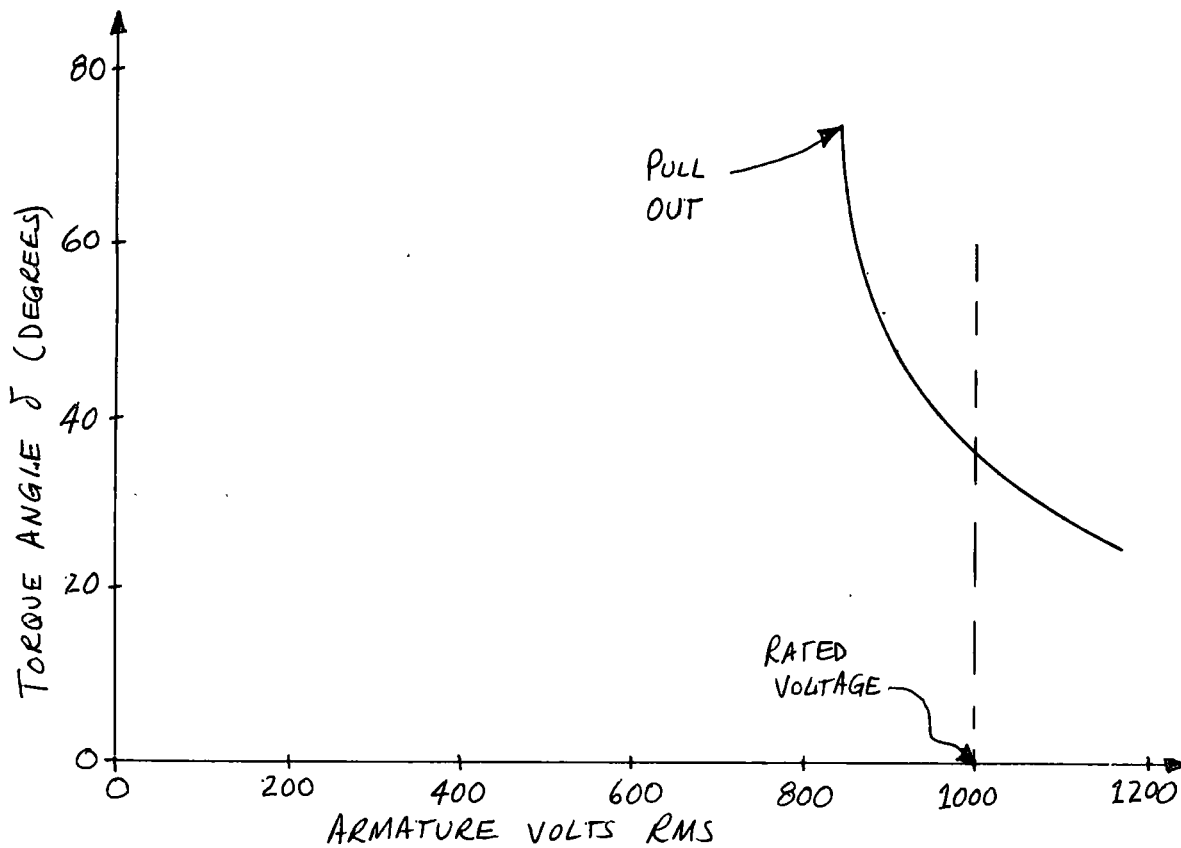
$$\frac{\partial P_L}{\partial \delta} = 0 = -0.500 \cos \delta - 0.166 \cos 2\delta$$

The resulting curve of δ as a function of V_s is shown in the attached graph.

Note that the voltage can only drop 15.5% before the motor pulls out of step.

ROTATING MACHINES

PROBLEM 4.27 (Continued)



Although it was not required for this problem, calculations will show that operation at reduced voltage will lead to excessive armature current, thus, operation in this range must be limited to transient conditions.

ROTATING MACHINES

PROBLEM 4.28

Part a

This is similar to part a of Prob. 4.24 except that now we are considering a number of pole pairs greater than two and we are treating a generator. Considering first the problem of pole pairs, reference to Sec. 4.1.8 and 4.2.4 shows that when we define electrical angles γ_e and δ_e as

$$\gamma_e = p\gamma \text{ and } \delta_e = p\delta$$

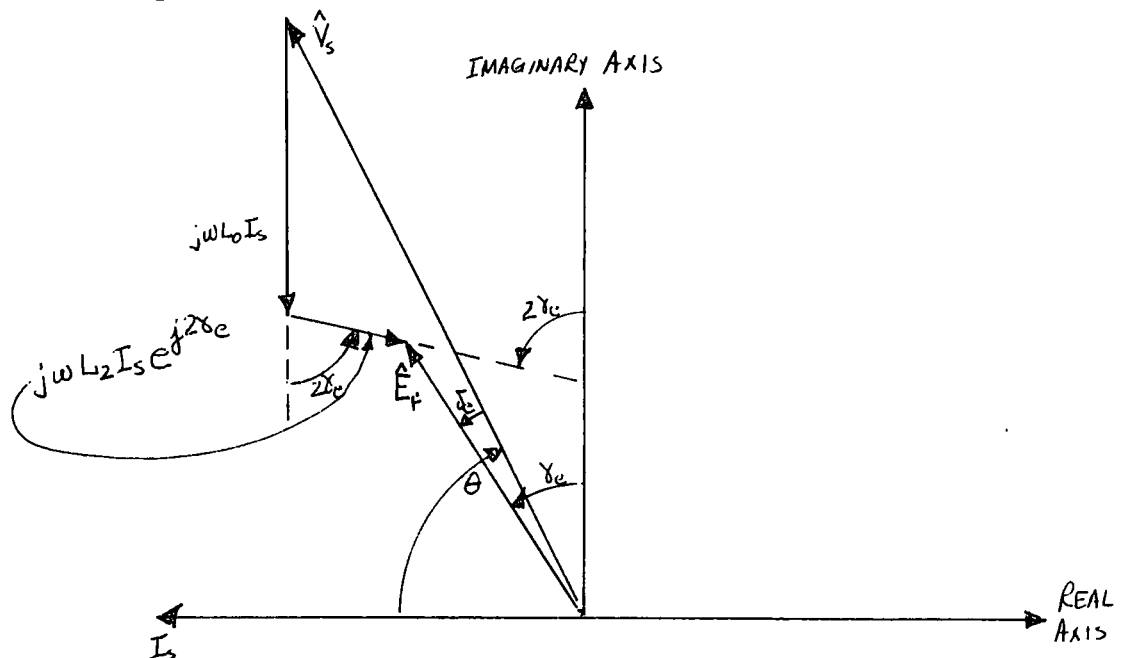
where p is number of pole pairs (36 in this problem) and when we realize that the electromagnetic torque was obtained as a derivative of inductances with respect to angle we get the results

$$T^e = -\frac{p}{\omega} \frac{V_s E_f}{X_d} \sin \delta_e - \frac{p(X_d - X_q)}{\omega 2 X_d X_q} \frac{V_s^2}{\sin 2\delta_e}$$

where $X_d = \omega(L_o + L_2)$ and $X_q = \omega(L_o - L_2)$, and, because the synchronous speed is ω/p (see 4.1.95) the electrical power output from the generator is

$$P = -\frac{\omega}{p} T^e = \frac{V_s E_f}{X_d} \sin \delta_e + \frac{(X_d - X_q) V_s^2}{2 X_d X_q} \sin 2\delta_e$$

Next, we are dealing with a generator so it is convenient to replace I_s by $-I_s$ in the equations. To make clear what is involved we redraw Fig. 4.2.5(a) with the sign of the current reversed.



ROTATING MACHINES

PROBLEM 4.28 (Continued)

Now, evaluating horizontal and vertical components of V_s we have

$$V_s \cos \theta - \omega L_2 I_s \sin 2\gamma_e = E_f \sin \gamma_e$$

$$-V_s \sin \theta = \omega L_o I_s + \omega L_2 I_s \cos 2\gamma_e + E_f \cos \gamma_e$$

From these equations we obtain

$$e_f = \frac{\cos \theta - \frac{\omega L_2 I_s}{V_s} \sin 2\gamma_e}{\sin \gamma_e}$$

$$e_f = \frac{-\sin \theta - \frac{\omega L_o I_s}{V_s} - \frac{\omega L_2 I_s}{V_s} \cos 2\gamma_e}{\cos \gamma_e}$$

where

$$e_f = \frac{E_f}{V_s} = \frac{\omega M I_r}{V_s}$$

with

V_s in volts peak

I_s in amps peak

ω is the electrical frequency

For the given constants

$$\cos \theta = \text{p.f.} = 0.850 \quad \sin \theta = 0.528$$

$$\frac{\omega L_2 I_s}{V_s} = 0.200 \quad \frac{\omega L_o I_s}{V_s} = 1.00$$

and

$$e_f = \frac{0.850 - 0.200 \sin 2\gamma_e}{\sin \gamma_e}$$

$$e_f = \frac{-1.528 - 0.200 \cos 2\gamma_e}{\cos \gamma_e}$$

Trial-and-error solution of these two equations to find a positive value of γ_e that satisfies both equations simultaneously yields

$$\gamma_e = 147.5^\circ \quad \text{and} \quad e_f = 1.92$$

From the definition of e_f we have

$$I_r = \frac{e_f V_s}{\omega M} = \frac{(1.92)(\sqrt{2})(10,000)}{(120)(\pi)(0.125)} = 576 \text{ amps}$$

ROTATING MACHINES

PROBLEM 4.28 (Continued)

Part b

From Prob. 4.14 the definition of complex power is

$$\hat{V}_s \hat{I}_s^* = P + jQ$$

where $\hat{V}_s$ and $\hat{I}_s$ are complex amplitudes.

The capability curve is not as easy to calculate for a salient-pole machine as it was for a smooth-air-gap machine in Prob. 4.14. It will be easiest to calculate the curve using the power output expression of part a

$$P = \frac{V_s E_f}{X_d} \sin \delta_e + \frac{(X_d - X_q) V_s^2}{2 X_d X_q} \sin 2\delta_e$$

the facts that

$$P = V_s I_s \cos \theta$$

$$Q = V_s I_s \sin \theta$$

and that I_s is given from (4.2.44) and (4.2.45) as

$$I_s = \sqrt{\left(\frac{V_s}{X_q} \sin \delta_e\right)^2 + \left(\frac{V_s}{X_d} \cos \delta_e - \frac{E_f}{X_d}\right)^2}$$

First, assuming operation at rated field current the power is

$$P = 320 \times 10^6 \sin \delta_e + 41.7 \times 10^6 \sin 2\delta_e \text{ watts.}$$

We assume values of δ_e starting from zero and calculate P ; then we calculate I_s for the same values of δ_e from

$$I_s = 11,800 \sqrt{(1.50 \sin \delta_e)^2 + (\cos \delta_e - 1.92)^2} \text{ amps peak}$$

Next, because we know P , V_s , and I_s we find θ from

$$\cos \theta = \frac{P}{V_s I_s}$$

From θ we then find Q from

$$Q = V_s I_s \sin \theta.$$

This process is continued until rated armature current

$$I_s = \sqrt{2} \ 10,000 \text{ amps peak}$$

is reached.

The next part of the capability curve is limited by rated armature current which defines the trajectory

ROTATING MACHINES

PROBLEM 4.28 (Continued)

$$\sqrt{P^2 + Q^2} = V_s I_s$$

where V_s and I_s are rated values.

For $Q < 0$, the capability curve is limited by pull-out conditions defined by the condition

$$\frac{dP}{d\delta_e} = 0 = \frac{V_s E_f}{X_d} \cos \delta_e + \frac{(X_d - X_q) V_s^2}{X_d X_q} \cos 2\delta_e$$

To evaluate this part of the curve we evaluate e_f in terms of δ_e from the power and current expressions

$$e_f = \frac{\frac{P X_d}{V_s^2} - \frac{(X_d - X_q)}{2X_q} \sin 2\delta_e}{\sin \delta_e}$$

$$e_f = \cos \delta_e - \sqrt{\left(\frac{I_s X_d}{V_s}\right)^2 - \left(\frac{X_d}{X_q} \sin \delta_e\right)^2}$$

For each level of power at a given power factor we find the value of δ_e that simultaneously satisfies both equations. The resulting values of e_f and δ_e are used in the stability criterion

$$\frac{dP}{d\delta_e} = \frac{V_s^2 e_f}{X_d} \cos \delta_e + \frac{(X_d - X_q) V_s^2}{X_d X_q} \cos 2\delta_e \geq 0$$

When this condition is no longer met (equal sign holds) the stability limit is reached. For the given constants

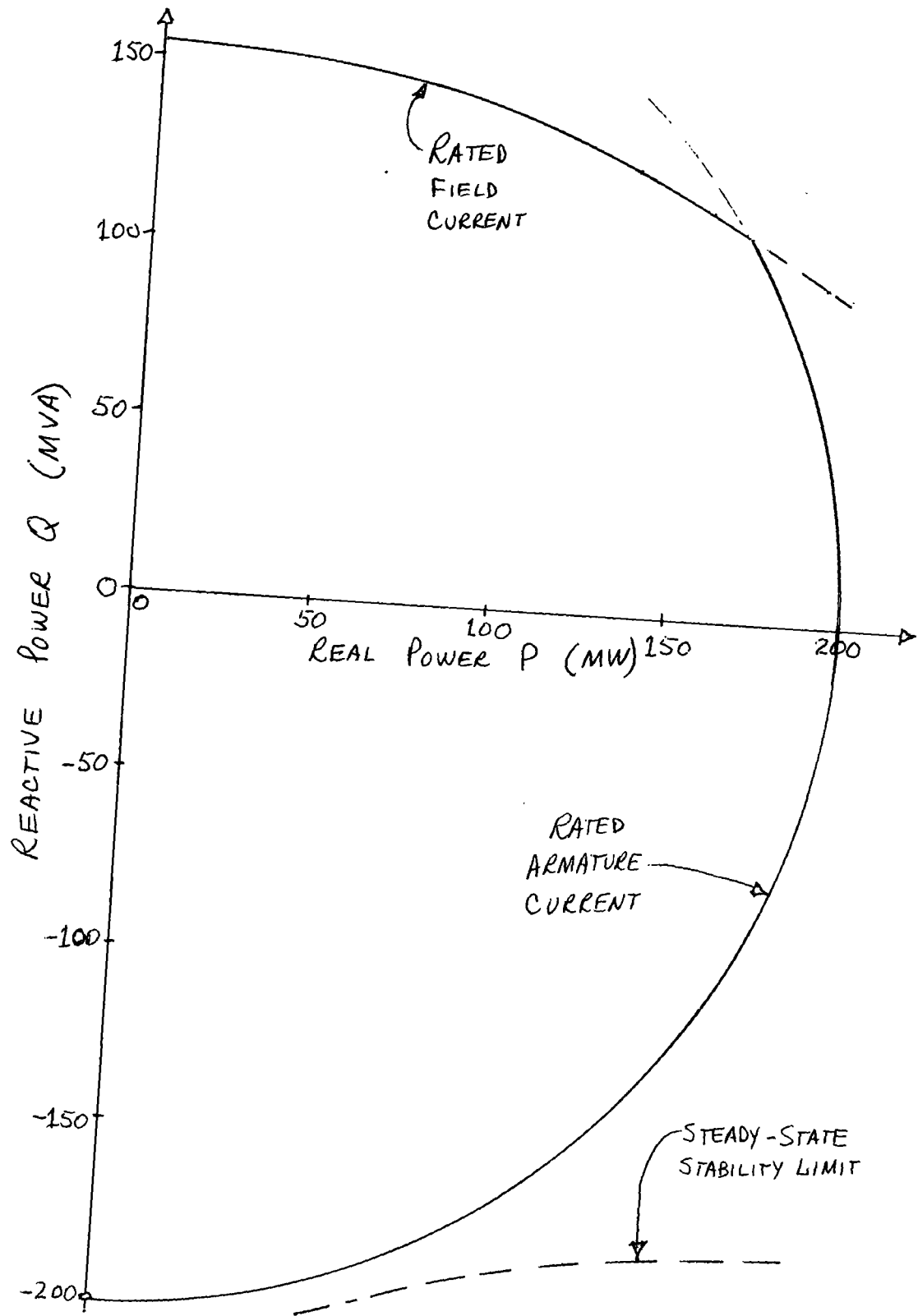
$$e_f = \frac{\frac{P}{167 \times 10^6} - 0.25 \sin 2\delta_e}{\sin \delta_e}$$

$$e_f = \cos \delta_e - \sqrt{\left(\frac{I_s}{11,800}\right)^2 - (1.5 \sin \delta_e)^2}$$

$$\frac{dP}{d\delta_e} = e_f \cos \delta_e + 0.5 \cos 2\delta_e \geq 0$$

The results of this calculation along with the preceding two are shown on the attached graph. Note that the steady-state stability never limits the capability. In practice, however, more margin of stability is required and the capability in the fourth quadrant is limited accordingly.

ROTATING MACHINES



ROTATING MACHINES

PROBLEM 4.29

Part a

For this electrically linear system the electric coenergy is

$$W'_e(v_1, v_2, \theta) = \frac{1}{2} C_o (1 + \cos 2\theta) v_1^2 + \frac{1}{2} C_o (1 + \sin 2\theta) v_2^2$$

The torque of electric origin is

$$T^e = \frac{\partial W'_e(v_1, v_2, \theta)}{\partial \theta} = C_o (v_2^2 \cos 2\theta - v_1^2 \sin 2\theta)$$

Part b

With $v_1 = V_o \cos \omega t$; $v_2 = V_o \sin \omega t$

$$T^e = C_o V_o^2 (\sin^2 \omega t \cos 2\theta - \cos^2 \omega t \sin 2\theta)$$

Using trig identities

$$T^e = \frac{C_o V_o^2}{2} [\cos 2\theta - \cos 2\omega t \cos 2\theta - \sin 2\theta - \cos 2\omega t \cos 2\theta]$$

$$T^e = \frac{C_o V_o^2}{2} (\cos 2\theta - \sin 2\theta) - \frac{C_o V_o^2}{2} [\cos(2\omega t - 2\theta) + \cos(2\omega t + 2\theta)]$$

Three possibilities for time-average torque:

Case I:

Shaft sitting still at fixed angle θ

Case II:

Shaft turning in positive θ direction

$$\theta = \omega t + \gamma$$

where γ is a constant

Case III:

Shaft turning in negative θ direction

$$\theta = -\omega t + \delta$$

where δ is a constant.

Part c

The time average torques are:

Case I: $\theta = \text{const.}$

$$\langle T^e \rangle = \frac{C_o V_o^2}{2} (\cos 2\theta - \sin 2\theta)$$

ROTATING MACHINES

PROBLEM 4.29 (Continued)

Case II: $\theta = \omega t + \gamma$

$$\langle T^e \rangle = - \frac{C_o V_o^2}{2} \cos 2\gamma$$

Case III: $\theta = -\omega t + \delta$

$$\langle T^e \rangle = - \frac{C_o V_o^2}{2} \cos 2\delta$$

PROBLEM 4.30

For an applied voltage $v(t)$ the electric coenergy for this electrically linear system is

$$W'_e(v, \theta) = \frac{1}{2}(C_o + C_1 \cos 2\theta)v^2$$

The torque of electric origin is then

$$T^e = \frac{\partial W'_e(v, \theta)}{\partial \theta} = -C_1 \sin 2\theta v^2$$

For $v = V_o \sin \omega t$

$$T^e = -C_1 V_o^2 \sin^2 \omega t \sin 2\theta$$

$$T^e = -\frac{C_1 V_o^2}{2} (\sin 2\theta - \cos 2\omega t \cos 2\theta)$$

$$T^e = -\frac{C_1 V_o^2}{2} \sin 2\theta + \frac{C_1 V_o^2}{4} [\cos(2\omega t - 2\theta) + \cos(2\omega t + 2\theta)]$$

For rotational velocity ω_m we write

$$\theta = \omega_m t + \gamma$$

and then

$$T^e = -\frac{C_1 V_o^2}{2} \sin 2(\omega_m t + \gamma) + \frac{C_1 V_o^2}{4} \{ \cos[2(\omega - \omega_m)t - 2\gamma] + \cos[2(\omega + \omega_m)t + 2\gamma] \}$$

This device can behave as a motor if it can produce a time-average torque for $\omega_m = \text{constant}$. This can occur when

$$\omega_m = \pm \omega$$

LUMPED-PARAMETER ELECTROMECHANICAL DYNAMICS

PROBLEM 5.1

Part a

The capacitance of the system of plane parallel electrodes is

$$C = (L+x)d\epsilon_0/s \quad (a)$$

and since the co-energy W' of an electrically linear system is simply $\frac{1}{2}Cv^2$ (remember v is the terminal voltage of the capacitor, not the voltage of the driving source)

$$f^e = \frac{\partial W'}{\partial x} = \frac{1}{2} \frac{d\epsilon_0}{s} v^2 \quad (b)$$

The plates tend to increase their area of overlap.

Part b

The force equation is

$$M \frac{d^2 x}{dt^2} = -Kx + \frac{1}{2} \frac{d\epsilon_0}{s} v^2 \quad (c)$$

while the electrical loop equation, written using the fact that the current dq/dt through the resistance can be written as Cv , is

$$V(t) = R \frac{d}{dt} \left[(L+x) \frac{d\epsilon_0}{s} v \right] + v \quad (d)$$

These are two equations in the dependent variables (x,v) .

Part c

This problem illustrates the important point that unless a system involving electromechanical components is either intrinsically or externally biased, its response will not in general be a linear reproduction of the input. The force is proportional to the square of the terminal voltage, which in the limit of small R is simply $V^2(t)$. Hence, the equation of motion is (c) with

$$v^2 = V^2(t) = u_{-1}(t) \frac{V_0^2}{2} (1 - \cos 2\omega t) \quad (e)$$

where we have used the identity $\sin^2 \omega t = \frac{1}{2}(1 - \cos 2\omega t)$. For convenience the equation of motion is normalized

PROBLEM 5.1 (Continued)

$$\frac{d^2x}{dt^2} + \omega_o^2 x = \alpha u_{-1}(t)(1 - \cos 2\omega t) \quad (f)$$

where

$$\omega_o^2 = K/M ; \alpha = V_o^2 d \epsilon_o / 4sM$$

To solve this equation, we note that there are two parts to the particular solution, one a constant

$$x = \frac{\alpha}{\omega_o^2} \quad (g)$$

and the other a cosinusoid having the frequency 2ω . To find this second part solve the equation

$$\frac{d^2x}{dt^2} + \omega_o^2 x = -\operatorname{Re} \alpha e^{2j\omega t} \quad (h)$$

for the particular solution

$$x = \frac{-\alpha \cos 2\omega t}{\omega_o^2 - 4\omega^2} \quad (i)$$

The general solution is then the sum of these two particular solutions and the homogeneous solution $t > 0$

$$x(t) = \frac{\alpha}{\omega_o^2} - \frac{\alpha \cos 2\omega t}{\omega_o^2 - 4\omega^2} + A \sin \omega_o t + B \cos \omega_o t \quad (j)$$

The constants A and B are determined by the initial conditions. At $t=0$, $dx/dt = 0$, and this requires that $A = 0$. The spring determines that the initial position is $x = 0$, from which it follows that

$$B = \alpha 4\omega^2 / \omega_o^2 (\omega_o^2 - 4\omega^2) \quad (k)$$

Finally, the required response is ($t > 0$)

$$x(t) = \frac{\alpha}{\omega_o^2} \left[1 - \frac{\cos 2\omega t}{[1 - (\frac{2\omega}{\omega_o})^2]} + \frac{(\frac{2\omega}{\omega_o})^2 \cos \omega_o t}{[1 - (\frac{2\omega}{\omega_o})^2]} \right] \quad (l)$$

LUMPED-PARAMETER ELECTROMECHANICAL DYNAMICS

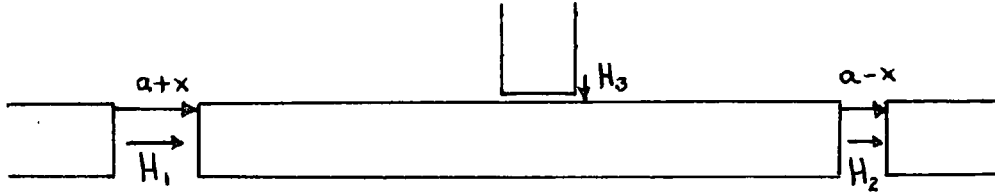
PROBLEM 5.1 (Continued)

Note that there are constant and double frequency components in this response, reflecting the effect of the drive. In addition, there is the response frequency ω_0 reflecting the natural response of the spring mass system. No part of the response has the same frequency as the driving voltage.

PROBLEM 5.2

Part a

The field intensities are defined as in the figure



Ampere's law, integrated around the outside magnetic circuit gives

$$2N_1 i_1 = H_1(a+x) + H_2(a-x) \quad (a)$$

and integrated around the left inner circuit gives

$$N_1 i_1 - N_2 i_2 = H_1(a+x) - H_3 a \quad (b)$$

In addition, the net flux into the movable plunger must be zero

$$0 = H_1 - H_2 + H_3 \quad (c)$$

These three equations can be solved for H_1 , H_2 and H_3 as functions of i_1 and i_2 . Then, the required terminal fluxes are

$$\lambda_1 = N_1 \mu_0 dW (H_1 + H_2) \quad (d)$$

$$\lambda_2 = N_2 \mu_0 dW H_3 \quad (e)$$

Hence, we have

$$\lambda_1 = \frac{N_1 \mu_0 dW}{3a^2 - x^2} [i_1 6aN_1 + i_2 2N_2 x] \quad (f)$$

$$\lambda_2 = \frac{N_2 \mu_0 dW}{3a^2 - x^2} [i_1 2N_1 x + i_2 2aN_2] \quad (g)$$

Part b

To use the device as a differential transformer, it would be excited at a frequency such that

PROBLEM 5.2 (Continued)

$$\frac{2\pi}{\omega} \ll T \quad (h)$$

where T is a period characterizing the movement of the plunger. This means that in so far as the signal induced at the output terminals is concerned, the effect of the motion can be ignored and the problem treated as though x is a constant (a quasi-static situation, but not in the sense of Chap. 1). Put another way, because the excitation is at a frequency such that (h) is satisfied, we can ignore idL/dt compared to Ldi/dt and write

$$v_2 = \frac{d\lambda_2}{dt} = - \frac{\omega 2N_1 N_2 \mu_o dWxI_o}{(3a^2 - x^2)} \sin \omega t$$

At any instant, the amplitude is determined by $x(t)$, but the phase remains independent of $x(t)$, with the voltage leading the current by 90° . By design, the output signal is zero at $x=0$ and tends to be proportional to x over a range of $x \ll a$.

PROBLEM 5.3

Part a

The potential function which satisfies the boundary conditions along constant θ planes is

$$\phi = \frac{V\theta}{\psi} \quad (a)$$

where differentiation shows that Laplace's equation is satisfied. The constant has been set so that the potential is V on the upper electrode where $\theta = \psi$, and zero on the lower electrode where $\theta = 0$. Then, the electric field is

$$\bar{E} = -\nabla\phi = -\bar{r}_\theta \frac{1}{r} \frac{\partial\phi}{\partial\theta} = -\bar{r}_\theta \frac{V}{r\psi} \quad (b)$$

Part b

The charge on the upper electrode can be written as a function of (V, ψ) by writing

$$q = D\epsilon_o \int_a^b \frac{V}{r\psi} dr = \frac{D\epsilon_o V}{\psi} \ln\left(\frac{b}{a}\right) \quad (c)$$

LUMPED-PARAMETER ELECTROMECHANICAL DYNAMICS

PROBLEM 5.3 (Continued)

Part c

Then, the energy stored in the electromechanical coupling follows as

$$W = \int V dq = \int \frac{q\psi dq}{D\epsilon_o \ln(\frac{b}{a})} = \frac{1}{2} q^2 \frac{\psi}{D\epsilon_o \ln(\frac{b}{a})} \quad (d)$$

and hence

$$T^e = - \frac{\partial W}{\partial \psi} = - \frac{1}{2} \frac{q^2}{D\epsilon_o \ln(\frac{b}{a})} \quad (e)$$

Part d

The mechanical torque equation for the movable plate requires that the inertial torque be balanced by that due to the torsion spring and the electric field

$$\frac{J d^2 \psi}{dt^2} = \alpha(\psi_o - \psi) - \frac{1}{2} \frac{q^2}{D\epsilon_o \ln(\frac{b}{a})} \quad (f)$$

The electrical equation requires that currents sum to zero at the current node, and makes use of the terminal equation (c).

$$\frac{dQ}{dt} = \frac{dq}{dt} + \frac{Gd}{dt} \left[\frac{q\psi}{D\epsilon_o \ln(\frac{b}{a})} \right] \quad (g)$$

Part e

With $G = 0$, $Q(t) = q(t)$. (This is true to within a constant, corresponding to charge placed on the upper plate initially. We will assume that this constant is zero.) Then, (f) reduces to

$$\frac{d^2 \psi}{dt^2} + \frac{\alpha}{J} \psi = \frac{\alpha}{J} \psi_o - \frac{1}{4} \frac{Q_o^2}{J D \epsilon_o \ln(\frac{b}{a})} (1 + \cos 2\omega t) \quad (h)$$

where we have used the identity $\cos^2 \omega t = \frac{1}{2}(1 + \cos 2\omega t)$. This equation has a solution with a constant part

$$\psi_1 = \psi_o - \frac{1}{4} \frac{Q_o^2}{\alpha D \epsilon_o \ln(\frac{b}{a})} \quad (i)$$

and a sinusoidal steady state part

$$\psi' = - \frac{Q_o^2 \cos 2\omega t}{J 4 D \epsilon_o \ln(\frac{b}{a}) \left[\frac{\alpha}{J} - (2\omega)^2 \right]} \quad (j)$$

PROBLEM 5.3 (Continued)

as can be seen by direct substitution. The plate responds with a d-c part and a part which has twice the frequency of the drive. As can be seen from the mathematical description itself, this is because regardless of whether the upper plate is positive or negative, it will be attracted toward the opposite plate where the image charges reside. The plates always attract. Hence, if we wish to obtain a mechanical response that is proportional to the driving signal, we must bias the system with an additional source and use the drive to simply increase and decrease the amount of this force.

PROBLEM 5.4

Part a

The equation of motion is found from (d) and (h) with $i=I_0$, as given in the solution to Prob. 3.4.

$$M \frac{d^2 x}{dt^2} = Mg - \frac{1}{2} I_0^2 \frac{(N^2 \mu_0 a w)}{(\frac{da}{b} + x)^2} \quad (a)$$

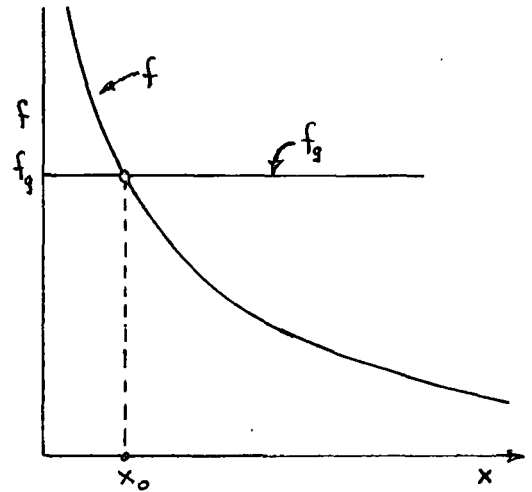
Part b

The mass M can be in static equilibrium if the forces due to the field and gravity just balance,

$$f_g = f$$

or

$$Mg = \frac{1}{2} I_0^2 \frac{(N^2 \mu_0 a w)}{(\frac{da}{b} + x)^2} \quad (b)$$



A solution to this equation is shown

graphically in the figure. The equilibrium is statically unstable because if the mass moves in the positive x direction from x_0 , the gravitational force exceeds the magnetic force and tends to carry it further from equilibrium.

Part c

Because small perturbations from equilibrium are being considered it is appropriate to linearize. We assume $x = x_0 + x'(t)$ and expand the last term in (a) to obtain

LUMPED-PARAMETER ELECTROMECHANICAL DYNAMICS

PROBLEM 5.4 (Continued)

$$- \frac{1}{2} I_o^2 \frac{(N^2 \mu_o a w)}{(\frac{da}{b} + x_o)^2} + I_o^2 \frac{(N^2 \mu_o a w)}{(\frac{da}{b} + x_o)^3} x' + \dots \quad (c)$$

(see Sec. 5.1.2a). The constant terms in the equation of motion cancel out by virtue of (b) and the equation of motion is

$$\frac{d^2 x'}{dt^2} - \alpha^2 x' = 0; \quad \alpha = \sqrt{\frac{I_o^2 (N^2 \mu_o a w)}{(\frac{da}{b} + x_o)^3 M}} \quad (d)$$

Solutions are $\exp \pm \alpha t$, and the linear combination which satisfies the given initial conditions is

$$x' = \frac{v_o}{\alpha} [e^{\alpha t} - e^{-\alpha t}] \quad (e)$$

PROBLEM 5.5

Part a

For small values of x relative to d , the equation of motion is

$$M \frac{d^2 x}{dt^2} = \frac{Q_o Q_1}{4\pi\epsilon} \left[\frac{1}{d^2} - \frac{2x}{d^3} - \frac{1}{d^2} - \frac{2x}{d^3} \right] \quad (a)$$

which reduces to

$$\frac{d^2 x}{dt^2} + \omega_o^2 x = 0 \text{ where } \omega_o^2 = \frac{Q_o Q_1}{M\pi\epsilon d^3} \quad (b)$$

The equivalent spring constant will be positive if

$$\frac{Q_o Q_1}{\pi\epsilon d^3} > 0 \quad (c)$$

and hence this is the condition for stability. The system is stable if the charges have like signs.

Part b

The solution to (b) has the form

$$x = A \cos \omega_o t + B \sin \omega_o t \quad (d)$$

and in view of the initial conditions, $B = 0$ and $A = x_o$.

LUMPED-PARAMETER ELECTROMECHANICAL DYNAMICS

PROBLEM 5.6

Part a

Questions of equilibrium and stability are of interest. Therefore, the equation of motion is written in the standard form

$$M \frac{d^2 x}{dt^2} = - \frac{\partial V}{\partial x} \quad (a)$$

where

$$V = Mgx - W' \quad (b)$$

Here the contribution of W' to the potential is negative because $F^e = \partial W' / \partial x$. The separate potentials are shown in the figure, together with the total potential. From this plot it is clear that there will be one point of static equilibrium as indicated.

Part b

An analytical expression for the point of equilibrium follows by setting the force equal to zero

$$\frac{\partial V}{\partial x} = Mg + \frac{2L_o x^3}{b^4} I^2 \quad (c)$$

Solving for x , we have

$$x = - \left[\frac{Mgb^4}{2L_o I^2} \right]^{1/3} \quad (d)$$

Part c

It is clear from the potential plot that the equilibrium is stable.

PROBLEM 5.7

From Prob. 3.15 the equation of motion is, for small θ

$$J \frac{d^2 \theta}{dt^2} = - K\theta + \frac{1}{2} \mu_o DN^2 \ln\left(\frac{b}{a}\right) I_o^2 \left[\frac{4\theta}{(\beta - \alpha)^3} \right] \quad (a)$$

Thus, the system will have a stable static equilibrium at $\theta = 0$ if the effective spring constant is positive, or if

$$K > \frac{2\mu_o DN^2}{(\beta - \alpha)^3} \ln\left(\frac{b}{a}\right) I_o^2 \quad (b)$$

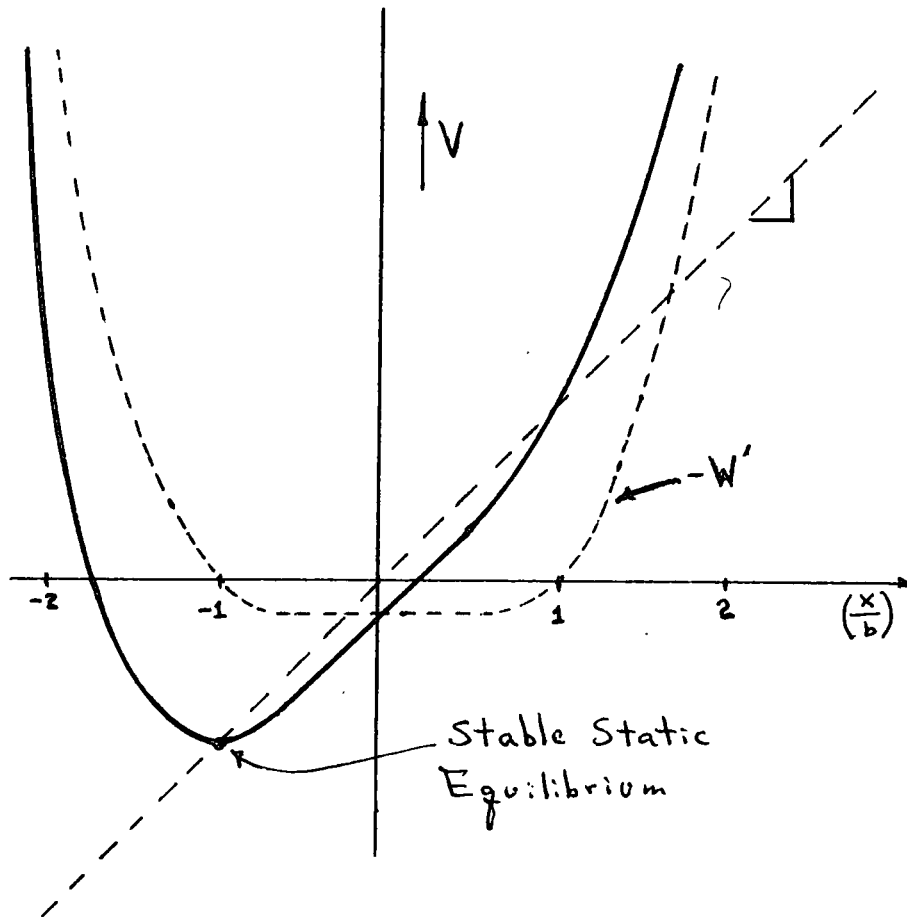


Figure for Prob. 5.6

PROBLEM 5.8

Part a

The coenergy is

$$W' = \int_0^{i_1} \lambda_1(i_1', 0, x) di_1' + \int_0^{i_2} \lambda_2(i_1, i_2', x) di_2' \quad (a)$$

which can be evaluated using the given terminal relations

$$W' = [\frac{1}{2} L_1 i_1^2 + M i_1 i_2 + \frac{1}{2} L_2 i_2^2] / (1 + \frac{x}{a})^3 \quad (b)$$

It follows that the force of electrical origin is

$$f^e = \frac{\partial W'}{\partial x} = - \frac{3}{2a} [L_1 i_1^2 + 2M i_1 i_2 + L_2 i_2^2] / (1 + \frac{x}{a})^4 \quad (c)$$

Part b

The static force equation takes the form

$$-f^e = Mg \quad (d)$$

or, with $i_2=0$ and $i_1=I$,

$$\frac{3}{2a} \frac{L_1 I^2}{[1 + \frac{x_0}{a}]^4} = Mg \quad (e)$$

Solution of this equation gives the required equilibrium position x_0

$$\frac{x_0}{a} = [\frac{3}{2a} \frac{L_1 I^2}{Mg}]^{1/4} - 1 \quad (f)$$

Part c

For small perturbations from the equilibrium defined by (e),

$$M_o \frac{d^2 x'}{dt^2} - \frac{6L_1 I^2 x'}{a^2 (1 + \frac{x_0}{a})^5} = f(t) \quad (g)$$

where $f(t)$ is an external force acting in the x direction on M .

With the external force an impulse of magnitude I_o and the mass initially at rest, one initial condition is $x(0) = 0$. The second is given by integrating the equation of motion from 0^- to 0^+

$$\int_{0^-}^{0^+} \frac{d}{dt} (M_o \frac{dx'}{dt}) dt - \text{constant} \int_{0^-}^{0^+} x' dt = I_o \int_{0^-}^{0^+} u_{-1}(t) dt \quad (h)$$

The first term is the jump in momentum at $t=0$, while the second is zero if x is to remain continuous. By definition, the integral on the right is I_o . Hence, from (h) the second initial condition is

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PROBLEM 5.8 (Continued)

$$M_o \frac{dx'}{dt}(0) = I_o, x'(0) = 0 \quad (i)$$

In view of these conditions, the response is

$$x'(t) = \frac{I_o}{2\alpha M_o} (e^{\alpha t} - e^{-\alpha t}) \quad (j)$$

where

$$\alpha = \left[6L_1 I^2 / a^2 M_o \left(1 + \frac{x_o}{a} \right)^5 \right]^{1/2}$$

Part d

With proportional feedback through the current i_2 , the mutual term in the force equation makes a linear contribution and the force equation becomes

$$M_o \frac{d^2 x'}{dt^2} - \left[\frac{6L_1 I^2}{a^2 \left(1 + \frac{x_o}{a} \right)^5} - \frac{3MI\alpha}{a} \right] x' = f(t) \quad (k)$$

$\uparrow \quad (1 + \frac{x_o}{a})^4$

The effective spring constant is positive if

$$\alpha I > 2L_1 I^2 / a \left(1 + \frac{x_o}{a} \right) M \quad (l)$$

and hence this is the condition for stability. However, once initiated oscillations remain undamped according to this model.

Part e (see insert)

With a damping term introduced by the feedback, the mechanical equation becomes

$$M_o \frac{d^2 x'}{dt^2} + \frac{3MI}{a} \beta \frac{dx'}{dt} + K_e x' = f(t) \quad (m)$$

$\uparrow \quad (1 + \frac{x_o}{a})^4$

where

$$K_e = \frac{3MI\alpha}{a} - \frac{6L_1 I^2}{a^2 \left(1 + \frac{x_o}{a} \right)^5} = \frac{3MI\alpha}{a \left(1 + \frac{x_o}{a} \right)^4} - \frac{6L_1 I^2}{a^2 \left(1 + \frac{x_o}{a} \right)^5}$$

This equation has solutions of the form $\exp st$, where substitution shows that

$$s = -\frac{3MI\beta}{2aM_o} \pm \sqrt{\left(\frac{3MI\beta}{2aM_o} \right)^2 - \frac{K_e}{M_o}} \quad (n)$$

$\uparrow \quad (1 + \frac{x_o}{a})^4$

For the response to decay, K_e must be positive (the system must be stable without damping) and β must be positive.

$$\frac{3}{2aM_o} \frac{MI\beta}{\left(1 + \frac{x_o}{a} \right)^4} > \sqrt{\left(\frac{3MI\beta}{2aM_o \left(1 + \frac{x_o}{a} \right)^4} \right)^2 - \frac{K_e}{M_o}}$$

PROBLEM 5.9

Part a

The mechanical equation of motion is

$$M \frac{d^2 x}{dt^2} = -K(x - \ell_0) - B \frac{dx}{dt} + f^e \quad (a)$$

Part b

where the force f^e is found from the coenergy function which is (because the system is electrically linear) $W' = \frac{1}{2} Li^2 = \frac{1}{2} Ax^3 i^2$

$$f^e = \frac{\partial W'}{\partial x} = \frac{3}{2} Ax^2 i^2 \quad (b)$$

Part c

We can both find the equilibrium points X_0 and determine if they are stable by writing the linearized equation at the outset. Hence, we let $x(t) = X_0 + x'(t)$ and (a) and (b) combine to give

$$M \frac{d^2 x'}{dt^2} = -K(X_0 - \ell_0) - Kx' - B \frac{dx'}{dt} + \frac{3}{2} Ai_0^2 (X_0^2 + 2X_0 x') \quad (c)$$

With the given condition on I_0 , the constant (equilibrium) part of this equation is

$$X_0 - \ell_0 = \frac{3X_0^2}{16\ell_0} \quad (d)$$

which can be solved for X_0/ℓ_0 to obtain

$$\frac{X_0}{\ell_0} = \left[\begin{array}{c} 12/3 \\ 4/3 \end{array} \right] \quad (e)$$

That is, there are two possible equilibrium positions. The perturbation part of (c) tells whether or not these are stable. That equation, upon substitution of X_0 and the given value of I_0 , becomes

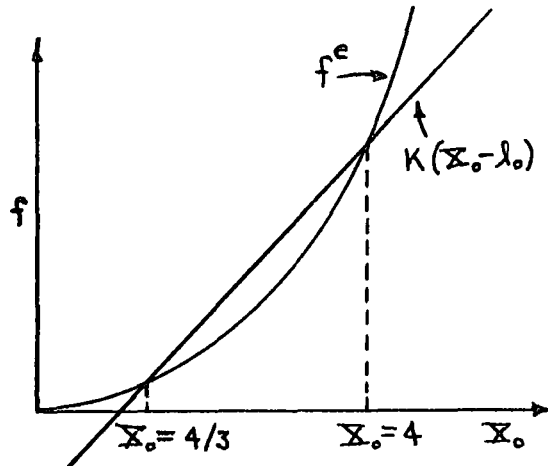
$$M \frac{d^2 x'}{dt^2} = -K[1 - (3/2)]x' - B \frac{dx'}{dt} \quad (f)$$

where the two possibilities correspond to the two equilibrium points. Hence, we conclude that the effective spring constant is positive (and the system is stable) at $X_0/\ell_0 = 4/3$ and the effective spring constant is negative (and hence the equilibrium is unstable) at $X_0/\ell_0 = 4$.

Part d

The same conclusions as to the stability of the equilibrium points can be made from the figure.

PROBLEM 5.9 (Continued)



Consider the equilibrium at $x_o = 4$. A small displacement to the right makes the force f^e dominate the spring force, and this tends to carry the mass further in the x direction. Hence, this point is unstable. Similar arguments show that the other point is stable.

PROBLEM 5.10

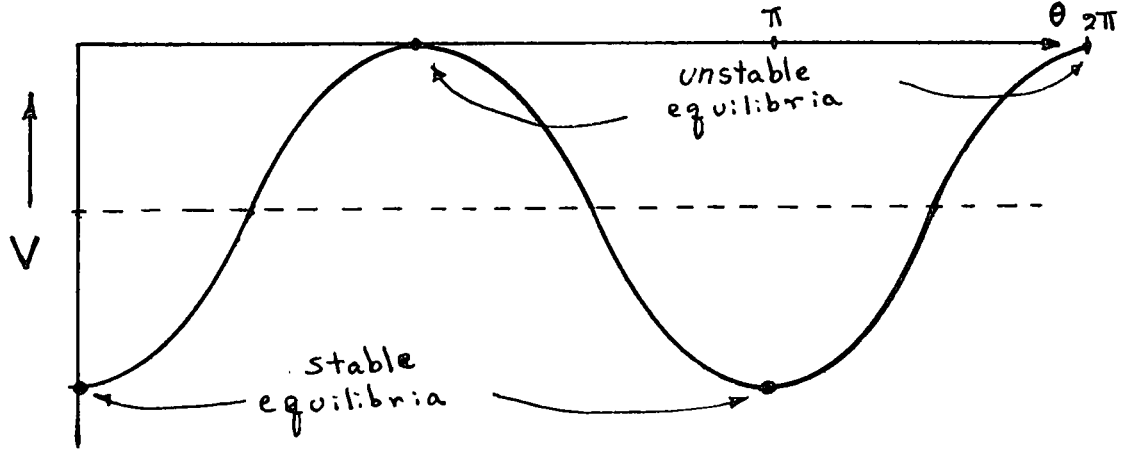
Part a

The terminals are constrained to constant potential, so use coenergy found from terminal equation as

$$W' = \int q dv = \frac{1}{2} C_o (1 + \cos 2\theta) V_o^2 \quad (a)$$

Then, since $T^e = \partial W' / \partial \theta$ and there are no other torques acting on the shaft, the total torque can be found by taking the negative derivative of a potential $V = -W'$, where V is the potential well. A sketch of this well is as shown in the figure.

PROBLEM 5.10 (Continued)



Here it is clear that there are points of zero slope (and hence zero torque and possible static equilibrium) at

$$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \dots \quad (b)$$

Part b

From the potential well it is clear that the first and third equilibria are stable, while the second and fourth are unstable.

PROBLEM 5.11

Part a

From the terminal pair relation, the coenergy is given by

$$W'_m(i_1, i_2, \theta) = \frac{1}{2}(L_o + M \cos 2\theta)i_1^2 + \frac{1}{2}(L_o - M \cos 2\theta)i_2^2 + M \sin 2\theta i_1 i_2 \quad (a)$$

so that the torque of electrical origin is

$$T^e = M[\sin 2\theta(i_2^2 - i_1^2) + 2 \cos 2\theta i_1 i_2] \quad (b)$$

Part b

For the two phase currents, as given,

$$\begin{aligned} i_2^2 - i_1^2 &= -I^2 \cos 2\omega_s t \\ i_1 i_2 &= I^2 \frac{1}{2} \sin 2\omega_s t \end{aligned} \quad (c)$$

so that the torque T^e becomes

PROBLEM 5.11 (Continued)

$$T^e = MI^2[-\sin 2\theta \cos 2\omega_s t + \sin 2\omega_s t \cos 2\theta] \quad (d)$$

or

$$T^e = MI^2 \sin(2\omega_s t - 2\theta) \quad (e)$$

Substitution of $\theta = \omega_m t + \delta$ obtains

$$T^e = -MI^2 \sin[2(\omega_m - \omega_s)t + 2\delta] \quad (f)$$

and for this torque to be constant, we must have the frequency condition

$$\omega_m = \omega_s \quad (g)$$

under which condition, the torque can be written as

$$T^e = -MI^2 \sin 2\delta \quad (h)$$

Part c

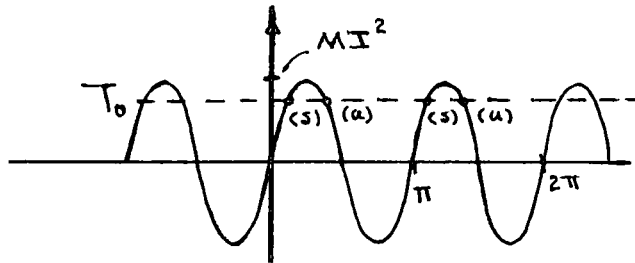
To determine the possible equilibrium angles δ_o , the perturbations and time derivatives are set to zero in the mechanical equations of motion.

$$T_o = MI^2 \sin 2\delta_o \quad (i)$$

Here, we have written the time dependence in a form that is convenient if $\cos 2\delta_o > 0$, as it is at the points marked (s) in the figure. Hence, these points are stable. At the points marked (u), the argument of the sin function and the denominator are imaginary, and the response takes the form of a sinh function. Hence, the equilibrium points indicated by (u) are unstable.

Graphical solutions of this expression are shown in the figure. For there to be equilibrium values of δ the currents must be large enough that the torque can be maintained with the rotor in synchronism with the rotating field.

($MI^2 > T_o$)



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PROBLEM 5.11 (Continued)

Returning to the perturbation part of the equation of motion with $\omega_m = \omega_s$,

$$J \frac{d^2}{dt^2} (\omega_m t + \delta_o + \delta') = T_o + T' - MI^2 \sin(2\delta_o + 2\delta') \quad (j)$$

linearization gives

$$J \frac{d^2 \delta'}{dt^2} + (2MI^2 \cos 2\delta_o) \delta' = T' \quad (k)$$

where the constant terms cancel out by virtue of (i). With $T' = \tau_o u_o(t)$ and initial rest conditions, the initial conditions are

$$\frac{d\delta'}{dt}(0^+) = \frac{\tau_o}{J} \quad (l)$$

$$\delta'(0^+) = 0 \quad (m)$$

and hence the solution for $\delta'(t)$ is

$$\delta'(t) = \frac{\tau_o}{J \sqrt{\frac{2MI^2 \cos 2\delta_o}{J}}} \sin \left[\sqrt{\frac{2MI^2 \cos 2\delta_o}{J}} t \right] \quad (n)$$

PROBLEM 5.12

Part a

The magnitude of the field intensity (H) in the gaps is the same. Hence, from Ampere's law,

$$H = NI/2x \quad (a)$$

and the flux linked by the terminals is N times that passing across either of the gaps.

$$\lambda = \frac{\mu_o a d N^2}{2x} i = L(x) i \quad (b)$$

Because the system is electrically linear, $W'(i, x) = \frac{1}{2} Li^2$, and we have

$$f^e = \frac{\partial W'}{\partial x} = - \frac{N^2 a d \mu_o}{4x^2} i^2 \quad (c)$$

as the required force of electrical origin acting in the x direction.

Part b

Taking into account the forces due to the springs, gravity and the magnetic field, the force equation becomes

PROBLEM 5.12 (Continued)

$$M \frac{d^2 x}{dt^2} = -2Kx + Mg - \frac{N^2 a \mu_o}{4x^2} i^2 + f(t) \quad (d)$$

where the last term accounts for the driving force.

The electrical equation requires that the currents sum to zero at the electrical node, where the voltage is $d\lambda/dt$, with λ given by (b).

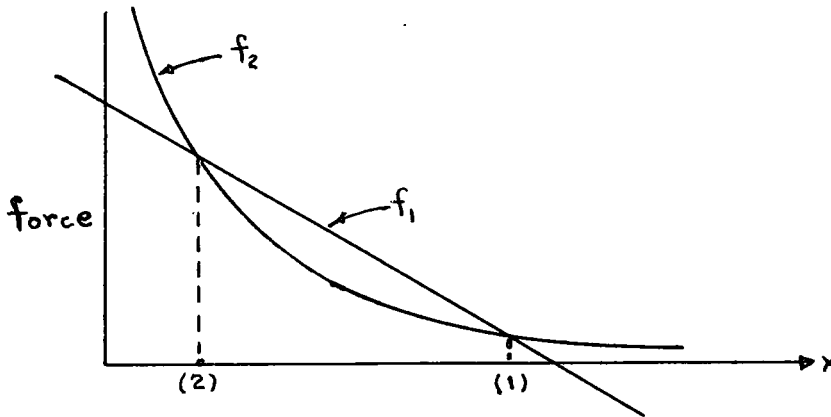
$$I = \frac{1}{R} \frac{d}{dt} \left[\frac{\mu_o a d N^2}{2x} i \right] + i \quad (e)$$

Part c

In static equilibrium, the electrical equation reduces to $i=I$, while the mechanical equation which takes the form $f_1 = f_2$ is satisfied if

$$-2Kx + Mg = \frac{N^2 a \mu_o I^2}{4x^2} \quad (f)$$

Here, f_2 is the negative of the force of electrical origin and therefore (if positive) acts in the $-x$ direction. The respective sides of (f) are shown in the sketch, where the points of possible static equilibrium are indicated. Point (1) is stable, because a small excursion to the right makes f_2 dominate over f_1 and this tends to return the mass in the minus x direction toward the equilibrium point. By contrast, equilibrium point (2) is characterized by having a larger force f_2 and f_1 for small excursions to the left. Hence, the dominate force tends to carry the mass even further from the point of equilibrium and the situation is unstable. In what follows, $x = X$ will be used to indicate the position of stable static equilibrium (1).



LUMPED-PARAMETER ELECTROMECHANICAL DYNAMICS

PROBLEM 5.12 (Continued)

Part d

If R is very large, then

$$i \approx I$$

even under dynamic conditions. This approximation allows the removal of the characteristic time L/R from the analysis as reflected in the reduction in the order of differential equation required to define the dynamics. The mechanical response is determined by the mechanical equation ($x = X + x'$)

$$M \frac{d^2 x'}{dt^2} = -2Kx' + \frac{N^2 a \mu_0}{2X^3} I^2 x' + f(t) \quad (g)$$

where the constant terms have been balanced out and small perturbations are assumed. In view of the form taken by the excitation, assume $x = \text{Re } \hat{x} e^{j\omega t}$ and define $K_e \equiv 2K - N^2 a \mu_0 I^2 / 2X^3$. Then, (g) shows that

$$\hat{x} = \hat{f} / (K_e - \omega^2 M) \quad (h)$$

To compute the output voltage

$$v_o \approx \left. \frac{d\lambda}{dt} \right|_{i=I} = - \frac{\mu_0 a d N^2 I}{2X^2} \frac{dx'}{dt} \quad (i)$$

or

$$\hat{v}_o = -j \frac{\omega \mu_0 a d N^2 I}{2X^2} \hat{x} \quad (j)$$

Then, from (h), the transfer function is

$$\frac{\hat{v}_o}{\hat{f}} = -j \frac{\omega \mu_0 a d N^2 I}{2X^2 (K_e - \omega^2 M)} \quad (k)$$

PROBLEM 5.13

Part a

The system is electrically linear. Hence, the coenergy takes the standard form

$$W' = \frac{1}{2} L_{11} i_1^2 + L_{12} i_1 i_2 + \frac{1}{2} L_{22} i_2^2 \quad (a)$$

and it follows that the force of electrical origin on the plunger is

$$f^e = \frac{\partial W'}{\partial x} = \frac{1}{2} i_1^2 \frac{\partial L_{11}}{\partial x} + i_1 i_2 \frac{\partial L_{12}}{\partial x} + \frac{1}{2} i_2^2 \frac{\partial L_{22}}{\partial x} \quad (b)$$

LUMPED-PARAMETER ELECTROMECHANICAL DYNAMICS

PROBLEM 5.13 (Continued)

which, for the particular terminal relations of this problem becomes

$$f^e = L_o \left\{ \frac{-i_1^2}{d} \left(1 + \frac{x}{d}\right) - \frac{i_1 i_2}{d} \frac{2x}{d} + \frac{i_2^2}{d} \left(1 - \frac{x}{d}\right) \right\} \quad (c)$$

Finally, in terms of this force, the mechanical equation of motion is

$$M \frac{d^2 x}{dt^2} = -Kx - B \frac{dx}{dt} + f^e \quad (d)$$

The circuit connections show that the currents i_1 and i_2 are related to the source currents by

$$\begin{aligned} i_1 &= I_o + i \\ i_2 &= I_o - i \end{aligned} \quad (e)$$

Part b

If we use (e) in (b) and linearize, it follows that

$$f^e = - \frac{4L_o I_o}{d} i - \frac{4L_o I_o^2}{d} \frac{x}{d} \quad (f)$$

and the equation of motion is

$$\frac{d^2 x}{dt^2} + \alpha \frac{dx}{dt} + \omega_o^2 x = -C i \quad (g)$$

where

$$\begin{aligned} \omega_o &= \sqrt{\left[K + \frac{4L_o I_o^2}{d^2}\right] / M} \\ \alpha &= B/M \\ C &= 4L_o I_o / dM \end{aligned}$$

Part c

Both the spring constant and damping in the equation of motion are positive, and hence the system is always stable.

Part d

The homogeneous equation has solutions of the form e^{pt} where

$$p^2 + \alpha p + \omega_o^2 = 0 \quad (h)$$

or, since the system is underdamped

$$p = -\frac{\alpha}{2} \pm j \sqrt{\omega_o^2 - \left(\frac{\alpha}{2}\right)^2} = -\frac{\alpha}{2} \pm j\omega_p \quad (i)$$

PROBLEM 5.13 (Continued)

The general solution is

$$x(t) = -\frac{CI_o}{\omega_o^2} + e^{-\frac{\alpha}{2}t} [A \sin \omega_p t + D \cos \omega_p t] \quad (j)$$

where the constants are determined by the initial conditions $x(0) = 0$ and $dx/dt(0) = 0$

$$D = \frac{CI_o}{\omega_o^2} ; \quad A = \frac{\alpha CI_o}{2\omega_p \omega_o^2} \quad (k)$$

Part e

With a sinusoidal steady state condition, assume $x = \text{Re } \hat{x} e^{j\omega t}$ and write $i(t) = \text{Re}(-jI_o)e^{j\omega t}$ and (g) becomes

$$\hat{x}(-\omega^2 + j\omega\alpha + \omega_o^2) = CjI_o \quad (l)$$

Thus, the required solution is

$$x(t) = \frac{\text{Re} jCI_o e^{j\omega t}}{(\omega_o^2 - \omega^2) + j\omega\alpha} \quad (m)$$

PROBLEM 5.14

Part a

From the terminal equations, the current i_1 is determined by Kirchhoff's current law

$$G L_1 \frac{di_1}{dt} + i_1 = I + GMI_2 \Omega \sin \Omega t \quad (a)$$

The first term in this expression is the current which flows through G because of the voltage developed across the self inductance of the coil, while the last is a current through G induced by the rotational motion. The terms on the right are known functions of time, and constitute a driving function for the linear equation.

Part b

We can divide the solution into particular solutions due to the two driving terms and a homogeneous solution. From the constant drive I we have the solution

$$i_1 = I \quad (b)$$

Because $\sin \Omega t = \text{Re}(-je^{j\Omega t})$, if we assume a particular solution for the sinusoidal drive of the form $i_1 = \text{Re}(\hat{I}_1 e^{j\Omega t})$, we have

LUMPED-PARAMETER ELECTROMECHANICAL DYNAMICS

PROBLEM 5.14 (Continued)

$$\hat{I}_1 (j\Omega GL_1 + 1) = -j\Omega GMI_2 \quad (c)$$

or, rearranging

$$\hat{I}_1 = \frac{-\Omega GMI_2 (\Omega GL_1 + j)}{1 + (\Omega GL_1)^2} \quad (d)$$

We now multiply this complex amplitude by $e^{j\Omega t}$ and take the real part to obtain the particular solution due to the sinusoidal drive

$$i_1 = \frac{-GMI_2\Omega}{1 + (\Omega GL_1)^2} (\Omega GL_1 \cos \Omega t - \sin \Omega t) \quad (e)$$

The homogeneous solution is

$$i_1 = A e^{-t/GL_1} \quad (f)$$

and the total solution is the sum of (b), (e) and (f) with the constant A determined by the initial conditions.

In view of the initial conditions, the complete solution for i_1 , normalized to the value necessary to produce a flux equal to the maximum mutual flux, is then

$$\begin{aligned} \frac{L_1 i_1}{MI_2} &= \left[\frac{(\Omega GL_1)^2}{1 + (\Omega GL_1)^2} - \frac{L_1 I}{MI_2} \right] e^{-\frac{\Omega t}{\Omega GL_1}} \\ &+ \frac{GL_1 \Omega}{1 + (\Omega GL_1)^2} (\sin \Omega t - \Omega GL_1 \cos \Omega t) + \frac{L_1 I}{MI_2} \end{aligned} \quad (g)$$

Part c

The terminal relation is used to find the flux linking coil 1

$$\begin{aligned} \frac{\lambda_1}{MI_2} &= \left[\frac{(\Omega GL_1)^2}{1 + (\Omega GL_1)^2} - \frac{L_1 I}{MI_2} \right] e^{-\frac{\Omega t}{\Omega GL_1}} \\ &+ \frac{GL_1 \Omega}{1 + (\Omega GL_1)^2} \sin \Omega t + \frac{\cos \Omega t}{1 + (\Omega GL_1)^2} + \frac{L_1 I}{MI_2} \end{aligned} \quad (h)$$

The flux has been normalized with respect to the maximum mutual flux (MI_2).

LUMPED-PARAMETER ELECTROMECHANICAL DYNAMICS

PROBLEM 5.14 (Continued)

Part d

In order to identify the limiting cases and the appropriate approximations it is useful to plot (g) and (h) as functions of time. These equations contain two constants, ΩGL_1 and $L_1 I / MI_2$. The time required for one rotation is $2\pi/\Omega$ and GL_1 is the time constant of the inductance L_1 and conductance G in series. Thus, ΩGL_1 is essentially the ratio of an electrical time constant to the time required for the coil to traverse the applied field one time. The quantity MI_2 is the maximum flux of the externally applied field that can link the rotatable coil and $L_1 I$ is the self flux of the coil due to current I acting alone. Thus, $L_1 I / MI_2$ is the ratio of self excitation to mutual excitation.

To first consider the limiting case that can be approximated by a current source we require that

$$\Omega GL_1 \ll 1 \text{ and } \Omega GL_1 \ll \frac{L_1 I}{MI_2} \quad (i)$$

To demonstrate this set

$$\Omega GL_1 = 0.1 \text{ and } \frac{L_1 I}{MI_2} = 1 \quad (j)$$

and plot current and flux as shown in Fig. (a). We note first that the transient dies out very quickly compared to the time of one rotation. Furthermore, the flux varies appreciably while the current varies very little compared to its average value. In the ideal limit ($G \rightarrow 0$) the transient would die out instantaneously and the current would be constant. Thus the approximation of the situation by an ideal current-source excitation would involve a small error; however, the saving in analytical time is often well worth the decrease in accuracy resulting from the approximation.

Part e

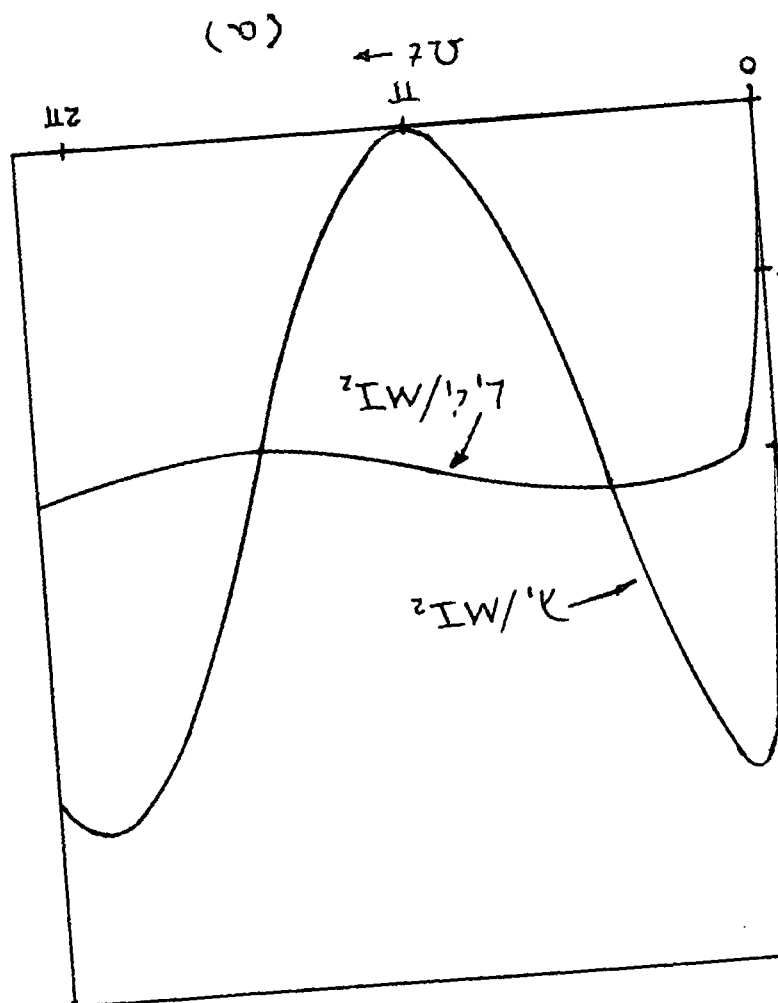
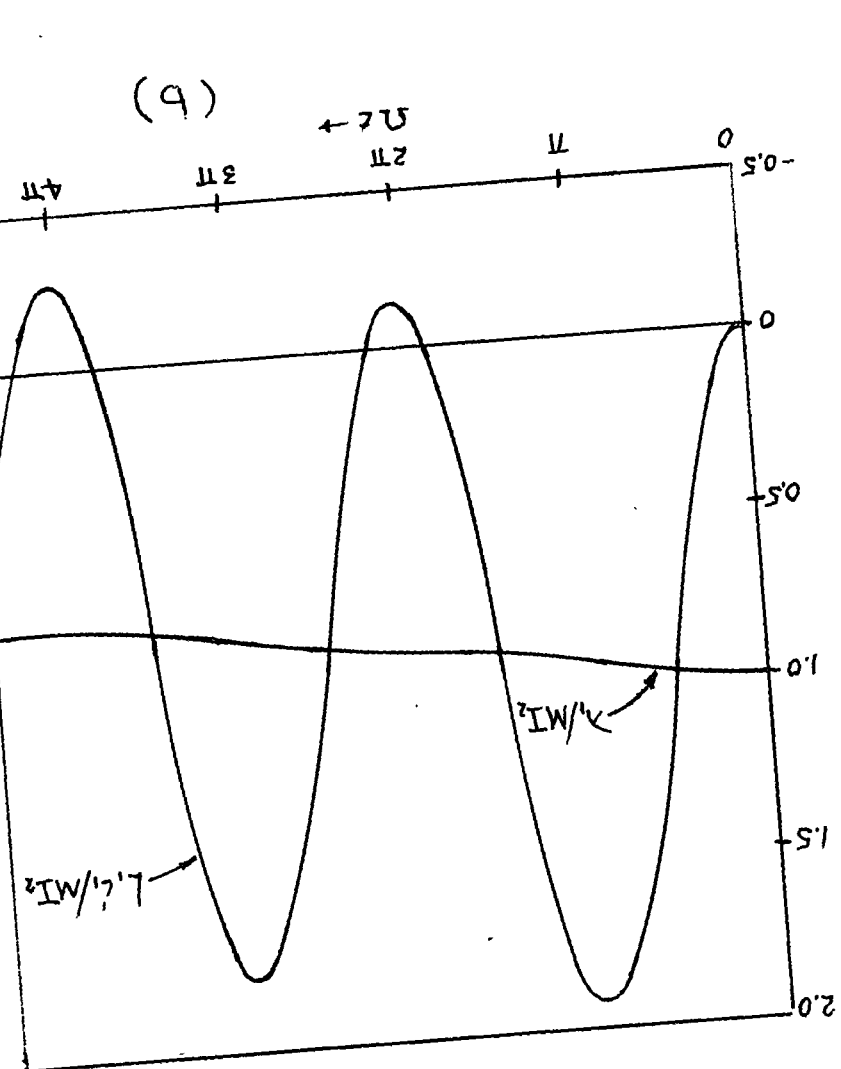
We next consider the limiting case that can be approximated by a constant-flux constraint. This requires that

$$\Omega GL_1 \gg 1 \quad (k)$$

To study this case, set

$$\Omega GL_1 = 50 \text{ and } I = 0 \quad (l)$$

The resulting curves of flux and current are shown plotted in Fig. (b). Note that with this constraint the current varies drastically but the flux pulsates only slightly about a value that decays slowly compared to a rotational period. Thus, when considering events that occur in a time interval comparable



PROBLEM 5.14 (Continued)

with the rotational period, we can approximate this system with a constant-flux constraint. In the ideal, limiting case, which can be approached with superconductors, $G \rightarrow \infty$ and λ_1 stays constant at its initial value. This initial value is the flux that links the coil at the instant the switch S is closed.

In the limiting cases of constant-current and constant flux constraints the losses in the electrical circuit go to zero. This fact allows us to take advantage of the conservative character of lossless systems, as discussed in Sec. 5.2.1.

Part f

Between the two limiting cases of constant-current and constant flux constraints the conductance G is finite and provides electrical damping on the mechanical system. We can show this by demonstrating that mechanical power supplied by the speed source is dissipated in the conductance G. For this purpose we need to evaluate the torque supplied by the speed source. Because the rotational velocity is constant, we have

$$T^m = -T^e \quad (m)$$

The torque of electrical origin T^e is in turn

$$T^e = - \frac{\partial W'(i_1, i_2, \theta)}{\partial \theta} \quad (n)$$

Because the system is electrically linear, the coenergy W' is

$$W' = \frac{1}{2} L_1 i_1^2 + M i_1 i_2 \cos \theta + \frac{1}{2} L_2 i_2^2 \quad (o)$$

and therefore,

$$T^e = -M i_1 I_2 \sin \theta \quad (p)$$

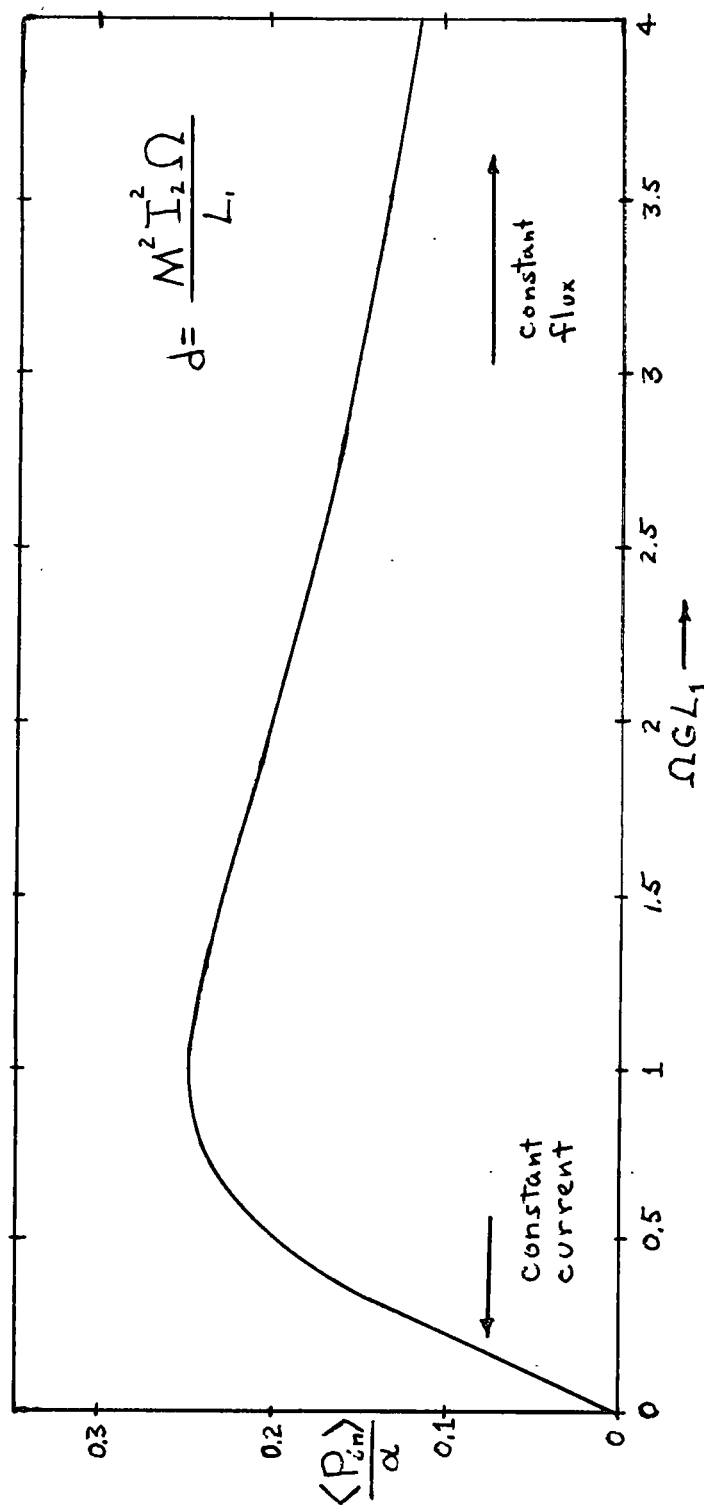
The power supplied by the torque T^m to rotate the coil is

$$P_{in} = -T^e \frac{d\theta}{dt} = M \Omega i_1 I_2 \sin \Omega t \quad (q)$$

Part g

Hence, from (p) and (q), it follows that in the sinusoidal steady state the average power $\langle P_{in} \rangle$ supplied by the external torque is

$$\langle P_{in} \rangle = \frac{1}{2} \left[\frac{G M^2 I_2^2 \Omega^2}{1 + (\Omega G L_1)^2} \right] \quad (r)$$



LUMPED-PARAMETER ELECTROMECHANICAL DYNAMICS

PROBLEM 5.14 (Continued)

This power, which is dissipated in the conductance G , is plotted as a function of ΩGL_1 in Fig. (c). Note that because Ω and L_1 are used as normalizing constants, ΩGL_1 can only be varied by varying G . Note that for both large and small values of ΩGL_1 the average mechanical power dissipated in G becomes small. The maximum in $\langle P_{in} \rangle$ occurs at $\Omega GL_1 = 1$.

PROBLEM 5.15

Part a

The coenergy of the capacitor is

$$W'_e = \frac{1}{2} C(x) V^2 = \frac{1}{2} \left(\epsilon_o \frac{A}{x} \right) V^2$$

The electric force in the x direction is

$$f_e = \frac{\partial W'_e}{\partial x} = - \frac{1}{2} \frac{\epsilon_o A}{x^2} V^2$$

If this force is linearized around $x = x_o$, $V = V_o$

$$f_e(x) = - \frac{1}{2} \frac{\epsilon_o A V_o^2}{x_o^2} - \frac{\epsilon_o A V_o^2}{x_o^3} x + \frac{\epsilon_o A V_o^2}{x_o^3} x$$

The linearized equation of motion is then

$$B \frac{dx}{dt} + \left(K - \frac{\epsilon_o A V_o^2}{x_o^3} \right) x = - \frac{\epsilon_o A}{2} \frac{V_o^2}{x_o} + f(t)$$

The equation for the electric circuit is

$$V + R \frac{d}{dt} (C(x)V) = V_o$$

Part b

We can keep the voltage constant if

$$R \longrightarrow 0$$

In this case

$$B \frac{dx}{dt} + K' x = f(t) = F u_{-1}(t); K' = K - \frac{\epsilon_o A V_o^2}{x_o^3}$$

The particular solution is

$$x(t) = F/K'$$

LUMPED-PARAMETER ELECTROMECHANICAL DYNAMICS

PROBLEM 5.15 (Continued)

The natural frequency S is the solution to

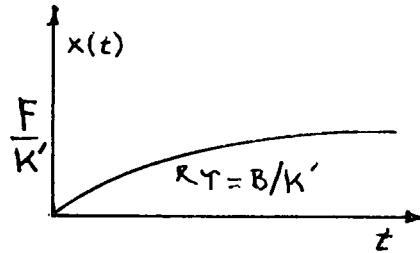
$$sB + K'x = 0$$

$$s = -K'/B$$

Notice that since

$$K'/B = (K - \frac{\epsilon_0 AV_0^2}{3x_0^3})/B$$

there is voltage V_0 above which the system is unstable. Assuming V_0 is less than this voltage



$$x(t) = F/K' (1 - e^{-(K'/B)t})$$

Now we can be more specific about the size of R . We want the time constant of the RC circuit to be small compared to the "action time" of the mechanical system

$$RC(x_0) \ll B/K'$$

$$R \ll \frac{B}{K'C(x_0)}$$

Part c

From part a we suspect that

$$RC(x_0) \gg \tau_{\text{mech}}$$

where τ_{mech} can be found by letting $R \rightarrow \infty$. Since the charge will be constant

$$\frac{dq}{dt} = 0 \quad q = C(x_0)V_0 = C(x_0+x)(V_0+v)$$

$$\approx C(x_0)V_0 + C(x_0)v + V_0 \frac{dC}{dx}(x_0)x$$

$$v = -\frac{V_0}{C(x_0)} \frac{dC}{dx}(x_0)x = +\frac{V_0 x}{\epsilon_0 A} \frac{\epsilon_0 A}{x_0^2} = V_0 \frac{x}{x_0}$$

Using this expression for induced v , the linearized equation of motion becomes

$$B \frac{dx}{dt} + (K - \frac{\epsilon_0 AV_0^2}{3x_0^3})x = -\frac{\epsilon_0 A}{3x_0^3} V_0^2 x + f(t)$$

$$B \frac{dx}{dt} + Kx = f(t)$$

PROBLEM 5.15 (Continued)

The electric effect disappears because the force of a capacitor with constant charge is independent of the plate separation. The solutions are the same as part (a) except that $K' = K$. The constraint on the resistor is then

$$R \gg \frac{1}{C(x_0)} B/K$$

PROBLEM 5.16

We wish to write the sum of the forces in the form

$$f = f_1 + f_2 = - \frac{\partial V}{\partial x} \quad (a)$$

For $x > 0$, this is done by making

$$V = - \frac{1}{2} Kx^2 + F_0 x \quad (b)$$

as shown in the figure. The potential is symmetric about the origin. The largest value of v_0 that can be contained by the potential well is determined by the peak value of potential which, from (b), comes at

$$x = F_0/K \quad (c)$$

where the potential is

$$V = \frac{1}{2} F_0^2/K \quad (d)$$

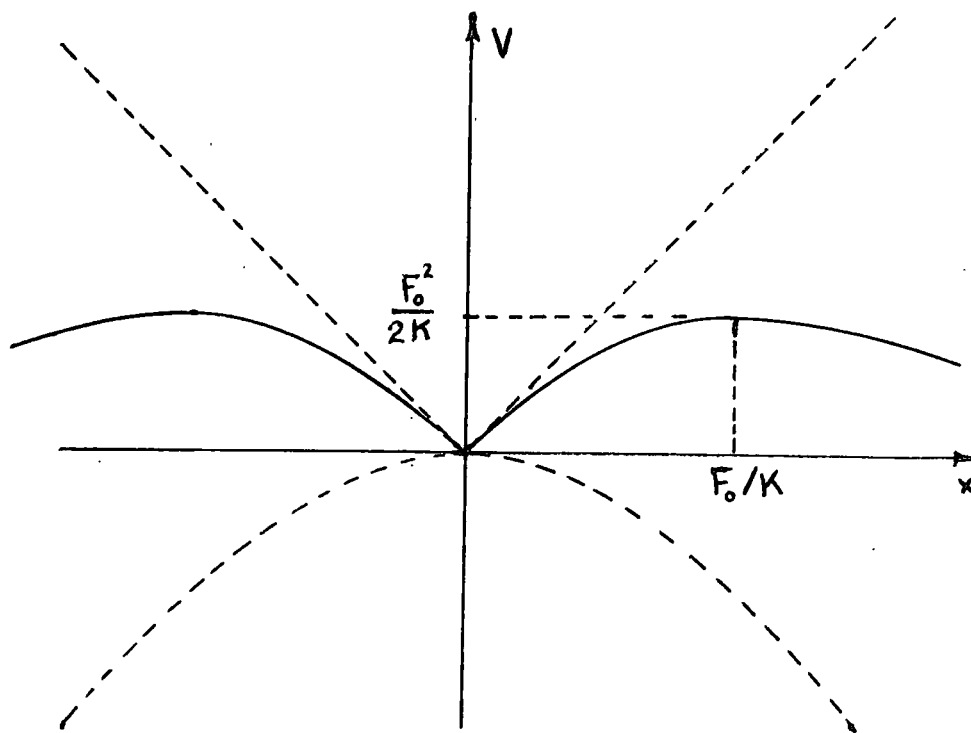
Because the minimum value of the potential is zero, this means that the kinetic energy must exceed this peak value to surmount the barrier. Hence,

$$\frac{1}{2} Mv_0^2 = \frac{1}{2} F_0^2/K \quad (e)$$

or

$$v_0 = \sqrt{\frac{F_0^2}{KM}} \quad (f)$$

PROBLEM 5.16 (Continued)



PROBLEM 5.17

Part a

The electric field intensities defined in the figure are

$$E_2 = (v_2 - v_1) / (d - x) \quad (a)$$

$$E_1 = v_1 / (d + x) \quad (b)$$

Hence, the total charge on the respective electrodes is

$$q_1 = v_1 \left[\frac{A_2 \epsilon_0}{d+x} + \frac{A_1 \epsilon_0}{d-x} \right] - \frac{v_2 A_1 \epsilon_0}{d-x} \quad (c)$$

$$q_2 = \frac{A_1 \epsilon_0 (v_2 - v_1)}{d-x} \quad (d)$$

Part b

Conservation of energy requires

$$v_1 dq_1 + v_2 dq_2 = dW + f^e dx \quad (e)$$

and since the charge q_1 and voltage v_2 are constrained, we make the transformation $v_2 dq_2 = d(v_2 q_2) - q_2 dv_2$ to obtain

$$v_1 dq_1 - q_2 dv_2 = dW'' + f^e dx \quad (f)$$

It follows from this form of the conservation of energy equation that $f^e = -\frac{\partial W''}{\partial x}$ and hence $W'' \equiv U$. To find the desired function we integrate (f) using the terminal relations.

$$U = W'' = \int v_1 dq_1 - q_2 dv_2 \quad (g)$$

The integration on q_1 makes no contribution since q_1 is constrained to be zero. We require $v_2(q_1=0, v_2)$ to evaluate the remaining integral

$$q_2(q_1=0, v_2) = \frac{v_2 A_1 \epsilon_0}{d-x} \left[1 - \frac{1}{\frac{A_2(d-x)}{A_1(d+x)} + 1} \right] \quad (h)$$

Then, from (g),

$$U = -\frac{1}{2} \frac{v_2^2 A_1 \epsilon_0}{d-x} \left[1 - \frac{1}{\frac{A_2(d-x)}{A_1(d+x)} + 1} \right] \quad (i)$$

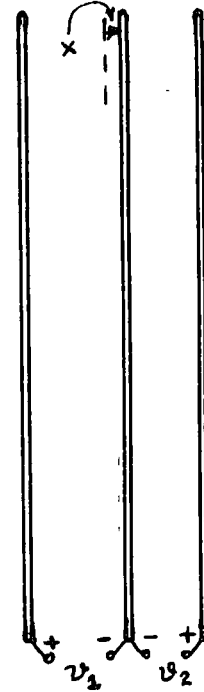
PROBLEM 5.18

Part a

Because the two outer plates are constrained differently once the switch is opened, it is convenient to work in terms of two electrical terminal pairs, defined as shown in the figure. The plane parallel geometry makes it straightforward to compute the terminal relations as being those for simple parallel plate capacitors, with no mutual capacitance.

$$q_1 = v_1 \epsilon_0 A/a + x \quad (a)$$

$$q_2 = v_2 \epsilon_0 A/a - x \quad (b)$$



LUMPED-PARAMETER ELECTROMECHANICAL DYNAMICS

PROBLEM 5.18 (Continued)

Conservation of energy for the electromechanical coupling requires

$$v_1 dq_1 + v_2 dq_2 = dW + f^e dx \quad (c)$$

This is written in a form where q_1 and v_2 are the independent variables by using the transformation $v_2 dq_2 = d(v_2 q_2) - q_2 dv_2$ and defining $W'' = W - v_2 q_2$

$$v_1 dq_1 - q_2 dv_2 = dW'' + f^e dx \quad (d)$$

This is done because after the switch is opened it is these variables that are conserved. In fact, for $t > 0$,

$$v_2 = V_o \text{ and (from (a)) } q_1 = V_o \epsilon_o A/a \quad (e)$$

The energy function W'' follows from (d) and the terminal conditions, as

$$W'' = \int v_1 dq_1 - \int q_2 dv_2 \quad (f)$$

or

$$\frac{1}{2} \frac{(a+x)}{\epsilon_o A} q_1^2 - \frac{1}{2} \frac{\epsilon_o AV_o^2}{a-x} \quad (g)$$

Hence, for $t > 0$, we have (from (e))

$$W'' = \frac{1}{2} \frac{(a+x)}{a^2} \epsilon_o AV_o^2 - \frac{1}{2} \frac{\epsilon_o AV_o^2}{a-x} \quad (h)$$

Part b

The electrical force on the plate is $f^e = -\frac{\partial W''}{\partial x}$. Hence, the force equation is (assuming a mass M for the plate)

$$M \frac{d^2 x}{dt^2} = -Kx - \frac{1}{2} \frac{\epsilon_o AV_o^2}{a^2} + \frac{1}{2} \frac{\epsilon_o AV_o^2}{(a-x)^2} \quad (i)$$

For small excursions about the origin, this can be written as

$$M \frac{d^2 x}{dt^2} = -Kx - \frac{1}{2} \frac{\epsilon_o AV_o^2}{a^2} + \frac{1}{2} \frac{\epsilon_o AV_o^2}{a^2} + \frac{\epsilon_o AV_o^2}{a^3} x \quad (j)$$

The constant terms balance, showing that a static equilibrium at the origin is possible. Then, the system is stable if the effective spring constant is positive.

$$K > \epsilon_o AV_o^2/a^3 \quad (k)$$

Part c

The total potential $V(x)$ for the system is the sum of W'' and the potential energy stored in the springs. That is,

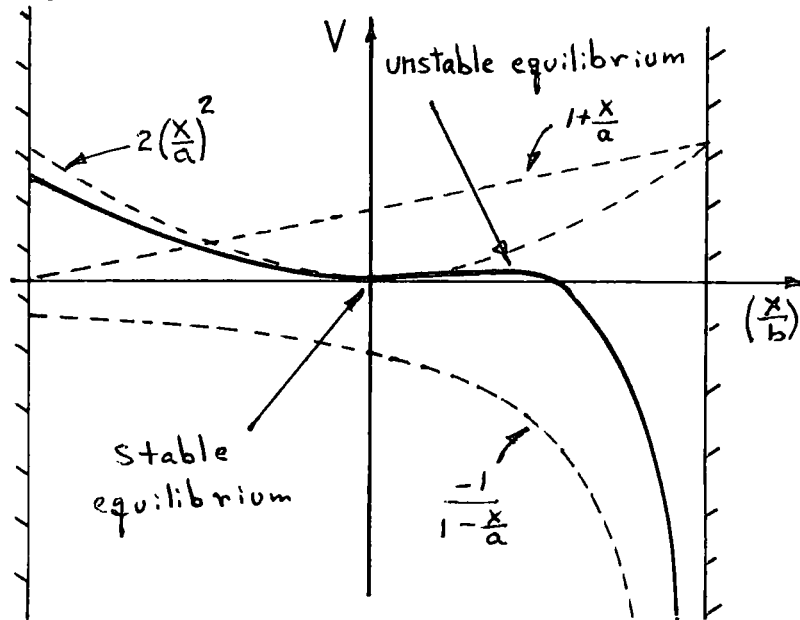
PROBLEM 5.18 (Continued)

$$V = \frac{1}{2} Kx^2 + \frac{1}{2} \frac{(a+x)^2}{a^2} \epsilon_o AV_o^2 - \frac{1}{2} \frac{\epsilon_o AV_o^2}{a-x} \quad (1)$$

or

$$\frac{a^2 K}{2} \left(\frac{x}{a}\right)^2 + \frac{1}{2} \frac{\epsilon_o AV_o^2}{a} \left[\left(1 + \frac{x}{a}\right)^2 - \frac{1}{1 - \frac{x}{a}} \right] \quad (m)$$

This is sketched in the figure for $a^2 K/2 = 2$ and $1/2 \epsilon_o AV_o^2/a = 1$. In addition to the point of stable equilibrium at the origin, there is also an unstable equilibrium point just to the right of the origin.



PROBLEM 5.19

Part a

The coenergy is

$$W' = \frac{1}{2} Li^2 = \frac{1}{2} L_o i^2 / \left[1 - \frac{x}{a}\right]^4 \quad (a)$$

and hence the force of electrical origin is

$$f^e = \frac{\partial W'}{\partial x} = 2L_o i^2 / a \left[1 - \frac{x}{a}\right]^5 \quad (b)$$

 Hence, the mechanical equation of motion, written as a function of (i, x) is

$$M \frac{d^2 x}{dt^2} = -Mg + \frac{2L_o i^2}{a \left[1 - \frac{x}{a}\right]^5} \quad (c)$$

LUMPED-PARAMETER ELECTROMECHANICAL DYNAMICS

PROBLEM 5.19 (Continued)

while the electrical loop equation, written in terms of these same variables (using the terminal relation for λ) is

$$V_o + v = Ri + \frac{d}{dt} \left[\frac{L_o i}{(1 - \frac{x}{a})^4} \right] \quad (d)$$

These last two expressions are the equations of motion for the mass.

Part b

In static equilibrium, the above equations are satisfied by (x, v, i) having the respective values (X_o, V_o, I_o) . Hence, we assume that

$$x = X_o + x'(t); \quad v = V_o + v(t); \quad i = I_o + i'(t) \quad (e)$$

The equilibrium part of (c) is then

$$0 = -Mg + \frac{2L_o I_o^2}{a} / (1 - \frac{X_o}{a})^5 \quad (f)$$

while the perturbations from this equilibrium are governed by

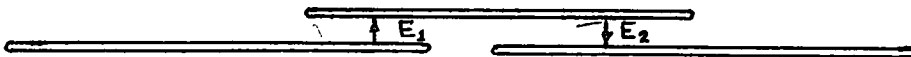
$$M \frac{d^2 x'}{dt^2} = + \frac{10 L_o I_o^2 x'}{a^2 (1 - \frac{X_o}{a})^6} + \frac{4 L_o I_o i'}{a (1 - \frac{X_o}{a})^5} \quad (g)$$

The equilibrium part of (d) is simply $V_o = I_o R$, and the perturbation part is

$$v = Ri' + \frac{L_o}{X_o} \frac{di'}{dt} + \frac{4 L_o I_o}{a [1 - \frac{X_o}{a}]^5} \frac{dx'}{dt} \quad (h)$$

Equations (g) and (h) are the linearized equations of motion for the system which can be solved given the driving function $v(t)$ and (if the transient is of interest) the initial conditions.

PROBLEM 5.20



Part a

The electric field intensities, defined as shown, are

$$E_1 = (v_1 - v_2)/s; \quad E_2 = v_2/s \quad (a)$$

PROBLEM 5.20 (Continued)

In terms of these quantities, the charges are

$$q_1 = \epsilon_0 \left(\frac{a}{2} - x \right) dE_1; \quad q_2 = -\epsilon_0 \left(\frac{a}{2} - x \right) dE_1 + \epsilon_0 \left(\frac{a}{2} + x \right) dE_2 \quad (b)$$

Combining (a) and (b), we have the required terminal relations

$$\begin{aligned} q_1 &= v_1 C_{11} - v_2 C_{12} \\ q_2 &= -v_1 C_{12} + v_2 C_{22} \end{aligned} \quad (c)$$

where

$$\begin{aligned} C_{11} &= \frac{\epsilon_0 d}{s} \left(\frac{a}{2} - x \right); \quad C_{22} = \frac{\epsilon_0 a d}{s} \\ C_{12} &= \frac{\epsilon_0 d}{s} \left(\frac{a}{2} - x \right) \end{aligned}$$

For the next part it is convenient to write these as $q_1(v_1, q_2)$ and $v_2(v_1, q_2)$.

$$\begin{aligned} q_1 &= v_1 \left[C_{11} - \frac{C_{12}^2}{C_{22}} \right] - q_2 \frac{C_{12}}{C_{22}} \\ v_2 &= \frac{q_2}{C_{22}} + v_1 \frac{C_{12}}{C_{22}} \end{aligned} \quad (d)$$

Part b

Conservation of energy for the coupling requires

$$v_1 dq_1 + v_2 dq_2 = dW + f^e dx \quad (e)$$

To treat v_1 and q_2 as independent variables (since they are constrained to be constant) we let $v_1 dq_1 = d(v_1 q_1) - q_1 dv_1$, and write (e) as

$$-q_1 dv_1 + v_2 dq_2 = -dW'' + f^e dx \quad (f)$$

From this expression it is clear that $f^e = \partial W'' / \partial x$ as required. In particular, the function W'' is found by integrating (f)

$$W'' = \int_0^{v_0} q_1(v_1', 0) dv_1' - \int_0^{q_2} v_2(v_0, q_2') dq_2' \quad (g)$$

to obtain

$$W'' = \frac{1}{2} v_0^2 \left[C_{11} - \frac{C_{12}^2}{C_{22}} \right] - \frac{Q^2}{2C_{22}} - \frac{v_0 Q C_{12}}{C_{22}} \quad (h)$$

Of course, C_{11} , C_{22} and C_{12} are functions of x as defined in (c).

PROBLEM 5.21

Part a

The equation of motion as developed in Prob. 3.8 but with $I(t)=I_0=\text{constant}$, is

$$J \frac{d^2 \theta}{dt^2} = - \frac{I L_m^2 I_0}{L_2} (1 - \cos \theta) \sin \theta \quad (a)$$

This has the required form if we define

$$V = - \frac{I L_m^2}{L_2} I_0 \left(\cos \theta + \frac{1}{2} \sin^2 \theta \right) \quad (b)$$

as can be seen by differentiating (b) and recovering the equation of motion. This potential function could also have been obtained by starting directly with the thermodynamic energy equation and finding a hybrid energy function (one having i_1, λ_2, θ as independent variables). See Example 5.2.2 for this more fundamental approach.

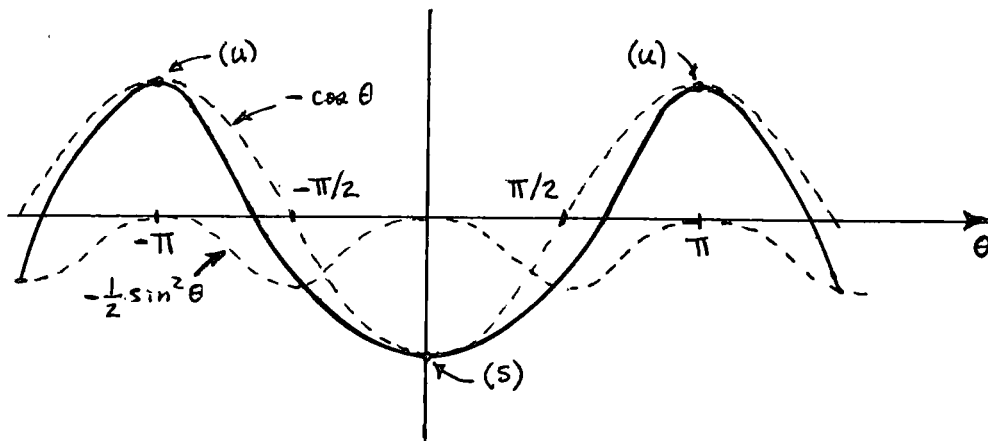
Part b

A sketch of the potential well is as shown below. The rotor can be in stable static equilibrium at $\theta = 0$ (s) and unstable static equilibrium at $\theta = \pi$ (u).

Part c

For the rotor to execute continuous rotory motion from an initial rest position at $\theta = 0$, it must have sufficient kinetic energy to surmount the peak in potential at $\theta = \pi$. To do this,

$$\frac{1}{2} J \left(\frac{d\theta}{dt} \right)^2 \geq \frac{2 I L_m^2 I_0}{L_2} \quad (c)$$



PROBLEM 5.22

Part a

The coenergy stored in the magnetic coupling is simply

$$W' = \frac{1}{2} L_o (1 + 0.2 \cos \theta + 0.05 \cos 2\theta) i^2 \quad (a)$$

Since the gravitational field exerts a torque on the pendulum given by

$$T_p = \frac{\partial}{\partial \theta} (-Mg \ell \cos \theta) \quad (b)$$

and the torque of electrical origin is $T^e = \partial W' / \partial \theta$, the mechanical equation of motion is

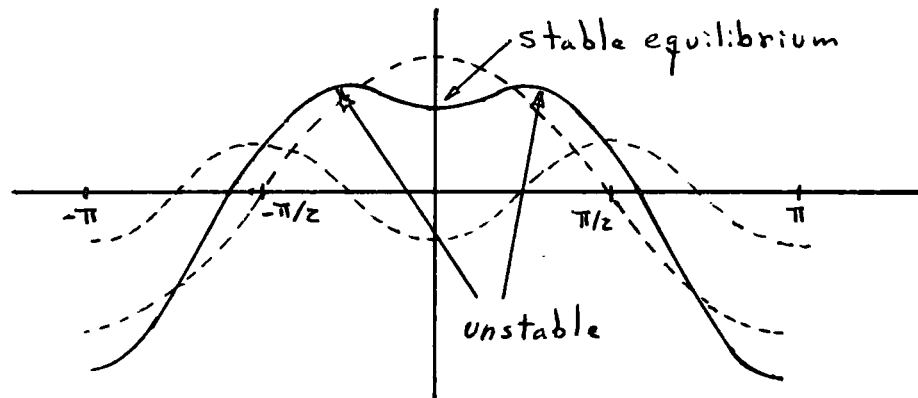
$$\frac{d}{dt} \left[\frac{1}{2} M \ell^2 \left(\frac{d\theta}{dt} \right)^2 + V \right] = 0 \quad (c)$$

where (because $I^2 L_o = 6Mg\ell$)

$$V = Mg\ell[0.4 \cos \theta - 0.15 \cos 2\theta - 3]$$

Part b

The potential distribution V is plotted in the figure, where it is evident that there is a point of stable static equilibrium at $\theta = 0$ (the pendulum straight up) and two points of unstable static equilibrium to either side of center. The constant contribution has been ignored in the plot because it is arbitrary.



PROBLEM 5.23

Part a

The magnetic field intensity is uniform over the cross section and equal to the surface current flowing around the circuit. Define H as into the paper and $H = i/D$. Then λ is H multiplied by μ_o and the area xd .

$$\lambda = \frac{\mu_o xd}{D} i \quad (a)$$

The system is electrically linear and so the energy is $W = \frac{1}{2} \lambda^2 / L$. Then, since $f^e = -\partial W / \partial x$, the equation of motion is

$$M \frac{d^2 x}{dt^2} = f = -Kx + \frac{1}{2} \frac{\Lambda^2 D}{\mu_o x^2 d} \quad (b)$$

Part b

Let $x = X_o + x'$ where x' is small and (b) becomes approximately

$$M \frac{d^2 x'}{dt^2} = -KX_o - Kx' + \frac{1}{2} \frac{\Lambda^2 D}{\mu_o X_o^2 d} - \frac{\Lambda^2 D x'}{\mu_o X_o^3 d} \quad (c)$$

The constant terms define the static equilibrium

$$X_o = \left[\frac{1}{2} \frac{\Lambda^2 D}{\mu_o dK} \right]^{1/3} \quad (d)$$

and if we use this expression for X_o , the perturbation equation becomes,

$$M \frac{d^2 x'}{dt^2} = -Kx' - 2Kx' \quad (e)$$

Hence, the point of equilibrium at X_o as given by (d) is stable, and the magnetic field is equivalent to the spring constant $2K$.

Part c

The total force is the negative derivative with respect to x of V where

$$V = \frac{1}{2} Kx^2 + \frac{1}{2} \frac{\Lambda^2 D}{\mu_o xd} \quad (f)$$

This makes it possible to integrate the equation of motion (b) once to obtain

$$\frac{dx}{dt} = \pm \sqrt{\frac{2}{M} (E - V)} \quad (g)$$

The potential well is as shown in figure (a). Here again it is apparent that the equilibrium point is one where the mass can be static and stable. The constant of integration E is established physically by releasing the mass from static

PROBLEM 5.23 (Continued)

positions such as (1) or (2) shown in Fig. (a). Then the bounded excursions of the mass can be pictured as having the level E shown in the diagram. The motions are periodic in nature regardless of the initial position or velocity.

Part d

The constant flux dynamics can be contrasted with those occurring at constant current simply by replacing the energy function with the coenergy function. That is, with the constant current constraint, it is appropriate to find the electrical force from $W' = \frac{1}{2} Li^2$, where $f^e = \partial W' / \partial x$. Hence, in this case

$$V = \frac{1}{2} Kx^2 - \frac{1}{2} \frac{\mu_0 x d}{D} I^2 \quad (h)$$

A plot of this potential well is shown in Fig. (b). Once again there is a point x_0 of stable static equilibrium given by

$$x_0 = \frac{1}{2} \frac{\mu_0 d I^2}{DK} \quad (i)$$

However, note that if oscillations of sufficiently large amplitude are initiated that it is now possible for the plate to hit the bottom of the parallel plate system at $x = 0$.

PROBLEM 5.25

Part a

Force on the capacitor plate is simply

$$f^e = \frac{\partial W'}{\partial x} = \frac{\partial}{\partial x} \left[\frac{1}{2} \frac{\pi a^2 \epsilon_0 v_o^2}{x} \right] \quad (a)$$

due to the electric field and a force f due to the attached string.

Part b

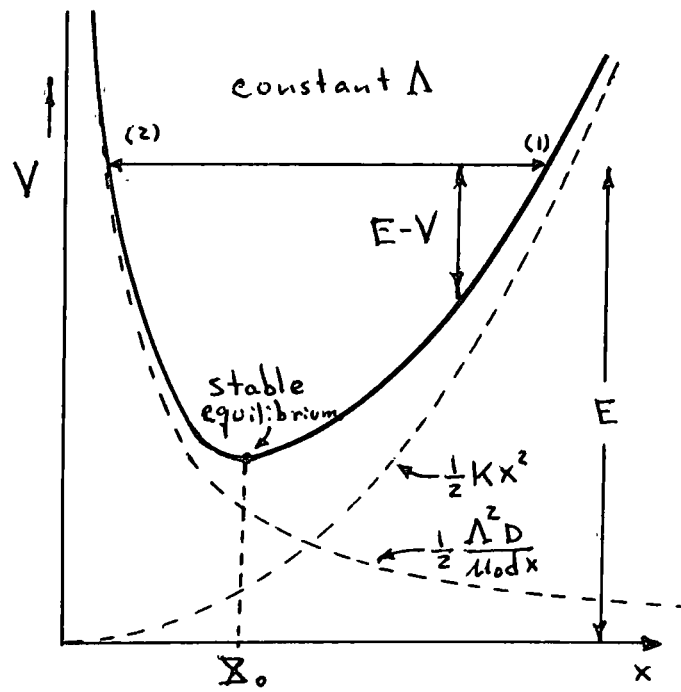
With the mass M_1 rotating at a constant angular velocity, the force f^e must balance the centrifugal force $\omega_m^2 r M_1$ transmitted to the capacitor plate by the string.

$$\frac{1}{2} \frac{\pi a^2 \epsilon_0 v_o^2}{\ell^2} = \omega_m^2 \ell M_1 \quad (b)$$

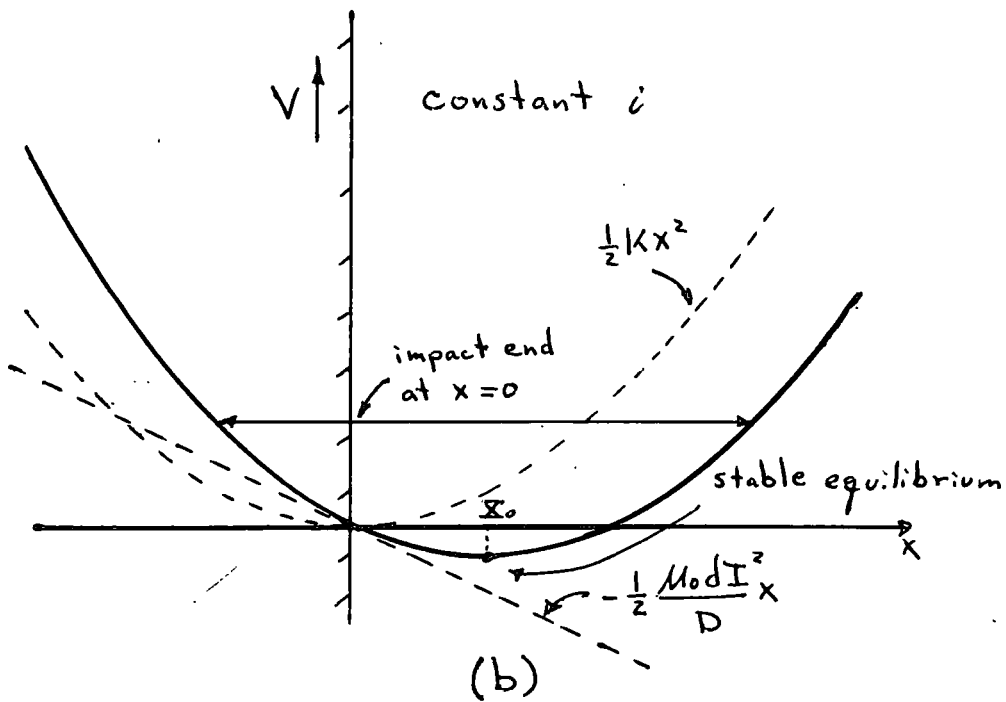
or

$$\omega_m = \sqrt{\frac{\pi a^2 \epsilon_0 v_o^2}{2 \ell^3 M_1}} \quad (c)$$

where ℓ is both the equilibrium spacing of the plates and the equilibrium radius of the trajectory for M_1 .



(a)



PROBLEM 5.25 (Continued)
Part c

The θ directed force equation is (see Prob. 2.8) for the acceleration on a particle in circular coordinates)

$$M_1 \left[r \frac{d^2 \theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} \right] = 0 \quad (d)$$

which can be written as

$$\frac{d}{dt} \left[M_1 r^2 \frac{d\theta}{dt} \right] = 0 \quad (e)$$

This shows that the angular momentum is constant even as the mass M_1 moves in and out

$$M_1 r^2 \frac{d\theta}{dt} = M_1 \ell^2 \omega_m = \text{constant of the motion} \quad (f)$$

This result simply shows that if the radius increases, the angular velocity must decrease accordingly

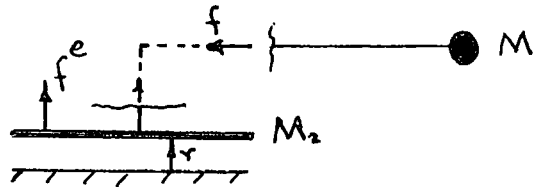
$$\frac{d\theta}{dt} = \frac{\ell^2}{r^2} \omega_m \quad (g)$$

Part d

The radial component of the force equation for M_1 is

$$M_1 \left[\frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \right] = -f \quad (h)$$

where f is the force transmitted by the string, as shown in the figure.



The force equation for the capacitor plate is

$$M_2 \frac{d^2 r}{dt^2} = f^e + f \quad (i)$$

where f^e is supplied by (a) with $v = v_0 = \text{constant}$. Hence, these last two expressions can be added to eliminate f and obtain

PROBLEM 5.25 (Continued)

$$(M_1 + M_2) \frac{d^2 r}{dt^2} - M_1 r \left(\frac{d\theta}{dt} \right)^2 = \frac{\partial}{\partial r} \left(\frac{1}{2} \frac{\pi a^2 \epsilon_0 V_o^2}{r} \right) \quad (j)$$

If we further use (g) to eliminate $d\theta/dt$, we obtain an expression for $r(t)$ that can be written in the standard form

$$(M_1 + M_2) \frac{d^2 r}{dt^2} + \frac{\partial}{\partial r} V = 0 \quad (k)$$

where

$$V = \frac{M_1 \ell^4 \omega_m^2}{2r^2} - \frac{1}{2} \frac{\pi a^2 \epsilon_0 V_o^2}{r} \quad (l)$$

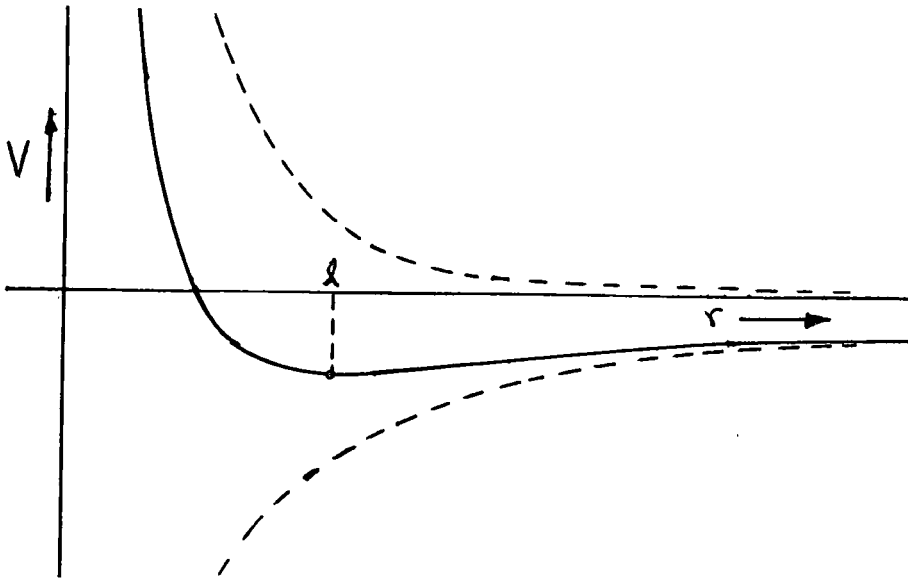
Of course, (k) can be multiplied by dr/dt and written in the form

$$\frac{d}{dt} \left[\frac{1}{2} (M_1 + M_2) \left(\frac{dr}{dt} \right)^2 + V \right] = 0 \quad (m)$$

to show that V is a potential well for the combined mass of the rotating particle and the plate.

Part e

The potential well of (l) has the shape shown in the figure. The minimum represents the equilibrium position found in (c), as can be seen by differentiating (l) with respect to r , equating the expression to zero and solving for ω_m assuming that $r = \ell$. In this example, the potential well is the result of a combination of the negative coenergy for the electromechanical system, constrained to constant potential, and the dynamic system with angular momentum conserved.



PROBLEM 5.26

Part a

To begin the analysis we first write the Kirchhoff voltage equations for the two electric circuits with switch S closed

$$V = i_1 R_1 + \frac{d\lambda_1}{dt} \quad (a)$$

$$0 = i_2 R_2 + \frac{d\lambda_2}{dt} \quad (b)$$

To obtain the electrical terminal relations for the system we neglect fringing fields and assume infinite permeability for the magnetic material to obtain*

$$\lambda_1 = N_1 \phi, \quad \lambda_2 = N_2 \phi \quad (c)$$

where the flux ϕ through the coils is given by

$$\phi = \frac{2\mu_o w d (N_1 i_1 + N_2 i_2)}{g(1 + \frac{x}{g})} \quad (d)$$

We can also use (c) and (d) to calculate the stored magnetic energy as**

$$W_m = \frac{g(1 + \frac{x}{g}) \phi^2}{4\mu_o w d} \quad (e)$$

We now multiply (a) by N_1/R_1 and (b) by N_2/R_2 , add the results and use (c) and (d) to obtain

$$\frac{N_1 V}{R_1} = \frac{g(1 + \frac{x}{g})}{2\mu_o w d} \phi + \left(\frac{N_1^2}{R_1} + \frac{N_2^2}{R_2} \right) \frac{d\phi}{dt} \quad (f)$$

Note that we have only one electrical unknown, the flux ϕ , and if the plunger is at rest ($x = \text{constant}$) this equation has constant coefficients.

*The neglect of fringing fields makes the two windings unity coupled. In practice there will be small fringing fields that cause leakage inductances. However, these leakage inductances affect only the initial part of the transient and neglecting them causes negligible error when calculating the closing time of the relay.

**Here we have used the equation (System is electrically linear)

$$W_m = \frac{1}{2} L_1 i_1^2 + L_{12} i_1 i_2 + \frac{1}{2} L_2 i_2^2$$

LUMPED-PARAMETER ELECTROMECHANICAL DYNAMICS

PROBLEM 5.26 (Continued)

Part b

Use the given definitions to write (f) in the form

$$\phi_o = (1 + \frac{x_o}{g})\phi + \tau_o \frac{d\phi}{dt} \quad (g)$$

Part c

During interval 1 the flux is determined by (g) with $x = x_o$ and the initial condition is $\phi = 0$. Thus the flux undergoes the transient

$$\phi = \frac{\phi_o}{1 + \frac{x_o}{g}} \left[1 - e^{-\left(1 + \frac{x_o}{g}\right) \frac{t}{\tau_o}} \right] \quad (h)$$

To determine the time at which interval 1 ends and to describe the dynamics of interval 2 we must write the equation of motion for the mechanical node. Neglecting inertia and damping forces this equation is

$$K(x - \ell) = f^e \quad (i)$$

In view of (c) (λ_1 and λ_2 are the independent variables implicit in ϕ) we can use (e) to evaluate the force f^e as

$$f^e = - \frac{\partial W_m(\lambda_1, \lambda_2, x)}{\partial x} = - \frac{\phi^2}{4\mu_o w d} \quad (j)$$

Thus, the mechanical equation of motion becomes

$$K(x - \ell) = - \frac{\phi^2}{4\mu_o w d} \quad (k)$$

The flux level ϕ_1 at which interval 1 ends is given by

$$K(x_o - \ell) = - \frac{\phi_1^2}{4\mu_o w d} \quad (l)$$

Part d

During interval 2, flux and displacement are related by (k), thus we eliminate x between (k) and (g) and obtain

$$\phi_o = \left[\left(1 + \frac{\ell}{g}\right) - \left(\frac{\ell - x_o}{g}\right) \frac{\phi^2}{\phi_1^2} \right] \phi + \tau_o \frac{d\phi}{dt} \quad (m)$$

where we have used (l) to write the equation in terms of ϕ_1 . This is the nonlinear differential equation that must be solved to find the dynamical behavior during interval 2.

PROBLEM 5.26 (Continued)

To illustrate the solution of (m) it is convenient to normalize the equation as follows

$$\frac{d(\frac{\phi}{\phi_o})}{d(\frac{t}{\tau_o})} = \left(\frac{l-x_o}{g}\right) \left(\frac{\phi_o}{\phi_1}\right)^2 \left(\frac{\phi}{\phi_o}\right)^3 - \left(1 + \frac{l}{g}\right) \frac{\phi}{\phi_o} + 1 \quad (n)$$

We can now write the necessary integral formally as

$$\int_{\frac{\phi_1}{\phi_o}}^{\frac{\phi}{\phi_o}} \frac{d(\frac{\phi'}{\phi_o})}{\left[\left(\frac{l-x_o}{g}\right) \left(\frac{\phi_o}{\phi_1}\right)^2 \left(\frac{\phi'}{\phi_o}\right)^3 - \left(1 + \frac{l}{g}\right) \frac{\phi'}{\phi_o} + 1 \right]} = \int_0^{\frac{t}{\tau_o}} d(\frac{t}{\tau_o}) \quad (o)$$

where we are measuring time t from the start of interval 2.

Using the given parameter values,

$$\int_{0.1}^{\frac{\phi}{\phi_o}} \frac{d(\frac{\phi'}{\phi_o})}{\left[400 \left(\frac{\phi'}{\phi_o}\right)^3 - 9 \frac{\phi'}{\phi_o} + 1 \right]} = \frac{t}{\tau_o} \quad (p)$$

We factor the cubic in the denominator into a first order and a quadratic factor and do a partial-fraction expansion* to obtain

$$\int_{0.1}^{\frac{\phi}{\phi_o}} \left[\frac{0.156}{5.29 \frac{\phi'}{\phi_o} + 1} + \frac{(-2.23 \frac{\phi'}{\phi_o} + 0.844)}{75.7 \left(\frac{\phi'}{\phi_o}\right)^2 - 14.3 \frac{\phi'}{\phi_o} + 1} \right] d(\frac{\phi'}{\phi_o}) = \frac{t}{\tau_o} \quad (q)$$

Integration of this expression yields

* Phillips, H.B., Analytic Geometry and Calculus, second edition, John Wiley and Sons, New York, 1946, pp. 250-253.

LUMPED-PARAMETER ELECTROMECHANICAL DYNAMICS

PROBLEM 5.26 (Continued)

$$\begin{aligned} \frac{t}{\tau_0} = & 0.0295 \ln \left[3.46 \left(\frac{\phi}{\phi_0} \right) + 0.654 \right] - 0.0147 \ln \left[231 \left(\frac{\phi}{\phi_0} \right)^2 - 43.5 \left(\frac{\phi}{\phi_0} \right) + 3.05 \right] \\ & + 0.127 \tan^{-1} \left[15.1 \left(\frac{\phi}{\phi_0} \right) - 1.43 \right] - 0.0108 \end{aligned} \quad (r)$$

Part e

During interval 3, the differential equation is (g) with $x = 0$, for which the solution is

$$\phi = \phi_2 + (\phi_0 - \phi_2) \left(1 - e^{-\frac{t}{\tau_0}} \right) \quad (s)$$

where t is measured from the start of interval 3 and where ϕ_2 is the value of flux at the start of interval 3 and is given by (k) with $x = 0$

$$K\ell = \frac{\phi_2^2}{4\mu_0 wd} \quad (t)$$

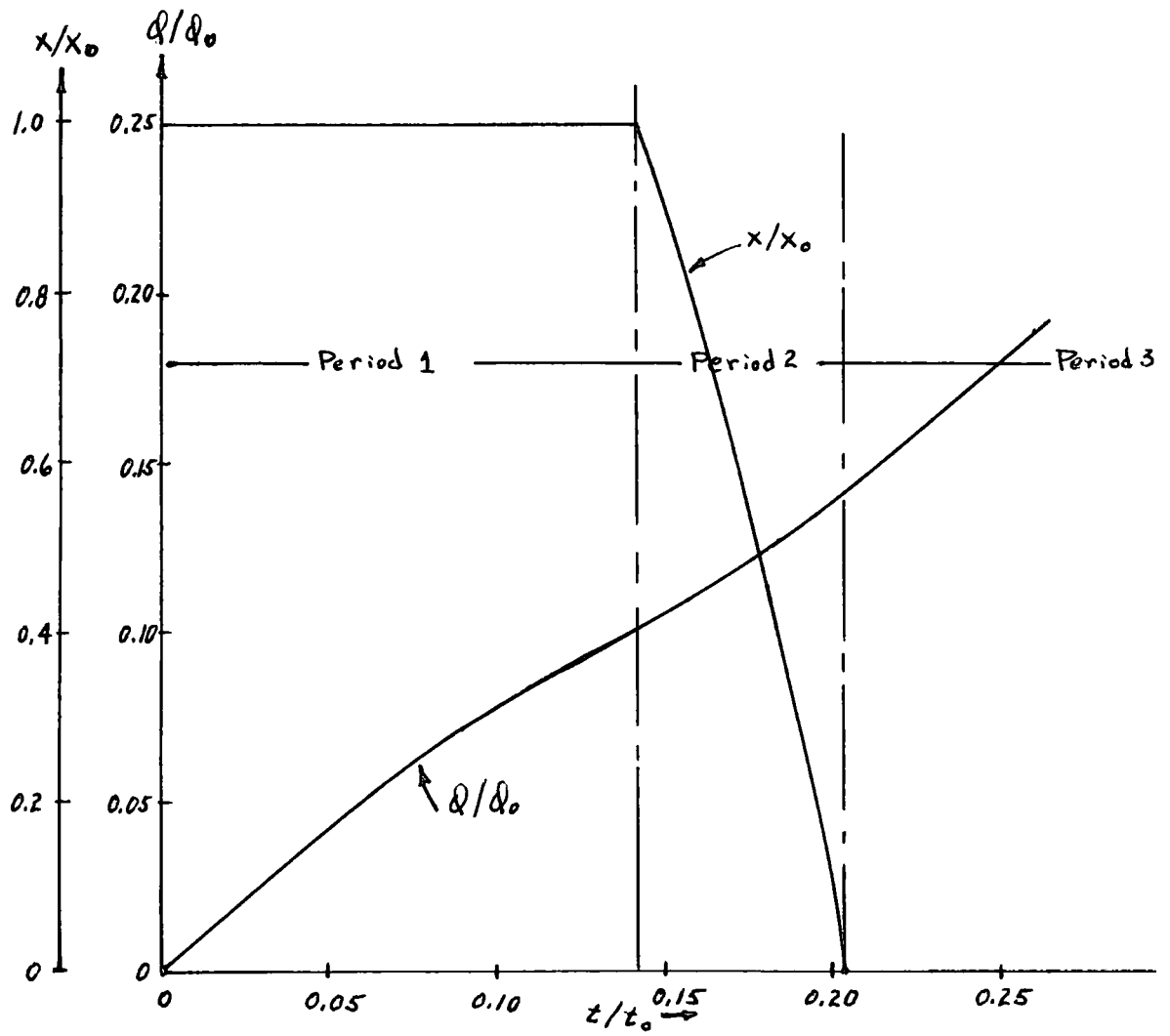
Part f

For the assumed constants in this problem

$$\frac{\phi_2}{\phi_1} = \sqrt{2} \quad (u)$$

The transients in flux and position are plotted in Fig. (a) as functions of time. Note that the mechanical transient occupies only a fraction of the time interval of the electrical transient. Thus, this example represents a case in which the electrical time constant is purposely made longer than the mechanical transient time.

LUMPED-PARAMETER ELECTROMECHANICAL DYNAMICS



PROBLEM 6.1
Part a

From Fig. 6P.1 we see the geometric relations

$$r' = r, \theta' = \theta - \Omega t, z' = z, t' = t \quad (a)$$

There is also a set of back transformations

$$r = r', \theta = \theta' + \Omega t', z = z', t = t' \quad (b)$$

Part b

Using the chain rule for partial derivatives

$$\frac{\partial \psi}{\partial t'} = \left(\frac{\partial \psi}{\partial r} \right) \left(\frac{\partial r}{\partial t'} \right) + \left(\frac{\partial \psi}{\partial \theta} \right) \left(\frac{\partial \theta}{\partial t'} \right) + \left(\frac{\partial \psi}{\partial z} \right) \left(\frac{\partial z}{\partial t'} \right) + \left(\frac{\partial \psi}{\partial t} \right) \left(\frac{\partial t}{\partial t'} \right) \quad (c)$$

From (b) we learn that

$$\frac{\partial r}{\partial t'} = 0, \frac{\partial \theta}{\partial t'} = \Omega, \frac{\partial z}{\partial t'} = 0, \frac{\partial t}{\partial t'} = 1 \quad (d)$$

Hence,

$$\frac{\partial \psi}{\partial t'} = \frac{\partial \psi}{\partial t} + \Omega \frac{\partial \psi}{\partial \theta} \quad (e)$$

We note that the remaining partial derivatives of ψ are

$$\frac{\partial \psi}{\partial r'} = \frac{\partial \psi}{\partial r}, \frac{\partial \psi}{\partial \theta'} = \frac{\partial \psi}{\partial \theta}, \frac{\partial \psi}{\partial z'} = \frac{\partial \psi}{\partial z} \quad (f)$$

PROBLEM 6.2
Part a

The geometric transformation laws between the two inertial systems are

$$x'_1 = x_1 - vt, x'_2 = x_2, x'_3 = x_3, t' = t \quad (a)$$

The inverse transformation laws are

$$x_1 = x'_1 + vt', x_2 = x'_2, x_3 = x'_3, t = t' \quad (b)$$

The transformation of the magnetic field when there is no electric field present in the laboratory frame is

$$\bar{B}' = \bar{B} \quad (c)$$

Hence the time rate of change of the magnetic field seen by the moving observer is

$$\frac{\partial B'}{\partial t'} = \frac{\partial B}{\partial t} = \left(\frac{\partial B}{\partial x_1} \right) \left(\frac{\partial x_1}{\partial t'} \right) + \left(\frac{\partial B}{\partial x_2} \right) \left(\frac{\partial x_2}{\partial t'} \right) + \left(\frac{\partial B}{\partial x_3} \right) \left(\frac{\partial x_3}{\partial t'} \right) + \left(\frac{\partial B}{\partial t} \right) \left(\frac{\partial t}{\partial t'} \right) \quad (d)$$

PROBLEM 6.2 (Continued)

From (b) we learn that

$$\frac{\partial x_1}{\partial t'} = v, \frac{\partial x_2}{\partial t'} = 0, \frac{\partial x_3}{\partial t'} = 0, \frac{\partial t}{\partial t'} = 1 \quad (e)$$

While from the given field we learn that

$$\frac{\partial B}{\partial x_1} = kB_0 \cos kx_1, \frac{\partial B}{\partial x_2} = \frac{\partial B}{\partial x_3} = \frac{\partial B}{\partial t} = 0 \quad (f)$$

Combining these results

$$\frac{\partial B'}{\partial t'} = \frac{\partial B}{\partial t'} = v \frac{\partial B}{\partial x_1} = v k B_0 \cos kx_1 \quad (g)$$

which is just the convective derivative of B.

Part b

Now (b) becomes

$$x_1 = x'_1, x_2 = x'_2 + vt', x_3 = x'_3, t = t' \quad (h)$$

When these equations are used with (d) we learn that

$$\frac{\partial B'}{\partial t'} = \frac{\partial B}{\partial t'} = v \frac{\partial B}{\partial x_2} + \frac{\partial B}{\partial t} = 0 \quad (i)$$

because both $\frac{\partial B}{\partial x_2}$ and $\frac{\partial B}{\partial t}$ are naught. The convective derivative is zero.

PROBLEM 6.3

Part a

The quasistatic magnetic field transformation is

$$\vec{B}' = \vec{B} \quad (a)$$

The geometric transformation laws are

$$x = x' + vt', y = y', z = z', t = t' \quad (b)$$

This means that

$$\begin{aligned} \vec{B}' &= \vec{B}(t, x) = \vec{B}(t', x' + vt') = \vec{i}_y B_0 \cos(\omega t' - k(x' + vt')) \\ &= \vec{i}_y B_0 \cos[(\omega - kv)t' - kx'] \end{aligned} \quad (c)$$

From (c) it is possible to conclude that

$$\omega' = \omega - kv \quad (d)$$

Part b

If $\omega' = 0$ the wave will appear stationary in time, although it will still have a spacial distribution; it will not appear to move.

PROBLEM 6.3 (Continued)

$$\omega' = 0 = \omega - kV; V = \omega/k = v_p \quad (e)$$

The observer must move at the phase velocity v_p to make the wave appear stationary.

PROBLEM 6.4

These three laws were determined in an inertial frame of reference, and since there is no a priori reason to prefer one inertial frame more than another, they should have the same form in the primed inertial frame.

We start with the geometric laws which relate the coordinates of the two frames

$$\bar{\mathbf{r}}' = \bar{\mathbf{r}} - \bar{\mathbf{v}}_r t, \quad t = t', \quad \bar{\mathbf{r}} = \bar{\mathbf{r}}' + \bar{\mathbf{v}}_r t' \quad (a)$$

We recall from Chapter 6 that as a consequence of (a) and the definitions of the operators

$$\nabla = \nabla', \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \bar{\mathbf{v}}_r \cdot \nabla', \quad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \bar{\mathbf{v}}_r \cdot \nabla \quad (b)$$

In an inertial frame of reference moving with the velocity $\bar{\mathbf{v}}_r$ we expect the equation to take the same form as in the fixed frame. Thus,

$$\rho' \frac{\partial \bar{\mathbf{v}}'}{\partial t'} + \rho' (\bar{\mathbf{v}}' \cdot \nabla') \bar{\mathbf{v}}' + \nabla' p' = 0 \quad (c)$$

$$\frac{\partial \rho'}{\partial t'} + \nabla' \cdot \rho' \bar{\mathbf{v}}' = 0 \quad (d)$$

$$p' = p'(\rho') \quad (e)$$

However, from (b) these become

$$\rho' \frac{\partial \bar{\mathbf{v}}'}{\partial t} + \rho' (\bar{\mathbf{v}}' + \bar{\mathbf{v}}_r) \cdot \nabla (\bar{\mathbf{v}}' + \bar{\mathbf{v}}_r) + \nabla p' = 0 \quad (f)$$

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot \rho' (\bar{\mathbf{v}}' + \bar{\mathbf{v}}_r) = 0 \quad (g)$$

$$p' = p'(\rho') \quad (h)$$

where we have used the fact that $\bar{\mathbf{v}}_r \cdot \nabla \rho' = \nabla \cdot (\bar{\mathbf{v}}_r \rho')$. Comparison of (1)-(3) with (f)-(h) shows that a self consistent transformation that leaves the equations invariant in form is

$$\rho' = \rho; \quad p' = p; \quad \bar{\mathbf{v}}' = \bar{\mathbf{v}} - \bar{\mathbf{v}}_r$$

FIELDS AND MOVING MEDIA

PROBLEM 6.5

Part a

$$\rho'(\bar{r}', t') = \rho(\bar{r}, t) = \rho_o(1 - \frac{r}{a}) = \rho_o(1 - \frac{r'}{a}) \quad (a)$$

$$\bar{J}' = \rho' \bar{v}' = 0 \quad (b)$$

Where we have chosen $\bar{v}_r = v_o \bar{i}_z$ so that

$$\bar{v}' = \bar{v} - \bar{v}_r = 0 \quad (c)$$

Since there are no currents, there is only an electric field in the primed frame

$$\bar{E}' = (\rho_o/\epsilon_o)(\frac{r'}{2} - \frac{r'^2}{3a})\bar{i}_r \quad (d)$$

$$\bar{H}' = 0, \bar{B}' = \mu_o \bar{H}' = 0 \quad (e)$$

Part b

$$\rho(r, t) = \rho_o(1 - \frac{r}{a}) \quad (f)$$

This charge distribution generates an electric field

$$\bar{E} = (\rho_o/\epsilon_o)(\frac{r}{2} - \frac{r^2}{3a})\bar{i}_r \quad (g)$$

In the stationary frame there is an electric current

$$\bar{J} = \rho \bar{v} = \rho_o(1 - \frac{r}{a})v_o \bar{i}_z \quad (h)$$

This current generates a magnetic field

$$\bar{H} = \rho_o v_o (\frac{r}{2} - \frac{r^2}{3a})\bar{i}_\theta \quad (i)$$

Part c

$$\bar{J} = \bar{J}' - \rho' \bar{v}_r = \rho_o(1 - \frac{r'}{a})v_o \bar{i}_z \quad (j)$$

$$\bar{E} = \bar{E}' - \bar{v}_r \times \bar{B}' = \bar{E}' = (\rho_o/\epsilon_o)(\frac{r'}{2} - \frac{r'^2}{3a})\bar{i}_r \quad (k)$$

$$\bar{H} = \bar{H}' + \bar{v}_r \times \bar{D}' = v_o \rho_o (\frac{r'}{2} - \frac{r'^2}{3a})\bar{i}_\theta \quad (l)$$

If we include the geometric transformation $r' = r, (j), (k),$ and (l) become $(h), (g),$ and (i) of part (b) which we derived without using transformation laws. The above equations apply for $r < a$. Similar reasoning gives the fields in each frame for $r > a$.

PROBLEM 6.6

Part a

In the frame rotating with the cylinder

$$\vec{E}'(r') = \frac{K}{r'}, \vec{i}_r \quad (a)$$

$$\vec{H}' = 0, \vec{B}' = \mu_0 \vec{H}' = 0 \quad (b)$$

But then since $r' = r$, $\vec{v}_r(r) = r\omega \vec{i}_\theta$

$$\vec{E} = \vec{E}' - \vec{v}_r \times \vec{B}' = \vec{E}' = \frac{K}{r} \vec{i}_r \quad (c)$$

$$V = \int_a^b \vec{E} \cdot d\vec{l} = \int_a^b \frac{K}{r} dr = K \ln(b/a) \quad (d)$$

$$\vec{E} = \frac{V}{\ln(b/a)} \frac{1}{r} \vec{i}_r = \vec{E}' = \frac{V}{\ln(b/a)} \frac{1}{r'} \vec{i}_r \quad (e)$$

The surface charge density is then

$$\sigma'_a = \vec{i}_r \cdot \epsilon_0 \vec{E}' = \frac{\epsilon_0 V}{\ln(b/a)} \frac{1}{a} = \sigma_a \quad (f)$$

$$\sigma'_b = -\vec{i}_r \cdot \epsilon_0 \vec{E}' = -\frac{\epsilon_0 V}{\ln(b/a)} \frac{1}{b} = \sigma_b \quad (g)$$

Part b

$$\vec{J} = \vec{J}' + \vec{v}_r \rho' \quad (h)$$

But in this problem we have only surface currents and charges

$$\vec{K} = \vec{K}' + \vec{v}_r \sigma' = \vec{v}_r \sigma' \quad (i)$$

$$\vec{K}(a) = \frac{a\omega\epsilon_0 V}{a \ln(b/a)} \vec{i}_\theta = \frac{\omega\epsilon_0 V}{\ln(b/a)} \vec{i}_\theta \quad (j)$$

$$\vec{K}(b) = -\frac{b\omega\epsilon_0 V}{b \ln(b/a)} \vec{i}_\theta = -\frac{\omega\epsilon_0 V}{\ln(b/a)} \vec{i}_\theta \quad (k)$$

Part c

$$\vec{H} = -\frac{\omega\epsilon_0 V}{\ln(b/a)} \vec{i}_z \quad (l)$$

Part d

$$\vec{H} = \vec{H}' + \vec{v}_r \times \vec{D}' = \vec{v}_r \times \vec{D}' \quad (m)$$

$$\vec{H} = r'\omega \left(\frac{\epsilon_0 V}{\ln(b/a)} \frac{1}{r'} \right) (\vec{i}_\theta \times \vec{i}_r) \quad (n)$$

PROBLEM 6.6 (Continued)

$$\vec{H} = - \frac{\omega \epsilon_0 V}{\ln(b/a)} \vec{i}_z \quad (o)$$

This result checks with the calculation of part (c).

PROBLEM 6.7
Part a

The equation of the top surface is

$$f(x,y,t) = y - a \sin(\omega t) \cos(kx) + d = 0 \quad (a)$$

The normal to this surface is then

$$\vec{n} = \frac{\nabla f}{|\nabla f|} \approx ak \sin(\omega t) \sin(kx) \vec{i}_x + \vec{i}_y \quad (b)$$

Applying the boundary condition $\vec{n} \cdot \vec{B} = 0$ at each surface and keeping only linear terms, we learn that

$$h_y(x,d,t) = -ak \sin(\omega t) \sin(kx) \frac{\Lambda}{\mu_0 d} \quad (c)$$

$$h_y(x,0,t) = 0 \quad (d)$$

We look for a solution for $\vec{h}$ that satisfies

$$\nabla \times \vec{h} = 0, \quad \nabla \cdot \vec{h} = 0 \quad (e)$$

$$\text{Let } \vec{h} = \nabla \psi, \quad \nabla^2 \psi = 0 \quad (f)$$

Now we must make an intelligent guess for a Laplacian ψ using the periodicity of the problem and the boundary condition $h_y = \partial \psi / \partial y = 0$ at $y = 0$. Try

$$\psi = \frac{\Lambda}{k} \cosh(ky) \sin(kx) \sin(\omega t) \quad (g)$$

$$\vec{h} = A \sin(\omega t) [\cos(kx) \cosh(ky) \vec{i}_x + \sin(kx) \sinh(ky) \vec{i}_y] \quad (h)$$

Equation (c) then requires the constant A to be

$$A = \frac{-ak \Lambda}{\sinh(kd) \mu_0 d} \quad (i)$$

Part b

$$\nabla \times \vec{E} = \vec{i}_x \left(\frac{\partial E}{\partial y} \right) - \vec{i}_y \left(\frac{\partial E}{\partial x} \right) = - \frac{\partial \vec{B}}{\partial t} \quad (j)$$

$$\frac{\partial \vec{B}}{\partial t} = \omega \mu_0 A \cos(\omega t) [\cos(kx) \cosh(ky) \vec{i}_x + \sin(kx) \sinh(ky) \vec{i}_y] \quad (k)$$

PROBLEM 6.7 (Continued)

$$\vec{E} = -\omega\mu_0 \frac{A}{k} \cos(\omega t) [\cos(kx) \sinh(ky)] \vec{i}_z \quad (1)$$

Now we check the boundary conditions. Because $\vec{v}(y=0) = 0$

$$\vec{n} \times \vec{E} = (\vec{n} \cdot \vec{v}) \vec{B} = 0 \quad (y=0) \quad (m)$$

But $\vec{E}(y=0) = 0$, so (m) is satisfied.

If a particle is on the top surface, its coordinates x, y, t must satisfy (a). It follows that

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f = 0 \quad (n)$$

Since $\vec{n} = \frac{\nabla f}{|\nabla f|}$ we have that

$$(\vec{n} \cdot \vec{v}) = \frac{-1}{|\nabla f|} \frac{\partial f}{\partial t} \sim \omega \cos(\omega t) \cos(kx) \quad (o)$$

Now we can check the boundary condition at the top surface

$$\vec{n} \times \vec{E} = -\omega\mu_0 \frac{A}{k} \cos(\omega t) \cos(kx) \sinh(kd) [\vec{i}_x - ak \sin(\omega t) \sin(kx) \vec{i}_y] \quad (p)$$

$$(\vec{n} \cdot \vec{v}) \vec{B} = \omega \cos(\omega t) \cos(kx) \left[\frac{-\sinh(kd)}{ak} \mu_0 A \vec{i}_x + \right. \quad (q)$$

$$\left. \mu_0 A \sin(\omega t) \sin(kx) \sinh(kd) \vec{i}_y \right]$$

Comparing (p) and (q) we see that the boundary condition is satisfied at the top surface.

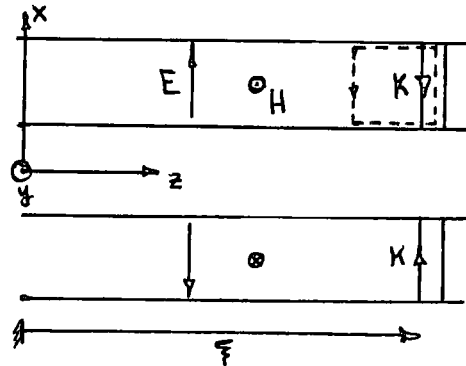
PROBLEM 6.8

Part a

Since the plug is perfectly conducting we expect that the current I will return as a surface current on the left side of the plug. Also E' , H' will be zero in the plug and the transformation laws imply that E, H will then also be zero.

Using ampere's law

$$\vec{H} = \begin{cases} \frac{-I}{2\pi r} \vec{i}_\theta & 0 < z < \xi \\ 0 & \xi < z \end{cases} \quad (a)$$



PROBLEM 6.8 (Continued)

Also we know that

$$\nabla \cdot \bar{\mathbf{E}} = 0, \quad \nabla \times \bar{\mathbf{E}} = - \frac{\partial \bar{\mathbf{B}}}{\partial t} = 0 \quad 0 < z < \xi \quad (b)$$

We choose a simple Laplacian $\bar{\mathbf{E}}$ field consistent with the perfectly conducting boundary conditions

$$\bar{\mathbf{E}} = \frac{K}{r} \hat{\mathbf{i}}_r \quad (c)$$

K can be evaluated from

$$\oint_C \bar{\mathbf{E}}'' \cdot d\bar{\mathbf{l}} = - \frac{d}{dt} \int_S \bar{\mathbf{B}} \cdot d\mathbf{a} \quad (d)$$

If we use the deforming contour shown above which has a fixed left leg at $z = z$ and a moving right leg in the conductor. The notation $\bar{\mathbf{E}}''$ means the electric field measured in a frame of reference which is stationary with respect to the local element of the deforming contour. Here

$$\bar{\mathbf{E}}''(z) = \bar{\mathbf{E}}(z), \quad \bar{\mathbf{E}}''(\xi + \Delta) = \bar{\mathbf{E}}'(\xi + \Delta) = 0 \quad (e)$$

$$\oint_C \bar{\mathbf{E}}'' \cdot d\bar{\mathbf{l}} = - \int_a^b E(z, r) dr = -K \ln(b/a) \quad (f)$$

The contour contains a flux

$$\int_S \bar{\mathbf{B}} \cdot d\mathbf{a} = (\xi - z) \int_a^b \mu_0 H_\theta dr = - \mu_0 \frac{I}{2\pi} \ln(b/a) (\xi - z) \quad (g)$$

So that

$$-K \ln(b/a) = - \frac{d}{dt} \int_S \bar{\mathbf{B}} \cdot d\mathbf{a} = + \mu_0 \frac{I}{2\pi} \ln(b/a) \frac{d\xi}{dt} \quad (h)$$

Since $v = \frac{d\xi}{dt}$,

$$\bar{\mathbf{E}} = \begin{cases} - \frac{v\mu_0 I}{2\pi} \frac{1}{r} \hat{\mathbf{i}}_r & 0 < z < \xi \\ 0 & \xi < z \end{cases} \quad (j)$$

Part b

The voltage across the line at $z = 0$ is

$$V = - \int_a^b E_r dr = \frac{v\mu_0 I}{2\pi} \ln(b/a) \quad (k)$$

FIELDS AND MOVING MEDIA

PROBLEM 6.8 (Continued)

$$I(R + \frac{v\mu_o}{2\pi} \ln(b/a)) = V_o \quad (1)$$

$$I = \frac{V_o}{R + \frac{v\mu_o}{2\pi} \ln(b/a)} \quad (m)$$

$$V = \left[\frac{1}{\frac{2\pi R}{v\mu_o \ln(b/a)} + 1} \right] V_o \quad (n)$$

$$\vec{H} = \begin{cases} \frac{-V_o}{R + \frac{v\mu_o}{2\pi} \ln(b/a)} \frac{1}{2\pi r} \vec{i}_\theta & 0 < z < \xi \\ 0 & \xi < z \end{cases} \quad (o)$$

$$\vec{E} = \begin{cases} - \left[\frac{1}{\frac{2\pi R}{v\mu_o} + \ln(b/a)} \right] \frac{V_o}{r} \vec{i}_r & 0 < z < \xi \\ 0 & \xi < z \end{cases} \quad (p)$$

Part c

Since $\vec{E} = 0$ to the right of the plug the voltmeter reads zero. The terminal voltage V is not zero because of the net change of magnetic flux in the loop connecting these two voltage points.

Part d

Using the results of part (b)

$$P_{in} = VI = \frac{v\mu_o \ln(b/a)}{2\pi} \left[\frac{1}{R + \frac{v\mu_o}{2\pi} \ln(b/a)} \right]^2 V_o^2$$

$$\begin{aligned} \frac{dW_m}{dt} &= v \int_a^b \frac{\mu_o}{2} H^2(r) 2\pi r dr \\ &= \frac{1}{2} \left[\frac{v\mu_o \ln(b/a)}{2\pi} \left(\frac{1}{R + \frac{v\mu_o}{2\pi} \ln(b/a)} \right)^2 V_o^2 \right] \end{aligned}$$

PROBLEM 6.8 (Continued)

There is a net electrical force on the block, the mechanical system that keeps the block traveling at constant velocity receives power at the rate

$$\frac{1}{2} \frac{\mu_o \ln(b/a)}{2\pi} \left[\frac{1}{R + \frac{\mu_o \ln(b/a)}{2\pi}} \right]^2 v_o^2$$

from the electrical system.

Part e

$$L(x) = \int \frac{\mu_o H(r, I) x \, dr}{I} = \frac{\mu_o}{2\pi} \ln(b/a) x$$

$$f^e = \frac{\partial W'}{\partial x} ; W'_m = \frac{1}{2} L(x) i^2$$

$$f^e = \frac{1}{2} \frac{\partial L}{\partial x} i^2 = \frac{1}{2} \frac{\mu_o}{2\pi} \ln(b/a) i^2$$

The power converted from electrical to mechanical is then

$$\bar{f}_e \frac{dx}{dt} = f_e v = \frac{1}{2} \frac{\mu_o v}{2\pi} \ln(b/a) \left[\frac{v_o}{R + \frac{\mu_o}{2\pi} \ln(b/a)} \right]^2$$

as predicted in Part (d).

PROBLEM 6.9

The surface current circulating in the system must remain

$$K = \frac{B_o}{\mu_o} \quad (a)$$

Hence the electric field in the finitely conducting plate is

$$E' = \frac{B_o}{\mu_o \sigma_s} \quad (b)$$

But then

$$E = E' - \bar{v} \times \bar{B} \quad (c)$$

$$= B_o \left(\frac{1}{\mu_o \sigma_s} - v \right)$$

v must be chosen so that $E = 0$ to comply with the shorted end, hence

$$v = \frac{1}{\mu_o \sigma_s} \quad (d)$$

FIELDS AND MOVING MEDIA

PROBLEM 6.10

Part a

Ignoring the effect of the induced field we must conclude that

$$\vec{E} = 0 \quad (a)$$

everywhere in the stationary frame. But then

$$\vec{E}' = \vec{E} + \vec{V} \times \vec{B} = \vec{V} \times \vec{B} \quad (b)$$

Since the plate is conducting

$$\vec{J}' = \vec{J} = \sigma \vec{V} \times \vec{B} \quad (c)$$

The force on the plate is then

$$\vec{F} = \int \vec{J} \times \vec{B} \, dv = DWd(\sigma \vec{V} \times \vec{B}) \times \vec{B} \quad (d)$$

$$\vec{F}_x = -DWd\sigma v B_o^2 \quad (e)$$

Part b

$$M \frac{dv}{dt} + (DWd\sigma B_o^2)v = 0 \quad (f)$$

$$v = v_o e^{-\frac{DWd\sigma B_o^2}{M} t} \quad (g)$$

Part c

The additional induced field must be small. From (e)

$$\vec{J}' \approx \sigma B_o v_o \quad (h)$$

$$\text{Hence } \vec{K}' \approx \sigma B_o dv_o \quad (i)$$

The induced field then has a magnitude

$$\frac{B'}{B_o} \approx \frac{\mu_o K'}{B_o} = \mu_o \sigma dv_o \ll 1 \quad (j)$$

$$\sigma d \ll \frac{1}{\mu_o v_o} \quad (k)$$

It must be a very thin plate or a poorly conducting one.

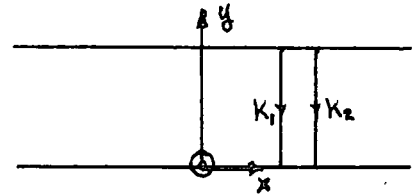
FIELDS AND MOVING MEDIA

PROBLEM 6.11

Part a

The condition $\frac{1}{W} \ll H_0$ means that the field induced by the current can be ignored. Then the magnetic field in the stationary frame is

$$\vec{H} = -H_0 \vec{i}_z \quad \text{everywhere outside the perfect conductors} \quad (a)$$



The surface currents on the sliding conductor are such that

$$K_1 + K_2 = I/W \quad (b)$$

The force on the conductor is then

$$\begin{aligned} \vec{F} &= \int \vec{J} \times \vec{B} \, dv = [(K_1 + K_2) \vec{i}_y \times B_0 \vec{i}_z] W d \\ &= \mu_0 H_0 d I \vec{i}_x \end{aligned} \quad (c)$$

Part b

The circuit equation is

$$RI + \frac{d\lambda}{dt} = V_0 \quad (d)$$

$$\frac{d\lambda}{dt} \approx \mu_0 H_0 d v \quad (e)$$

$$\text{Since } F = M \frac{dv}{dt} \quad (f)$$

$$\left(\frac{MR}{\mu_0 H_0 d} \right) \frac{dv}{dt} + (\mu_0 H_0 d) v = V_0 \quad (g)$$

$$v = \frac{V_0}{\mu_0 H_0 d} \left(1 - e^{-\frac{(\mu_0 H_0 d)^2}{MR} t} \right) u_{-1}(t) \quad (h)$$

PROBLEM 6.12

Part a

We assume the simple magnetic field

$$\vec{H} = \begin{cases} -\frac{1}{D} \vec{i}_3 & 0 < x_1 < x \\ 0 & x < x_1 \end{cases} \quad (a)$$

$$\lambda(x) = \int \vec{B} \cdot d\vec{a} = \frac{\mu_0 W x}{D} \quad (b)$$

Part b

$$L(x) = \frac{\lambda(x, I)}{I} = \frac{\mu_0 W x}{D} \quad (c)$$

FIELDS AND MOVING MEDIA

PROBLEM 6.12 (Continued)

Since the system is linear

$$W'_m(i, x) = \frac{1}{2} L(x) i^2 = \frac{1}{2} \frac{\mu_o W x}{D} i^2 \quad (d)$$

Part c

$$f^e = \frac{\partial W'_m}{\partial x} = \frac{1}{2} \frac{\mu_o W}{D} i^2 \quad (e)$$

Part d

The mechanical equation is

$$M \frac{dx^2}{dt^2} + B \frac{dx}{dt} = \frac{1}{2} \frac{\mu_o W}{D} i^2 \quad (f)$$

The electrical circuit, equation is

$$\frac{d\lambda}{dt} = \frac{d}{dt} \left(-\frac{\mu_o W x}{D} i \right) = V_o \quad (g)$$

Part e

From (f) we learn that

$$\frac{dx}{dt} = \frac{\mu_o W}{2BD} i^2 = \text{const} \quad (h)$$

while from (g) we learn that

$$\frac{\mu_o W i}{D} \frac{dx}{dt} = V_o \quad (i)$$

Solving these two simultaneously

$$\frac{dx}{dt} = \left[\frac{DV_o^2}{2\mu_o WB} \right]^{1/3} \quad (j)$$

Part f

From (e)

$$i = \sqrt{\frac{2BD}{\mu_o W} \frac{dx}{dt}} = \left(\frac{D}{\mu_o W} \right)^{2/3} (2B)^{1/3} V_o^{1/3} \quad (k)$$

Part g

As in part (a)

$$\vec{H} = \begin{cases} -\frac{i(t)}{D} \vec{i}_3 & 0 < x_1 < x \\ 0 & x < x_1 \end{cases} \quad (l)$$

Part h

The surface current $\vec{K}$ is

PROBLEM 6.12 (Continued)

$$\bar{\mathbf{K}} = -\frac{i(t)}{D} \vec{i}_2 \quad (m)$$

The force on the short is

$$\begin{aligned} \bar{\mathbf{F}} &= \int \bar{\mathbf{J}} \times \bar{\mathbf{B}} \, dv = DW \bar{\mathbf{K}} \times \left(\frac{\mu_o \bar{\mathbf{H}}_1 + \mu_o \bar{\mathbf{H}}_2}{2} \right) \\ &= \frac{\mu_o W}{2D} i^2(t) \vec{i}_1 \end{aligned} \quad (n)$$

Part i

$$\nabla \times \bar{\mathbf{E}} = \frac{\partial \bar{\mathbf{E}}_2}{\partial x_1} \vec{i}_3 - \frac{\partial \bar{\mathbf{B}}}{\partial t} = \frac{\mu_o}{D} \frac{di}{dt} \vec{i}_3 \quad (o)$$

$$\begin{aligned} \bar{\mathbf{E}}_2 &= \left[\frac{\mu_o}{D} \times \frac{di}{dt} + C \right] \vec{i}_3 \\ &= \left[\frac{\mu_o}{D} \times \frac{di}{dt} - \frac{V(t)}{W} \right] \vec{i}_3 \end{aligned} \quad (p)$$

Part j

Choosing a contour with the right leg in the moving short, the left leg fixed at $x_1 = 0$

$$\oint_C \bar{\mathbf{E}}' \cdot d\vec{\ell} = -\frac{d}{dt} \int_S \bar{\mathbf{B}} \cdot d\vec{a} \quad (q)$$

Since $\mathbf{E}' = 0$ in the short and we are only considering quasistatic fields

$$\oint_C \bar{\mathbf{E}}' \cdot d\vec{\ell} = V(t) = W \times \mu_o \frac{\partial H_o}{\partial t} + W \frac{dx}{dt} \mu_o H_o \quad (r)$$

$$= \frac{d}{dt} \left(\frac{\mu_o W x}{D} i(t) \right) \quad (s)$$

Part k

$$\bar{\mathbf{n}} \times (\bar{\mathbf{E}}^b) = v_n \bar{\mathbf{B}}^b \quad (t)$$

Here

$$\bar{\mathbf{n}} = \vec{i}_1, \quad v_n = \frac{dx}{dt}, \quad \bar{\mathbf{B}}^b = -\frac{\mu_o i}{D} \vec{i}_3 \quad (u)$$

$$\bar{\mathbf{E}}_b = \left(\frac{\mu_o x}{D} \frac{di}{dt} - \frac{V(t)}{W} \right) \vec{i}_2 = \left(-\frac{dx}{dt} \frac{\mu_o W}{D} i \right) \vec{i}_2 \quad (v)$$

$$-\frac{dx}{dt} \frac{\mu_o W}{D} i = \left(\frac{dx}{dt} \right) \left(-\frac{\mu_o i}{D} \right) \quad (w)$$

Part l

Equations (n) and (e) are identical. Equations (s) and (g) are identical if $V(t) = V_o$. Since we used (e) and (g) to solve the first part we would get the same answer using (n) and (s) in the second part.

FIELDS AND MOVING MEDIA

PROBLEM 6.12 (Continued)

Part m

Since $\frac{di}{dt} = 0$,

$$\bar{E}_2(x) = -\frac{V(t)}{w} \vec{i}_y = -\frac{V_0}{W} \vec{i}_y \quad (x)$$

PROBLEM 6.13

Part a

$$K \frac{d^2\psi}{dt^2} = T_1^e(\psi) + T_2(\psi) \quad (a)$$

Part b

$$J_1 = \frac{i_1 \vec{i}_r}{D2\alpha r}; \quad \bar{F}_1 = \bar{J}_1 \times \bar{B} = -\frac{\mu_o H_o i_1}{D2\alpha R} \vec{i}_\theta \quad (b)$$

Similarly

$$\bar{F}_2 = -\frac{\mu_o H_o i_2}{D2\alpha R} \vec{i}_\theta \quad (c)$$

Part c

$$T_1^e = \left[\int (\vec{r} \times \vec{f}) dv \right]_z = -\mu_o H_o (R_2 - R_1) i_1 \quad (d)$$

$$T_2^e = -\mu_o H_o (R_2 - R_1) i_2 \quad (e)$$

Part d

$$v_1 = E_1(R_2 - R_1); \quad v_2 = E_2(R_2 - R_1) \quad (f)$$

Part e

$$J_1 = J_1' = \sigma E_1' = \sigma(\bar{E}_1 + \bar{V} \times \bar{B}) = \sigma(E_1 + R\mu_o H_o \frac{d\psi}{dt}) \quad (g)$$

$$E_1 = \frac{1}{\sigma} \frac{i_1}{2\alpha DR} - R\mu_o H_o \frac{d\psi}{dt} \quad (h)$$

$$v_1 = \frac{1}{\sigma} \frac{(R_2 - R_1)}{2\alpha RD} i_1 - \mu_o H_o R(R_2 - R_1) \frac{d\psi}{dt} \quad (i)$$

$$v_2 = \frac{1}{\sigma} \frac{R_2 - R_1}{2\alpha RD} i_2 - \mu_o H_o R(R_2 - R_1) \frac{d\psi}{dt} \quad (j)$$

Part f

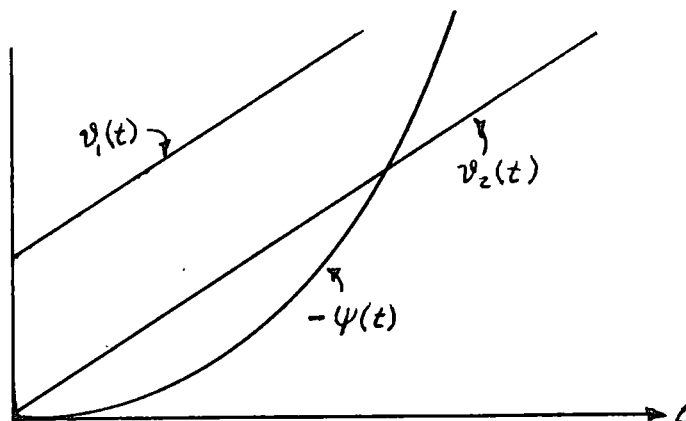
$$K \frac{d^2\psi}{dt^2} = -\mu_o H_o (R_2 - R_1) i_0 u_{-1}(t) \quad (k)$$

$$\psi(t) = -\frac{\mu_o H_o}{2K} (R_2 - R_1) i_0 t^2 u_{-1}(t) \quad (l)$$

$$v_2(t) = (\mu_o H_o (R_2 - R_1))^2 \frac{R}{K} i_0 t u_{-1}(t) \quad (m)$$

PROBLEM 6.13 (Continued)

$$v_1(t) = \left[\frac{1}{\sigma} \frac{(R_2 - R_1)}{2\alpha R D} + (\mu_o H_o (R_2 - R_1))^2 \frac{R}{K} t \right] i_o u_{-1}(t) \quad (n)$$



Part g

$$K \frac{d^2 \psi}{dt^2} = - \mu_o H_o (R_2 - R_1) i_1 \quad (o)$$

$$= - \frac{\mu_o H_o (R_2 - R_1) \sigma 2\alpha R D}{(R_2 - R_1)} \left[v_1 + \mu_o H_o R (R_2 - R_1) \frac{d\psi}{dt} \right]$$

$$\frac{d^2 \psi}{dt^2} + K_1 \frac{d\psi}{dt} = - K_2 v_1(t) \quad (p)$$

$$K_1 = [(\mu_o H_o R)^2 2\alpha D (R_2 - R_1) \sigma] / K \quad (q)$$

$$K_2 = \frac{\mu_o H_o 2\alpha D R \sigma}{K}$$

Find the particular solution

$$\psi_p(\omega, t) = \text{Re} \left[\frac{-j K_2 v_o}{\omega^2 - K_1 j \omega} e^{j \omega t} \right] \quad (r)$$

$$= \frac{K_2 v_o}{\omega \sqrt{K_1^2 + \omega^2}} \sin(\omega t + \tan^{-1} \frac{K_1}{\omega}) u_{-1}(t) \quad (s)$$

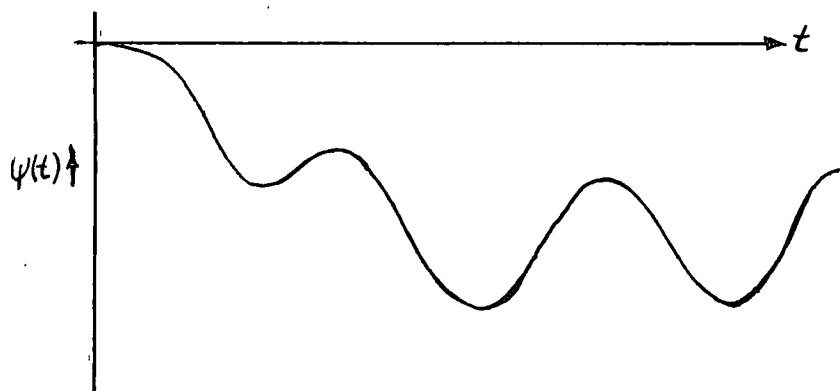
$$\psi(t) = A + \frac{B}{K_1} e^{-K_1 t} + \psi_p(\omega, t) \quad (t)$$

We must choose A and B so that

$$\psi(0) = 0; \quad \frac{d\psi}{dt}(0) = 0 \quad (u)$$

$$A = -\frac{K_2}{K_1 \omega} v_o \quad B = + \frac{K_2 \omega}{(K_1^2 + \omega^2)} v_o \quad (v)$$

PROBLEM 6.13(Continued)



Part h

The secondary terminals are constrained so that $v_2 = -i_2 R_2$. Thus, (j) becomes

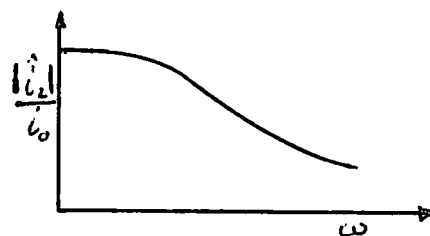
$$\frac{d\psi}{dt} = \frac{R_3}{RK_4} i_2; R_3 = R_o + \frac{1}{\sigma} \frac{(R_2 - R_1)}{2RD\alpha}; K_4 = \mu_o H_o (R_2 - R_1) \quad (w)$$

Then, it follows from (a), (d) and (e) that

$$\frac{di_2}{dt} + \frac{RK_4^2}{KR_3} i_2 = - \frac{K_4^2 R_1 o}{KR_3} \cos \omega t$$

from which it follows that

$$\frac{|\hat{i}_2|}{i_o} = \frac{K_4^2 R}{KR_3 \sqrt{\omega^2 + \left(\frac{RK_4^2}{KR_3}\right)^2}}$$



PROBLEM 6.14

Part a

The electric field in the moving laminations is

$$\vec{E}' = \frac{\vec{J}'}{\sigma} = \frac{\vec{J}}{\sigma} = \frac{1}{\sigma A} \vec{i}_z \quad (a)$$

The electric field in the stationary frame is

$$\vec{E} = \vec{E}' - \vec{v} \times \vec{B} = \left(\frac{1}{\sigma A} + r\omega B_y \right) \vec{i}_z \quad (b)$$

$$B_y = - \frac{\mu_o N i}{S} \quad (c)$$

$$\vec{v} = \left(\frac{2D}{\sigma A} - \frac{\mu_o 2Dr_o N}{S} \right) \vec{i} \quad (d)$$

PROBLEM 6.14 (Continued)

Now we have the V-i characteristic of the device. The device is in series with an inductance and a load resistor $R_t = R_L + R_{int}$.

$$\left[R_t + \frac{2D}{\sigma A} - \frac{\mu_o 2DrN}{S} \omega \right] i + \frac{\mu_o N^2 aD}{S} \frac{di}{dt} = 0 \quad (e)$$

Part b

Let

$$R_1 = R_t + \frac{2D}{\sigma A} - \frac{2D\mu_o rN\omega}{S}, \quad L = \frac{\mu_o N^2 aD}{S} \quad (f)$$

$$i = I_o e^{-R_1/L t} \quad (g)$$

$$P_d = i^2/R_L = \frac{I_o^2}{R_L} \left[e^{-R_1/L t} \right]^2$$

If

$$R_1 = R_t + \frac{2D}{\sigma A} - \frac{2D\mu_o rN\omega}{S} < 0 \quad (h)$$

the power delivered is unbounded as $t \rightarrow \infty$.

Part c

As the current becomes large, the electrical nonlinearity of the magnetic circuit will limit the exponential growth and determine a level of stable steady state operation (see Fig. 6.4.12).

PROBLEM 6.15

After the switch is closed, the armature circuit equation is

$$(R_L + R_a) i_L + L_a \frac{di_L}{dt} = G \dot{\theta} i_f \quad (a)$$

Since $G \dot{\theta} i_f$ is a constant and $i_L(0) = 0$ we can solve for the load current and shaft torque

$$i_L(t) = \frac{G \dot{\theta} i_f}{(R_L + R_a)} \left(1 - e^{-\frac{(R_L + R_a)}{L_a} t} \right) u_{-1}(t) \quad (b)$$

$$\begin{aligned} T^e(t) &= i_L(t) G i_f \\ &= \frac{(G i_f)^2 \dot{\theta}}{(R_L + R_a)} \left(1 - e^{-\frac{(R_L + R_a)}{L_a} t} \right) u_{-1}(t) \end{aligned} \quad (c)$$

* Note: $i_a = -i_L$

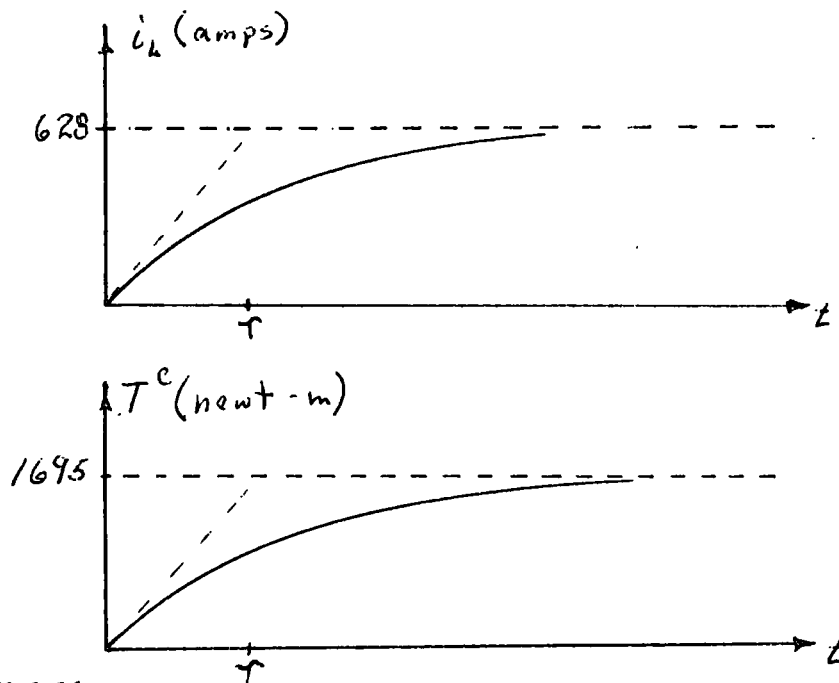
PROBLEM 6.15 (Continued)

From the data given

$$\tau = L_a / R_L + R_a \approx 2.5 \times 10^{-3} \text{ sec} \quad (d)$$

$$i_{L \max} = \frac{G \dot{\theta} i_f}{R_L + R_a} = 628 \text{ amps} \quad (e)$$

$$T_{\max} = \frac{(G i_f)^2 \dot{\theta}}{R_L + R_a} \approx 1695 \text{ newton-meters} \quad (f)$$



PROBLEM 6.16

Part a

 With S_1 closed the equation of the field circuit is

$$R_f i_f + L_f \frac{di_f}{dt} = V_f \quad (a)$$

 Since $i_f(0) = 0$

$$i_f(t) = \frac{V_f}{R_f} \left(1 - e^{-\frac{R_f}{L_f} t} \right) u_{-1}(t) \quad (b)$$

Since the armature circuit is open

$$V_a = G \dot{\theta} i_f = \frac{V_f G \dot{\theta}}{R_f} \left(1 - e^{-\frac{R_f}{L_f} t} \right) u_{-1}(t) \quad (c)$$

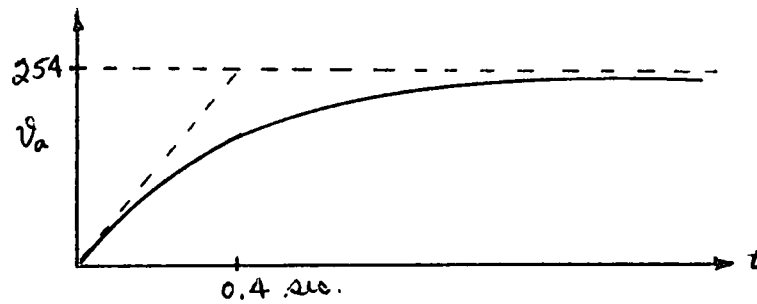
FIELDS AND MOVING MEDIA

PROBLEM 6.16 (Continued)

From the given data

$$\tau = L_f/R_f = 0.4 \text{ sec}$$

$$v_{a_{\max}} = \frac{v_f G \dot{\theta}}{R_f} = 254 \text{ volts}$$



Part b

Since there is no coupling of the armature circuit to the field circuit i_f is still given by (b).

Because S_2 is closed, the armature circuit equation is

$$(R_L + R_a)v_L + L_a \frac{dv_L}{dt} = R_L G \dot{\theta} i_f \quad (d)$$

Since the field current rises with a time constant

$$\tau = 0.4 \text{ sec} \quad (e)$$

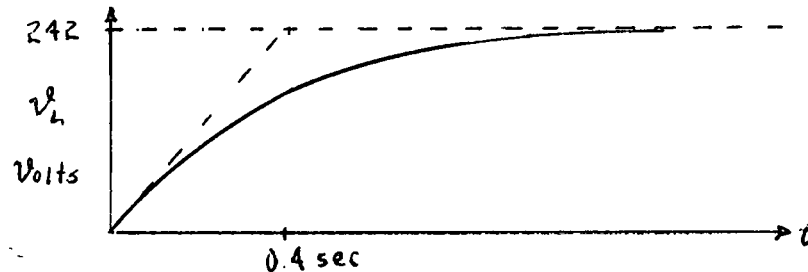
while the time constant of the armature circuit is

$$\tau = L_a / (R_L + R_a) = 0.0025 \text{ sec} \quad (f)$$

we will only need the particular solution for $v_L(t)$

$$v_L(t) = \frac{R_L G \dot{\theta}}{R_L + R_a} i_f = \left(\frac{R_L}{R_L + R_a} \right) G \dot{\theta} \frac{v_f}{R_f} \left(1 - e^{-\frac{R_f}{L_f} t} \right) u_{-1}(t) \quad (g)$$

$$v_{L_{\max}} = \left(\frac{R_L}{R_L + R_a} \right) \left(\frac{G \dot{\theta}}{R_f} \right) v_f = 242 \text{ volts} \quad (h)$$



PROBLEM 6.17

The equation of motion of the shaft is

$$J_r \frac{d\omega}{dt} + \frac{T_o}{\omega_o} \omega = T_o + T_e(t) \quad (a)$$

If $T_e(t)$ is thought of as a driving term, the response time of the mechanical circuit is

$$\tau = \frac{J_r \omega_o}{T_o} = 0.0785 \text{ sec} \quad (b)$$

In Probs. 6.15 to 6.16 we have already calculated the armature circuit time constant to be

$$\tau = \frac{L_a}{R_a + R_L} \approx 2.5 \times 10^{-3} \text{ sec} \quad (c)$$

We conclude that the rise time of the armature circuit may be neglected, this is equivalent to ignoring the armature inductance. The circuit equation for the armature is then

$$(R_a + R_L) i_L = G \omega i_f \quad (d)$$

Then

$$T_e = G i_f i_L = \frac{(G i_f)^2 \omega}{R_a + R_L} \quad (e)$$

Plugging into (a)

$$J_r \frac{d\omega}{dt} + K \omega = T_o \quad (f)$$

Here

$$K = \left(\frac{T_o}{\omega_o} + \frac{(G i_f)^2}{R_a + R_L} \right); \quad i_f = \frac{V_f}{R_f} \quad (g)$$

Using the initial condition that $\omega(0) = \omega_o$

$$\omega(t) = \frac{T_o}{K} + \left(\omega_o - \frac{T_o}{K} \right) e^{-\frac{(K/J_r)t}{\tau}} \quad t \geq 0 \quad (h)$$

From which we can calculate the net torque on the shaft as

$$T = J_r \frac{d\omega}{dt} = (T_o - K \omega_o) e^{-\frac{(K/J_r)t}{\tau}} u_{-1}(t) \quad (i)$$

and the armature current $i_L(t)$

$$i_L(t) = \left(\frac{G i_f}{R_a + R_L} \right) \omega(t) \quad t \geq 0 \quad (j)$$

FIELDS AND MOVING MEDIA

PROBLEM 6.17 (Continued)

From the given data

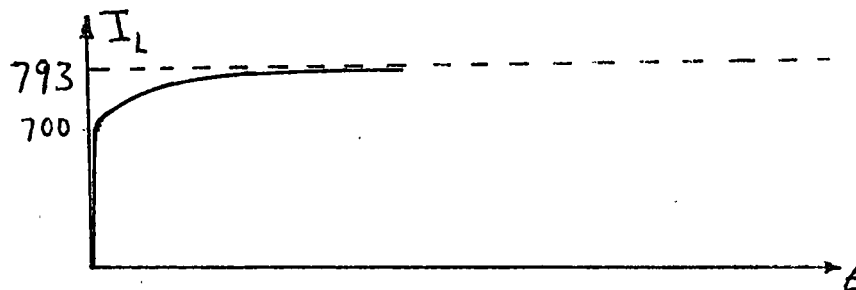
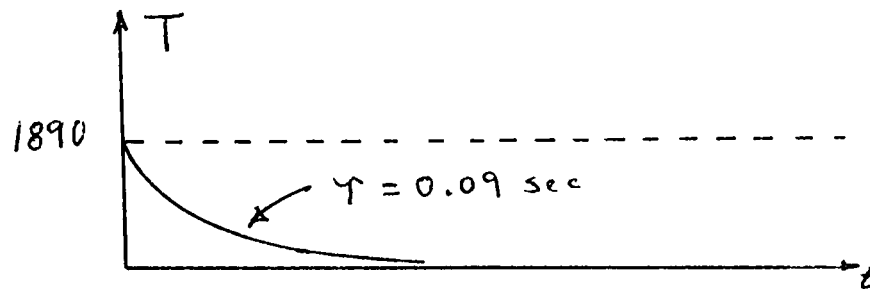
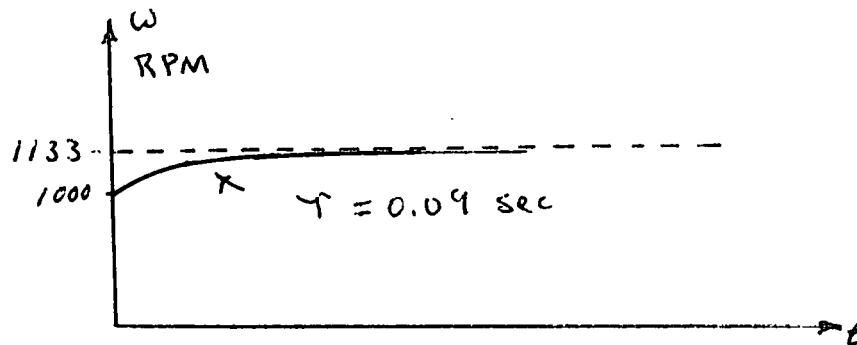
$$\omega_{\text{final}} = \frac{T_o}{K} = 119.0 \text{ rad/sec} = 1133 \text{ RPM} \quad (k)$$

$$T_{\text{max}} = (T_o - K\omega_o) \approx 1890 \text{ newton-m} \quad (l)$$

$$i_{L_{\text{min}}} = \frac{G I_f}{R_a + R_L} \omega_o \approx 700 \text{ amps} \quad (m)$$

$$i_{L_{\text{max}}} = \left(\frac{G I_f}{R_a + R_L} \right) \omega_{\text{final}} \approx 793 \text{ amps} \quad (n)$$

$$K = 134.5 \text{ newton-meters}, \tau = J_r / K \approx 0.09 \text{ sec} \quad (o)$$



FIELDS AND MOVING MEDIA

PROBLEM 6.18

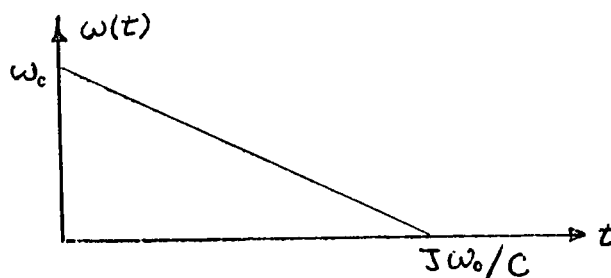
Part a

Let the coulomb torque be C , then the equation of motion is

$$J \frac{d\omega}{dt} + C = 0 \quad (a)$$

Since $\omega(0) = \omega_0$

$$\omega(t) = \omega_0 \left(1 - \frac{C}{J\omega_0} t\right) \quad 0 \leq t \leq (J/C) \omega_0 \quad (b)$$

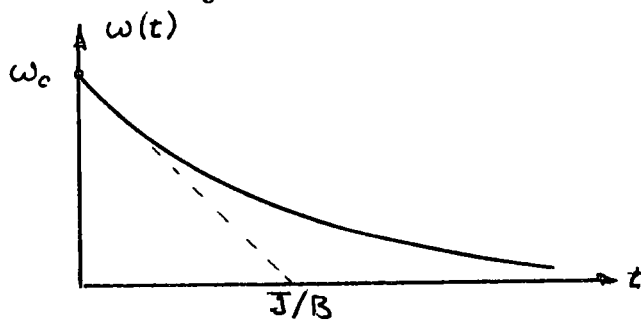


Part b

Now the equation of motion is

$$J \frac{d\omega}{dt} + B\omega = 0 \quad (c)$$

$$\omega(t) = \omega_0 e^{-(B/J)t} \quad (d)$$



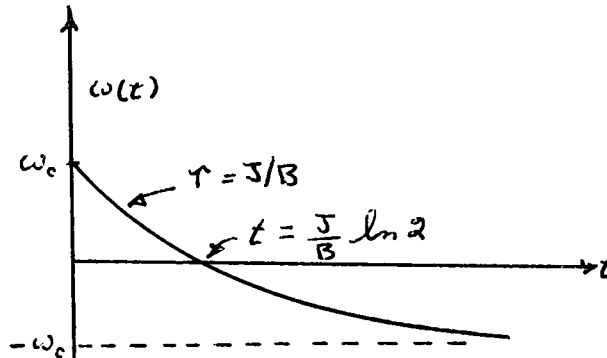
Part c

Let $C = B\omega_0$, the equation of motion is now

$$J \frac{d\omega}{dt} + B\omega = -B\omega_0 \quad (e)$$

$$\left\{ \omega(t) = -\omega_0 + 2\omega_0 e^{-\frac{B}{J}t} \quad 0 < t \leq \frac{J}{B} \ln 2 \right\} \quad (f)$$

PROBLEM 6.18 (Continued)



PROBLEM 6.19

Part a

The armature circuit equation is

$$R_a i_L + L_a \frac{di_L}{dt} = G \omega i_f - V_a u_{-1}(t) \quad (a)$$

Differentiating

$$L_a \frac{d^2 i_L}{dt^2} + R_a \frac{di_L}{dt} = G i_f \frac{d\omega}{dt} - V_a u_o(t) \quad (b)$$

The mechanical equation of motion is

$$J_r \frac{d\omega}{dt} = -G i_L i_f \quad (c)$$

Thus, (b) becomes

$$L_a \frac{d^2 i_L}{dt^2} + R_a \frac{di_L}{dt} + \frac{(G i_f)^2}{J_r} i_L = -V_a u_o(t) \quad (d)$$

Initial conditions are

$$i_L(0^+) = 0, \quad \frac{di_L}{dt}(0^+) = -\frac{V_a}{L_a} \quad (e)$$

and it follows from (d) that

$$i_L(t) = \left(-\frac{a}{L_a \beta}\right) e^{-\alpha t} \sin \beta t u_{-1}(t) \quad (f)$$

where

$$\alpha = \frac{R_a}{2L_a} = 7.5 / \text{sec} \quad (g)$$

$$\beta = \sqrt{\frac{(G i_f)^2}{J_r L_a} - \left(\frac{R_a}{2L_a}\right)^2} \approx 19.9 \text{ rad/sec} \quad (h)$$

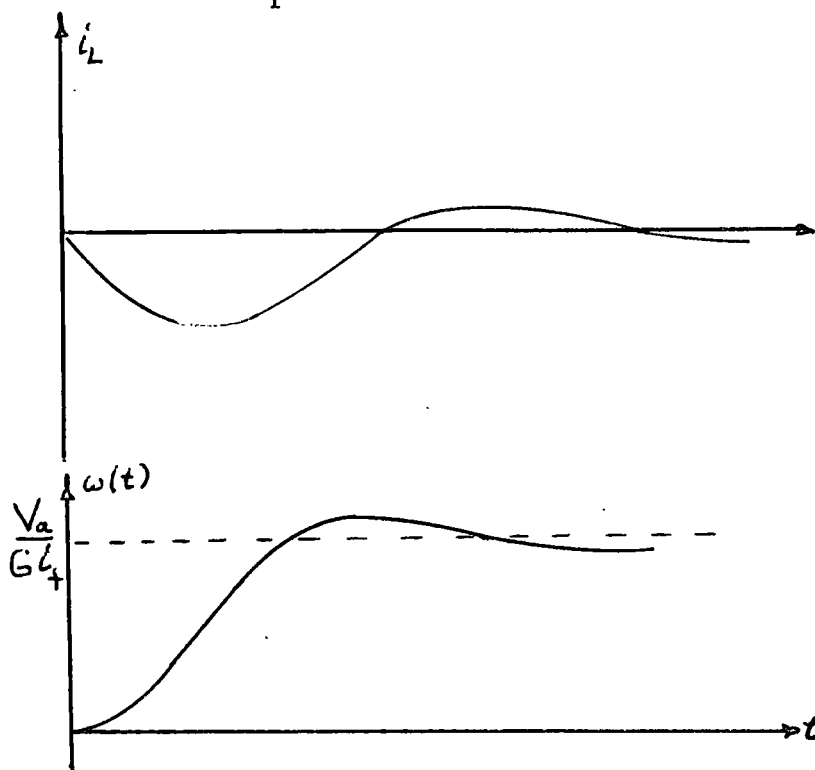
FIELDS AND MOVING MEDIA

PROBLEM 6.19 (Continued)

$$\frac{V_a}{L_a \beta} = 1160 \text{ amps} \quad (i)$$

$$\omega(t) = -\frac{V_a}{G I_f} \left[\frac{d}{\beta} e^{-\alpha t} \sin \beta t + (e^{-\alpha t} \cos \beta t - 1) \right]$$

$$\frac{V_a}{G I_f} = 153.3 \text{ rad/sec} \quad (k)$$



Part b

Now we replace R_a by $R_a + R_L$ in part (a). Because of the additional damping

$$i_L(t) = -\frac{V_a}{2L_a \gamma} (e^{-(\alpha-\gamma)t} - e^{-(\alpha+\gamma)t}) u_{-1}(t) \quad (l)$$

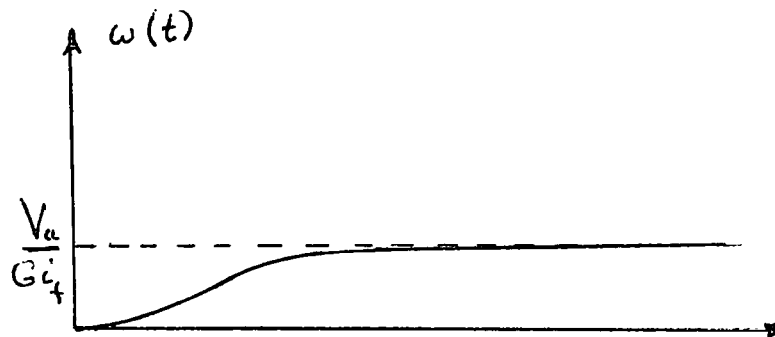
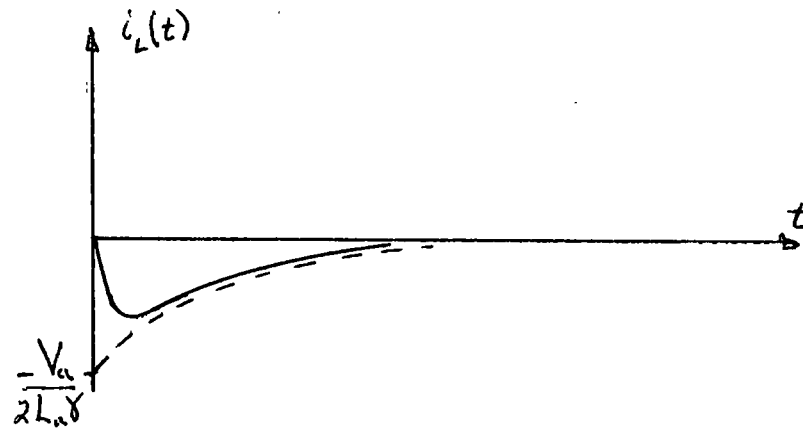
where

$$\alpha = \frac{R_a + R_L}{2L_a} = 75/\text{sec} \quad (m)$$

$$\gamma = \sqrt{\left(\frac{R_a + R_L}{2L_a}\right)^2 - \frac{(G I_f)^2}{J_r L_a}} = 10.6/\text{sec}. \quad (n)$$

PROBLEM 6.19 (Continued)

$$\omega(t) = \frac{V_a}{2L_a \gamma} \frac{GI_f}{J_r} \left[\frac{1}{(\gamma - \alpha)} e^{-(\alpha - \gamma)t} + \frac{1}{\gamma + \alpha} e^{-(\alpha + \gamma)t} + \frac{2\gamma}{\alpha^2 - \gamma^2} \right] \quad (o)$$



PROBLEM 6.20

Part a

The armature circuit equation is

$$v_a = R_a i_a + GI_f \omega \quad (a)$$

The equation of motion is

$$J \frac{d\omega}{dt} = GI_f i_a \quad (b)$$

Which may be integrated to yield

$$\omega(t) = \frac{G}{J} \int_{-\infty}^t i_a(t) \quad (c)$$

FIELDS AND MOVING MEDIA

PROBLEM 6.20 (Continued)

Combining (c) with (a)

$$v_a = R_a i_a + \frac{(GI_f)^2}{J_r} \int_{-\infty}^t i_a(t) dt \quad (d)$$

We recognize that

$$C = \frac{J_r}{(GI_f)^2} \quad (e)$$

Part b

$$C = \frac{J_r}{(GI_f)^2} = \frac{(0.5)}{(1.5)^2(1)} = 0.22 \text{ farads}$$

PROBLEM 6.21

According to (6.4.30) the torque of electromagnetic origin is

$$T^e = GI_f i_a$$

For operation on a-c, maximum torque is produced when i_f and i_a are in phase, a situation assured for all loading conditions by a series connection of field and armature. Parallel operation, on the other hand, will yield a phase relation between i_f and i_a that varies with loading. This gives reduced performance unless phase connecting means are employed. This is so troublesome and expensive that the series connection is used almost exclusively.

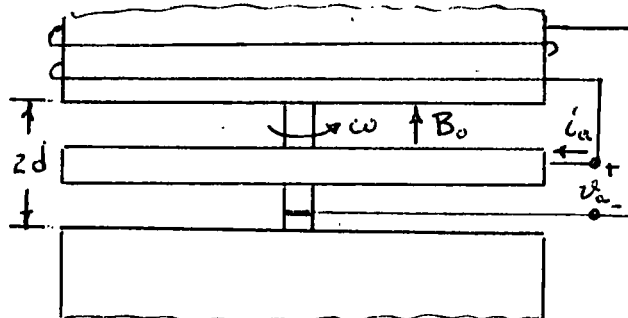
PROBLEM 6.22

From (6.4.50) et. seq. the homopolar machine, viewed from the disk terminals in the steady state, has the volt ampere relation

$$v_a = R_a i_a + G\omega i_f$$

$$R_a = \frac{\ln(b/a)}{2\pi\sigma d}$$

For definition of v_a and i_a shown to the right and with the interconnection with the coil shown in Fig. 6P.22



$$B_o = \frac{\mu_o N i_a}{2d}$$

Then from (6.4.52)

$$G\omega i_f = \frac{\omega B_o}{2} (b^2 - a^2) = \frac{\omega \mu_o N i_a}{4d} (b^2 - a^2)$$

FIELDS AND MOVING MEDIA

PROBLEM 6.22 (Continued)

Substitution of this into the voltage equation yields for steady state (because the coil resistance is zero).

$$0 = R_a i_a + \frac{\omega \mu_o N^2 a}{4d} (b^2 - a^2)$$

for self-excitation with $i_a \neq 0$

$$\frac{\omega \mu_o N^2}{4d} (b^2 - a^2) = -R_a$$

Because all terms on the left are positive except for ω , we specify $\omega < 0$ (it rotates in the direction opposite to that shown). With this provision the number of turns must be

$$N = \frac{4dR_a}{|\omega| \mu_o (b^2 - a^2)} = \frac{4d \ln(b/a)}{2\pi \sigma d |\omega| \mu_o (b^2 - a^2)}$$

$$N = \frac{2 \ln(b/a)}{\pi \sigma \mu_o |\omega| (b^2 - a^2)}$$

PROBLEM 6.23

Part a

Denoting the left disk and magnet as 1 and the right one as 2, the flux densities defined as positive upward are

$$B_1 = -\frac{\mu_o N}{\ell} (i_1 - i_2)$$

$$B_2 = -\frac{\mu_o N}{\ell} (i_1 + i_2)$$

Adding up voltage drops around the loop carrying current i_1 we have:

$$-N\pi a^2 \frac{dB_2}{dt} - N\pi a^2 \frac{dB_1}{dt} + i_1 R_L + i_1 R_a + \frac{\Omega B_1}{2} (b^2 - a^2) = 0$$

where $R_a = \frac{\ln(b/a)}{2\pi \sigma h}$

Part b

Substitution of the expression for B_1 and B_2 into this voltage expression and simplification yield

$$L \frac{di_1}{dt} + i_1 (R_L + R_a) - G\Omega i_1 + G\Omega i_2 = 0$$

PROBLEM 6.23 (Continued)

where

$$L = \frac{\frac{1}{2} \mu_o N^2 \pi a^2}{\ell}$$

$$G = \frac{-\mu_o N(b^2 - a^2)}{2\ell}$$

The equation for the circuit carrying current i_2 can be written similarly as

$$L \frac{di_2}{dt} + i_2(R_L + R_a) - G\Omega i_1 = 0$$

These are linear differential equations with constant coefficients, hence, assume

$$i_1 = I_1 e^{st}; \quad i_2 = I_2 e^{st}$$

Then

$$[Ls + R_L + R_a - G\Omega] I_1 + G\Omega I_2 = 0$$

$$[Ls + R_L + R_a - G\Omega] I_2 - G\Omega I_1 = 0$$

Elimination of I_1 yields

$$\left[\frac{[Ls + R_L + R_a - G\Omega]^2}{G\Omega} + G\Omega \right] I_2 = 0$$

If $I_2 \neq 0$ as it must be if we are to supply current to the load resistances, then

$$[Ls + R_L + R_a - G\Omega]^2 + (G\Omega)^2 = 0$$

For steady-state sinusoidal operation s must be purely imaginary. This requires

$$R_L + R_a - G\Omega = 0$$

or

$$G = \frac{-\mu_o N(b^2 - a^2)}{2\ell} = \frac{R_L + \frac{\ln(b/a)}{2\pi\sigma h}}{\Omega}$$

This is the condition required.

Part c

When the condition of (b) is satisfied

$$s = \pm j\omega = \pm j \frac{G\Omega}{L}$$

$$\omega = \frac{-\mu_o N(b^2 - a^2)\ell\Omega}{2\ell\mu_o N^2\pi a^2} = \frac{-(\frac{b^2}{a^2} - 1)\Omega}{2\pi N}$$

PROBLEM 6.23 (Continued)

Thus the system will operate in the sinusoidal steady-state with amplitudes determined by initial conditions. With the condition of part (b) satisfied the voltage equations show that

$$I_1 = jI_2$$

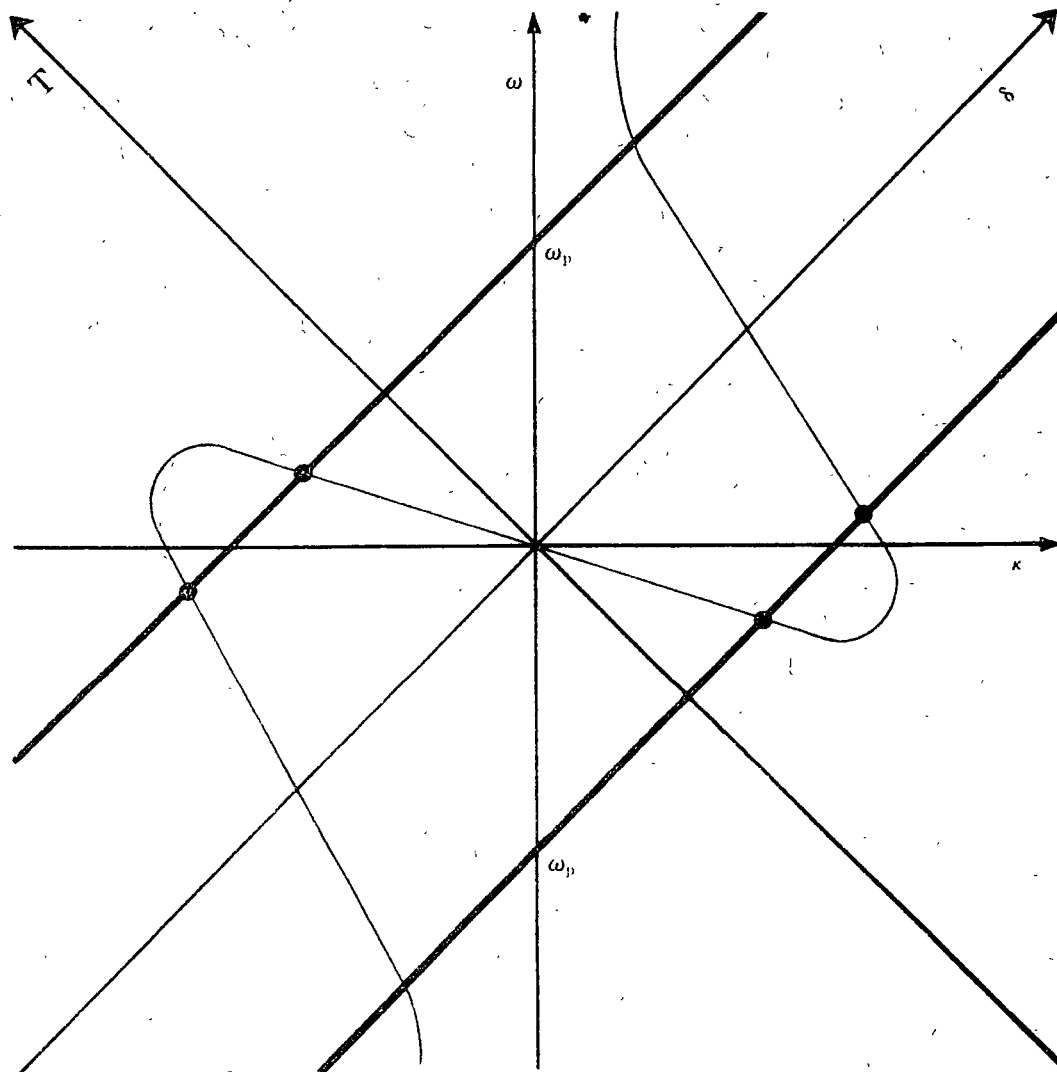
and the currents form a balanced two-phase set.

SOLUTIONS MANUAL FOR

ELECTROMECHANICAL DYNAMICS

PART II: Fields, Forces, and Motion

HERBERT H. WOODSON JAMES R. MELCHER



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Mark Zabin

ELECTROMECHANICAL DYNAMICS

Part II: Fields, Forces, and Motion

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PREFACE TO: SOLUTIONS MANUAL FOR
ELECTROMECHANICAL DYNAMICS, PART II:
FIELDS, FORCES, AND MOTION

This manual presents in an informal format solutions to the problems found at the ends of chapters in Part II of the book, Electromechanical Dynamics. It is intended as an aid for instructors, and in special circumstances for use by students. We have included a sufficient amount of explanatory material that solutions, together with problem statements, are in themselves a teaching aid. They are substantially as found in our records for the course 6.06, as taught at M.I.T. over a period of several years.

Typically, the solutions were originally written up by graduate student tutors, whose responsibility it was to conduct one-hour tutorials once a week with students in pairs. These tutorials focused on the homework, with the problem solutions reproduced and given to the students upon receipt of their own homework solutions.

It is difficult to give proper credit to all of those who contributed to these solutions, because the individuals involved range over teaching assistants, instructors, and faculty, who have taught the material over a period of more than four years. However, significant contributions were made by D.S. Guttman, Dr. K.R. Edwards, M. Zahn, F.A. Centanni, and T.B. Jones, Jr. The manuscript was typed by Mrs. Barbara Morton, whose patience and expertise were invaluable.

H.H. Woodson

J.R. Melcher

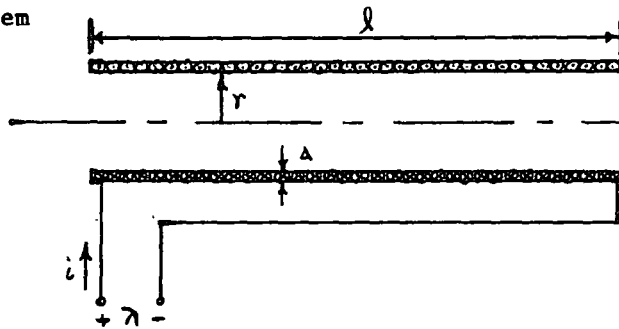
Cambridge, Massachusetts

September, 1968

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.1

It is the purpose of this problem to illustrate the limitations inherent to common conductors in achieving long magnetic time constants. (Diffusion times.) For convenience in making this point consider the solenoid shown with



l = length

Δ = cross-sectional dimensions of single layer of wire (square-cross section).

r = radius ($r \gg \Delta$) but $r \ll l$.

Then there are l/Δ turns, each having a length $2\pi r$, and the total d-c resistance is directly proportional to the length and inversely proportional to the area and electrical conductivity σ .

$$R = \frac{2\pi r}{\sigma(\Delta^2)} \left(\frac{l}{\Delta}\right)$$

The H field in the axial direction, by Ampere's law, is $H = \frac{i}{\Delta}$ and the flux linked by one turn is $\mu_o H(\pi r^2)$ so that

$$\lambda = \mu_o H(\pi r^2) \left(\frac{l}{\Delta}\right) = \mu_o (\pi r^2) \frac{l}{\Delta^2} i$$

and it follows that

$$L = \mu_o (\pi r^2) \left(\frac{l}{\Delta^2}\right)$$

Finally, the time constant is

$$\frac{L}{R} = \frac{1}{2} \mu_o r \Delta \sigma$$

Thus, the diffusion time (see Eq. 7.1.28) is based on an equivalent length $\sqrt{r\Delta}$. Consider using copper with

$$\sigma = 5.9 \times 10^7 \text{ mhos/m}$$

$$\Delta = 10 \text{ m}$$

and find Δ required to give $L/R = 10^2$

$$\begin{aligned} \Delta &= 2 \left(\frac{L}{R}\right) \frac{1}{\mu_o \sigma} = \frac{(200)}{(4\pi \times 10^{-7})(10)(5.9 \times 10^7)} \\ &= 2.7 \times 10^{-1} \text{ m or } 27 \text{ cm} \end{aligned}$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.1 (Continued)

Note that to satisfy the condition that $\ell \gg r$, the length must be greater than 10 meters also. The coil is larger than the average class-room! Of course, if magnetic materials are used, the dimensions of the coil can be reduced considerably, but long L/R time constants are difficult to obtain on a laboratory scale with ordinary conductors.

PROBLEM 7.2

Part a

Our solution will parallel the one in the text, only now the $\vec{B}$ field will be trapped in the slab until it diffuses away. The fundamental equations are

$$\begin{aligned}\nabla \times \vec{B} &= \mu \vec{J} = \mu \sigma \vec{E}; & \nabla \times \vec{E} &= - \frac{\partial \vec{B}}{\partial t} \\ \nabla \times \nabla \times \vec{B} &= \nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = \mu \sigma \nabla \times \vec{E} = - \mu \sigma \frac{\partial \vec{B}}{\partial t}\end{aligned}$$

Because $\nabla \cdot \vec{B} = 0$,

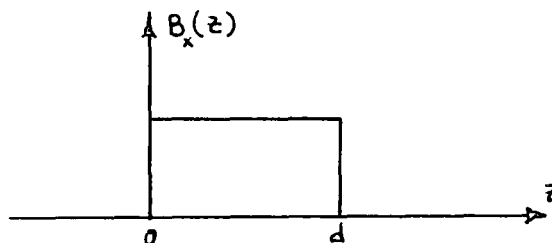
$$\nabla^2 \vec{B} = \mu \sigma \frac{\partial \vec{B}}{\partial t}$$

or in one dimension

$$\frac{1}{\mu \sigma} \frac{\partial^2 B_x}{\partial z^2} = \frac{\partial B_x}{\partial t} \quad (a)$$

at $t = 0^+$

$$B_x = \begin{cases} 0 & z < 0 \\ B_0 & 0 < z < d \\ 0 & z > d \end{cases}$$



This suggests that between 0 and d, we can write $B_x(z)$ as

$$B_x(z) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi z}{d}\right) \quad 0 < z < d$$

To solve for the coefficients a_n , we take advantage of the orthogonal property of the sine functions.

$$\int_0^d B_x(z) \sin\left(\frac{m\pi z}{d}\right) dz = \sum_{n=1}^{\infty} \int_0^d a_n \sin\left(\frac{n\pi z}{d}\right) \sin\left(\frac{m\pi z}{d}\right) dz$$

But

$$\int_0^d B_x(z) \sin\left(\frac{m\pi z}{d}\right) dz = \int_0^d B_0 \sin\left(\frac{m\pi z}{d}\right) dz = \begin{cases} \frac{2dB_0}{m\pi} & m \text{ odd} \\ 0 & m \text{ even} \end{cases}$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.2 (Continued)

Also

$$\int_0^d a_n \sin\left(\frac{n\pi z}{d}\right) \sin\left(\frac{m\pi z}{d}\right) dz = \begin{cases} \frac{a_m d}{2} & n = m \\ 0 & n \neq m \end{cases}$$

Hence,

$$a_m = \begin{cases} \frac{4B_0}{m\pi} & m \text{ odd} \\ 0 & m \text{ even} \end{cases}$$

$$B_x(t=0, z) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} B_0 \sin\left(\frac{n\pi z}{d}\right) \quad 0 < z < d$$

We assume that for $t > 0$, $0 < z < d$

$$B_x(t, z) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} B_0 \sin\left(\frac{n\pi z}{d}\right) e^{-\alpha_n^2 t}$$

Plugging into (a) we find that $\frac{1}{\mu\sigma} \left(\frac{n\pi}{d}\right)^2 = \alpha_n^2$. Let's define $\tau = \mu\sigma \left(\frac{d}{\pi}\right)^2$ as the fundamental diffusion time. Then

$$B_x(t, z) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} B_0 \sin\left(\frac{n\pi z}{d}\right) e^{-n^2 t / \tau} \quad \begin{matrix} 0 < z < d \\ t > 0 \end{matrix}$$

Part b

$$\vec{J} = \frac{\nabla \times \vec{B}}{\mu} = \frac{1}{\mu} \frac{\partial B_x}{\partial z} \vec{i}_y = \frac{4B_0}{\mu_0 d} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \cos\left(\frac{n\pi z}{d}\right) e^{-n^2 t / \tau} \vec{i}_y \quad \begin{matrix} 0 < z < d \\ t > 0 \end{matrix}$$

PROBLEM 7.3

Part a

If we neglect the capacitance of the block, the current we put in at $t=0$ will have to return by means of the block. This can be seen from the magnetic field system equation

$$\nabla \times \vec{H} = \vec{J} \quad (a)$$

which implies

$$\nabla \cdot \vec{J} = 0 \quad (b)$$

or "what goes in must come out".

If the current penetrated the block at $t=0^+$ there would be a magnetic field within the block at $t=0^+$, a situation we cannot allow since some time must elapse (relative to the diffusion time) before the fields in the block can change significantly.

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.3 (Continued)

We conclude that the source current returns as a surface current down the left side of the block. This current must be

$$K_y = -I_o/D \quad (c)$$

where y is the upwards vertical direction. The current loop between $x = -L$ and $x = 0$ thus provides a magnetic field

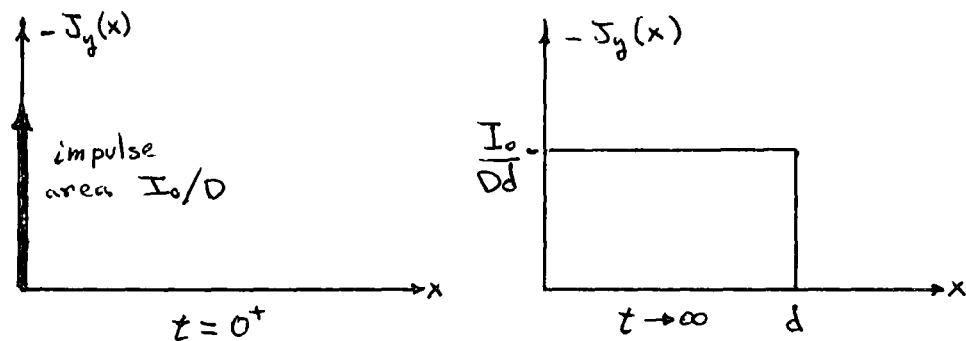
$$H_z(t=0^+) = \begin{cases} -I_o/D & -L < x < 0 \\ 0 & 0 < x \end{cases} \quad (d)$$

where z points out of sketch.

Part b

As $t \rightarrow \infty$ the system will reach a static state with input current I_o/D per unit length. The current will return uniformly through the block. Hence,

$$J_y(x) = -\frac{I_o}{Dd} \quad (e)$$



Part c

As a diffusion problem this system is very much like the system of Fig. 7.1.1 of the text except for the fact that here diffusion occurs on only one side of the block instead of two. This suggests a fundamental diffusion time constant of

$$\tau = \frac{\mu_o \sigma (2d)^2}{\pi^2} \quad (f)$$

where we have replaced the term d^2 by $(2d)^2$ in Eq. 7.1.28 of the text.

PROBLEM 7.4

Part a

This is a magnetic field system characterized by a diffusion equation. With $B_z = \text{Re} \hat{B}_z(x) e^{j\omega t}$,

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.4 (Continued)

$$\frac{1}{\mu\sigma} \frac{d^2 \hat{B}_z}{dx^2} = j\omega \hat{B}_z \quad (a)$$

Let $\hat{B}_z(x) = \hat{B}_0 e^{\alpha x}$, then

$$\alpha^2 = j\omega\mu\sigma \quad (b)$$

or

$$\alpha = \pm \frac{1}{\delta} (1 + j), \quad \delta = \sqrt{\frac{2}{\omega\mu\sigma}} \quad (c)$$

The boundary conditions are

$$\hat{B}_z(x=0) = -\mu \hat{i}/D \quad (d)$$

$$\hat{B}_z(x \rightarrow \infty) \rightarrow 0$$

which means that we use only the (-) sign

$$\hat{B}_z(x,t) = -\text{Re} \frac{\mu \hat{i}}{D} e^{-x/\delta} e^{j(\omega t - \frac{x}{\delta})} \quad (e)$$

Part b

$$\nabla \times \vec{B} = \mu \vec{J} \quad (f)$$

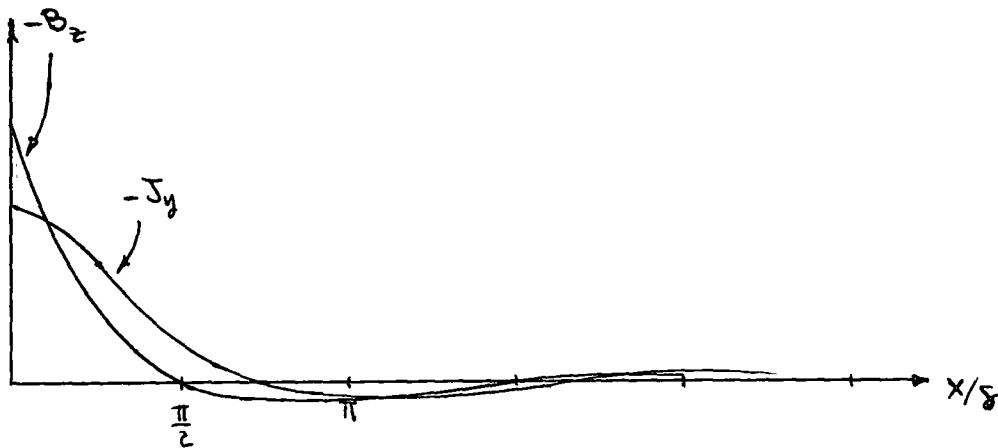
or

$$\frac{\partial B_z}{\partial x} = -\mu J_y \quad (g)$$

so that

$$J_y = -\text{Re} \frac{\hat{i}}{D} \frac{(1+j)}{\delta} e^{-\frac{x}{\delta}} e^{j(\omega t - \frac{x}{\delta})} \quad (h)$$

Part c



MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.4 (Continued)

Part d

The electric field is given by

$$\nabla \times \vec{E} = \hat{i}_z \frac{\partial E_y}{\partial x} = -j\omega \hat{i}_z B_z \quad (i)$$

$$E_y(x=0) = -\frac{\omega \delta \mu}{2D} (1+j) \hat{i} \quad (j)$$

Faraday's law (Eq. 1.1.23, Table 1.2, Appendix E) written for a counter-clockwise contour through the source and left edge of the block, gives

$$\hat{V} + \hat{E}_y d = \frac{j\omega \mu_o (Ld)}{D} \hat{i} \quad (k)$$

where from (j)

$$\hat{E}_y d = -\frac{1}{\sigma} \left(\frac{d}{\delta}\right) \frac{1}{D} (1+j) \hat{i} \quad (l)$$

Hence, assuming that $\hat{V} = \hat{i}[R(\omega) + j\omega L(\omega)]$, (don't confuse the L's)

$$R(\omega) = \frac{1}{\sigma} \left(\frac{d}{\delta}\right) \frac{1}{D} = \frac{d}{D} \sqrt{\frac{\omega \mu}{2\sigma}} \quad (m)$$

$$L(\omega) = \frac{\mu_o Ld}{D} + \frac{d}{D} \sqrt{\frac{\mu}{2\omega\sigma}} \quad (n)$$

Thus, as $\omega \rightarrow \infty$ the inductance becomes just that due to the free-space portion of the circuit between $x=0$ and $x=-L$. The resistance becomes infinite because the currents crowd to the left edge of the block.

In the opposite extreme, as $\omega \rightarrow 0$, the resistance approaches zero because the currents have an infinite x - z area of the block through which to flow. Similarly, the inductance becomes large because the x - y area enclosed by the current paths increases without limit. At low frequencies it would be necessary to include the finite extent of the block in the x direction in the analysis to obtain a realistic estimate of the resistance and inductance.

PROBLEM 7.5

Part a

This is a magnetic field system characterized by a diffusion equation.

Place origin of coordinates at left edge of block, x to right and z out of paper.

With $B_x = \text{Re} \hat{B}_x(x) e^{j\omega t}$

$$\frac{1}{\mu\sigma} \frac{\partial^2 \hat{B}_x}{\partial x^2} = j\omega \hat{B}_z \quad (a)$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.5 (Continued)

Let $\hat{B}_z(x) = \hat{B}_0 e^{\alpha x}$, then

$$\alpha^2 = j\omega\mu\sigma \quad (b)$$

$$\alpha = \pm \frac{1}{\delta} (1+j), \quad \delta = \sqrt{\frac{2}{\omega\mu\sigma}} \quad (c)$$

The boundary conditions are

$$\hat{B}_z(x=0) = -\mu \frac{\hat{I}}{D} \quad (d)$$

$$\hat{B}_z(x=l) = 0 \quad (e)$$

because all of the current $I_0(t)$ is returned through the block. Thus the appropriate linear combination of solutions to satisfy the boundary conditions is

$$B_z(x,t) = \text{Re} \frac{\mu \hat{I}}{D} \frac{\sinh[\alpha(x-l)] e^{j\omega t}}{\sinh(\alpha l)} \quad (f)$$

where α is a complex quantity, (c). The current is related to $\hat{B}_z$ by

$$\nabla \times \vec{B} = -\frac{\partial B}{\partial x} \hat{i}_y = \mu \vec{J} = \mu J_y \hat{i}_y \quad (g)$$

From (f) and (g),

$$J_y = -\frac{\hat{I}\alpha}{D} \frac{\cosh[\alpha(x-l)] e^{j\omega t}}{\sinh \alpha l} \quad (h)$$

Part b

The time average magnetic force on the block is given by

$$f_x = \text{Re} \left[D d \int_0^l \frac{\hat{J}_y(x) \hat{B}_z^*(x) dx}{2} \right] \quad (i)$$

where we have taken advantage of the identity

$$\langle \text{Re} \hat{A} e^{j\omega t} \text{Re} \hat{B} e^{j\omega t} \rangle = \frac{1}{2} \text{Re} \hat{A} \hat{B}^*$$

to integrate the force density $(\vec{J} \times \vec{B})_x$ over the volume of the block. Note that a detailed calculation is required to complete (i), because α in (f) and (h) is complex.

This example is one where the total force is more easily computed using the Maxwell stress tensor. See Probs. 8.16, 8.17 and 8.22 for this approach.

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.6

As an example of electromagnetic phenomena that occur in conductors at rest we consider the system of Fig. 7.1.1 with the constant-current source and switch replaced by an alternating current source.

$$i(t) = I \cos \omega t \quad (a)$$

We make all of the assumptions of Sec. 7.1.1 and adopt the coordinate system of Fig. 7.1.2. Interest is now confined to a steady-state problem.

The equation that describes the behavior of the flux density in this system is Eq. 7.1.15

$$\frac{1}{\mu_o \sigma} \frac{\partial^2 B_x}{\partial z^2} = \frac{\partial B_x}{\partial t} \quad (b)$$

and the boundary conditions are now, at $z = 0$ and $z = d$,

$$B_x = B_o \cos \omega t = \left[\text{Re } B_o e^{j\omega t} \right] \quad (c)$$

where

$$B_o = \frac{\mu_o NI}{w} \quad (d)$$

The boundary condition of (c) coupled with the linearity of (b) lead us to assume a solution

$$B_x = \text{Re} \left[\hat{B}(z) e^{j\omega t} \right] \quad (e)$$

We substitute this form of solution into (b), cancel the exponential factor, and drop the Re to obtain

$$\frac{d^2 \hat{B}}{dz^2} = j \mu_o \sigma \hat{B} \quad (f)$$

Solutions to this equation are of the form

$$\hat{B}(z) = e^{rz} \quad (g)$$

where substitution shows that

$$r = \pm \sqrt{j\omega\mu_o\sigma} = \pm \sqrt{\frac{j\omega\mu_o\sigma}{2}} (1+j) \quad (h)$$

It is convenient to define the skin depth δ as (see Sec. 7.1.3a)

$$\delta = \sqrt{\frac{2}{\omega\mu_o\sigma}} \quad (i)$$

We use this definition and write the solution, (g) as

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.6 (Continued)

$$\hat{B}(z) = c_1 e^{(1+j)\frac{z}{\delta}} + c_2 e^{-(1+j)\frac{z}{\delta}} \quad (j)$$

The boundary conditions at $z = 0$ and $z = d$ (c) require that

$$\begin{aligned} B_0 &= c_1 + c_2 \\ B_0 &= c_1 e^{(1+j)d/\delta} + c_2 e^{-(1+j)d/\delta} \end{aligned}$$

Solution of these equations for c_1 and c_2 yields

$$c_1 = \frac{B_0 \left[1 - e^{-(1+j)\frac{d}{\delta}} \right]}{D} \quad (k)$$

$$c_2 = \frac{-B_0 \left[1 - e^{(1+j)\frac{d}{\delta}} \right]}{D} \quad (l)$$

$$\text{where } D = 2 \left(\cos \frac{d}{\delta} \sinh \frac{d}{\delta} + j \sin \frac{d}{\delta} \cosh \frac{d}{\delta} \right)$$

We now substitute (k) and (l) into (j); and, after manipulation, obtain

$$\hat{B}(z) = B_0 [f(z) + j g(z)] \quad (m)$$

where

$$f(z) = \frac{M}{F} \cos \frac{d}{\delta} \sinh \frac{d}{\delta} + \frac{N}{F} \sin \frac{d}{\delta} \cosh \frac{d}{\delta}$$

$$g(z) = \frac{N}{F} \cos \frac{d}{\delta} \sinh \frac{d}{\delta} - \frac{M}{F} \sin \frac{d}{\delta} \cosh \frac{d}{\delta}$$

$$M = \cos \frac{z}{\delta} \sinh \frac{z}{\delta} + \cos \left(\frac{d-z}{\delta} \right) \sinh \left(\frac{d-z}{\delta} \right)$$

$$N = \sin \frac{z}{\delta} \cosh \frac{z}{\delta} + \sin \left(\frac{d-z}{\delta} \right) \cosh \left(\frac{d-z}{\delta} \right)$$

$$F = \cos^2 \frac{d}{\delta} \sinh^2 \frac{d}{\delta} + \sin^2 \frac{d}{\delta} \cosh^2 \frac{d}{\delta}$$

Substitution of (m) into (e) yields

$$B_x = B_m(z) \cos[\omega t + \theta(z)] \quad (n)$$

where

$$B_m(z) = B_0 \sqrt{[f(z)]^2 + [g(z)]^2} \quad (o)$$

$$\theta(z) = \tan^{-1} \frac{g(z)}{f(z)} \quad (p)$$

It is clear from the form of (n) that both the amplitude and phase of the flux density vary as functions of z .

To illustrate the nature of the distribution of flux density predicted

PROBLEM 7.6 (Continued)

by this set of equations the maximum flux density is plotted as a function of position for several values of d/δ in the figure. Recalling the definition of the skin depth δ in (1), we realize that for a system of fixed geometry and fixed properties $\frac{d}{\delta} \propto \sqrt{\omega}$, thus, as $\frac{d}{\delta}$ increases, the frequency of the excitation increases. From the curves of the figure we see that as the frequency increases the flux density penetrates less and less into the specimen until at high frequencies ($\frac{d}{\delta} \gg 1$) the flux density is completely excluded from the conductor. At very low frequencies ($\frac{d}{\delta} \ll 1$) the flux density penetrates completely and is essentially unaffected by the presence of the conducting material.

It is clear that at high frequencies ($\frac{d}{\delta} \gg 1$) when the flux penetrates very little into the slab, the induced (eddy) currents flow near the surfaces. In this case it is often convenient, when considering electromagnetic phenomena external to the slab, to assume $\sigma \rightarrow \infty$ and treat the induced currents as surface currents.

It is informative to compare the flux distribution of the figure for a steady-state a-c problem with the distribution of Fig. 7.1.4 for a transient problem. We made the statement in Sec. 7.1.1 that when we deal with phenomena having characteristic times that are short compared to the diffusion time constant, the flux will not penetrate appreciably into the slab. We can make this statement quantitative for the steady-state a-c problem by defining a characteristic time as

$$\tau_c = \frac{1}{\omega}$$

We now take the ratio of the diffusion time constant given by Eq. 7.1.28 to this characteristic time and use the definition of skin depth in (1).

$$\frac{\tau}{\tau_c} = \omega \tau = \frac{2}{\pi} \left(\frac{d}{\delta} \right)^2 \quad (q)$$

Thus, for our steady-state a-c problem, this statement that the diffusion time constant is long compared to a characteristic time is the same as saying that the significant dimension d is much greater than the skin depth δ .

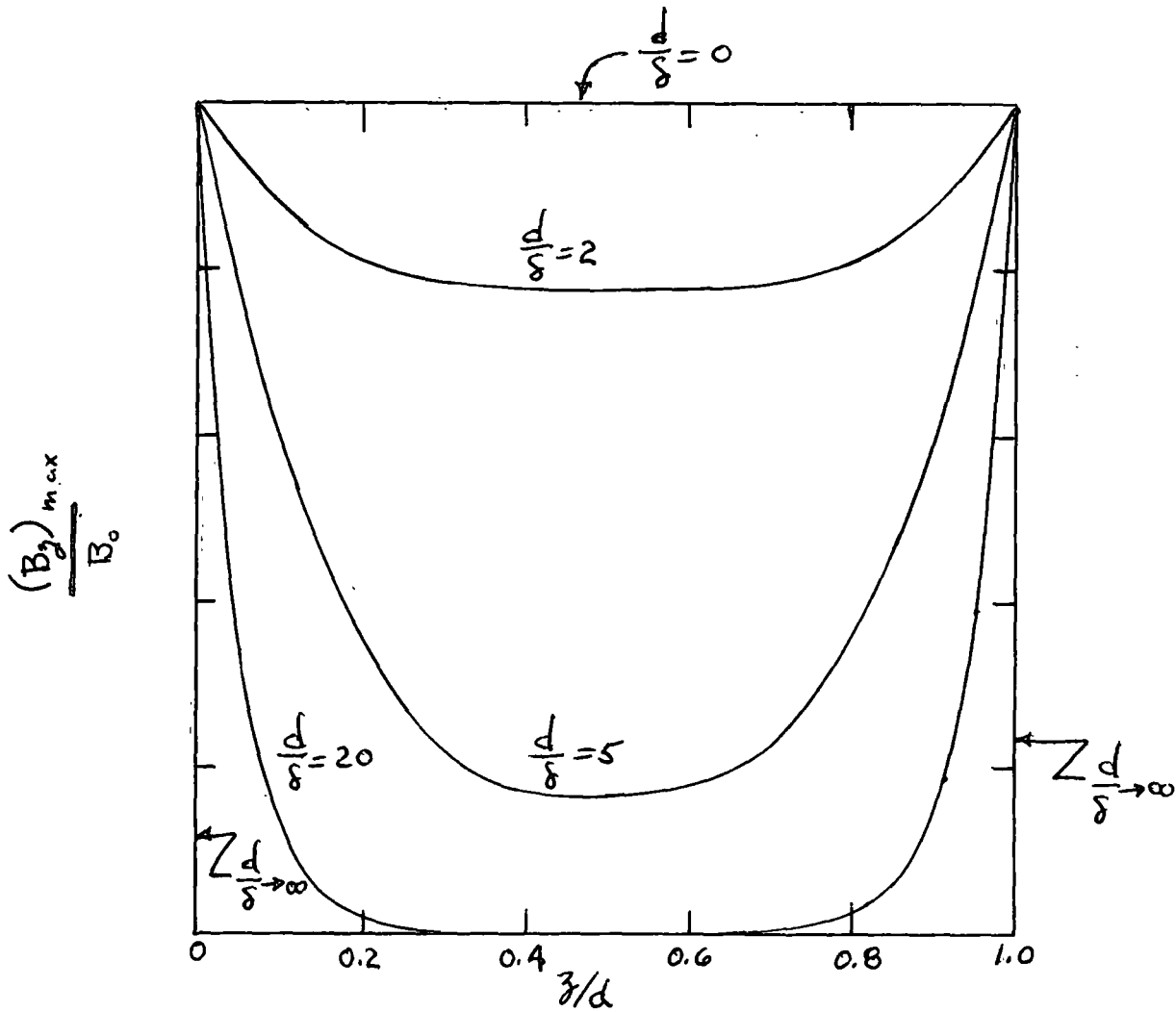
The current distribution follows from the magnetic flux density by using Ampere's law;

$$J_y = \frac{1}{\mu_0} \frac{\partial B_x}{\partial z} \quad (r)$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.6 (Continued)

Thus the distribution of $|J_y|$ is somewhat as shown in the figure for B_x .
The instantaneous J_y has odd symmetry about $z = 0.5 d$.



DISTRIBUTION OF FLUX DENSITY WITH SKIN EFFECT

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.7

Part a

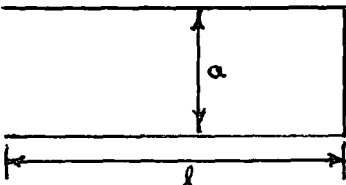
Assume the resistors in the circuit model each have approximately their D.C. resistance

$$R \sim R_{D.C.} = \frac{a}{\sigma \Delta D} \quad (a)$$

The inductance is the "loop" of metal

Depth D

$$L \approx \frac{\mu_0 a \ell}{D}$$



(b)

Hence the time constant involved is

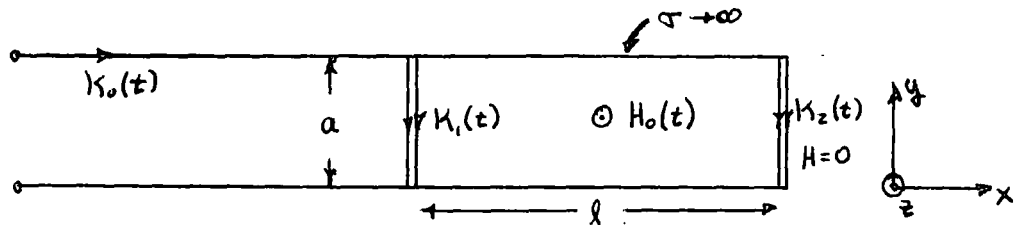
$$\tau \approx \frac{L}{2R} = \frac{\mu_0 \Delta \ell \sigma}{2} \quad (c)$$

The equivalent length in the diffusion time is $\sqrt{\Delta \ell} \gg \Delta$.

Part b

By adding the vacuum space of region 2 we have increased the amount of magnetic field that must be stored in the region before equilibrium is reached while the dissipation is confined to the two slabs. In the problem of Fig. 7.1.1, the slab stores a magnetic field only in a region of thickness Δ , the same region occupied by the currents, while here the magnetic field region is of thickness ℓ .

Part c



MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.7 (Continued)

Since diffusion in the slabs takes negligible time compared to the main problem, each slab could be modeled as a conducting sheet with

$$\vec{K} = (\sigma\Delta)\vec{E} \quad (d)$$

In region 2.

$$\nabla \times \vec{H} = 0 \quad \text{or} \quad \vec{H} = H_0(t)\vec{i}_z = -K_2(t)\vec{i}_z \quad (e)$$

From

$$\oint \vec{E} \cdot d\vec{l} = - \frac{d}{dt} \int \vec{B} \cdot \vec{n} \, da \quad (f)$$

we learn that

$$\frac{a}{\sigma\Delta} [K_1(t) - K_2(t)] = + \frac{d}{dt} [\mu_0 a \ell K_2(t)] \quad (g)$$

Since $K_0(t) = K_1(t) + K_2(t)$ we know that

$$K_0(t) = \frac{I}{D} u_{-1}(t) = 2K_2(t) + \sigma\mu_0 \Delta \ell \frac{dK_2(t)}{dt} \quad (h)$$

The solution is therefore

$$K_2(t) = \frac{I}{2D} (1 - e^{-t/\tau}) u_{-1}(t); \quad \tau = \frac{\sigma\mu_0 \Delta \ell}{2} \quad (i)$$

and, because $K_2 = -H_0$, the magnetic field fills region (2) with the time constant τ .

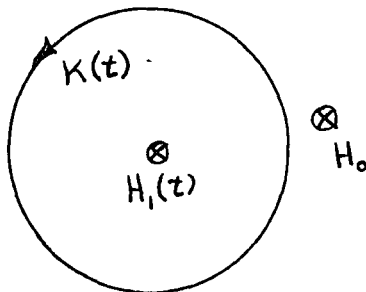
PROBLEM 7.8

As in Prob. 7.7, the diffusion time associated with the thin conducting shell is small compared to the time required for the field to fill the region $r < R$. Modeling the thin shell as having the property

$$\vec{K} = \Delta\sigma \vec{E} \quad (a)$$

and assuming that

$$H_1(t)\vec{i}_z = [H_0 - K(t)]\vec{i}_z \quad (b)$$



MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.8 (Continued)

We can use the induction equation

$$\oint \vec{E} \cdot d\vec{\ell} = - \frac{d}{dt} \int \vec{B} \cdot \vec{n} da \quad (c)$$

to learn that, because $H_0 = \text{constant}$ for $t > 0$

$$\frac{2\pi R}{\Delta\sigma} K(t) = - \pi R^2 \mu_0 \frac{dK(t)}{dt} \quad (d)$$

The solution to (d) is

$$K(t) = H_0 e^{-t/\tau} u_{-1}(t); \quad \tau = \frac{\mu_0 \sigma R \Delta}{2} \quad (e)$$

and from (b), it follows that

$$H_1(t) = H_0 - K(t) = H_0 (1 - e^{-t/\tau}) u_{-1}(t) \quad (f)$$

The $\vec{H}$ field is finally distributed uniformly for $r < a$, with a diffusion time based on the length $\sqrt{R\Delta}$.

PROBLEM 7.9

Part a

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad (a)$$

$$\nabla \times \vec{B} = \mu\sigma \vec{E} \quad (b)$$

So

$$\nabla \times \nabla \times \vec{B} = - \mu\sigma \frac{\partial \vec{B}}{\partial t} \quad (c)$$

But

$$\nabla \times (\nabla \times \vec{B}) = \nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = - \nabla^2 \vec{B} \quad (d)$$

So

$$\nabla^2 \vec{B} = \mu\sigma \frac{\partial \vec{B}}{\partial t}$$

Part b

Since $\vec{B}$ only has a z component

$$\nabla^2 B_z = \mu\sigma \frac{\partial B_z}{\partial t} \quad (e)$$

In cylindrical coordinates

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \quad (f)$$

Here $B_z = B_z(r, t)$ so

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.9 (Continued)

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \hat{B}}{\partial r} \right) + \mu \sigma \alpha \hat{B} = 0 \quad (g)$$

Part c

We want the magnetic field to remain finite at $r = 0$, hence $C_2 = 0$.

Part d

At $r = a$

$$B(a, t) = \mu_o H_o - C_1 J_o(\sqrt{\mu_o \sigma \alpha} a) = \mu_o H_o \quad (h)$$

Hence if $C_1 \neq 0$

$$J_o(\sqrt{\mu_o \sigma \alpha} a) = 0 \quad (i)$$

Part e

Multiply both sides of expression for $B(r, t=0) = 0$ by $r J_o(v_j r/a)$ and integrate from 0 to a . Then,

$$\int_0^a \mu_o H_o r J_o(v_j r/a) dr = \mu_o H_o \frac{a^2}{v_j} J_1(v_j) \quad (j)$$

$$\int_0^a \sum_{i=1}^{\infty} C_i J_o(v_i r/a) r J_o(v_j r/a) dr = C_j \frac{a^2}{2} J_1^2(v_j) \quad (k)$$

from which it follows that

$$C_j = \frac{2 \mu_o H_o}{v_j J_1(v_j)} \quad (l)$$

The values of v_j and $J_1(v_j)$ given in the table lead to the coefficients

$$\frac{C_1}{2 \mu_o H_o} = .802; \frac{C_2}{2 \mu_o H_o} = -.535; \frac{C_3}{2 \mu_o H_o} = 0.425 \quad (m)$$

Part f

$$\alpha_1 = \frac{1}{\mu_o \sigma} \left(\frac{v_1}{a} \right)^2$$

$$\tau_1 = \frac{\mu_o \sigma a^2}{v_1^2} = 0.174 \mu_o \sigma a^2 \quad (n)$$

$$\tau_1 = (0.174) (4\pi \times 10^{-7}) \frac{10^4}{4\pi} (25) \times 10^{-4}$$

$$\approx 4.35 \times 10^{-7} \text{ seconds} \quad (o)$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.10

Part a

$$\nabla \times \vec{E} = \vec{i}_z \frac{\partial E_y}{\partial x} = - \frac{\partial}{\partial t} B_z \vec{i}_z = 0 \quad (a)$$

$$\nabla \times \vec{B} = - \vec{i}_y \frac{\partial B_z}{\partial x} = \mu_0 \vec{J} = \mu_0 \sigma (E_y - UB_z) \vec{i}_y \quad (b)$$

$$\begin{aligned} \nabla \times \nabla \times \vec{B} &= \nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} \\ &= - \frac{\partial^2 B_z}{\partial x^2} \vec{i}_z = \mu_0 \sigma \left(\frac{\partial E_y}{\partial x} - U \frac{\partial B_z}{\partial x} \right) \vec{i}_z \end{aligned} \quad (c)$$

But $\frac{\partial E_y}{\partial x} = 0$ from (a), so

$$\frac{\partial^2 B_z}{\partial x^2} = \mu_0 \sigma U \frac{\partial B_z}{\partial x} \quad (d)$$

Part b

At

$$x = 0 \quad B_z = - \mu_0 K \quad (e)$$

$$\text{at } x = L \quad B_z = 0 \quad (f)$$

Part c

Let

$$B_z(x) = C e^{\alpha x}, \text{ then}$$

$$\alpha(\alpha - \mu_0 \sigma U) = 0 \quad (g)$$

$$\alpha = 0, \alpha = \mu_0 \sigma U \quad (h)$$

Using the boundary conditions

$$B_z(x) = - \mu_0 K \frac{(1 - e^{\mu_0 \sigma U (x-L)})}{1 - e^{-\mu_0 \sigma UL}} \quad (i)$$

Note that as $U \rightarrow 0$

$$B_z(x) = - \mu_0 K \left(\frac{L-x}{L} \right) \quad (j)$$

as expected.

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.11

Part a

$$\begin{aligned}\vec{F} &= \vec{J} \times \vec{B} = -J_y B_x \vec{i}_z \\ &= -\frac{\mu_o I^2}{w^2 \ell} \frac{R_m e^{R_m z/\ell} (e^{R_m z/\ell} - 1)}{(e^{R_m} - 1)^2} \vec{i}_z\end{aligned}\quad (a)$$

Part b

$$f_z = \int -F_z w dz = -\frac{\mu_o I^2 d}{2w} \quad (b)$$

This result can be found more simply by using the Maxwell Stress Tensor by methods similar to those used with Probs. 8.16 and 8.17.

Part c

The power supplied by the velocity source is

$$P_U = -f_z U = \frac{\mu_o I^2 d U}{2w} = \frac{I^2 d}{\sigma w \ell} \frac{R_m}{2} \quad (c)$$

The electric field at the current source is

$$E_y(z=\ell) = \frac{J_y(z=\ell)}{\sigma} - U B_x(z=\ell) \quad (d)$$

$$= \frac{I}{\sigma \ell w} \frac{R_m}{(e^{R_m} - 1)} \quad (e)$$

Power supplied by the current source is then

$$-V_g I = +E_y dI = +\frac{I^2 d}{\sigma \ell w} \left(\frac{R_m}{e^{R_m} - 1} \right) \quad (f)$$

Power dissipated in the moving conductor is then

$$P_d = P_U + V_g I = \frac{I^2 d}{\sigma \ell w} \frac{R_m}{2} \left(\frac{e^{R_m} + 1}{e^{R_m} - 1} \right) \quad (g)$$

which is just what is obtained from

$$P_d = w d \int_0^\ell \frac{J^2}{\sigma} dx \quad (h)$$

PROBLEM 7.12

If a point in the reference frame is outside the block it must satisfy

$$\nabla \times \bar{E} = - \frac{\partial \bar{B}}{\partial t} \quad (a)$$

$$\bar{J} = 0 \quad \text{and} \quad \nabla \times \bar{B} = 0 \quad (b)$$

Since the points outside the block have $\bar{J}=0$, and uniform static fields (for differential changes in time), (a) and (b) are satisfied.

Points inside the block must satisfy

$$\frac{1}{\mu_0} \frac{\partial B}{\partial z} = J_y \quad (c)$$

$$- \frac{1}{\mu_0 \sigma} \frac{\partial^2 B}{\partial z^2} + \frac{\partial B}{\partial t} = - V \frac{\partial B}{\partial z} \quad (d)$$

Since these points see

$$J_y = \sigma V B_0, \quad \frac{\partial B}{\partial z} = \frac{B_0}{\ell}, \quad \frac{\partial^2 B}{\partial z^2} = 0 \quad (e)$$

$$\frac{\partial B}{\partial t} = - V \frac{B_0}{\ell} \quad \text{and} \quad V \mu_0 \sigma \ell = 1$$

these conditions are satisfied.

Points on the block boundaries are satisfied because the field quantities $\bar{E}$ and $\bar{B}$ are continuous.

PROBLEM 7.13
Part a

Because $\nabla \cdot \bar{B}=0$ the magnetic flux lines run in closed loops. The field lines prefer to run through the high μ material near the source, hence very few lines will close beyond the edge of the material at $z=0$. Currents in the slab will tend to remain between the pole pieces.

Part b

$$\frac{1}{\mu \sigma} \frac{\partial^2 B}{\partial z^2} = \frac{\partial B}{\partial t} + V \frac{\partial B}{\partial z} \quad (a)$$

Let $B_y(z, t) = C e^{j(\omega t - kz)}$, then

$$k^2 - j\mu \sigma V k + j\mu \sigma \omega = 0; \quad (b)$$

A quadratic equation with roots

PROBLEM 7.13 (Continued)

$$k = \mu\sigma \left[j \frac{V}{2} \pm j \sqrt{\left(\frac{V}{2}\right)^2 + j \frac{\omega}{\mu\sigma}} \right] \quad (c)$$

or in terms of $R_m = \mu\sigma VL$ and $\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$

$$(k \pm) L = j \frac{R_m}{2} \pm j \sqrt{\left(\frac{R_m}{2}\right)^2 + j 2 \left(\frac{L}{\delta}\right)^2} \quad (d)$$

From Fig. 7.1.16 of the text we see that

$$k^+ = k_r^+ + jk_i^+, \quad k^- = k_r^- + jk_i^-$$

where

$$-k_r^- = k_r^+ > 0 \quad \text{and} \quad k_i^- > -k_i^+ > 0 \quad (e)$$

To meet the boundary condition of part (a) we must have

$$B_y(z, t) = C \left[e^{-jk^+ z} - e^{-jk^- z} \right] e^{j\omega t} \quad (f)$$

Using the boundary condition at $z = -L$

$$B_y(z, t) = \frac{B_0}{(e^{jk^+ L} - e^{jk^- L})} (e^{-jk^+ z} - e^{-jk^- z}) e^{j\omega t} \quad (g)$$

Part c

$$\nabla_x \bar{B} = -\bar{i}_x \frac{\partial B}{\partial z} = J_x \bar{i}_x / \mu_0 \quad (h)$$

$$J_x = \frac{jB_0 / \mu_0}{(e^{jk^+ L} - e^{jk^- L})} (k^+ e^{-jk^+ z} - k^- e^{-jk^- z}) e^{j\omega t} \quad (i)$$

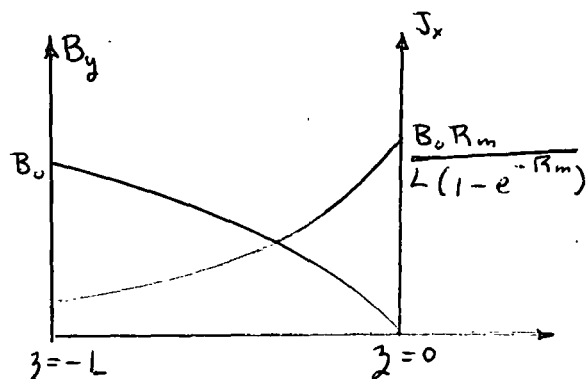
Part d

$$\text{As } \omega \rightarrow 0 \quad k^+ \rightarrow 0, \quad k^- \rightarrow j \frac{R_m}{L}$$

$$B_y = \frac{B_0}{(1 - e^{-R_m})} (1 - e^{(R_m/L)z}) \quad (j)$$

$$J_x = \frac{B_0/L}{(1 - e^{-R_m})} R_m e^{(R_m/L)z} \quad (k)$$

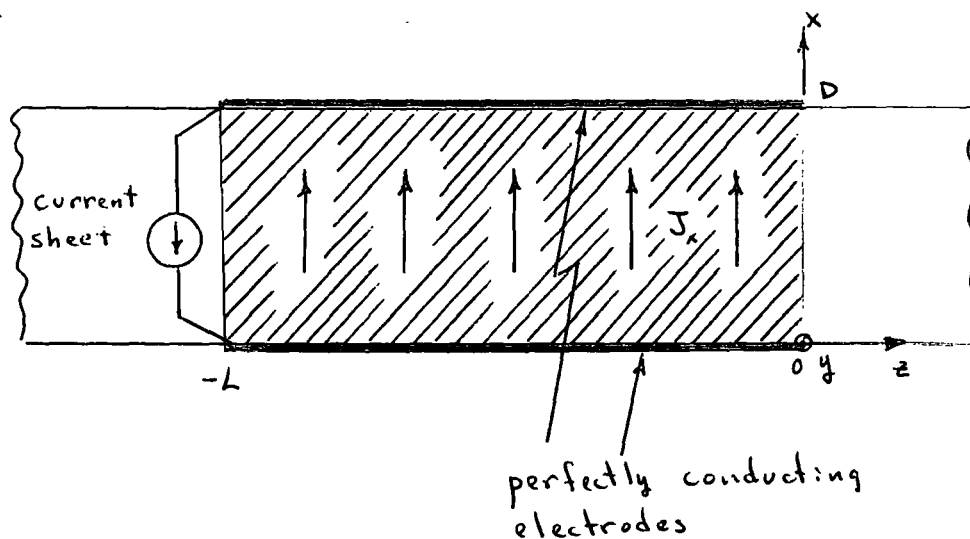
PROBLEM 7.13 (Continued)



As the sketch Fig. 7.1.9 of the text suggests, we could realize this problem by placing a current sheet source

$$\bar{K} = -\frac{B_0}{\mu_0} \bar{i}_x e^{j\omega t}$$

across the end $z = -L$ and providing perfect conductors to slide against the slab at $x = 0, D$. The top view of the slab then appears as shown in the figure.



Note from (j) and (k) that as $R_m \rightarrow 0$, the current density J_x is uniform and B_y is a linear function of z . This limiting case is as would be obtained with the given driving arrangement.

PROBLEM 7.14

Part a

Since $\bar{J}' = \bar{J}$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.14 (Continued)

$$\bar{\mathbf{K}} = \bar{\mathbf{i}}_z K_o \cos(kUt - kx) \quad (a)$$

$$= \bar{\mathbf{i}}_z K_o \cos(\omega t - kx); \omega = kU$$

Part b

The track can be taken as large in the y direction when it is many skin depths thick

$$L = \text{track thickness} \gg \delta = \sqrt{\frac{2}{\omega \mu_o \sigma}} = \sqrt{\frac{2}{kU \mu_o \sigma}} \quad (b)$$

In the track we have the diffusion equation

$$\frac{1}{\mu_o \sigma} \nabla^2 \bar{\mathbf{B}} = \frac{\partial \bar{\mathbf{B}}}{\partial t} \quad (c)$$

or, with $\bar{\mathbf{B}} = \text{Re } \hat{\mathbf{B}} \exp j(\omega t - kx)$,

$$\frac{1}{\mu_o \sigma} \left(\frac{\partial^2 \hat{\mathbf{B}}_x}{\partial y^2} - k^2 \hat{\mathbf{B}}_x \right) = j\omega \hat{\mathbf{B}}_x \quad (d)$$

Let $\hat{\mathbf{B}}_x(y) = C e^{\alpha y}$, then

$$\frac{1}{\mu_o \sigma} \alpha^2 = j\omega + \frac{k^2}{\mu_o \sigma} \quad (e)$$

$$\alpha = k \sqrt{1 + jS}; S = \frac{\omega \mu_o \sigma}{k^2} = \frac{U \mu_o \sigma}{k} \quad (f)$$

Since the track is modeled as infinitely thick

$$\mathbf{B}_x = C e^{\alpha y} e^{j(\omega t - kx)} \quad (g)$$

The gap between track and train is very thin; thus,

$$-\bar{\mathbf{i}}_y \times \frac{\bar{\mathbf{B}}}{\mu_o} = \bar{\mathbf{K}} = K_o e^{j(\omega t - kx)} \bar{\mathbf{i}}_z \quad (h)$$

which yields

$$\mathbf{B}_x(x, y, t) = \mu_o K_o e^{\alpha y} e^{j(\omega t - kx)} \quad (i)$$

We must also have $\nabla \cdot \bar{\mathbf{B}} = \partial \mathbf{B}_x / \partial x + \partial \mathbf{B}_y / \partial y = 0$ or

$$\mathbf{B}_y = \frac{j k}{\alpha} \mathbf{B}_x(x, y, t) \quad (j)$$

PROBLEM 7.14 (Continued)

To compute the current in the track we note that

$$\nabla \times \vec{B} = \vec{I}_z \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) = \mu_0 \vec{J} \quad (k)$$

$$\vec{J} = - \left(j \frac{s}{\alpha} k^2 \right) \frac{B_x}{\mu_0} (x, y, t) \vec{I}_z \quad (l)$$

Part c

The time average force density in the track is (see footnote, page 368)

$$\langle F_y \rangle = \frac{1}{2} \operatorname{Re}(J_z B_x^*) \quad (m)$$

Hence the time average lifting force per unit x-z area on the train is

$$\langle T_y \rangle = - \int_{-\infty}^0 \langle F_y \rangle dy = - \operatorname{Re} \int_{-\infty}^0 \frac{1}{2} J_z B_x^* dy \quad (n)$$

$$= \frac{1}{4} \mu_0 K_0^2 \left(\frac{\sqrt{1+s^2} - 1}{\sqrt{1+s^2}} \right) > 0 \quad (o)$$

See Fig. 7.1.21 of the text for a plot of this lifting force.

Part d

The time average force density in the track in the x direction is

$$\langle F_x \rangle = - \frac{1}{2} \operatorname{Re}(J_z B_y^*) \quad (p)$$

The force on the train in the x direction is then

$$\begin{aligned} \langle T_x \rangle &= - \int_{-\infty}^0 \langle F_x \rangle dy = \frac{1}{2} \operatorname{Re} \int_{-\infty}^0 J_z B_y^* dy \\ &= - \frac{\mu_0 K_0^2}{4} \frac{s}{\sqrt{1+s^2} \operatorname{Re} \sqrt{1+jS}} < 0 \end{aligned} \quad (q)$$

The problem is that this force drags the train instead of propelling it in the x direction. (See Fig. 7.1.20 of the text for a plot of the magnitude of this drag force). To make matters worse, if the train stops, the magnetic levitation force becomes zero.

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.15

Part a

Let the current sheet lie in the plane $y = -s$. In the region $-s < y < 0$ we have the "diffusion equation"

$$\nabla^2 B_z = 0 \quad (a)$$

If $B_z(x, y, t) = B_z(y) e^{j(\omega t - kx)}$ this equation yields

$$\frac{\partial^2 B_z}{\partial y^2} = k^2 B_z \quad (b)$$

Hence we can conclude that

$$B_z = [A \cosh k(y+s) + B \sinh k(y+s)] e^{j(\omega t - kz)} \quad (c)$$

At $y = -s$ we have the boundary condition

$$\bar{i}_y \times \bar{B}_z = \mu_0 K_0 \cos(\omega t - kz) \bar{i}_x \quad (d)$$

Thus

$$B_z = [\mu_0 K_0 \cosh k(y+s) + B \sinh k(y+s)] e^{j(\omega t - kz)} \quad (e)$$

Since $\nabla \cdot \bar{B} = \partial B_y / \partial y + \partial B_z / \partial z = 0$ we must have

$$B_y = [j(\mu_0 K_0 \sinh k(y+s) + B \cosh k(y+s))] e^{j(\omega t - kz)} \quad (f)$$

In the conductor the diffusion equation is

$$\frac{1}{\mu_0 \sigma} \nabla^2 \bar{B} = \frac{\partial \bar{B}}{\partial t} + v \frac{\partial \bar{B}}{\partial z} \quad (g)$$

Then

$$\frac{\partial^2 B_z}{\partial y^2} = (j\mu_0 \sigma (\omega - kv) + k^2) B_z \quad (h)$$

which suggests a solution

$$B_z(y) = C e^{-\alpha y}, \quad \alpha = k\sqrt{1+jS}, \quad S = \frac{\mu_0 \sigma (\omega - kv)}{k^2} \quad (i)$$

Since $\nabla \cdot \bar{B} = 0$ in the conductor too, we must have

$$B_y = -j \frac{k}{\alpha} B_z \quad (j)$$

As the boundary $y = 0$ we must have

$$B_{y1} = B_{y2}, \quad H_{z1} = H_{z2} \quad (k)$$

Note that

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.15 (Continued)

$$\begin{aligned} & \cosh ks B_{z2} + j \sinh ks B_{y2} \\ & = \mu_o K_o (\cosh^2 ks - \sinh^2 ks) = \mu_o K_o \end{aligned} \quad (l)$$

Then we must also have

$$\begin{aligned} \mu_o K_o & = \cosh ks B_{z1} + j \sinh ks B_{y1} \\ & = C (\cosh ks + \frac{k}{\alpha} \sinh ks) \end{aligned} \quad (m)$$

It follows that the $\bar{B}$ field for $y > 0$ is

$$\bar{B} = \frac{\mu_o K_o}{\cosh ks + \frac{k}{\alpha} \sinh ks} \left(-j \frac{k}{\alpha} \bar{i}_y + \bar{i}_z \right) e^{-\alpha y} e^{j(\omega t - kz)} \quad (n)$$

Comparing with Eq. 7.1.91 of Sec. 7.1.4 of the text we see that it is only necessary to replace

$$K_o \text{ by } \frac{K_o}{\cosh ks + \frac{k}{\alpha} \sinh ks}$$

starting with Eq. 7.1.90. The average forces depend on the magnitude, not the phase, of K_o , which is reduced by this substitution.

Part b

We note that if $ks \ll 1$

$$\frac{K_o}{\cosh ks + \frac{k}{\alpha} \sinh ks} \approx K_o \quad (o)$$

which shows that the results of Sec. 7.1.4 are valid when $ks \ll 1$.

Part c

When $ks \rightarrow \infty$

$$\frac{K_o}{\cosh ks + \frac{k}{\alpha} \sinh ks} \rightarrow 0$$

No fields will then be present in the conductor.

PROBLEM 7.16

Part a

Because the charge needs time to move through the conductor, at $t=0^+$ there is only free charge on the plates. The electric fields are directed in the negative vertical direction and satisfy

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.16 (Continued)

$$E_{\ell} b + E_g a = V_o \quad (a)$$

at the interface at $t=0^+$

$$\epsilon E_{\ell} = \epsilon_o E_g \quad (b)$$

Hence at $t=0^+$

$$E_{\ell} = \frac{V_o}{b + \frac{\epsilon}{\epsilon_o} a}, \quad E_g = \frac{V_o}{\frac{\epsilon_o}{\epsilon} b + a} \quad (c)$$

Part b

As $t \rightarrow \infty$ the charge on the interface excludes the fields from the conducting liquid, hence

$$E_{\ell} = 0 \quad E_g = \frac{V_o}{a} \quad (f)$$

Part c

The charge on the interface at any time is

$$\sigma_f = \epsilon E_{\ell} - \epsilon_o E_g \quad (g)$$

Conservation of charge requires

$$\frac{d\sigma_f}{dt} = -\sigma E_{\ell} \quad (h)$$

The voltage across the plates is V_o for $t > 0$

$$V_o = E_{\ell} b + E_g a \quad (i)$$

Solving g, h, i we find that the charge obeys

$$\frac{(\epsilon + \epsilon_o b/a)}{\sigma} \frac{d\sigma_f}{dt} + \sigma_f = -\frac{\epsilon_o}{a} V_o \quad (j)$$

Let $\tau = \frac{\epsilon + \epsilon_o b/a}{\sigma}$, then

$$\sigma_f = -\frac{\epsilon_o V_o}{a} (1 - e^{-t/\tau}), \quad t \geq 0 \quad (k)$$

$$q_f = -\frac{\epsilon_o A V_o}{a} (1 - e^{-t/\tau}), \quad t \geq 0 \quad (l)$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.17

Part a

In the inner sphere

$$\frac{\sigma_1}{\epsilon_0} \rho_f + \frac{\partial \rho_f}{\partial t} = 0 \quad (a)$$

So we find that

$$\rho_f(r, t) = \rho_0(r) e^{-\sigma_1 / \epsilon_0 t}, \quad t \geq 0 \quad r < R_1 \quad (b)$$

A similar equation holds for the charge in the outer sphere, but it has no initial charge distribution at $t = 0$, so

$$\rho_f(r, t) = 0, \quad t \geq 0 \quad R_1 < r < R_0 \quad (c)$$

Part b

Let

$$Q_0 = \int_0^{R_1} 4\pi r^2 \rho_0(r) dr \quad (d)$$

Also define

σ_A = the surface charge density at $r = R_1$

σ_B = the surface charge density at $r = R_0$

The field at R_0^+ is, by Gauss' law

$$E(R_0^+) = \frac{Q_0}{4\pi \epsilon_0 R_0^2} \quad (e)$$

Then, conservation of charge requires that the electric field at $r = R_0^-$ obey

$$\sigma_0 E(R_0^-) + \epsilon_0 \frac{\partial E}{\partial t}(R_0^-) = 0 \quad (f)$$

$$E(R_0^-) = \frac{Q_0}{4\pi \epsilon_0 R_0^2} e^{-\sigma_0 / \epsilon_0 t}, \quad t \geq 0 \quad (g)$$

We can thus conclude that

$$\sigma_B = \frac{Q_0}{4\pi R_0^2} (1 - e^{-\sigma_0 / \epsilon_0 t}), \quad t \geq 0 \quad (h)$$

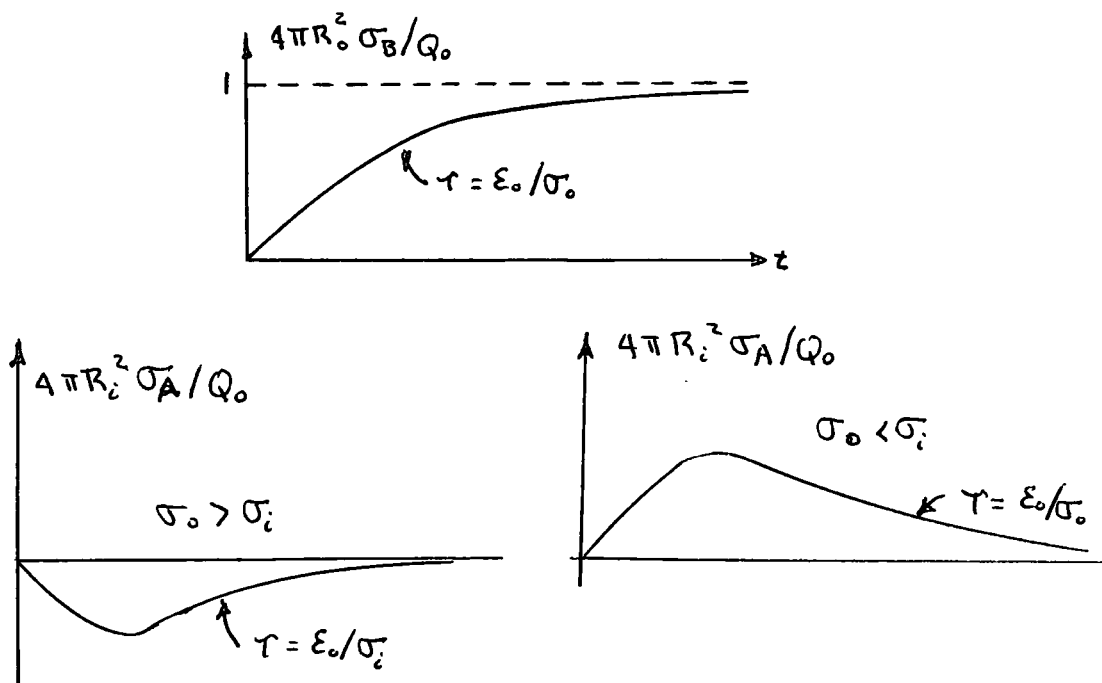
Since charge is conserved we now know that

$$\sigma_A = \frac{Q_0}{4\pi R_1^2} (e^{-\sigma_0 / \epsilon_0 t} - e^{-\sigma_1 / \epsilon_0 t}), \quad t \geq 0 \quad (i)$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.17 (Continued)

Part c



PROBLEM 7.18

Part a

At the radius b

$$\epsilon[E(b^+) - E(b^-)] = \sigma_f \quad (a)$$

$$\sigma[E(b^+) - E(b^-)] = -\frac{\partial \sigma_f}{\partial t} = -\epsilon \frac{\partial}{\partial t} [E(b^+) - E(b^-)] \quad (b)$$

For $t < 0$ when the system has come to rest

$$\nabla \cdot \vec{J} = (\sigma/\epsilon) \nabla \cdot \vec{E} = -\frac{\partial \rho_f}{\partial t} = 0 \quad (c)$$

For cylindrical geometry this has the solution

$$\vec{E} = +\frac{A}{r} \vec{i}_r; \quad V_o = + \int_a^b E_r dr = A \ln(b/a) \quad (d)$$

PROBLEM 7.18 (Continued)

then

$$\left. \begin{aligned} E(r=b^-) &= + \frac{V_o}{\ln(b/a)} \frac{1}{b} \\ E(r=b^+) &= 0 \end{aligned} \right\} t = 0^- \quad (e)$$

 Since $E(b^+) - E(b^-) = \sigma_f/\epsilon$ it cannot change instantaneously, so

$$E(b^+) - E(b^-) = - \frac{V_o}{b \ln(b/a)} e^{-(\sigma/\epsilon)t}, \quad t \geq 0 \quad (f)$$

 Because there is no initial charge between the shells, there will be no charge between the shells for $t \geq 0$, thus

$$E_r = \begin{cases} + \frac{C_1(t)}{r} & a < r < b \\ + \frac{C_2(t)}{r} & b < r < c \end{cases} \quad t \geq 0 \quad (g)$$

The battery adds the constraint

$$V_o = C_1 \ln(b/a) + C_2 \ln(c/b) \quad (h)$$

while (f) becomes

$$C_1 - C_2 = \frac{V_o}{\ln(b/a)} e^{-(\sigma/\epsilon)t} \quad (i)$$

 Solving (h) and (i) for C_1, C_2

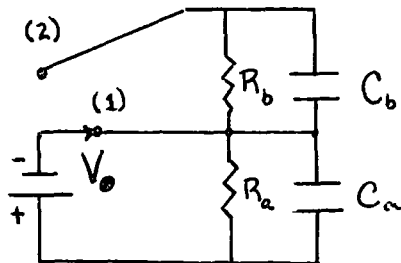
$$C_2 = \frac{V_o}{\ln c/a} (1 - e^{-(\sigma/\epsilon)t}) \quad (j)$$

$$C_1 = \frac{V_o}{\ln c/a} (1 + \frac{\ln(c/b)}{\ln(b/a)} e^{-(\sigma/\epsilon)t}) \quad (k)$$

Part b

$$\sigma_f = \epsilon(E(b^+) - E(b^-)) = - \frac{\epsilon}{b \ln(b/a)} V_o e^{-(\sigma/\epsilon)t} \quad (l)$$

Part c



$$R_b = \frac{\ln c/b}{2\pi\sigma}, \quad C_b = \frac{2\pi\epsilon}{\ln c/b}$$

$$R_a = \frac{\ln(b/a)}{2\pi\sigma}, \quad C_a = \frac{2\pi\epsilon}{\ln b/a}$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.19

While the potential v is applied the system reaches an equilibrium. During this time

$$\nabla \cdot \bar{J} = \frac{\sigma}{\epsilon} \rho_f = - \frac{\partial \rho_f}{\partial t} \quad (a)$$

in the bulk of the liquid. If the potential V is applied for many time constants ($\tau = \epsilon/\sigma$) any charge in the fluid decays away. For $t > 0$ if the fluid is incompressible ($\nabla \cdot \bar{v} = 0$) and $\bar{J} = \sigma \bar{E} + \rho_f \bar{v}$ we know that

$$\nabla \cdot \bar{J} = (\sigma/\epsilon) \rho_f + \bar{v} \cdot \nabla \rho_f = - \frac{\partial \rho_f}{\partial t} \quad (b)$$

But in a frame moving with the particles of fluid

$$\frac{d}{dt} \rho_f = \frac{\partial \rho_f}{\partial t} + \bar{v} \cdot \nabla \rho_f = - (\sigma/\epsilon) \rho_f \quad (c)$$

$$\rho_f(t) = \rho_f(t=0) e^{-(\sigma/\epsilon) t} \quad t \geq 0 \quad (d)$$

where $\rho_f(t)$ is the local charge seen by a moving particle. But for all fluid particles

$$\rho_f(t = 0) = 0 \quad (e)$$

Hence the charge remains zero everywhere for $t \geq 0$.

Now draw a volume around the upper sphere big enough to enclose it for a few seconds even though it is moving.

$$\oint_S \bar{J} \cdot d\bar{a} = - \frac{d}{dt} \int_V \rho_f dV \quad (f)$$

Now because $\rho_f = 0$ in the fluid

$$\bar{J} = \sigma \bar{E}, \quad \oint_S \bar{J} \cdot d\bar{a} = (\sigma/\epsilon) \oint_S \epsilon \bar{E} \cdot d\bar{a} = (\sigma/\epsilon) Q(t) \quad (g)$$

Then

$$(\sigma/\epsilon) Q(t) = - \frac{d}{dt} \int_V \rho_f dV = - \frac{d}{dt} Q(t) \quad (h)$$

which has solution

$$Q(t) = Q e^{-t/\tau}; \tau = \epsilon/\sigma$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.20

Part a

We can use Gauss' law

$$\oint_S \epsilon_0 \vec{E} \cdot d\vec{a} = \int_V \rho_f dV \quad (a)$$

to determine the electric field if we note that there is no net charge in the system, which means that

$$\vec{E} = E_x \hat{i}_x = 0 \quad x < 0 \text{ and } x > 3d \quad (b)$$

$$\epsilon_0 E_x(x) = \int_0^x \frac{Q}{D^2 d} dx = \frac{Q}{D^2} \frac{x}{d} \quad \begin{cases} 0 < x < d \\ t = 0 \end{cases} \quad (c)$$

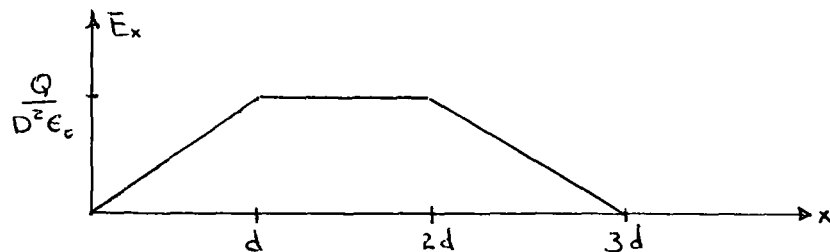
There is no charge in the middle region so

$$E_x = \frac{Q}{D^2 \epsilon_0} \quad d < x < 2d; \quad t = 0 \quad (d)$$

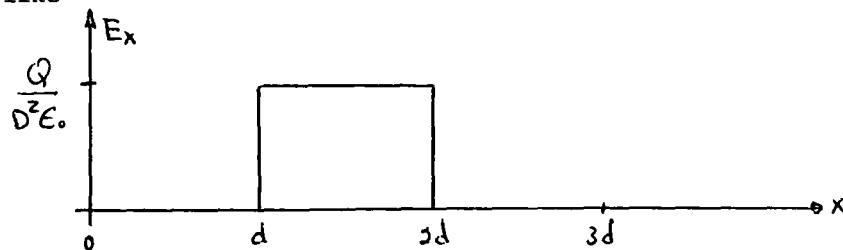
In the region $2d < x < 3d$

$$\epsilon_0 (E_x(x) - E_x(2d)) = \int_{2d}^x \frac{-Q}{D^2 d} dx = -\frac{Q}{D^2} \frac{(x-2d)}{d} \quad (e)$$

$$E_x(x) = \frac{Q}{D^2 \epsilon_0} \frac{(3d-x)}{d} \quad \begin{cases} 2d < x < 3d \\ t = 0 \end{cases} \quad (f)$$



As $t \rightarrow \infty$ all the charge on the lower plate relaxes to the surface $x = d$, while the charge on the upper plate relaxes to the surface $x = 2d$. The electric field then looks like

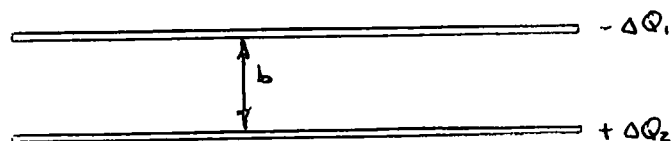


MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.20 (Continued)

Part b

Each charge distribution can be thought of as made up of many thin charges sheets; any two such sheets,



one located somewhere in the top conductor, one located somewhere in the bottom conductor, attract each other with a force

$$\Delta F = \frac{\Delta Q_1 \Delta Q_2}{2\epsilon_0 D^2} \quad (g)$$

which is independent of their separation, hence the net attractive force between plates does not change with time. At $t \rightarrow \infty$ there is a surface charge

$$\sigma_T = -\frac{Q}{D^2} \quad x = 2d \quad (h)$$

$$\sigma_B = +\frac{Q}{D^2} \quad x = d$$

and the force per unit area T_x is simply that found for a pair of capacitor plates having separation d and supporting surface charge densities $\pm Q$. (See Sec. 3.1.2b).

$$T_x = \frac{Q^2}{2\epsilon_0 D^2} \quad t \geq 0 \quad (i)$$

This force can be easily seen to be constant from the viewpoint taken in Chapter 8, where the force on the lower plate can be found from the Maxwell Stress Tensor.

The only contribution comes from $T_{xx} = \frac{1}{2} \epsilon_0 E_x^2$ evaluated at $x = d$, and thus $T_{xx}(x = d) = T_x$ as given by (i) regardless of t . Problem 8.23 is worked out following the stress-tensor approach.

PROBLEM 7.21

Part a

If the electric field beyond the plates is zero the conservation of charge equation

$$\oint_S \vec{J} \cdot d\vec{a} = -\frac{\partial}{\partial t} \int_V \rho_f dV = -\frac{\partial}{\partial t} \oint_S \epsilon \vec{E} \cdot d\vec{a} \quad (a)$$

MAGNETIC DIFFUSION OF CHARGE RELAXATION

PROBLEM 7.21 (Continued)

becomes

$$\sigma \hat{E}_x(x) - \frac{\hat{I}}{A} = -j\omega \epsilon \hat{E}_x(x) \quad (b)$$

That is, the equation for $\hat{E}_x$ is as given by (f) of Example 7.2.3, with ϵ now a function of x .

$$\hat{E}_x(x) = \frac{\hat{I}}{A(\sigma + j\omega \epsilon)} = \frac{\hat{I}/A}{[\sigma_1 + \frac{\sigma_2}{\ell} x + j\omega(\epsilon_1 + \frac{\epsilon_2}{\ell} x)]} \quad (c)$$

From Coulomb's law

$$\hat{\rho}_f = \frac{d}{dx}(\epsilon \hat{E}_x) = -\frac{\epsilon \hat{I}}{A} \frac{(j\omega \frac{d\epsilon}{dx} + \frac{d\sigma}{dx})}{(j\omega \epsilon + \sigma)^2} + \frac{\hat{I}}{A} \frac{\frac{d\epsilon}{dx}}{(j\omega \epsilon + \sigma)} \quad (d)$$

$$\frac{\hat{\rho}_f A}{\hat{I}} = -\frac{(\epsilon_1 + \frac{\epsilon_2}{\ell} x)(j\omega \frac{\epsilon_2}{\ell} + \frac{\sigma_2}{\ell})}{[(\sigma_1 + \frac{\sigma_2}{\ell} x) + j\omega(\epsilon_1 + \frac{\epsilon_2}{\ell} x)]^2} + \frac{\frac{\epsilon_2}{\ell}}{[(\sigma_1 + \frac{\sigma_2}{\ell} x) + j\omega(\epsilon_1 + \frac{\epsilon_2}{\ell} x)]} \quad (e)$$

Part b

Consider the effect of a small change in ϵ alone

$$\sigma_2 = 0; \epsilon_2/\epsilon_1 \ll 1$$

then

$$\hat{\rho}_f \approx \frac{\sigma_1 \epsilon_2 \hat{I}}{A \ell (j\omega \epsilon_1 + \sigma_1)^2} \quad (f)$$

It is seen from (f) that in the presence of conduction the gradient of ϵ causes free charge to be stored in the bulk of the fluid. This effect is highly dependent on frequency, being greatest at zero frequency and disappearing when the cycle time is short compared to the relaxation time of the material.

PROBLEM 7.22

Part a

In the fluid the constitutive law for conduction is

$$\bar{J} = \sigma \bar{E} + \rho_f \bar{v} \quad (a)$$

Since the given velocity distribution has the property

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.22 (Continued)

$$\nabla \cdot \vec{v} = 0 \quad (b)$$

$$\nabla \cdot \vec{J} = \frac{\sigma}{\epsilon} \nabla \cdot (\epsilon E) + \vec{v} \cdot \nabla \rho_f = \frac{\sigma}{\epsilon} \rho_f + U \frac{\partial}{\partial x} \rho_f = - \frac{\partial \rho_f}{\partial t} \quad (c)$$

or

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \rho_f = - \frac{\sigma}{\epsilon} \rho_f \quad (d)$$

The charge is relaxing in the frame of the moving fluid. The solution has the form

$$\rho_f = \text{Re } \hat{\rho}_0 e^{-\frac{\sigma}{\epsilon} \frac{x}{U}} e^{j\omega(t - \frac{x}{U})}, \quad -\frac{\Delta}{2} < y < \frac{\Delta}{2} \quad (e)$$

= 0 elsewhere in the channel

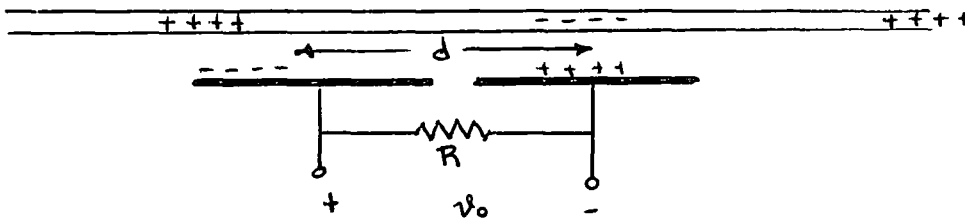
where $y = 0$ is the channel center. Note that (e) satisfies the boundary condition at $x = 0$ and states that a charge at x at time t has been decaying $\frac{x}{U}$ seconds (since it left the source) and was dumped in the channel at time

$$t' = t - \frac{x}{U}$$

Substitution of (e) into (d) verifies that it is a solution.

Part b

From (e) it is clear that the wave length of the sinusoidally (and decaying) charge stream is $2\pi U/\omega$. Thus, the wave length can be altered simply by changing ω . One technique for measuring the flow velocity would consist in measuring the voltage induced across the resistance R (as shown in the figure) as a function of the frequency. With the distance between electrode centers d equal to $1/2$ wave length, a peak in the output signal would be expected. If we call the frequency at which this peak occurs ω_p , then



MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.22 (Continued)

$$\frac{2\pi U}{\omega_p} = 2d$$

or

$$U = \frac{d\omega_p}{\pi}$$

Thus, a determination of ω_p gives U . There are, of course, problems with this approach. For example, there would be lesser peaks in the output at harmonic frequencies that could be mistaken for the desired peak. Alternatives are to use the decay rate, but such techniques are vulnerable to conductivity variations which are likely to be large.

PROBLEM 7.23

Part a

Current is carried by the conductor because of normal conduction and also because of convection of a net charge.

$$\bar{J} = \sigma \bar{E} + \rho_f \bar{v}$$

Also

$$\nabla \cdot \bar{E} = \rho_f / \epsilon = \nabla \cdot \frac{(\bar{J} - \rho_f \bar{v})}{\sigma}$$

But

$$\nabla \cdot \bar{J} = - \frac{\partial \rho_f}{\partial t} = 0 \text{ in steady state}$$

$$\nabla \cdot \bar{v} = \nabla \cdot (U \bar{i}_x) = 0 \text{ also, so that}$$

$$\rho_f / \epsilon = - \frac{\bar{v} \cdot \nabla \rho_f}{\sigma} = - \frac{U}{\sigma} \frac{\partial \rho_f}{\partial x}$$

The solution to this last equation is

$$\rho = \rho_o e^{-\left(\frac{\sigma}{\epsilon}\right)\left(\frac{x}{U}\right)}$$

i.e., the charge relaxes in the conductor; the time $\tau = \frac{x}{U}$ is a measure of how long since the charge left the source at the first screen.

Part b

Let

$$E_x(x=0) = E_o$$

$$\frac{\partial E_x}{\partial x} = \frac{\rho(x)}{\epsilon} = \frac{\rho_o}{\epsilon} e^{-\left(\frac{\sigma}{\epsilon}\right)\left(\frac{x}{U}\right)}$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.23 (Continued)

$$E_x(x) = E_o + \int_o^x \frac{\rho_f(x)}{\epsilon} dx = E_o + \frac{\rho_o U}{\sigma} \left(1 - e^{-\frac{\sigma}{\epsilon} \frac{x}{U}}\right)$$

Note that since $J_x(x=0) = \sigma E_o + \rho_o U = \frac{V}{RA}$

$$E_x(x) = \frac{V}{RA\sigma} - \frac{\rho_o U}{\sigma} e^{-\left(\frac{\sigma}{\epsilon}\right)\left(\frac{x}{U}\right)}$$

We must finish the problem to know V

Part c

$$V = - \int_o^l E_x(x) dx = - \frac{Vl}{RA\sigma} + \rho_o \epsilon_o \left(\frac{U}{\sigma}\right)^2 \left(1 - e^{-\left(\frac{\sigma}{\epsilon}\right) \frac{l}{U}}\right)$$

$$V = \left(\frac{1}{1 + \frac{l}{RA\sigma}}\right) \rho_o \epsilon \left(\frac{U}{\sigma}\right)^2 \left(1 - e^{-\left(\frac{\sigma}{\epsilon}\right) \left(\frac{l}{U}\right)}\right)$$

PROBLEM 7.24

Part a

The model for this problem is similar to that used in Example 7.2.6 of the text. Each ring induces a charge on the stream having opposite polarity to its potential. Thus, conservation of charge for the can at potential v_3 (under the ring at potential v_1) is

$$-C_1 n v_1 = C \frac{dv_3}{dt} + \frac{v_3}{R} \quad (a)$$

Similarly, for the other two cans,

$$-C_1 n v_2 = C \frac{dv_1}{dt} + \frac{v_1}{R} \quad (b)$$

$$-C_1 n v_3 = C \frac{dv_2}{dt} + \frac{v_2}{R} \quad (c)$$

To solve these three equations, we assume solutions of the form

$$v_1 = \hat{v}_1 e^{st} \quad (d)$$

and the complex amplitudes $\hat{v}_1$ are governed by the conditions that follow from substitution of (d) into (a)-(c)

$$\begin{bmatrix} C_1 n & 0 & (Cs + \frac{1}{R}) \\ (Cs + \frac{1}{R}) & C_1 n & 0 \\ 0 & (Cs + \frac{1}{R}) & C_1 n \end{bmatrix} = 0 \quad (e)$$

PROBLEM 7.24 (Continued)

The solution for s is

$$s = -\frac{1}{RC} + \frac{C_1 n}{C} \left[\frac{1}{2} \pm \frac{j\sqrt{3}}{2}, -1 \right] \quad (f)$$

Part b

Thus, the system is unstable if

$$\frac{1}{RC} < \frac{C_1 n}{2C} \quad (g)$$

Part c

In particular, from (g), the system is self-excited as

$$\frac{1}{R} = \frac{C_1 n}{2} \quad (h)$$

Part d

The frequency of oscillation under condition (h) follows from (f) and (h) as

$$\omega = \frac{C_1 n \sqrt{3}}{2C} = \frac{\sqrt{3}}{RC} \quad (i)$$

PROBLEM 7.25

The crucial quantities in the respective systems are the magnetic diffusion time (Eq. 7.1.28) and the charge relaxation time (Eq. 7.2.11) relative to the period of excitation $T = 1/f$. The conductivities required to make these respective times equal to the excitation period T are

$$\sigma = \pi^2 T / \mu_0 d^2 \quad (a)$$

$$\sigma = \epsilon / T \quad (b)$$

In terms of the given numbers,

$$\begin{aligned} \sigma &= (3.14)^2 (10^{-5}) / (4) (3.14 \times 10^{-7}) (10^{-4}) \\ &= 7.85 \times 10^7 \text{ mhos/m} \end{aligned} \quad (c)$$

and

$$\sigma = (81) (8.85 \times 10^{-12}) / 10^{-5} = 7.16 \times 10^{-5} \text{ mhos/m} \quad (d)$$

For the change in depth to have a large effect on the inductance, the conductivity must be greater than that given by (c). Thus, the magnetic device would not be satisfactory. By contrast, (d) indicates that the conductivity of the electric apparatus is more than sufficient to make a change in capacitance with liquid depth apparent even if $\epsilon = \epsilon_0$. Both devices would be attractive for this application only if the conductivity exceeded that given by (c).

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.26

This problem depends on the same physical reasoning as used in connection with Prob. 7.25. There are two modes in which either device can operate. Consider configuration (a): the inductance can change either because of the magnetization of the water, or because of currents induced in the water. However, water is only weakly magnetic and so the first mode of operation is not attractive. Moreover, the frequency is too low to induce appreciable currents, as can be seen by comparing the magnetic diffusion time to the period of excitation. Hence, configuration (a) does not represent an attractive approach to the engineering problem.

On the other hand, configuration (b) can operate either because of a change in capacitance between the electrodes due to the change in position of the polarized liquid (at high frequencies) or due to a change in position of a perfectly conducting liquid (low frequencies). As the calculations of Prob. 7.26 show, it is this last mode of operation that is appropriate in this case.

PROBLEM 7.27

Part a

Because we have changed only a boundary condition, the potentials in regions (a) and (b) are still of the general form

$$\begin{aligned}\hat{\phi}_a &= A \sinh kx + B \cosh kx \\ \hat{\phi}_b &= C \sinh kx + D \cosh kx\end{aligned}\tag{a}$$

There are now four boundary conditions:

$$\hat{\phi}_a(d) = \hat{V}\tag{b}$$

$$\hat{\phi}_a(0) = \hat{\phi}_b(0)\tag{c}$$

$$-\sigma \frac{\partial \phi_b(0)}{\partial x} = \left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial z} \right) \left(-\epsilon_0 \frac{\partial \phi_a(0)}{\partial x} + \epsilon \frac{\partial \phi_b(0)}{\partial x} \right)\tag{d}$$

$$\phi_b(-f) = 0\tag{e}$$

Only boundary condition (e) is new; it has replaced the assumption that ϕ_b must go to zero as $x \rightarrow -\infty$.

Solving for A, B, C and D we find that

PROBLEM 7.27 (Continued)

$$\phi_a = \text{Re} \frac{\hat{V}}{\Delta} [(1+jS \epsilon/\epsilon_o) \sinh kx + jS \tanh kf \cosh kx] e^{j(\omega t - kz)} \quad (f)$$

$$\phi_b = \text{Re} \frac{\hat{V}}{\Delta} [jS \sinh kx + jS \tanh kf \cosh kx] e^{j(\omega t - kz)} \quad (g)$$

where $\Delta = (1+jS \epsilon/\epsilon_o) \sinh kd + jS \tanh kf \cosh kd$.

Part b

If

$$|fk| \gg 1$$

$$\tanh kf \rightarrow 1 \quad (h)$$

A comparison shows that in this limit the results agree with Sec. 7.2.4 if we note that

$$e^{kx} = \cosh kx + \sinh kx \quad (i)$$

PROBLEM 7.28

Part a

The regions between the traveling wave electrodes and the moving sheet are free space, and therefore the fields are governed by

$$\nabla^2 \phi = 0 \quad (a)$$

where

$$\bar{E} = -\nabla \phi \quad (b)$$

Moreover, solutions that have the same (z-t) dependence as the imposed traveling wave potentials, and that satisfy (a) are

$$\phi_a = \text{Re} [A_1 \cosh kx + A_2 \sinh kx] e^{j(\omega t - kz)} \quad (c)$$

$$\phi_b = \text{Re} [B_1 \cosh kx + B_2 \sinh kx] e^{j(\omega t - kz)} \quad (d)$$

The constants A_1, A_2, B_1, B_2 must be adjusted to make these solutions satisfy the boundary conditions

$$\hat{\phi}_a = \hat{V}_o \quad \text{at} \quad x = c \quad (e)$$

$$\hat{\phi}_b = \hat{V}_o \quad \text{at} \quad x = -c \quad (f)$$

$$\hat{\phi}_a = \hat{\phi}_b \quad \text{at} \quad x = 0 \quad (g)$$

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) (\epsilon_a E_x^a - \epsilon_b E_x^b) + \sigma_s \frac{\partial E_z^a}{\partial z} = 0 \quad (h)$$

Part b

The symmetry requires that

PROBLEM 7.28 (Continued)

$$\phi_a(x, z, t) = \phi_b(-x, z, t) \quad (i)$$

and this implies that $A_1 = B_1$, $A_2 = -B_2$. The boundary conditions become

$$A_1 \cosh kc + A_2 \sinh kc = \hat{V}_0 \quad (j)$$

$$jS (2A_2) = A_1 \quad (k)$$

where

$$S = (\omega - kU)\epsilon_0 / k\sigma_s \quad (l)$$

Thus,

$$A_1 = B_1 = 2j \hat{V}_0 / (\sinh kc + 2j S \cosh kc) \quad (m)$$

and

$$A_2 = -B_2 = \hat{V}_0 / (\sinh kc + 2j S \cosh kc) \quad (n)$$

Part b

A section of the sheet can be enclosed by a thin volume of small area in the y-z plane to give the force per unit area as

$$T_z = 2T_{zx}^a \quad (x = 0) \quad (o)$$

where the symmetry has been used to set

$$T_{zx}^a = -T_{zx}^b \quad (p)$$

Thus, the time average force per unit area is

$$\langle T_z \rangle = \text{Re}[\epsilon_0 \hat{E}_x^a(0) \hat{E}_z^{a*}(0)] \quad (q)$$

and from (m) and (n),

$$\langle T_z \rangle = \text{Re}[\epsilon_0 (-jk) A_1^* (-k) A_2] \quad (r)$$

$$= \text{Re} \left[\frac{2\epsilon_0 k^2 |\hat{V}_0|^2 S}{\sinh^2 kc + 4S^2 \cosh^2 kc} \right] \quad (s)$$

$$= \frac{2\epsilon_0 k^2 |\hat{V}_0|^2 S}{(\sinh^2 kc + 4S^2 \cosh^2 kc)} \quad (t)$$

It follows from (t) that the maximum occurs as

$$S^* = \frac{1}{2} \tanh kc \quad (u)$$

or

$$\omega = kU + \frac{\sigma_s k}{2\epsilon_0} \tanh kc \quad (v)$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.28 (Continued)

Part c

Note that if S is held fixed at the value given by (u), the force per unit area remains fixed. Thus, as $\sigma_s \rightarrow 0$, the velocities of the potential wave and the sheet must become equal to retain the force at a constant value

$$\omega \rightarrow kU$$

(w)

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.1

The identity to be verified is

$$\nabla \cdot (\psi \bar{A}) = \psi \nabla \cdot \bar{A} + \bar{A} \cdot \nabla \psi \quad (a)$$

First express the identity in index notation.

$$\frac{\partial}{\partial x_m} [\psi A_m] = \psi \frac{\partial A_m}{\partial x_m} + A_m \frac{\partial \psi}{\partial x_m} \quad (b)$$

The repeated subscript indicates summation. Thus, expanding the first term on the left yields:

$$\psi \frac{\partial A_m}{\partial x_m} + A_m \frac{\partial \psi}{\partial x_m} \equiv \psi \nabla \cdot \bar{A} + \bar{A} \cdot \nabla \psi \quad (c)$$

PROBLEM 8.2

We wish to show that

$$\bar{B} \cdot \nabla (\psi \bar{A}) = \psi \bar{B} \cdot \nabla \bar{A} + \bar{A} \bar{B} \cdot \nabla \psi \quad (a)$$

First, the identity is expressed in index notation, considering the m^{th} component of this vector equation. Note that the equation relates two vectors.

$$(\bar{B} \cdot [\nabla (\psi \bar{A})])_m = (\psi \bar{B} \cdot [\nabla \bar{A}])_m + A_m \bar{B} \cdot \nabla \psi \quad (b)$$

Now, consider each term separately

$$(\bar{B} \cdot [\nabla (\psi \bar{A})])_m = B_k \frac{\partial}{\partial x_k} (\psi A_m) = A_m B_k \frac{\partial \psi}{\partial x_k} + \psi B_k \frac{\partial A_m}{\partial x_k} \quad (c)$$

$$(\psi \bar{B} \cdot [\nabla \bar{A}])_m = \psi B_k \frac{\partial A_m}{\partial x_k} \quad (d)$$

$$A_m \bar{B} \cdot \nabla \psi = A_m B_k [\nabla \psi]_k = A_m B_k \frac{\partial \psi}{\partial x_k} \quad (e)$$

The sum of (d) and (e) give (c) so that the identity is verified.

PROBLEM 8.3

Part a

a_{ik} is the cosine of the angle between the x'_i axis and the x_k axis (see page 435). Thus for our geometry

$$a_{ik} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (a)$$

PROBLEM 8.3 (Continued)

Now, we may apply the transformation law for vectors (Eq. 8.2.10)

$$A'_i = a_{ik} A_k \quad (b)$$

where the components of $\bar{A}$ in the (x_1, x_2, x_3) system are given as

$$A_1 = 1; A_2 = 2; A_3 = -1 \quad (c)$$

Thus:

$$A'_1 = a_{1k} A_k = a_{11} A_1 + a_{12} A_2 + a_{13} A_3 \quad (d)$$

$$A'_1 = 1/2 + \sqrt{3} \quad (e)$$

$$A'_2 = a_{2k} A_k = -\frac{\sqrt{3}}{2} + 1 \quad (f)$$

$$A'_3 = a_{3k} A_k = -1 \quad (g)$$

Using matrix algebra, we can write a more concise solution. That is:

$$\begin{bmatrix} A'_1 \\ A'_2 \\ A'_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} (\frac{1}{2} + \sqrt{3}) \\ (-\frac{\sqrt{3}}{2} + 1) \\ (-1) \end{bmatrix} \quad (h)$$

Part b

The tensor a_{ik} is associated with coordinate transforms involving the direction of force while the tensor a_{jl} is associated with coordinate transforms involving the direction of the area normal vectors. The tensor transformation is (Eq. 8.2.17), page 437;

$$T'_{ij} = a_{ik} a_{jl} T_{kl} \quad (i)$$

For example,

$$\begin{aligned} T'_{11} &= a_{1k} a_{1l} T_{kl} = a_{11} a_{11} T_{11} + a_{12} a_{11} T_{21} + a_{13} a_{11} T_{31} \\ &\quad + a_{11} a_{12} T_{12} + a_{12} a_{12} T_{22} + a_{13} a_{12} T_{32} \\ &\quad + a_{11} a_{13} T_{13} + a_{12} a_{13} T_{23} + a_{13} a_{13} T_{33} \end{aligned} \quad (j)$$

Thus:

$$T'_{11} = \frac{7}{4} + \frac{6\sqrt{3}}{4} \quad (k)$$

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.3 (Continued)

Similarly

$$T'_{12} = -\frac{3}{2} + \frac{\sqrt{3}}{4} \quad (\ell)$$

$$T'_{13} = 0 \quad (\text{m})$$

$$T'_{21} = -\frac{3}{2} + \frac{\sqrt{3}}{4} \quad (\text{n})$$

$$T'_{22} = \frac{5}{4} - \frac{3\sqrt{3}}{2} \quad (\text{o})$$

$$T'_{23} = 0 \quad (\text{p})$$

$$T'_{31} = 0 \quad (\text{q})$$

$$T'_{32} = 0 \quad (\text{r})$$

$$T'_{33} = 1 \quad (\text{s})$$

Written in matrix algebra, the problem is solved below:

$$\begin{bmatrix} T'_{11} & T'_{12} & T'_{13} \\ T'_{21} & T'_{22} & T'_{23} \\ T'_{31} & T'_{32} & T'_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \quad (\text{t})$$

Note that the third matrix on the right is the transpose of a_{ij} . Matrix multiplication of (t) gives

$$T'_{ij} = \begin{bmatrix} (\frac{7}{4} + \frac{6\sqrt{3}}{2}) & (-\frac{3}{2} + \frac{\sqrt{3}}{4}) & 0 \\ (-\frac{3}{2} + \frac{\sqrt{3}}{4}) & (\frac{5}{4} - \frac{3\sqrt{3}}{2}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{u})$$

PROBLEM 8.4

The m^{th} component of the force density at a point is (Eq. 8.1.10)

$$F_i = \frac{\partial T_{ij}}{\partial x_j} \quad (\text{a})$$

Thus in the $\vec{i}_1$ direction,

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.4 (Continued)

$$F_1 = \left(\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} \right) = \left(-\frac{p_o^2}{a^2} x_1 - \frac{p_o^2}{a^2} x_1 + 0 \right) = 0 \quad (b)$$

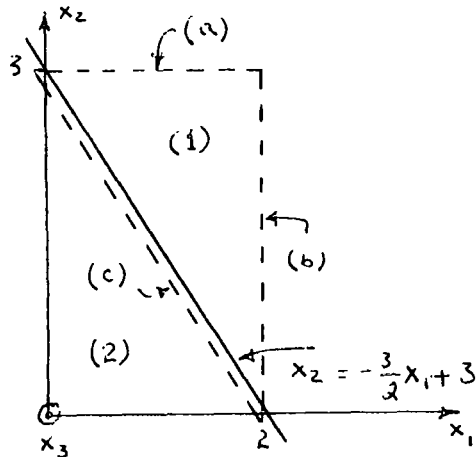
Similarly in the $\vec{i}_2$ and $\vec{i}_3$ directions we find

$$F_2 = \left(\frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} \right) = 0 \quad (c)$$

$$F_3 = \left(\frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} \right) = 0 \quad (d)$$

Hence, the total volume force density resulting from the given stress tensor is zero.

PROBLEM 8.5



in region (1) $\vec{E} = E_o \left(\frac{3}{2} \vec{i}_1 + \vec{i}_2 \right)$

in region (2) $\vec{E} = 0$

$$T_{ij} = \epsilon E_i E_j - \frac{\delta_{ij}}{2} \epsilon_o E_k E_k \quad (a)$$

Thus in region (2)

$$T_{ij} = [0] \quad (b)$$

in region (1)

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.5 (Continued)

$$T_{ij} = \begin{bmatrix} \frac{5}{8} \epsilon_o E_o^2 & \frac{3}{2} \epsilon_o E_o^2 & 0 \\ \frac{3}{2} \epsilon_o E_o^2 & -\frac{5}{8} \epsilon_o E_o^2 & 0 \\ 0 & 0 & -\frac{13}{8} \epsilon_o E_o^2 \end{bmatrix} \quad (c)$$

The total contribution to the forces found by integrating the stress tensor over surface (c) is zero, because surface (c) lies in region (2) where the stress tensor is zero. By symmetry the sum of contributions to the force resulting from integrations over the two surfaces perpendicular to the x_3 axis is zero.

Now let us note the fact that:

$$\text{area (a)} = 2 \quad (d)$$

$$\text{area (b)} = 3 \quad (e)$$

Thus:

$$f_1 = \oint T_{1j} n_j da \quad (f)$$

$$f_1 = \int_{(b)} T_{11} da + \int_{(a)} T_{12} da + \int T_{13} da$$

$$= \frac{5}{8} \epsilon_o E_o^2 (3) + \frac{3}{2} \epsilon_o E_o^2 (2) \quad (g)$$

$$f_1 = 4 \frac{7}{8} \epsilon_o E_o^2 \quad (h)$$

$$f_2 = \int_{(b)} T_{21} da + \int_{(a)} T_{22} da + \int T_{23} da$$

$$= \frac{3}{2} \epsilon_o E_o^2 (3) - \frac{5}{8} \epsilon_o E_o^2 (2) \quad (i)$$

$$f_2 = 3 \frac{1}{4} \epsilon_o E_o^2 \quad (j)$$

$$f_3 = \int T_{31} da + \int T_{32} da + \int T_{33} da$$

$$= 0 \quad (k)$$

Hence, the total force is:

$$\vec{f} = 4 \frac{7}{8} \epsilon_o E_o^2 \vec{i}_1 + 3 \frac{1}{4} \epsilon_o E_o^2 \vec{i}_2 \quad (l)$$

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.6

Part a

At point A, the electric field intensity is a superposition of the imposed field and the field due to the surface charges; $\bar{E} = (\sigma_f/\epsilon_o)\bar{i}_y$. Thus at A,

$$\bar{E} = \bar{i}_x(E_o) + \bar{i}_y(E_o + \frac{\sigma_f}{\epsilon_o}) \quad (a)$$

while at B,

$$\bar{E} = \bar{i}_x(E_o) + \bar{i}_y(E_o) \quad (b)$$

Thus, from Eq. 8.3.10, at A,

$$T_{ij} = \begin{bmatrix} \frac{1}{2} \epsilon_o [E_o^2 - (E_o + \frac{\sigma_f}{\epsilon_o})^2] & \epsilon_o E_o (E_o + \frac{\sigma_f}{\epsilon_o}) & 0 \\ \epsilon_o E_o (E_o + \frac{\sigma_f}{\epsilon_o}) & \frac{1}{2} \epsilon_o [(E_o + \frac{\sigma_f}{\epsilon_o})^2 - E_o^2] & 0 \\ 0 & 0 & -\frac{1}{2} \epsilon_o [E_o^2 + (E_o + \frac{\sigma_f}{\epsilon_o})^2] \end{bmatrix} \quad (c)$$

while at B the components are given by (c) with $\sigma_f \rightarrow 0$.

Part b

In the x direction, because the fields are independent of x and z,

$$f_x = (b-a) \left[(T_{xy}) \Big|_A - (T_{xy}) \Big|_B \right] D = (b-a) D \epsilon_o \sigma_f \quad (d)$$

or simply the area multiplied by the surface charge density and x component of electric field intensity.

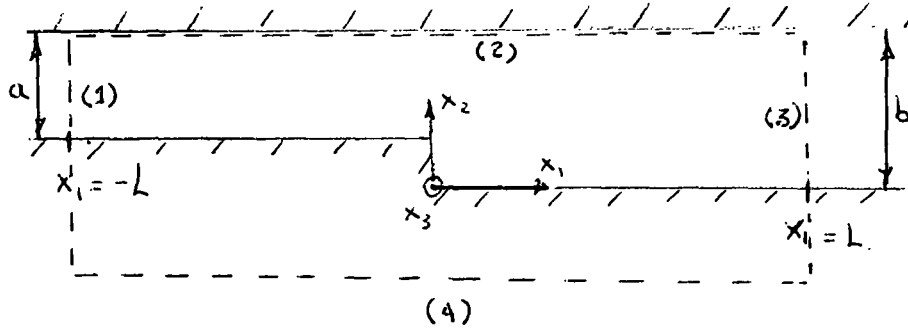
In the y direction

$$f_y = (b-a) (T_{yy} \Big|_A - T_{yy} \Big|_B) D = (b-a) D [E_o \sigma_f + \frac{\sigma_f^2}{2\epsilon_o}] \quad (e)$$

Note that both (d) and (e) could be found by multiplying the surface charge density by the average electric field intensity and the area, as shown by Eq. 8.4.8.

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.7



Before finding the force, we must calculate the $\vec{H}$ field at $x_1 = L$. To find this field let us use

$$\oint \vec{B} \cdot \vec{n} da = 0 \quad (a)$$

over the dotted surface. At $x_1 = +L$,

$$\vec{H}(x_1=L) = H_0 \vec{i}_1 \quad (b)$$

over surface (4) $\vec{H} = 0$, and over surface (2), $\vec{H}$ is in the $\vec{i}_1$ direction, where $\vec{n} = \vec{i}_2$. Thus over surface (2) $\vec{B} \cdot \vec{n} = 0$.

Hence, the integral in (a) reduces to

$$\begin{aligned} - \int_{(1)} \mu_0 H_0 da + \int_{(3)} \mu_0 H(x_1 = +L) da &= 0 \\ - \mu_0 H_0 a + \mu_0 H b &= 0 \quad \text{per unit depth} \end{aligned} \quad (d)$$

Thus:

$$\vec{H}(x_1 = +L) = (a/b) H_0 \vec{i}_1 \quad (e)$$

$$T_{ij} = \mu_0 H_i H_j - \frac{\delta_{ij}}{2} \mu_0 H_k H_k \quad (f)$$

Hence, the stress tensor over surfaces (1), (2) and (3) is:

$$T_{ij} = \begin{bmatrix} \frac{\mu_0}{2} H_1^2 & 0 & 0 \\ 0 & -\frac{\mu_0}{2} H_1^2 & 0 \\ 0 & 0 & -\frac{\mu_0}{2} H_1^2 \end{bmatrix} \quad (g)$$

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.7 (Continued)

over surface (4)

$$T_{ij} = [0] \quad (h)$$

Thus the force in the 1 direction is

$$f_1 = \int T_{ij} n_j da \quad (i)$$

$$f_1 = - \int_{(1)} T_{11} da + \int_{(3)} T_{11} da + \int_{(2)} T_{12} da \quad (j)$$

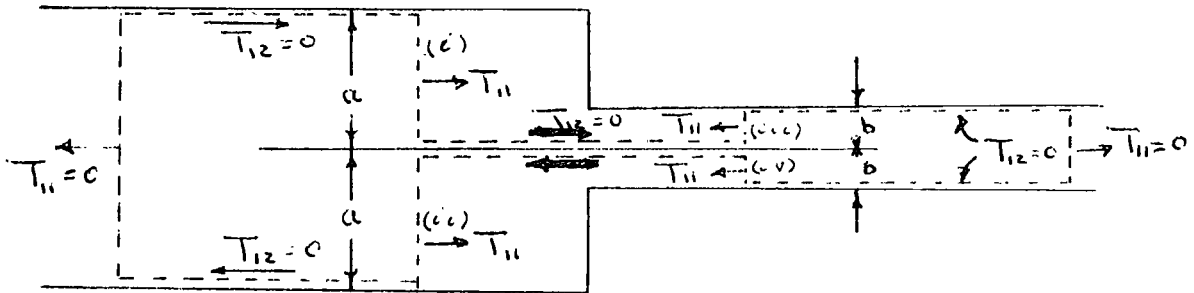
Thus, since the last integral makes no contribution,

$$f_1 = - \frac{\mu_0}{2} H_0^2 (a) + \frac{\mu_0}{2} H_0^2 \left(\frac{a}{b}\right)^2 \cdot b = \frac{\mu_0}{2} H_0^2 a \left\{ \frac{a}{b} - 1 \right\} \quad (k)$$

Since $T_{ij} = 0$ over surface (4) there is no contribution to the force from this surface, and by symmetry, there is no contribution to the force from the surfaces perpendicular to the x_3 axis. Thus, the force per unit depth in 1 direction is (k).

PROBLEM 8.8

The appropriate surface of integration is shown in the figure



The stresses acting in the x direction on the respective surfaces are as shown. Because the plates are perfectly conducting, all shear stresses required to complete the integration of Eq. 8.1.17 vanish. The only contributions are from surfaces (i), (ii), (iii) and (iv), where the fields

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.8 (Continued)

are known to be

$$\begin{aligned} \bar{E} &= -\frac{V_0}{a} \bar{i}_y & (i) \quad ; \quad \bar{E} &= -\frac{V_0}{b} \bar{i}_y & (iii) \\ \bar{E} &= \frac{V_0}{a} \bar{i}_y & (ii) \quad ; \quad \bar{E} &= \frac{V_0}{b} \bar{i}_y & (iv) \end{aligned} \quad (a)$$

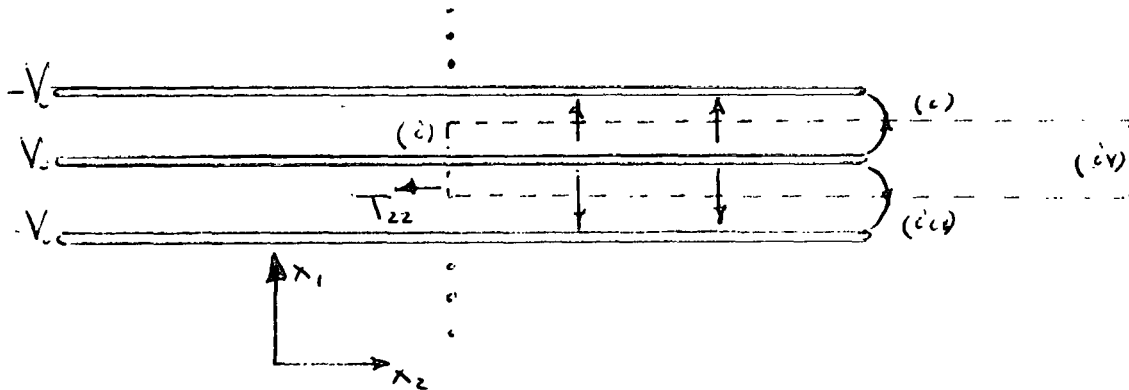
Thus,

$$f_1 = (T_{11})_i ad + (T_{11})_{ii} ad - (T_{11})_{iii} bd - (T_{11})_{iv} bd \quad (b)$$

$$= dV_0^2 \epsilon_0 \left[\frac{1}{b} - \frac{1}{a} \right] \quad (c)$$

The plate tends to be drawn to the right, where the fields are greater.

PROBLEM 8.9



The volume enclosing the half of the plate is arbitrary so long as it is defined so that it does not include additional charge. Thus the volume shown in the figure encloses no more than the desired distribution of charge. Moreover, surfaces (i) and (iii) pass through the fringing fields half way between the plates where by symmetry there is no x_2 component of $\bar{E}$. Thus surfaces (i) and (iii) support no shear stress T_{21} . There is no field at surface (iv) and hence the only contribution is from surface (i), where the square of the field is known to be

$$E_1^2 = \frac{4V_0^2}{s^2} \quad (a)$$

and it follows that because T_{22} on (i) is $-\frac{1}{2} \epsilon_0 E_1^2$ and the normal vector is negative

$$f_2 = \frac{4ws\epsilon_0}{2} \frac{V_0^2}{s^2} \quad (b)$$

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.9 (Continued)

The fringing field tends to pull the end of the plate in the $+x_2$ direction.

PROBLEM 8.10

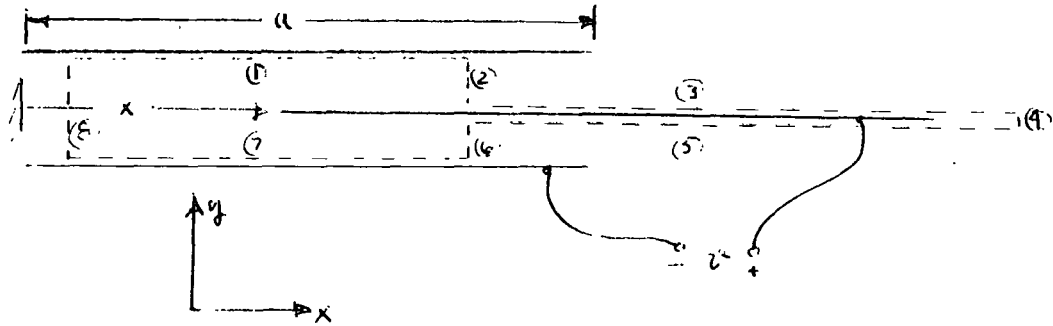


Figure 1

Part a

Consider the surface shown in Figure 1. The total force in the x direction is:

$$f_x = \int_{1,3} T_{xy} da - \int_{5,7} T_{xy} da + \int_4 T_{xx} da - \int_8 T_{xx} da + \int_{2,6} T_{xx} da \quad (a)$$

The first four integrals disappear because:

$T_{xy} = \epsilon E_x E_y = 0$ on 1, 3, 5 and 7 because we are next to the conducting plates ($E_x = 0$)

$T_{xx} = 0$ on 4 and 8 because the $\vec{E}$ field = 0 there

Hence

$$f_x = \int_{2,6} T_{xx} da = \int_{2,6} -\frac{1}{2} E_y^2 da \quad (b)$$

where T_{ij} is evaluated using Eq. 8.3.10.

$$E_y = \frac{v}{s} \quad (c)$$

and hence:

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.10 (Continued)

$$f_x = \int_{2,6} -\frac{\epsilon}{2} \left(\frac{v}{s}\right)^2 da = -\frac{\epsilon d}{s} v^2 \quad (d)$$

Part b

The coenergy of the system is

$$W' = \frac{1}{2} C(x) v^2 \quad (e)$$

where $C(x) = \frac{2(a-x)d\epsilon}{s} \quad (f)$

Thus, (see Sec. 3.1.2b)

$$f_x = \frac{\partial W'}{\partial x} = \frac{1}{2} \frac{\partial C(x)}{\partial x} v^2 = -\frac{d\epsilon}{s} v^2 \quad (g)$$

which is the same value determined in part (a).

Part c

The equation of motion of the plate is:

$$M \frac{d^2 x}{dt^2} + K(x-a) = f_x = -\frac{d\epsilon}{s} v_o^2 \quad (h)$$

When the system reaches equilibrium with the switch closed,

$$K(X_o - a) = -\frac{d\epsilon}{s} v_o^2 \quad (i)$$

thus

$$X_o = a - \frac{d\epsilon}{sK} v_o^2 \quad (j)$$

After the switch is opened,

$$M \frac{d^2 x}{dt^2} + K(x-a) = -\frac{d\epsilon}{s} v^2(t) \quad (k)$$

The electrical circuit is like an R-C circuit with time varying elements



$$v + R(x)i(t) = 0 \quad (l)$$

$$v + R(x) \frac{d}{dt} [C(x)v] = 0 \quad (m)$$

$$v + R(x)C(x) \frac{dv}{dt} + R(x) \frac{dC(x)}{dx} \frac{dx}{dt} v = 0 \quad (n)$$

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.10 (Continued)

where:

$$R(x) = \frac{s}{2\sigma d(a-x)} \quad \text{and} \quad C(x) = \frac{2d(a-x)\epsilon}{s} \quad (o)$$

Hence

$$v + \frac{\epsilon}{\sigma} \frac{dv}{dt} - \left[\frac{\epsilon}{\sigma} \frac{1}{(a-x)} \frac{dx}{dt} \right] v = 0 \quad (p)$$

Part d

Dropping the inertial term from (h) leaves:

$$K(x-a) = - \frac{d\epsilon}{s} v^2(t) \quad \text{from (k)} \quad (q)$$

But we may write the identity

$$- \frac{1}{(a-x)} \frac{dx}{dt} = \frac{1}{K(x-a)} \frac{d}{dt} K(x-a) \quad (r)$$

and then, from (q)

$$\begin{aligned} - \frac{1}{(a-x)} \frac{dx}{dt} &= - \frac{s}{d\epsilon v^2(t)} \frac{d}{dt} \cdot \left[- \frac{d\epsilon}{s} v^2(t) \right] \\ &= \frac{1}{v^2(t)} \frac{d}{dt} v^2(t) = \frac{2}{v} \frac{dv}{dt} \end{aligned} \quad (s)$$

Substituting back into (p) we have

$$v + \frac{\epsilon}{\sigma} \frac{dv}{dt} + \frac{2\epsilon}{\sigma} \frac{dv}{dt} = 0 \quad (t)$$

Solving we find

$$v = v_o e^{-(\sigma/3\epsilon)t} \quad (u)$$

and substituting back into (q),

$$x = a - \frac{d\epsilon}{sK} v_o^2 e^{-\frac{2}{3} \frac{\sigma}{\epsilon} t} \quad (v)$$

A long relaxation time is consistent with neglecting the inertial terms, as then $x(t)$ varies slowly.

Part e

Proceed as in (c), and record the time constant τ of $a-x(t)$ by measuring the mechanical displacement. Then,

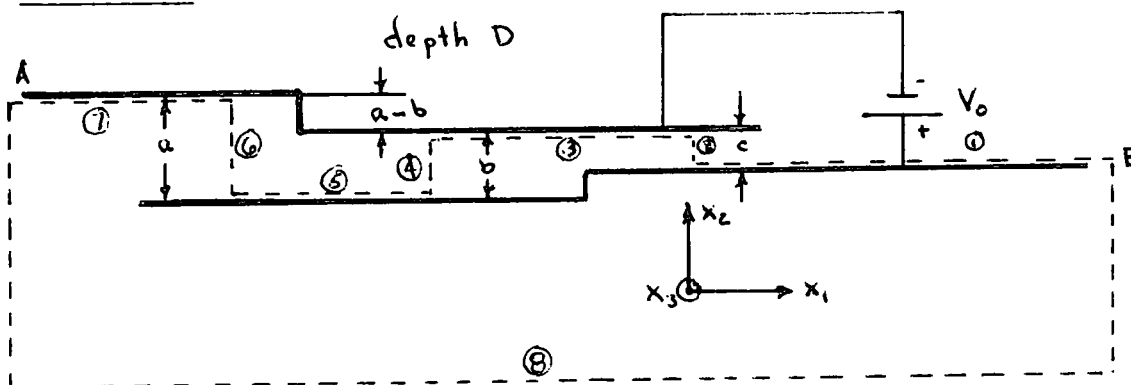
$$\frac{\epsilon}{\sigma} = \frac{2}{3} \tau \quad (w)$$

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.10 (Continued)

This problem should raise questions as to the appropriate form of T_{ij} used in (b). Note that the surface of integration encloses liquid as well as the plate. We want only the force on the plate, so our calculation is correct only if there is no net force on the enclosed liquid. The electrical force density in the liquid is given by Eq. 8.5.45. There is no free charge or gradient of permittivity in the bulk of the liquid and hence the first two of the three contributions to this force density vanish in the liquid. However, there remains the electrostriction force density. Note that it is ignored in our calculation because the electrostriction term was not included in the stress tensor (we used Eq. 8.3.10 rather than 8.5.46). Our reason for ignoring the electrostriction is this: it gives rise to a force density that takes the form of the gradient of a pressure. Hence, it simply alters the distribution of liquid pressure around the plate. Because each element of the liquid is in static equilibrium and can give way to motions of the plate without changing its volume, the "hydrostatic pressure" of the liquid is altered by the electric field so as to exactly cancel the effect of the electrostriction force density. Hence, to correctly include the effect of electrostriction in integrating the stresses over the surface, we must also include the hydrostatic pressure of the liquid. If this is done, the effect of the electrostriction will cancel out, leaving the force on the plate we have derived by two alternative methods here.

PROBLEM 8.11



FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.11 (Continued)

First, let us note the $\vec{E}$ fields on each of the surfaces of the figure over surfaces (1), (3), (5) and (7), $E_1 = 0$ (a)

over surface

$$(6) \quad E_2 = \frac{V_0}{a} \quad E_1 = 0 \quad (b)$$

$$(4) \quad E_2 = \frac{V_0}{b} \quad E_1 = 0 \quad (c)$$

$$(2) \quad E_2 = \frac{V_0}{c} \quad E_1 = 0 \quad (d)$$

From Eq. 8.3.10,

$$T_{ij} = \epsilon_0 E_i E_j - \frac{\delta_{ij}}{2} \epsilon_0 E_k E_k \quad (e)$$

Hence, over surfaces (1), (3), (5) and (7)

$$T_{12} = 0 \quad (f)$$

and over surfaces

$$(6) \quad T_{11} = -\frac{\epsilon_0}{2} \left(\frac{V_0}{a}\right)^2 \quad (g)$$

$$(4) \quad T_{11} = -\frac{\epsilon_0}{2} \left(\frac{V_0}{b}\right)^2 \quad (h)$$

$$(2) \quad T_{11} = -\frac{\epsilon_0}{2} \left(\frac{V_0}{c}\right)^2 \quad (i)$$

Now;

$$f_1 = \int T_{ij} n_j da = \int T_{11} n_1 da + \int T_{12} n_2 da + \int T_{13} n_3 da \quad (j)$$

$$\int T_{13} n_3 da = 0 \quad \text{because the problem is two dimensional.} \quad (k)$$

Let us consider each of the other integrals:

$$\int T_{12} n_2 da = 0 \quad (l)$$

because the surfaces which have normal n_2 are (1), (3), (5) and (7) and by (f) we have shown that $T_{12} = 0$ over these surfaces. Also, we get no contribution to the force over surface (8), because $\vec{E} \rightarrow 0$ faster than the area $\rightarrow \infty$.

Hence the calculation of the force reduces to

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.11 (Continued)

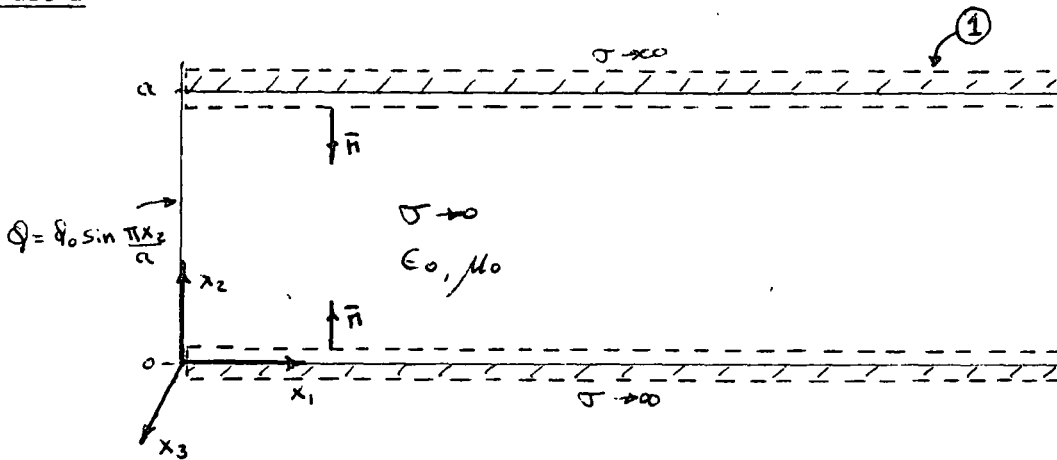
$$f_1 = \int_{(6)} T_{11}^{(6)} da_6 - \int_{(4)} T_{11}^{(4)} da_4 - \int_{(2)} T_{11}^{(2)} da_2 \quad (m)$$

$$f_1 = - \frac{\epsilon_o DV_o^2}{2} \left\{ \frac{1}{a} - \frac{1}{b} + \frac{1}{c} \right\} \quad (n)$$

Note: by symmetry, there is no contribution to the force from the surfaces perpendicular to the x_3 axis.

PROBLEM 8.12

Part a



From elementary field theory, we find that

$$\phi = \phi_o \sin \frac{\pi x_2}{a} e^{-\pi x_1/a} \quad (a)$$

satisfies $\nabla^2 \phi = 0$ in the region between the plates and the required boundary conditions. The distribution of $\bar{E}$ follows from

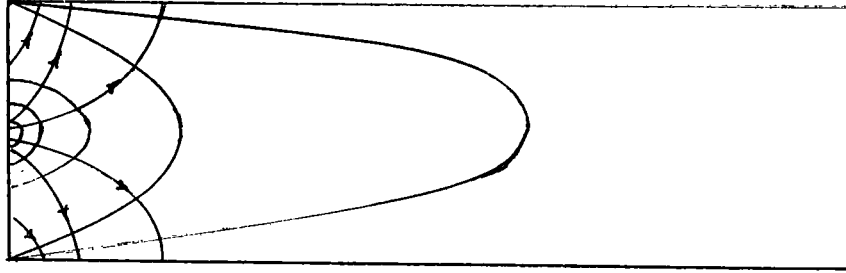
$$\bar{E} = - \nabla \phi \quad (b)$$

Hence,

$$\bar{E} = \frac{\pi \phi_o}{a} e^{-\pi x_1/a} \left[\sin \frac{\pi x_2}{a} \bar{i}_1 - \cos \frac{\pi x_2}{a} \bar{i}_2 \right] \quad (c)$$

The sketch of the $\bar{E}$ field is obtained by recognizing that $\bar{E}$ is directed perpendicular to contours of constant ϕ .

PROBLEM 8.12 (Continued)



Part b

To find the force as the bottom plate, we use surface (2). $\vec{E} = 0$ everywhere except on the upper side where the normal $\vec{n} = \vec{i}_2$ (d)

and the field is

$$\vec{E} = -\frac{\pi\phi_o}{a} e^{-\pi x_1/a} \vec{i}_2 \quad (e)$$

Hence,

$$f_1 = \int T_{1j} n_j da = 0 \quad (f)$$

$$f_2 = \int T_{2j} n_j da = \int T_{22} n_2 da_2 \quad (g)$$

per unit x_3 , this reduces to

$$f_2 = \int_0^\infty T_{22} dx_1 \quad (h)$$

$$\text{but, } T_{22} = \frac{1}{2} \epsilon_o E_2 E_2 = \frac{1}{2} \epsilon_o \frac{\pi^2 \phi_o^2}{a^2} e^{-\frac{2\pi x_1}{a}} \quad (i)$$

and thus

$$f_2 = \frac{\epsilon_o \pi^2 \phi_o^2}{2a^2} \int_0^\infty e^{-\frac{2\pi x_1}{a}} dx_1 \quad (j)$$

$$f_2 = \frac{\epsilon_o \pi \phi_o^2}{4a} \quad (k)$$

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.12 (Continued)

Part c

On the top plate, use surface (1). Only the sign of the normal changes, and the result is

$$f_1 = 0 \quad (l)$$

$$f_2 = -\frac{\epsilon_0 \pi \phi_0^2}{4a} \quad (m)$$

or the force is equal and opposite to that on the bottom plate.

PROBLEM 8.13

Part a

$$T_{ij} = \epsilon E_i E_j - \frac{\delta_{ij}}{2} \epsilon E_k E_k \quad (a)$$

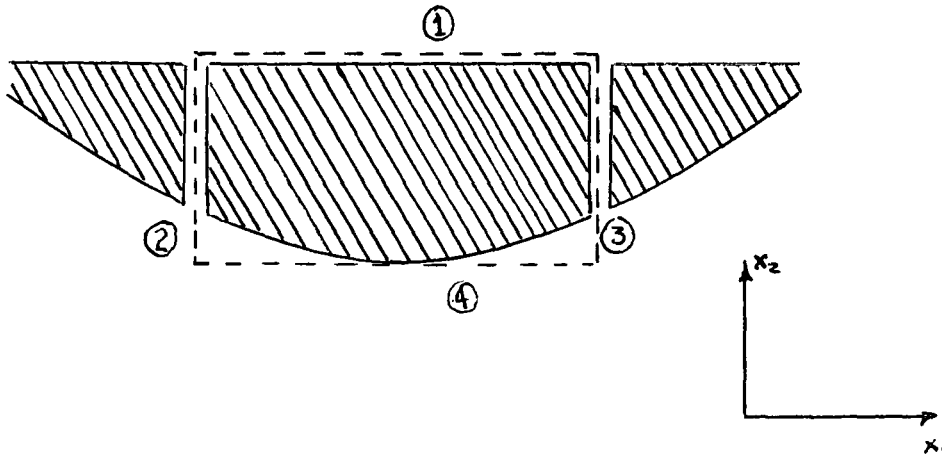
Hence:

$$T_{22} = \frac{1}{2} \epsilon_0 \left(\frac{2V_0}{3a^2} \right)^2 (x_2^2 - x_1^2) \quad (b)$$

$$T_{21} = \epsilon_0 E_2 E_1 = -\epsilon_0 \left(\frac{2V_0}{3a^2} \right)^2 x_1 x_2 \quad (c)$$

Part b

Consider the surface of integration shown in the figure.



$$f_2 = \int T_{2j} n_j da = \int_{(2)(3)} T_{21} n_1 da + \int_{(1)(4)} T_{22} n_2 da + \int_{\text{by symmetry}} T_{33} n_3 da \quad (d)$$

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.13 (Continued)

Let us look at each of these integrals separately

$$\int_{(1)(4)} T_{22} n_2 da = \int_{(1)} T_{22} da_1 - \int_{(4)} T_{22} da_4 \quad (e)$$

over surface (1), $\bar{E} \approx 0$; $T_{22} \approx 0$ and hence, the integral is merely:

$$\begin{aligned} - \int_{(4)} T_{22} da_4 &= - \int_{x_1=-a}^{x_1=a} \frac{\epsilon_0}{2} \left(\frac{2V_0}{3a^2} \right)^2 (x_2^2 - x_1^2) w dx_1 \\ &\quad x_2 = 2a \\ &= - \frac{44}{27} \frac{\epsilon_0 V_0^2 w}{a} \end{aligned} \quad (f)$$

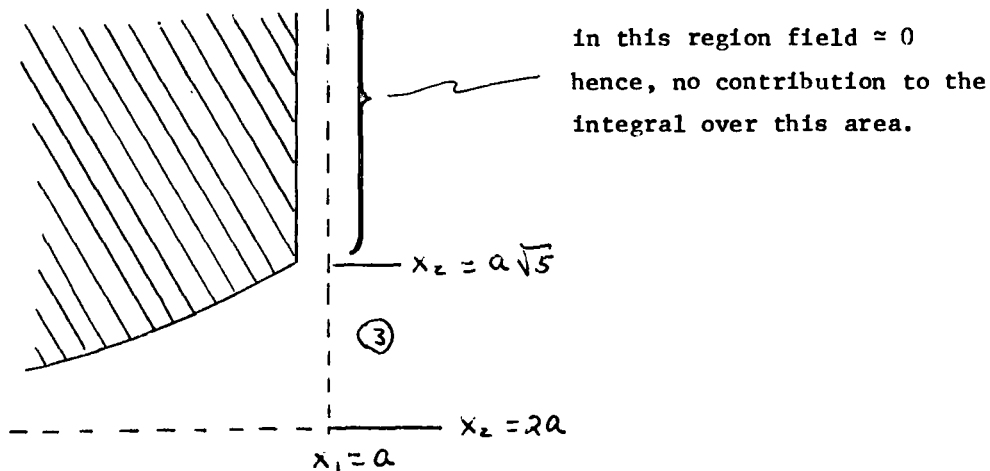
Thus,

$$\int_{(1)(4)} T_{22} n_2 da = - \frac{44}{27} \frac{\epsilon_0 V_0^2 w}{a} \quad (g)$$

Let us now evaluate:

$$\int_{(2)(3)} T_{21} n_1 da$$

Consider the surface shown.



FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.13 (Continued)

Thus;

$$\begin{aligned} \int_{(3)} T_{21} da_3 &= \int_{x_2=2a}^{x_2=a\sqrt{5}} -\epsilon_o \left(\frac{2V_o}{3a^2}\right)^2 a w x_2 dx_2 \\ &= -\frac{2}{9} \frac{\epsilon_o V_o^2 w}{a} \end{aligned} \quad (h)$$

Over surface (2), we have essentially the same thing, except $\bar{n} = -\bar{i}_1$ and $x_1 = -a$. Hence:

$$\int_{(2)} T_{21} da_2 = -\frac{2}{9} \frac{\epsilon_o V_o^2 w}{a} \quad (i)$$

Therefore, the total force in the f_2 direction is

$$f_2 = -\frac{56}{27} \frac{\epsilon_o V_o^2 w}{a} \quad (j)$$

Part c

$$f_1 = \int_{(2)(3)} T_{11} n_1 da + \int_{(1)(4)} T_{12} n_2 da + \int_{(1)(3)} T_{13} n_3 da \quad (k)$$

by symmetry

$$\int_{(1)(4)} T_{12} n_2 da = - \int_{(4)} T_{12} da_4 \quad \text{over (1) we get 0 as before}$$

$$= \epsilon_o \left(\frac{2V_o}{3a^2}\right)^2 \int_{x_2=2a}^a x_1 x_2 w dx_1 = 0 \quad (l)$$

Now, over surfaces, 2 and 3

$$\int_{(2)(3)} T_{11} n_1 da = - \int_{(2)} T_{11} da_2 + \int_{(3)} T_{11} da_3 = 0 \quad (m)$$

because,

$$T_{11}|_2 = T_{11}|_3$$

$$\text{hence } f_1 = 0. \quad (n)$$

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.13 (Continued)

Part d

$$\sigma_f = \bar{n} \cdot \epsilon_0 \bar{E} \quad (o)$$

at the lower surface of the movable conductor. The functional relation, $f(x_1, x_2)$, for the lower surface if the movable conductor is given as

$$f(x_1, x_2) = \sqrt{4a^2 + x_1^2} - x_2 = 0 \quad (p)$$

the outward unit normal to this surface is

$$\bar{n} = \frac{\nabla f(x_1, x_2)}{|\nabla f(x_1, x_2)|} = \left[\frac{x_1}{x_2} \bar{i}_1 - \bar{i}_2 \right] \left[\frac{1}{\sqrt{\frac{x_1^2}{x_2^2} + 1}} \right] \quad (q)$$

$$\text{at } x_2 = \sqrt{4a^2 + x_1^2},$$

$$\sigma_f = \epsilon_0 [n_1 E_1 + n_2 E_2] = \frac{2V_0 \epsilon_0}{3a^2} \left[\frac{x_1^2}{x_2} + x_2 \right] \left[\frac{4a^2 + x_1^2}{4a^2 + 2x_1^2} \right]^{1/2} \quad (r)$$

The surface force density (see Eq. 8.4.8) is equal to:

$$\bar{T} = \sigma_f \frac{\bar{E}^b + \bar{E}^a}{2} \quad (s)$$

where, $\bar{E}^b$ = field just below the charge sheet

$\bar{E}^a$ = field just above the charge sheet

Since

$$\bar{E}^a = 0, \quad \bar{T} = \frac{1}{2} \sigma_f \bar{E}^b \quad (t)$$

thus

$$\bar{T} = \frac{\epsilon_0}{2} \left(\frac{2V_0}{3a^2} \right)^2 \left[\frac{x_1^2}{x_2} + x_2 \right] \left[x_1 \bar{i}_1 - x_2 \bar{i}_2 \right] \left[\frac{4a^2 + x_1^2}{4a^2 + 2x_1^2} \right]^{1/2} \quad (u)$$

To find the total force, the surface force density must be integrated over the surface. Hence, we find

$$f_2 = -2\epsilon_0 \left(\frac{V_0}{3a^2} \right)^2 \int_{-a}^{+a} \{2x_1^2 + 4a^2\}^{1/2} \{x_1^2 + 4a^2\}^{1/2} dx_1 \quad (v)$$

If the student wishes, he may **carry** out this integral, but the complexity of the integration shows the value of the stress tensor in calculating such a

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.13 (Continued)

force. We realize that by using the stress tensor, we have essentially carried out this difficult integral by an integration by parts.

PROBLEM 8.14

Part a

$$\phi = \frac{V_o}{a} x_1 x_2 \quad (a)$$

$$\vec{E} = -\nabla\phi \quad (b)$$

hence,

$$\vec{E} = \vec{i}_1 \left(-\frac{V_o}{a} x_2\right) + \vec{i}_2 \left(-\frac{V_o}{a} x_1\right) \quad (c)$$

and, from Eq. 8.3.10

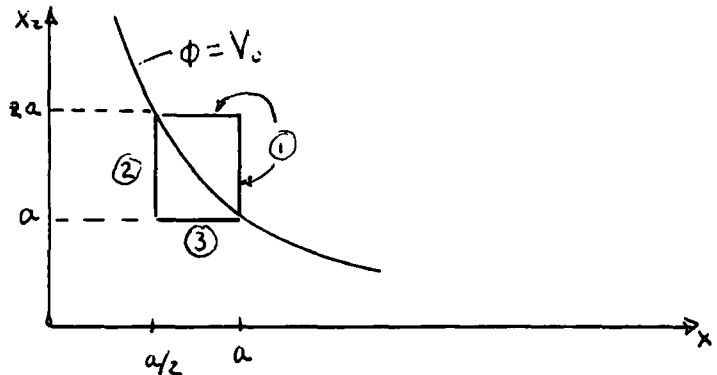
$$T_{ij} = \epsilon E_i E_j - \delta_{ij} \frac{\epsilon}{2} E_k E_k \quad (d)$$

Thus, the stress tensor becomes:

$$T_{ij} = \begin{bmatrix} \left(\frac{V_o}{a}\right)^2 \frac{\epsilon_o}{2} (x_2^2 - x_1^2) & \left(\frac{V_o}{a}\right)^2 \epsilon_o (x_1 x_2) & 0 \\ \left(\frac{V_o}{a}\right)^2 \epsilon_o (x_1 x_2) & \left(\frac{V_o}{a}\right)^2 \frac{\epsilon_o}{2} (x_1^2 - x_2^2) & 0 \\ 0 & 0 & -\left(\frac{V_o}{a}\right)^2 \frac{\epsilon_o}{2} (x_1^2 + x_2^2) \end{bmatrix} \quad (e)$$

Part b

Consider the surface shown, bounded by the line segment $x_2 = 2a$, $x_2 = a$, and $x_1 = a/2$ and $x_1 = a$.



FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.14 (Continued)

As before, because the geometry and fields are two-dimensional, the force in the $\bar{i}_3$ direction is zero. Also, since along surface (1) $\phi = \text{constant}$, then the $\bar{E}$ field = 0, and hence $T_{1j} = 0$ along this surface. Thus the calculation of the force on AB reduces to:

$$f_1 = - \int_{(2)} T_{11} da - \int_{(3)} T_{12} da \quad (f)$$

$$f_2 = - \int_{(2)} T_{21} da - \int_{(3)} T_{22} da \quad (g)$$

$$f_1 = - \left(\frac{V_0}{a} \right)^2 \epsilon_0 D \left[\frac{1}{2} \int_a^{2a} [x_2^2 - \left(\frac{a}{2} \right)^2] dx_2 + \int_{a/2}^a x_1 a dx_1 \right] \quad (h)$$

and hence

$$f_1 = - \epsilon_0 D \left(\frac{V_0}{a} \right)^2 a^3 \left[\frac{17}{12} \right] \quad (i)$$

Similarly:

$$f_2 = - \left(\frac{V_0}{a} \right)^2 D \epsilon_0 \left[\int_a^{2a} \frac{a}{2} x_2 dx_2 + \frac{1}{2} \int_{a/2}^a (x_1^2 - a^2) dx_1 \right] \quad (j)$$

and hence

$$f_2 = - \epsilon_0 D a^3 \left(\frac{V_0}{a} \right)^2 \left[\frac{31}{48} \right] \quad (k)$$

Thus,

$$\bar{f} = - \epsilon_0 \frac{V_0^2}{a} D \left[\bar{i}_1 \frac{17}{12} + \bar{i}_2 \frac{31}{48} \right] \quad (l)$$

PROBLEM 8.15

Part a

The $\bar{E}$ field in the laboratory frame is zero since the two perfectly conducting plates are shorted. This can be seen by integrating $\bar{E}$ around a fixed contour through the block and short and recognizing that the enclosed flux is constant. Hence,

$$\bar{E}' = \bar{E} + \bar{v} \times \bar{B}, \quad \bar{E} = 0 \quad (a)$$

and thus

$$\bar{E}' = \bar{v} \times \bar{B} = - V \mu_0 H_0 \bar{i}_2 \quad (b)$$

Therefore we may now calculate $\bar{J}$ in the moving block.

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.15 (Continued)

$$\vec{J}' = \sigma \vec{E}' = -\sigma \mu_o V H_o \vec{i}_2 \quad (c)$$

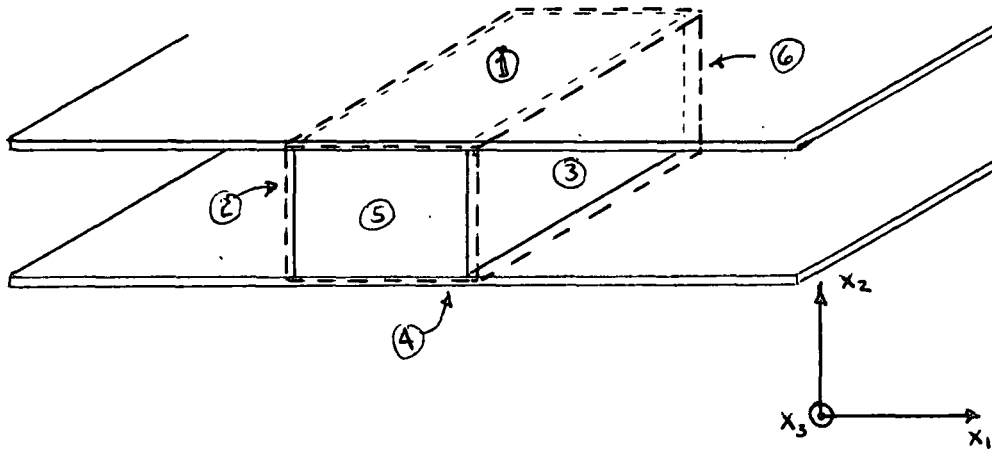
Thus:

$$\vec{F} = \vec{J} \times \vec{B} = -\sigma \mu_o^2 V H_o^2 \vec{i}_1 \quad (d)$$

$$\vec{F} = \int_{\text{volume}} (\vec{J} \times \vec{B}) dV = -\mu_o^2 \sigma V H_o^2 (abD) \vec{i}_1 \quad (e)$$

Part b

The closed surface of integration is shown in the figure below.



Since the field is uniform everywhere, the only non-zero components of the stress tensor are the diagonal elements

$$T_{11} = T_{22} = -\frac{1}{2} \mu_o H_o^2 \quad T_{33} = \frac{1}{2} \mu_o H_o^2 \quad (f)$$

Thus

$$\begin{aligned} f_1 &= \int_{(3)} T_{11} da_3 - \int_{(2)} T_{11} da_2 \\ &= \frac{\mu_o}{2} H_o^2 bD - \frac{\mu_o}{2} H_o^2 bD = 0 \end{aligned} \quad (g)$$

Similarly

$$f_2 = \int_{(1)} T_{22} da_1 - \int_{(4)} T_{22} da_4 = 0 \quad (h)$$

$$f_3 = \int_{(5)} T_{33} da_5 - \int_{(6)} T_{33} da_6 = 0 \quad (i)$$

Hence:

$$\vec{F} = 0 \quad (j)$$

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.15 (Continued)

Part c

The magnetic field strength and the current density are inconsistent. The quasi-static magnetic field cannot be uniform and irrotational in a region where a finite current density exists. The Maxwell stress tensor was developed with the aid of Ampere's Law (quasi-static) which relates current density and magnetic field rotation.

$$\vec{J} = \nabla \times \vec{H} \quad (k)$$

$$\vec{F} = \mu_0 \vec{J} \times \vec{H} = \mu_0 (\nabla \times \vec{H}) \times \vec{H} \quad (l)$$

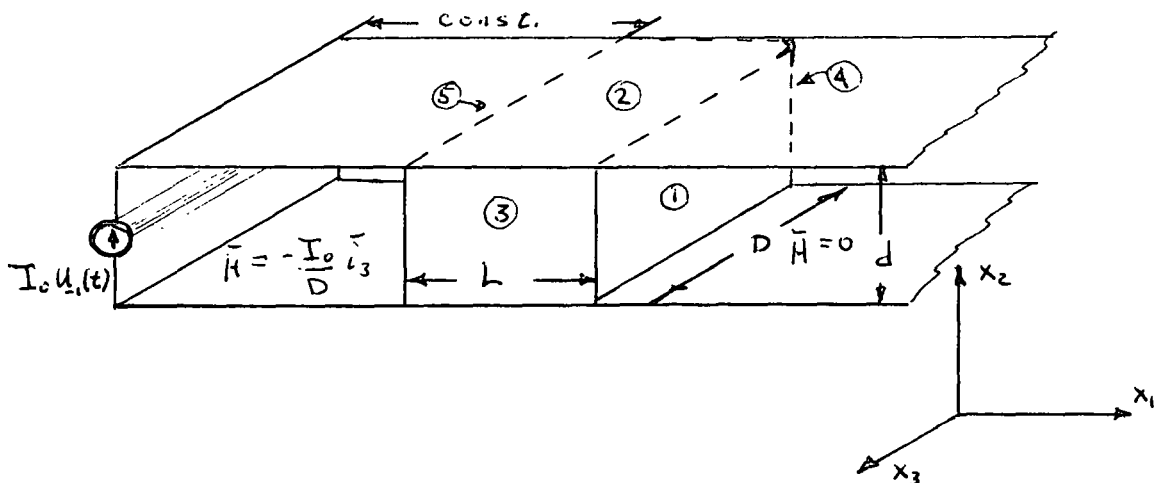
For this case, we have assumed that

$$\nabla \times \vec{H} = 0 \quad (m)$$

In the limit of small magnetic Reynold's number, ($R_m \ll 1$), the motion does not appreciably affect the field, and the answer found in part a is a good approximation. There are some problems more easily handled with the stress tensor. This problem illustrates that in other cases it is easiest to use the force density $\vec{J} \times \vec{B}$ directly. Note that we could compute the field induced by $\vec{J}$ and then use the Maxwell stress tensor and the self-consistent fields to find the same force as given by (e).

PROBLEM 8.16

To find the force on the block, we will use the stress tensor over the surface shown in the figure. Note that the surface is just outside the block.



PROBLEM 8.16 (Continued)

In the region to the left of the block

$$\bar{H} = -\frac{I_o}{D} \bar{i}_3, \text{ and to the right } \bar{H}=0$$

Thus:

$$f_1 = \int T_{11} n_1 da + \int T_{12} n_2 da + \int T_{13} n_3 da \quad (a)$$

but, since

$$H_1 = H_2 = 0; T_{12} = T_{13} = 0 \quad (b)$$

hence,

$$f_1 = - \int_{(5)} T_{11} da_5 + \int_{(1)} T_{11} da_1 \quad (c)$$

on surface (5),

$$T_{11} = -\frac{\mu_o}{2} \frac{I_o^2}{D^2} \quad (d)$$

on surface (1)

$$T_{11} = 0 \quad (e)$$

therefore

$$f_1 = + \frac{\mu_o}{2} \frac{I_o^2}{D^2} \cdot Dd = + \frac{\mu_o I_o^2 d}{2D} \quad (f)$$

Similarly, f_2 reduces to

$$f_2 = \int_2 T_{22} da_2 - \int_6 T_{22} da_6 \quad (g)$$

But, since T_{22} is a function of x_1 alone ($\bar{H}$ is a function of x_1 alone) the two surface integrals are identical, and hence $f_2 = 0$. Similar reasoning shows that $f_3 = 0$ and thus the total force is

$$\bar{f} = \frac{\mu_o I_o^2 d}{2D} \bar{i}_1$$

PROBLEM 8.17

Part a

$$\nabla^2 \bar{H} = \mu_o \sigma \frac{\partial \bar{H}}{\partial t} \quad (a)$$

Assume a solution of the form:

$$\bar{H} = R_e [\underline{H}_z(x) e^{j\omega t} \bar{i}_z] \quad (b)$$

$$\frac{\partial^2 \underline{H}_z}{\partial x^2} = j\omega \sigma \mu_o \underline{H}_z \quad (c)$$

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.17 (Continued)

Try

$$\underline{H}_z(x) = \underline{H}_0 e^{Kx} \quad (d)$$

where

$$K^2 = j\omega\sigma\mu_0 \quad (e)$$

and hence

$$K = \pm \sqrt{\frac{j\omega\mu_0\sigma}{2}} (1+j) \quad (f)$$

Let us define the skin depth as:

$$\delta \equiv \sqrt{\frac{2}{\omega\mu_0\sigma}} \quad (g)$$

And thus

$$\underline{\bar{H}} = \left[\underline{H}_1 e^{-\frac{x}{\delta}(1+j)} + \underline{H}_2 e^{+\frac{x}{\delta}(1+j)} \right] e^{j\omega t} \underline{\bar{i}}_z \quad (h)$$

Because the skin depth δ is assumed to be small, and the excitation is on the left,

$$\underline{H}_2(\text{large } x) \rightarrow 0 \quad \text{which implies } \underline{H}_2 = 0$$

Hence,

$$\underline{\bar{H}}(x_1, t) = \underline{H}_1 e^{-\frac{x}{\delta}(1+j)} e^{j\omega t} \underline{\bar{i}}_z \quad (i)$$

But, our boundary condition at $x = 0$ is

$$\underline{H}(x=0, t) = \text{Re} \underline{H}_1 e^{-j\omega t} \underline{\bar{i}}_z = -\text{Re} \frac{I}{D} e^{j\omega t} \underline{\bar{i}}_z \quad (j)$$

and thus

$$\underline{\bar{H}}(x_1, t) = -\frac{I}{D} e^{-\frac{x}{\delta}(1+j)} e^{j\omega t} \underline{\bar{i}}_z \quad (k)$$

$$\underline{\bar{J}} = \nabla \times \underline{\bar{H}} = -\left(\frac{\partial \underline{H}}{\partial x}\right) \underline{\bar{i}}_y = -\frac{I}{D} \frac{(1+j)}{\delta} e^{-\frac{x}{\delta}(1+j)} e^{j\omega t} \underline{\bar{i}}_y \quad (l)$$

Part b

$$\underline{\bar{f}} = \int \underline{\bar{J}} \times \underline{\bar{B}} dV = \int \underline{\bar{J}} \times \mu_0 \underline{\bar{H}} dV \quad (m)$$

$$\underline{\bar{f}} = \text{Re} \left[\frac{\int \underline{\bar{J}} \times \mu_0 \underline{\bar{H}}^* dV}{2} \right] + \text{Re} \left[\frac{\int \underline{\bar{J}} \times \mu_0 \underline{\bar{H}} dV}{2} e^{2j\omega t} \right] \quad (n)$$

Now, solving each of these integrals:

$$\begin{aligned} \frac{\int \underline{\bar{J}} \times \mu_0 \underline{\bar{H}}^* dV}{2} &= \frac{1}{2} \mu_0 a D \left(\frac{I}{D} \right)^2 \frac{1}{\delta} (1+j) \int_0^\infty e^{-\frac{2x}{\delta}} dx \underline{\bar{i}}_x \\ &= \frac{1}{4} \frac{\mu_0 a}{D} I^2 (1+j) \underline{\bar{i}}_x \end{aligned} \quad (o)$$

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.17 (Continued)

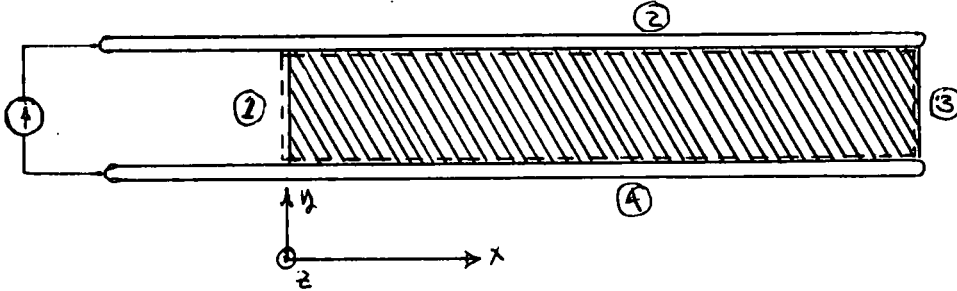
$$\begin{aligned} \frac{\int \bar{\mathbf{J}} \times \mu_0 \bar{\mathbf{H}}}{2} e^{2j\omega t} dv &= \frac{1}{2} \mu_0 a D \left(\frac{I}{D}\right)^2 \frac{1}{\delta} (1+j) e^{2j\omega t} \int_0^\infty e^{-\frac{2x}{\delta}(1+j)} dx \bar{\mathbf{i}}_x \\ &= \frac{1}{4} \frac{\mu_0 a}{D} I^2 e^{2j\omega t} \bar{\mathbf{i}}_x \end{aligned} \quad (p)$$

Hence, taking the real part, the force as in equation (n) is:

$$\bar{\mathbf{f}} = \frac{1}{4} \frac{\mu_0 a}{D} I^2 (1 + \cos 2\omega t) \bar{\mathbf{i}}_x \quad (q)$$

Part c

Using the Maxwell stress tensor, we choose the surface shown in the figure,



$$f_x = \int T_{xj} n_j da = \int_{(1)(3)} T_{xx} n_x da + \int_{(2)(4)} T_{xy} n_y da \quad (r)$$

Along surfaces (2) and (4), $H_x = 0$ along the interface between the perfect conductors and the finite conductivity block. Thus,

$$T_{xy} = \mu_0 H_x H_y = 0 \quad (s)$$

At surface (3), the field is zero since all current filaments complete a closed loop circuit with the source through the block. Hence

$$T_{xx} = 0 \text{ on surface (3)} \quad (t)$$

Therefore the calculation of the force reduces to

$$f_x = - \int_1 T_{xx} da \quad (u)$$

$$T_{xx} = - \frac{\mu_0}{2} H_z^2 \quad (v)$$

And thus,

$$f_x = \frac{a D \mu_0}{2} H_z^2 \quad (w)$$

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.17 (Continued)

where the field H_z is evaluated on surface 1, i.e. $x = 0$, and is simply given by the boundary condition (j). Thus it follows

$$\bar{f} = \frac{a\mu_0}{4D} I^2 \{1 + \cos 2\omega t\} \bar{I}_x \quad (x)$$

which checks with (q). Note that the distribution of $\bar{J}$ and $\bar{H}$, as found in part (a), are not required to find the total force in this problem. Even more, (x) is not limited to $\delta \ll x$ block dimension, while the detailed integration is.

Note: We have made use of the rule for products, namely of:

$$a(t) = \text{Re}[Ae^{j\omega t}] = \frac{Ae^{j\omega t} + A^*e^{-j\omega t}}{2}$$

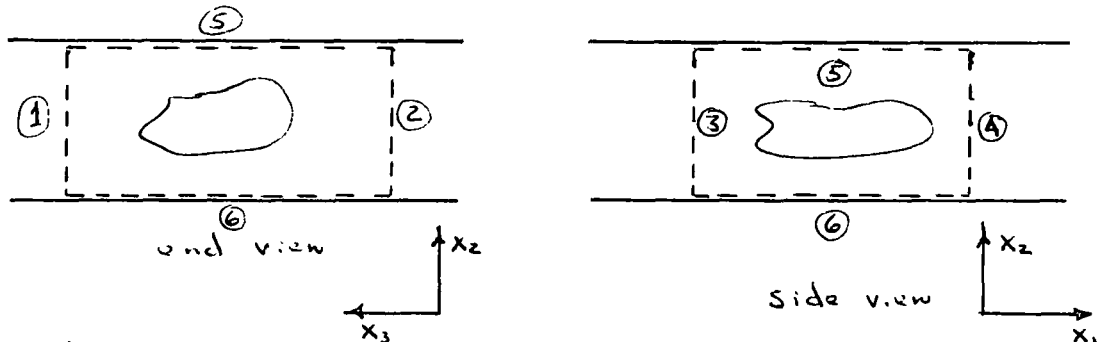
$$b(t) = \text{Re}[Be^{j\omega t}] = \frac{Be^{j\omega t} + B^*e^{-j\omega t}}{2}$$

then

$$\begin{aligned} a(t)b(t) &= \frac{AB^* + A^*B}{4} + \frac{ABe^{2j\omega t} + A^*B^*e^{-2j\omega t}}{4} \\ &= \underbrace{\text{Re}\left[\frac{AB^*}{2}\right]}_{\text{avg. value}} + \underbrace{\text{Re}\left[\frac{AB}{2}e^{2j\omega t}\right]}_{\text{time varying part}} \end{aligned}$$

PROBLEM 8.18

Choose the surface shown in the figure.



$$f_1 = \int_{3,4} T_{1j} n_j da = \int_{5,6} T_{11} n_1 da + \int_{1,2} T_{12} n_2 da + \int_{1,2} T_{13} n_3 da \quad (a)$$

Since the plates are perfectly conducting, $E_1 = 0$ at surfaces (5) and (6) and hence $T_{12} = 0$ on surfaces (5) and (6). Surfaces (1), (2), (3) and (4) are far from the body so

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.18 (Continued)

$$\bar{E} = \frac{V_0}{d} \bar{i}_z \quad (b)$$

at each of them, and thus, on surfaces (1) and (3), $T_{13} = 0$. Therefore,

$$f_1 = - \int_{(3)} T_{11} da_3 + \int_{(4)} T_{11} da_4 \quad (c)$$

$$T_{11}^{(3)} = T_{11}^{(4)} = - \frac{\epsilon_0}{2} \left(\frac{V_0}{d} \right)^2 \quad (d)$$

and $a_3 = a_4$ (areas). Hence,

$$f_1 = 0 \quad (e)$$

PROBLEM 8.19

Part a

Since the system is electrically linear,

$$\bar{B} = \bar{B}_\ell + \bar{B}_r \quad (a)$$

where $\bar{B}_\ell$ and $\bar{B}_r$ are respectively the fields from the left and right wires.

The force on a unit length of the right wire is

$$\bar{f}_r = \int \bar{J}_r \times \bar{B} da = \int \bar{J}_r \times \bar{B}_\ell da + \int \bar{J}_r \times \bar{B}_r da \quad (b)$$

but,

$$\int \bar{J}_r \times \bar{B}_r da = 0 \quad (c)$$

and hence,

$$\bar{f}_r = \int \bar{J}_r \times \bar{B}_\ell da \quad (d)$$

Since, we don't need the fields near the wire,

$$\bar{B}_\ell \approx \frac{\mu_0 I}{2\pi} \left[\frac{x_2 \bar{i}_1 - (x_1 + a) \bar{i}_2}{(x_1 + a)^2 + x_2^2} \right] \quad (e)$$

$$\bar{B}_r \approx \frac{\mu_0 I}{2\pi} \left[\frac{-x_2 \bar{i}_1 + (x_1 - a) \bar{i}_2}{(x_1 - a)^2 + x_2^2} \right] \quad (f)$$

Hence,

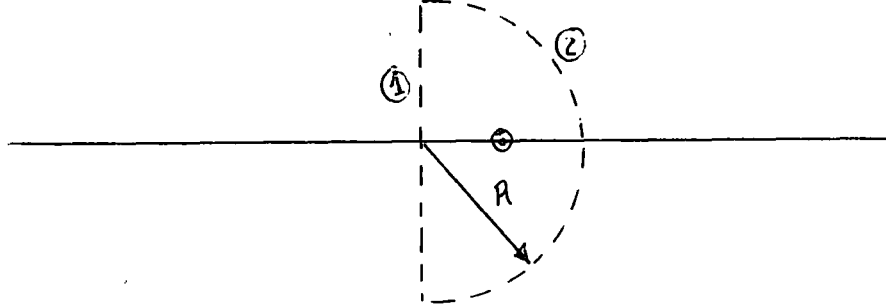
$$\bar{f}_r = \int \bar{J}_r \times \bar{B}_\ell da \approx I \bar{i}_3 \times \bar{B}_\ell (x_1 = a, x_2 = 0) \quad (g)$$

$$\bar{f}_r = \frac{\mu_0 I^2}{2\pi} \frac{(2a) \bar{i}_1}{(2a)^2} = \frac{\mu_0 I^2}{4\pi a} \bar{i}_1 \quad (h)$$

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.19 (Continued)

Part b



Along the symmetry plane of the surface shown in the figure

$$\vec{B} = \frac{\mu_0 I}{2\pi} \frac{(-2a)}{(a^2 + x_2^2)} \vec{i}_2 \quad (i)$$

The terms of T_{ij} go as B^2 , but $B^2 \propto \frac{1}{R^2}$ and the surface area goes as $2\pi R$ on surface (2), hence the contributions of the stress tensor will vanish on surface (2) as $R \rightarrow \infty$; we need only compute the integral on surface (1). Because $H_1 = 0$ in the plane $x_1 = 0$

$$\begin{aligned} f_1 &= \int -T_{11} da = \int_{-\infty}^{+\infty} \frac{\mu_0 H_2^2}{2} dx_2 \\ &= \frac{\mu_0}{2} \left(\frac{Ia}{\pi}\right)^2 \int_{-\infty}^{+\infty} \frac{dx_2}{(a^2 + x_2^2)^2} \end{aligned} \quad (j)$$

Solving this integral, we find

$$f_1 = \frac{\mu_0 I^2}{4\pi a} \quad (k)$$

also

$$f_2 = f_3 = 0 \quad (l)$$

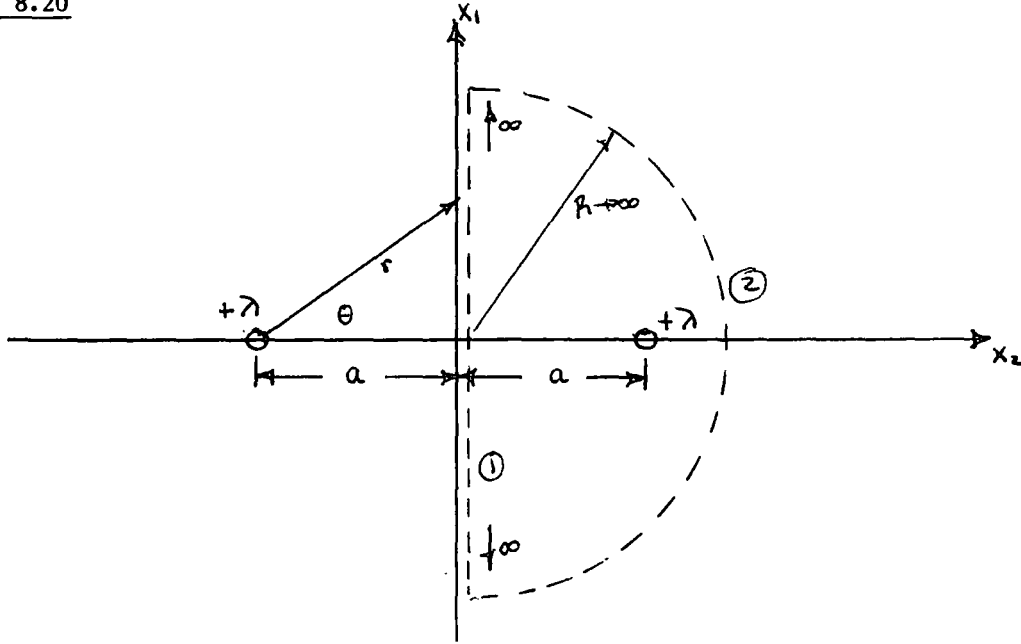
since

$$T_{21} = T_{31} = 0 \quad (m)$$

and hence the total force is that of (k) and it agrees with that determined in part (a).

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.20



Part a

Use the contour indicated in the figure. At infinity the fields will go to zero, and hence there will be no contribution to the force from the semi-circular part of the area, i.e. surface (2).

Along the line $x_2 = 0$, $E_2 = 0$ by symmetry and

$$E_1 = \frac{2}{\epsilon_0} \left(\frac{\lambda}{2\pi r} \right) \sin\theta \quad (a)$$

$$r^2 = a^2 + x_1^2 \quad (b)$$

$$\sin\theta = \frac{x_1}{r} = \frac{x_1}{\sqrt{a^2 + x_1^2}} \quad (c)$$

Hence

$$E_1 = \frac{\lambda}{\epsilon_0 \pi} \frac{x_1}{a^2 + x_1^2} \quad (d)$$

$$f_2 = \int T_{2j} n_j da = \int_{(1)} T_{21} n_1 da + \int_{(1)} T_{22} n_2 da + \int_{(1)} T_{23} n_3 da \quad (e)$$

first and last integrals = 0, $\bar{n}_1$ and $\bar{n}_3 = 0$ on surface 1

$$T_{22} = -\frac{\epsilon_0}{2} E_1^2 = -\frac{\epsilon_0}{2} \left(\frac{\lambda^2}{\epsilon_0^2 \pi^2} \right) \frac{x_1^2}{(a^2 + x_1^2)^2} \quad (f)$$

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.20 (Continued)

Thus

$$f_2 = -2 \frac{\lambda^2}{2\epsilon_0 \pi^2} \int_0^\infty \frac{x_1^2 dx_1}{(a^2 + x_1^2)^2} \quad (g)$$

$$f_2 = \frac{\lambda^2}{4\pi\epsilon_0 a} \quad (h)$$

Part b

From electrostatics,

$$\vec{f} = \lambda \vec{E}$$

From the figure, we see that

$$\vec{E}(x_2=a) = \frac{\lambda}{2\pi\epsilon_0(2a)} \vec{i}_2 \quad (i)$$

Hence,

$$\vec{F} = \frac{\lambda^2}{4\pi\epsilon_0 a} \vec{i}_2 \quad (j)$$

which is the same as we obtained using the stress tensor - (see equation (h)).

PROBLEM 8.21

Part a

From Eq. 8.1.11,

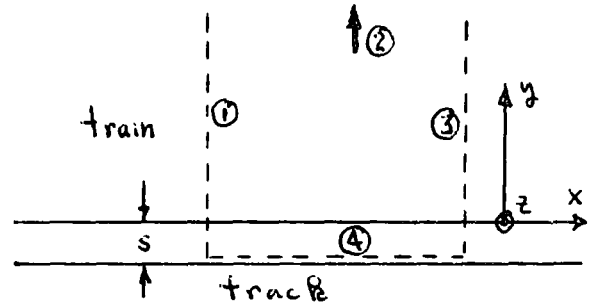
$$T_{ij} = \begin{bmatrix} \frac{1}{2\mu_0}(B_x^2 - B_y^2) & \frac{B_x B_y}{\mu_0} & 0 \\ \frac{B_x B_y}{\mu_0} & \frac{1}{2\mu_0}(-B_x^2 + B_y^2) & 0 \\ 0 & 0 & \frac{1}{2\mu_0}(-B_x^2 - B_y^2) \end{bmatrix} \quad (a)$$

where the components of $\vec{B}$ are given in the problem.

Part b

The appropriate surface of integration, which is fixed with respect to the fixed frame, is shown in the figure.

We compute the time average force, and hence contributions from surfaces (1) and (3) cancel. Fields go to zero on surface (2), which is at $y \rightarrow \infty$. Thus, there remains the stress on surface (4). The time average value of the surface force density $\vec{T}$



FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.21 (Continued)

is independent of x . Hence,

$$T_y = - \langle T_{yy}(y=0) \rangle \quad (b)$$

$$T_y = - \frac{1}{2\mu_0} \langle -B_x^2 + B_y^2 \rangle \quad (c)$$

Observe that

$$\langle \text{Re } \hat{A} e^{-jkUt} \text{Re } \hat{B} e^{-jkUt} \rangle \equiv \frac{1}{2} \text{Re } \hat{A} \hat{B}^* \quad (d)$$

where $\hat{B}^*$ is complex conjugate of $\hat{B}$, and (c) becomes

$$T_y = - \frac{1}{4\mu_0} \text{Re} \left\{ -(\mu_0 K_0 e^{jkx})(\mu_0 K_0 e^{-jkx}) + \frac{(-jk\mu_0 K_0)}{\alpha} e^{jkx} \frac{(jk\mu_0 K_0)}{\alpha^*} e^{-jkx} \right\}$$

$$= \frac{\mu_0 K_0^2}{4} \left(1 - \frac{k^2}{\alpha\alpha^*} \right) \quad (e)$$

$$= \frac{\mu_0 K_0^2}{4} \left(1 - \frac{k^2}{\alpha\alpha^*} \right) \quad (f)$$

Finally, use the given definition of α to write (f) as

$$T_y = \frac{\mu_0 K_0^2}{4} \left[1 - \frac{1}{\sqrt{1 + \left(\frac{\mu_0 \sigma U}{k} \right)^2}} \right] \quad (g)$$

Note that T_y is positive so that the train is supported by the magnetic field.

However, as $U \rightarrow 0$ (the train is stopped) the levitation force goes to zero.

Part c

For the force per unit area in the x direction;

$$T_x = - \frac{1}{2\mu_0} \langle B_x B_y(y=0) \rangle \quad (h)$$

$$= - \frac{1}{2\mu_0} \text{Re} \left[\mu_0 K_0 e^{jkx} \left(\frac{jk\mu_0}{\alpha^*} \right) K_0 e^{-jkx} \right] \quad (i)$$

Thus

$$T_x = - \frac{\mu_0 K_0^2}{2 \left[1 + \left(\frac{\mu_0 \sigma U}{k} \right)^2 \right]^{1/2}} \text{Re } j \sqrt{1 - j \left(\frac{\mu_0 \sigma U}{k} \right)} \quad (j)$$

As must be expected, the force on the train in the x directions vanishes as $U \rightarrow 0$. Note that in any case the force always tends to retard the motion and hence could hardly be used to propel the train.

The identity $\sin(\theta/2) = \pm \sqrt{(1 - \cos\theta)/2}$ is helpful in reducing (j) to the form

$$T_x = \frac{-\mu_0 K_0^2}{2 \left[1 + \left(\frac{\mu_0 \sigma U}{k} \right)^2 \right]^{1/2}} \sqrt{\frac{1}{2} \left(\sqrt{1 + \left(\frac{\mu_0 \sigma U}{k} \right)^2} - 1 \right)} \quad (k)$$

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.22

This problem makes the same point as Probs. 8.16 and 8.17, with the additional effect of material motion included. Regardless of the motion, with the current constrained as given, the magnetic field intensity is zero to the right of the block and uniform into the paper (z direction) to the left of the block, where

$$\vec{H} = \vec{i}_z \frac{I_o}{d} \quad (a)$$

The only contribution to an integration of the stress tensor over a surface enclosing the block is on the left surface. Thus

$$f_x = ds T_{xx} = - ds \frac{1}{2} \mu_o H_z^2 \quad (b)$$

$$= - \frac{ds}{2} \mu_o \left(\frac{I_o}{d} \right)^2 \quad (c)$$

The magnetic force is to the right and independent of the magnetic Reynolds number.

PROBLEM 8.23

In plane geometry, a knowledge of the charge on the upper plate is equivalent to knowing the electric field intensity on the surface of the plate. Thus, the surface charge density on the upper plate is

$$\sigma_f = \frac{1}{A} \int_0^t I_o \cos \omega t \, dt = \frac{I_o}{A\omega} \sin \omega t \quad (a)$$

and

$$E_x(x=a) = - \frac{\sigma_f}{\epsilon_o} = - \frac{I_o}{A\epsilon_o \omega} \sin \omega t \quad (b)$$

Now, we enclose the upper plate with a surface just outside the electrode surface. The only contribution to the integration of Eq. 8.1.17 using the stress tensor 8.3.10 is

$$f_x = - AT_{xx}(x=a) = - \frac{A\epsilon_o}{2} E_x^2(x=a) \quad (c)$$

which we can evaluate from (b) as

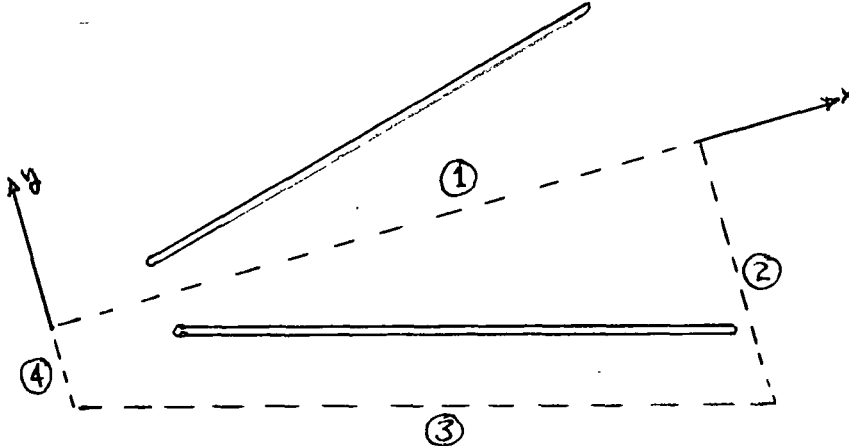
$$f_x = - \frac{A\epsilon_o}{2} \left(\frac{I_o}{A\epsilon_o \omega} \right)^2 \sin^2 \omega t \quad (d)$$

The force of attraction between the conducting slab and upper electrode is not dependent on σ_1 or σ_o .

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.24

The force on the lower electrode in the x direction is zero, as can be seen by integrating the Maxwell stress tensor over the surface shown.



The fields are zero on surfaces (2), (3) and (4). Hence, the total force per unit depth into the paper is

$$f_x = \int_0^{\infty} T_{xy} dx \quad (a)$$

where contributions from surfaces in the plane of the paper cancel because the problem is two-dimensional. Moreover, by symmetry the electric field intensity on the surface (1), even in the fringing regions, is in the y direction only and $T_{xy} = \epsilon_0 E_x E_y$ in (a) is zero. Thus, the total x directed force is zero.

PROBLEM 8.25

The force density in the dielectric slab is Eq. 8.5.45. Not only is the first term zero, but because the block moves as a rigid body (we are interested only in the net force giving rise to a rigid body displacement) the last term, which originates in changes in volume of the material, does not give a contribution. Hence, the force density is

$$\bar{F} = -\frac{1}{2} \bar{E} \cdot \bar{E} \nabla \epsilon \quad (a)$$

and the stress tensor is

$$T_{ij} = \epsilon E_i E_j - \frac{\delta_{ij}}{2} \epsilon E_k E_k \quad (b)$$

Note that, from (a), the force density in the x_1 direction is confined to the right edge of the block, where it acts as a surface force. Thus, we obtain the total force by simply integrating over a surface that encloses the right edge;

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.25 (Continued)

$$f_1 = aD \left[-\frac{1}{2} \epsilon_0 (E_2^a)^2 + \frac{1}{2} \epsilon (E_2^b)^2 \right] \quad (c)$$

where a and b are to the right and left of the right edge of the slab. Also $E_2^a = E_2^b = -V_0/a$. Hence (c) becomes

$$f_1 = \frac{aD}{2} \left(\frac{V_0}{a} \right)^2 (\epsilon - \epsilon_0) \quad (d)$$

The force acts to the right, as could be computed by the energy method.

PROBLEM 8.26

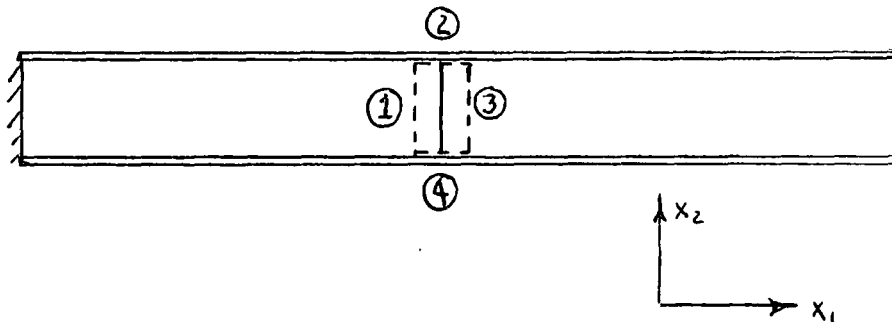
Part a

The force density for polarizable materials is:

$$\vec{F} = -\frac{1}{2} \vec{E} \cdot \vec{E} \nabla \epsilon + \frac{1}{2} \nabla (\vec{E} \cdot \vec{E} \rho \frac{\partial \epsilon}{\partial \rho}) \quad (a)$$

The second term on the right side represents electrostriction. Note that this is a case where the material volume must change, and hence the effect of electrostriction is important. Since free space and the elastic bulk are homogeneous, changes in permittivity and $\partial \epsilon / \partial \rho$ occur only at the boundary where the permittivity is discontinuous. The upper and lower elastic bulk surfaces are constrained by the plates. Thus only the x_1 component of force is pertinent. Since the left-hand edge is fixed, any stress arising from the discontinuity in permittivity at that boundary is counterbalanced by the rigidity of the wall. Therefore, all of the force arises at the right-hand boundary which is free to move.

The closed surface of integration is shown in the figure.



FIELD DESCRIPTION ON MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.26 (Continued)

$$T_{ij} = \epsilon E_i E_j - \frac{1}{2} \delta_{ij} \left[\epsilon - \rho \frac{\partial \epsilon}{\partial \rho} \right] E_k E_k \quad (b)$$

Since $a/c \ll 1$ and $b \sim \frac{1}{2} a$, the field at the dielectric interface is essentially uniform.

$$\vec{E} = - \vec{i}_2 \frac{V_o}{a} \quad (c)$$

The relevant components of the stress tensor are:

$$T_{11} = - \frac{\epsilon}{2} E_2^2 + \frac{1}{2} \rho \frac{\partial \epsilon}{\partial \rho} E_2^2 \quad (d)$$

$$T_{12} = \epsilon E_1 E_2 = 0 \quad (e)$$

$$f_1 = \int_{(1)(3)} T_{11} n_1 da + \int_{(2)(4)} T_{12} n_2 da \quad (f)$$

Hence

$$\begin{aligned} f_1 &= \int_{(3)} T_{11} da_3 - \int_{(1)} T_{11} da_1 \\ &= - \frac{\epsilon_o}{2} \left(\frac{V_o}{a} \right)^2 (ab) - \left[- \frac{\epsilon_1}{2} \left(\frac{V_o}{a} \right)^2 (ab) + \frac{\rho}{2} \frac{\partial \epsilon}{\partial \rho} \left(\frac{V_o}{a} \right)^2 (ab) \right] \end{aligned} \quad (g)$$

Thus;

$$f_1 = \frac{(\epsilon_1 - \epsilon_o) V_o^2 D}{2a} - \frac{\rho}{2} \frac{\partial \epsilon}{\partial \rho} \left(\frac{V_o^2 D}{a} \right) \quad (h)$$

Part b

In order to use lumped parameter energy methods, the charge on the upper plate will be found. The permittivity of the dielectric bulk is a function of the displacement of the right-hand edge. That is, if mass conservation is to hold,

$$\rho_o abD = (\rho_o + \Delta\rho) aD(b+\xi) \quad (i)$$

where

$$\rho = \rho_o + \Delta\rho, \Delta\rho = 0 \text{ if } \xi = 0 \quad (j)$$

Thus, if $\Delta\rho \ll \rho_o$ and $\xi \ll b$, to first order

$$\Delta\rho = -\rho_o \frac{\xi}{b} \quad (k)$$

(see Eqs. 8.5.9 and 8.5.10)

Furthermore, to first order, using a Taylor series,

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.26 (Continued)

$$\epsilon = \epsilon_1 + \frac{\partial \epsilon}{\partial \rho} \Delta \rho = \epsilon_1 - \frac{\rho_o}{b} \frac{\partial \epsilon}{\partial \rho} \xi \quad (l)$$

Also, the electric field will be assumed as uniform everywhere between the plates. Hence; in the block

$$\bar{D} = -\bar{i}_2 \left\{ \frac{V_o}{2} \left[\epsilon_1 + \frac{\partial \epsilon}{\partial \rho} \Delta \rho \right] \right\} \quad (m)$$

to the right of the block

$$\bar{D} = -\bar{i}_2 \frac{V_o}{a} \epsilon_o \quad (n)$$

By employing Gauss's law, we find the charge on the upper plate as:

$$q = \left(\frac{V_o}{a} \right) \left\{ \epsilon_1 - \frac{\rho_o}{b} \frac{\partial \epsilon}{\partial \rho} \xi \right\} (b+\xi) D + \epsilon_o \left(\frac{V_o}{a} \right) (c-b-\xi) D \quad (o)$$

$$\int dw'_e = \int q dv + \int f_e dx \quad (p)$$

integrating we find

$$w'_e = \frac{1}{2} \left(\frac{V_o^2}{a} \right) \left[\epsilon_1 - \frac{\rho_o}{b} \frac{\partial \epsilon}{\partial \rho} \xi \right] (b+\xi) D + \frac{1}{2} \frac{V_o^2}{a} (c-b-\xi) D \quad (q)$$

Thus,

$$f_e = \frac{\partial w'_e}{\partial \xi} \bigg|_{V=V_o} = \frac{(\epsilon_1 - \epsilon_o) V_o^2 D}{2a} - \frac{1}{2} \frac{V_o^2}{a} \rho_o D \frac{\partial \epsilon}{\partial \rho} \quad (r)$$

Second order terms have been dropped in the co-energy expression (alternatively, first order terms can be dropped in the force expression).

Part c

If the result of part (a) is written for $\rho = \rho_o + \Delta \rho$, where $\rho_o \gg \Delta \rho$, then the answers to part (a) and (b) are identical to first order. This should be expected since the lumped parameter approach assumed a value for permittivity which was correct only to first order.

PROBLEM 8.27

The surface force density is

$$T_m = [T_{mn}^a - T_{mn}^b] n_n \quad (a)$$

For this problem, we require $m = 1$ and $\bar{n} = \bar{i}_2$. Thus

$$T_1 = (T_{12}^a - T_{12}^b) \quad (b)$$

From Eq. 8.5.46,

FIELD DESCRIPTION OF MAGNETIC AND ELECTRIC FORCES

PROBLEM 8.27 (Continued)

$$T_1 = \epsilon_0 E_1^a E_2^a - \epsilon E_1^b E_2^b \quad (c)$$

Note that $E_2^a = E_2^b$ (see Eq. 6.2.31). Moreover, because there is no free charge $\epsilon_0 E_1^a = \epsilon E_1^b$ (see Eq. 6.2.33). Thus, (c) becomes

$$T_1 = E_2^a [\epsilon_0 E_1^a - \epsilon E_1^b] = 0 \quad (d)$$

That the shear surface force density is zero in the x_3 direction follows the same reasoning.

PROBLEM 8.28

The force density, Eq. 8.5.45, written in component form, is

$$F_i = E_i \frac{\partial \epsilon}{\partial x_j} - \frac{1}{2} E_k E_k \frac{\partial \epsilon}{\partial x_i} + \frac{\partial}{\partial x_i} \left(\frac{1}{2} E_k E_k \frac{\partial \epsilon}{\partial \rho} \rho \right) \quad (a)$$

The first term can be rewritten as two terms, one of which is in the desired form

$$F_i = \frac{\partial}{\partial x_j} (\epsilon E_i E_j) - \epsilon E_j \frac{\partial E_i}{\partial x_j} - \frac{1}{2} E_k E_k \frac{\partial \epsilon}{\partial x_i} + \frac{\partial}{\partial x_i} \left(\frac{1}{2} E_k E_k \frac{\partial \epsilon}{\partial \rho} \rho \right) \quad (b)$$

Because $\nabla \times \vec{E} = 0$, $\partial E_i / \partial x_j = \partial E_j / \partial x_i$, so that the second term can be rewritten and combined with the third. (Note the j is a dummy summation variable.)

$$F_i = \frac{\partial}{\partial x_j} (\epsilon E_i E_j) - \frac{1}{2} \frac{\partial}{\partial x_i} (\epsilon E_k E_k) + \frac{\partial}{\partial x_i} \left(\frac{1}{2} E_k E_k \frac{\partial \epsilon}{\partial \rho} \rho \right) \quad (c)$$

Finally, we introduce δ_{ij} (see Eq. 8.1.7) to write (c) in the required form

$$F_i = \frac{\partial T_{ij}}{\partial x_j} \quad (d)$$

where

$$T_{ij} = \epsilon E_i E_j - \frac{1}{2} \delta_{ij} E_k E_k \left(\epsilon - \rho \frac{\partial \epsilon}{\partial \rho} \right) \quad (e)$$

This is identical to Eq. 8.5.46.

SIMPLE ELASTIC CONTINUA

PROBLEM 9.1

The equation of motion for a static rod is

$$0 = E \frac{d^2\delta}{dx^2} + F_x \text{ where } F_x = \rho g \quad (a)$$

We can integrate this equation directly and get

$$\delta(x) = - \frac{\rho g}{E} \left(\frac{x^2}{2} \right) + Cx + D, \quad (b)$$

where C and D are arbitrary constants.

Part a

The stress function is $T(x) = E \frac{d\delta}{dx}$, and therefore

$$T(x) = - \rho g x + CE. \quad (c)$$

We have a free end at $x = \ell$ and this implies $T(x=\ell)=0$. Now we can write the stress as

$$T(x) = - \rho g x + \rho g \ell. \quad (d)$$

The maximum stress occurs at $x = 0$ and is $T_{\max} = \rho g \ell$. Equating this to the maximum allowable stress, we have

$$2 \times 10^9 = (7.8 \times 10^3)(9.8)\ell$$

hence

$$\ell = 2.6 \times 10^4 \text{ meters.}$$

Part b

From part (a)

$$T(x) = - \rho g x + \rho g \ell \quad (e)$$

The fixed end at $x = 0$ implies that $D = 0$, so now we can write the displacement

$$\delta(x) = - \frac{\rho g}{E} \left(\frac{x^2}{2} \right) + \frac{\rho g \ell}{E} (x) \quad (f)$$

Part c

$$\delta(\ell) = - \frac{\rho g}{E} \frac{\ell^2}{2} + \frac{\rho g \ell}{E}(\ell) = \frac{\rho g \ell^2}{2E} \quad (g)$$

For $\ell = 2.6 \times 10^4$ meters, $\delta(\ell) = 129$ meters. This appears to be a large displacement, but note that the total unstressed length is 26,000 meters.

SIMPLE ELASTIC CONTINUA

PROBLEM 9.2

Part a

The equation of motion for a static rod is

$$0 = E \frac{d^2 \delta}{dx^2} + \rho g \quad (a)$$

If we define $x' = x - L_1$, we can write the solutions for δ in rod 1 and in rod 2 as

$$\delta_1(x) = - \frac{\rho_1 g}{E_1} \left(\frac{x^2}{2} \right) + C_1 x + D_1 \quad (b)$$

and

$$\delta_2(x') = - \frac{\rho_2 g}{E_2} \left(\frac{x'^2}{2} \right) + C_2 x' + D_2 \quad (c)$$

where C_1, C_2, D_1 , and D_2 are arbitrary constants. Since $T = E \frac{d\delta}{dx}$ we can also write the tensions,

$$T_1(x) = - \rho_1 g x + E_1 C_1 \quad (d)$$

and

$$T_2(x') = - \rho_2 g x' + E_2 C_2 \quad (e)$$

We must have four boundary conditions to evaluate the constants and they are:

$$\delta_1(x=0) = 0, \quad (f)$$

$$\delta_2(x'=0) = \delta_1(x=L_1) \quad (g)$$

$$0 = - A_1 T_1(x=L_1) + A_2 T_2(x'=0) + mg, \quad (h)$$

and

$$0 = - A_2 T_2(x'=L_2) + Mg + f_x^e \quad (i)$$

where f_x^e is found using the Maxwell stress tensor

$$f_x^e = \frac{\epsilon_0 A v_o^2}{2d^2} \quad (j)$$

where we assume $d \gg \delta^{(2)}(x'=L_2)$.

Equations (f), (g), (h) and (i) serve to define the constants of integration.

Substitution of (b)-(e) shows that

$$D_1 = 0 \quad (k)$$

$$- \frac{\rho_1 g}{E_1} \left(\frac{L_1^2}{2} \right) + C_1 L_1 + D_1 - D_2 = 0 \quad (l)$$

SIMPLE ELASTIC CONTINUA

PROBLEM 9.2 (Continued)

$$-A_1[-\rho_1 g L_1 + E_1 C_1] + A_2[E_2 C_2] + mg = 0 \quad (m)$$

$$-A_2[-\rho_2 g L_2 + E_2 C_2] + Mg + \frac{\epsilon_o A_M V_o^2}{2d^2} = 0 \quad (n)$$

Solution of these expressions, beginning with (n), gives

$$C_2 = \left[Mg + \frac{\epsilon_o A_M V_o^2}{2d^2} + \rho_2 g L_2 A_2 \right] \frac{1}{A_2 E_2} \quad (o)$$

and hence

$$C_1 = \left[mg + \rho_1 g L_1 A_1 + A_2 E_2 C_2 \right] \frac{1}{A_1 E_1} \\ = \left\{ [(M+m) + \rho_1 L_1 A_1 + \rho_2 L_2 A_2] g + \frac{\epsilon_o A_M V_o^2}{2d^2} \right\} \frac{1}{A_1 E_1} \quad (p)$$

$$D_2 = \frac{L_1}{A_1 E_1} \left\{ [(M+m) + \frac{\rho_1 L_1 A_1}{2} + \rho_2 L_2 A_2] g + \frac{\epsilon_o A_M V_o^2}{2d^2} \right\} \quad (q)$$

$$D_1 = 0 \quad (r)$$

Thus, (b) and (c) are determined.

PROBLEM 9.3

Part a

Longitudinal displacements on the rod satisfy the wave equation

$$\rho \frac{\partial^2 \delta}{\partial t^2} = E \frac{\partial^2 \delta}{\partial x^2} \text{ and the stress } T = E \frac{\partial \delta}{\partial x} \quad (a)$$

We can write $\delta(x,t) = \text{Re}[\hat{\delta}(x)e^{j\omega t}]$ for sinusoidal excitations. $\hat{\delta}(x)$ can be written as $\hat{\delta}(x) = C_1 \sin \beta x + C_2 \cos \beta x$ where $\beta = \omega\sqrt{\rho/E}$. The two constants are found from the boundary conditions

$$M \frac{\partial^2 \delta}{\partial t^2}(\ell, t) = -AT(\ell, t) + f(t) \quad (b)$$

$$\delta(0, t) = 0. \quad (c)$$

These conditions become

$$-M\omega^2 \hat{\delta}(\ell) = -AE \frac{d\hat{\delta}}{dx}(\ell) + f_o \quad (d)$$

and

$$\hat{\delta}(0) = 0 \quad (e)$$

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PROBLEM 9.3 (Continued)

for sinusoidal excitations.

Now we find $C_2 = 0$ and

$$C_1 = \frac{f_o}{AE\beta\cos\beta l - M\omega^2\sin\beta l} \quad (f)$$

Hence,

$$\delta(x,t) = \frac{\sin\beta x}{AE\beta\cos\beta l - M\omega^2\sin\beta l} \operatorname{Re}[f_o e^{j\omega t}] \quad (g)$$

and

$$T(x,t) = E \frac{\partial \delta}{\partial x} = \frac{E\beta\cos\beta x}{AE\beta\cos\beta l - M\omega^2\sin\beta l} \operatorname{Re}[f_o e^{j\omega t}] \quad (h)$$

Part b

At $x = l$,

$$\delta(l,t) = \frac{1}{AE\beta\cot\beta l - M\omega^2} \operatorname{Re}[f_o e^{j\omega t}] \quad (i)$$

where $\beta\cot\beta l = \omega\sqrt{\rho/E} \cot(\omega l\sqrt{\rho/E})$.

For small ω , $\cot(\omega l\sqrt{\rho/E}) \rightarrow \frac{1}{\omega l\sqrt{\rho/E}}$ and

$$\delta(l,t) \rightarrow \frac{1}{\frac{AE}{l} - M\omega^2} f(t) \quad (j)$$

This equation is as used to describe a mass on the end of a massless spring:

$$M_o \frac{d^2 x}{dt^2} = -Kx + f(t) \quad (k)$$

and

$$x = \operatorname{Re}[\hat{x} e^{j\omega t}],$$

$$-M_o \omega^2 \hat{x} = -K\hat{x} + f_o \quad (l)$$

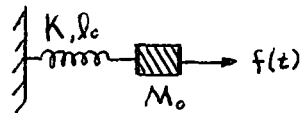
or

$$x = \frac{1}{K - M_o \omega^2} f(t) \quad (m)$$

Comparing (j) and (l) we note that

$$K = \frac{AE}{l} \text{ and } l_o = l. \quad (n)$$

Our comparison is complete and since $M \gg \rho A l$ we can use the massless spring model with a mass $M_o = M$ on the end.



SIMPLE ELASTIC CONTINUA

PROBLEM 9.4

A response that can be represented purely as a wave traveling in the negative x direction implies that there be no wave reflection at the left-hand boundary.

We must have

$$v(0,t) + \frac{1}{\sqrt{\rho E}} T(0,t) = 0 \quad (a)$$

as seen in Sec. 9.1.1b.

This condition can be satisfied by a viscous damper alone:

$$AT(0,t) + Bv(0,t) = 0 \quad (b)$$

Hence, we can write

$$\begin{aligned} B &= A\sqrt{\rho E} \\ M &= 0 \\ K &= 0. \end{aligned} \quad (c)$$

PROBLEM 9.5

Part a

At $x = \ell$ the boundary condition is

$$0 = -AT(\ell,t) - B \frac{\partial \delta}{\partial t}(\ell,t) + f(t) \quad (a)$$

Part b

We can write the solution as

$$\hat{\delta}(x) = C_1 \sin \beta x + C_2 \cos \beta x, \quad (b)$$

where $\beta = \omega \sqrt{\frac{\rho}{E}}$. At $x = 0$ there is a fixed end, hence $\hat{\delta}(x=0) = 0$ and $C_2 = 0$.

At $x = \ell$ our boundary condition becomes

$$F_o = j\omega B \hat{\delta}(x=\ell) + AE \frac{d\hat{\delta}}{dx}(x=\ell), \quad (c)$$

or in terms of C_1 ;

$$F_o = j\omega B C_1 \sin \beta \ell + AE \beta C_1 \cos \beta \ell \quad (d)$$

After solving for C_1 , we can write our solution as

$$\hat{\delta}(x) = \frac{F_o \sin \beta x}{AE \beta \cos \beta \ell + j\omega B \sin \beta \ell} \quad (e)$$

Part c

For ω real and $B > 0$, δ cannot be infinite with a finite-applied force, because the denominator of $\hat{\delta}(x)$ can never be zero.

Physically, $B > 0$ implies that the system is damped and energy would be

SIMPLE ELASTIC CONTINUA

PROBLEM 9.5 (Continued)

dissipated for each cycle of operation, hence a perfect resonance cannot occur. However, there will be frequencies which will maximize the amplitude.

PROBLEM 9.6

First, we can calculate the force of magnetic origin, f_x , on the rod. If we define $\delta(\ell, t)$ to be the a.c. deflection of the rod at $x = \ell$, then using Ampere's law and the Maxwell stress tensor (Eq. 8.5.41 with magnetostriction ignored) we find

$$f_x = \frac{\mu_o AN^2 I^2}{2[d - \delta(\ell, t)]^2} \quad (a)$$

This result can also be obtained using the energy methods of Chap. 3 (See Appendix E, Table 3.1). Since $d \gg \delta(\ell, t)$, we may linearize f_x :

$$f_x \approx \frac{\mu_o AN^2 I^2}{2d^2} + \frac{\mu_o AN^2 I^2}{d^3} \delta(\ell, t) \quad (b)$$

The first term represents a constant force which is balanced by a static deflection on the rod. If we assume that this static deflection is included in the equilibrium length ℓ , then we need only use the last term of f_x to compute the dynamic deflection $\delta(\ell, t)$. In the bulk of the rod we have the wave equation; for sinusoidal variations

$$\delta(x, t) = \text{Re}[\hat{\delta}(x)e^{j\omega t}] \quad (c)$$

we can write the complex amplitude $\hat{\delta}(x)$ as

$$\hat{\delta}(x) = C_1 \sin \beta x + C_2 \cos \beta x \quad (d)$$

where $\beta = \omega \sqrt{\frac{\rho}{E}}$. At $x = 0$ we have a fixed end, so $\hat{\delta}(0) = 0$ and $C_2 = 0$. At $x = \ell$ the boundary condition is

$$0 = f_x - AE \frac{\partial \delta}{\partial x}(\ell, t), \quad (e)$$

or

$$0 = \frac{\mu_o AN^2 I^2}{d^3} \hat{\delta}(x=\ell) - AE \frac{d\hat{\delta}}{dx}(x=\ell) \quad (f)$$

Substituting we obtain

$$\frac{\mu_o AN^2 I^2}{d^3} C_1 \sin \beta \ell = C_1 AE \beta \cos \beta \ell \quad (g)$$

Our solution is $\hat{\delta}(x) = C_1 \sin \beta x$ and for a non-trivial solution we must have $C_1 \neq 0$. So, divide (g) by C_1 and obtain the resonance condition:

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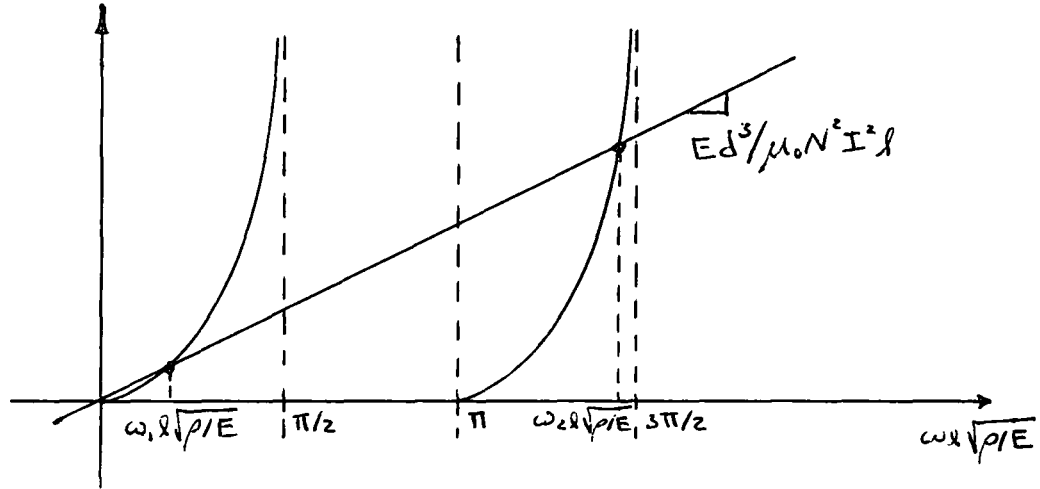
PROBLEM 9.6 (Continued)

$$\left(\frac{\mu_0 A N^2 I^2}{d^3}\right) \sin \beta l = A E \beta \cos \beta l \quad (h)$$

Substituting $\beta = \omega \sqrt{\frac{\rho}{E}}$ and rearranging, we have

$$\frac{E d^3}{\mu_0 N^2 I^2 l} \left(\omega l \sqrt{\frac{\rho}{E}}\right) = \tan(\omega l \sqrt{\frac{\rho}{E}}) \quad (i)$$

which, when solved for ω , yields the eigenfrequencies. Graphically, the first two eigenfrequencies are found from the sketch.



Notice that as the current I is increased, the slope of the straight line decreases and the first eigenfrequency (denoted by ω_1) goes to zero and then seemingly disappears for still higher currents. Actually ω_1 now becomes imaginary and can be found from the equation

$$\times \quad \frac{E d^3}{\mu_0 N^2 I^2 l} (|\omega_1| l \sqrt{\frac{\rho}{E}}) = \tanh (|\omega_1| l \sqrt{\frac{\rho}{E}}) \quad (j)$$

Just as there are negative solutions to (i), $-\omega_1, -\omega_2 \dots$ etc., so there are now solutions $\pm j|\omega_1|$. Thus, because ω_1 is imaginary, the system is unstable,

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PROBLEM 9.6 (Continued)

(amplitude of one solution growing in time).

Hence when the slope of the straight line becomes less than unity, the system is unstable. This condition can be stated as:

$$\text{STABLE} \longrightarrow \frac{Ed^3}{\mu_o N^2 I^2 l} > 1 \quad (k)$$

or

$$\text{UNSTABLE} \longrightarrow \frac{Ed^3}{\mu_o N^2 I^2 l} < 1 \quad (l)$$

PROBLEM 9.7

Part a

$\delta(x, t)$ satisfies the wave equation

$$\rho \frac{\partial^2 \delta}{\partial t^2} = E \frac{\partial^2 \delta}{\partial x^2} \quad (a)$$

and the stress is $T = E \frac{\partial \delta}{\partial x}$. We can write

$$\delta(x, t) = \text{Re}[\hat{\delta}(x)e^{j\omega t}] \quad (b)$$

and substitution into the wave equation gives

$$\frac{d^2 \hat{\delta}}{dx^2} + \beta^2 \hat{\delta} = 0. \quad (c)$$

$$\beta = \omega \sqrt{\frac{\rho}{E}}$$

For $x > 0$ we have,

$$\hat{\delta}_a(x) = \hat{C}_1 \sin \beta x + \hat{C}_2 \cos \beta x \quad (d)$$

and

$$\hat{T}_a(x) = \hat{C}_1 E \beta \cos \beta x - \hat{C}_2 E \beta \sin \beta x \quad (e)$$

and for $x < 0$ we have,

$$\hat{\delta}_b(x) = \hat{C}_3 \sin \beta x + \hat{C}_4 \cos \beta x \quad (f)$$

and

$$\hat{T}_b(x) = \hat{C}_3 E \beta \cos \beta x - \hat{C}_4 E \beta \sin \beta x \quad (g)$$

Part b

There are four constants to be determined; thus we need four boundary conditions. At the right end ($x=L$), we have

$$\hat{\delta}_a(x=L) = 0 \quad (h)$$

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PROBLEM 9.7 (Continued)

and the left end,

$$\hat{\delta}_b(x=-L) = \delta_o e^{-j(\frac{\pi}{2})} \quad (i)$$

There are two conditions at the middle ($x=0$),

$$\hat{\delta}_a(x=0^+) = \hat{\delta}_o(x=0^-) \quad (j)$$

and

$$-M\omega^2 \hat{\delta}_a(x=0) = \hat{A}T_a(x=0^+) - \hat{A}T_b(x=0^-) - 4K\hat{\delta}_a(x=0) \quad (k)$$

Part c

Solving for $\hat{C}_1, \hat{C}_2, \hat{C}_3$, and $\hat{C}_4$ we obtain

$$\hat{C}_1 = \frac{-\delta_o AE\beta e^{-j\frac{\pi}{2}} \cot\beta L}{\sin\beta L(4K+2AE\beta \cot\beta L - M\omega^2)} \quad (l)$$

$$\hat{C}_2 = \frac{\delta_o AE\beta e^{-j\frac{\pi}{2}}}{\sin\beta L(4K+2AE\beta \cot\beta L - M\omega^2)} \quad (m)$$

$$\hat{C}_3 = \frac{\delta_o AE\beta e^{-j\frac{\pi}{2}} \cot\beta L}{\sin\beta L(4K+2AE\beta \cot\beta L - M\omega^2)} - \frac{\delta_o e^{-j\frac{\pi}{2}}}{\sin\beta L} \quad (n)$$

$$\hat{C}_4 = \hat{C}_2 \quad (o)$$

Thus, (b), (e), and (g) with these constants give the desired stress distribution.

PROBLEM 9.8

In terms of the complex amplitudes, (l) and (r) become

$$\hat{T}'(0) = \frac{L_o I}{aA} \hat{i}_1 \quad (l) - \text{text} \quad (a)$$

and

$$\hat{T}'(l) = \frac{L_o I}{aA} \hat{i}' \quad (r) - \text{text} \quad (b)$$

where $\hat{i}' = -Gv_o$.

Equation (t) without the approximation becomes

$$\hat{v}_o = -j\omega \frac{GL_o(\mu+\mu_o)}{\mu-\mu_o} \hat{v}_o + j\omega \frac{L_o I}{a} \hat{\delta}_o \quad (c)$$

Using the steady-state solutions for the rod, we can solve for $T(x)$ in terms of the boundary values $\hat{T}(0)$ and $\hat{T}(l)$:

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PROBLEM 9.8 (Continued)

$$\hat{T}(x) = \hat{T}(0) \frac{\sin[k(\ell-x)]}{\sin[k\ell]} + \hat{T}(\ell) \frac{\sin[kx]}{\sin[k\ell]} \quad (d)$$

then

$$\hat{\delta} = \frac{1}{\omega\sqrt{\rho E}} \left[\hat{T}(0) \frac{\cos[k(\ell-x)]}{\sin[k\ell]} - \hat{T}(\ell) \frac{\cos[kx]}{\sin[k\ell]} \right] \quad (e)$$

From (a) and (b), this becomes

$$\hat{\delta}(\ell) = \hat{\delta}_0 = \frac{1}{\omega\sqrt{\rho E}} \left[\frac{\bar{L}_0 I}{aA} \hat{i}_1 \frac{1}{\sin[k\ell]} + \frac{GL_0 I}{aA} \hat{v}_0 \frac{\cos[k\ell]}{\sin[k\ell]} \right] \quad (f)$$

Thus, in view of (c) solved for $\hat{\delta}_0$, we obtain the system function

$$H(\omega) = \frac{\hat{v}_0}{\hat{i}_1} = \frac{-\frac{1}{G}}{\cos[k\ell] + j\sqrt{\rho E} \left(\frac{A}{G} \right) \left(\frac{a}{L_0 I} \right)^2 \sin[k\ell] - \left[\frac{\omega GL_0 (\mu + \mu_0)}{(\mu - \mu_0)} \right] \sqrt{\rho E} \left(\frac{A}{G} \right) \left(\frac{a}{L_0 I} \right)^2 \sin[k\ell]} \quad (g)$$

PROBLEM 9.9

Part a

First of all, $y(t) = \delta(-L, t)$ where $\delta(x, t) = \text{Re}[\hat{\delta}(x)e^{j\omega t}]$. We can write the solution for $\hat{\delta}$ as $\hat{\delta}(x) = C_1 \sin \beta x + C_2 \cos \beta x$, where $\beta = \omega\sqrt{\rho/E}$. The C_2 is zero because of the fixed end at $x = 0$ ($\hat{\delta}(0) = 0$). At the other end we have

$$M \frac{\partial^2 \delta}{\partial t^2} (-L, t) = A_2 E \frac{\partial \delta}{\partial x} (-L, t) + f^e(t) \quad (a)$$

Using the Maxwell stress tensor, (or the energy method of Chap. 3) we find

$$f^e(t) = \frac{A\mu_0 N^2}{2} \left\{ \frac{[I_0 - I(t)]^2}{[d-D+\delta(-L, t)]^2} - \frac{[I_0 + I(t)]^2}{[d-D-\delta(-L, t)]^2} \right\} \quad (b)$$

which when linearized becomes,

$$f^e(t) \approx -C_I I(t) - C_y \delta(-L, t), \quad (c)$$

where

$$C_I = \frac{2N^2 \mu_0 A I_0}{(d-D)^2}; \quad C_y = \frac{2N^2 \mu_0 A I_0^2}{(d-D)^3}$$

Our boundary condition (a) becomes

$$-M\omega^2 \hat{\delta}(-L) = A_2 E \frac{d\hat{\delta}}{dx}(-L) - C_I \hat{I} - C_y \hat{\delta}(-L) \quad (d)$$

Solving for C_1 we obtain

SIMPLE ELASTIC CONTINUA

PROBLEM 9.9 (Continued)

$$C_1 = \frac{C_I \hat{I}}{A_2 E \beta \cos \beta L - (M \omega^2 - C_y) \sin \beta L}, \quad (e)$$

and we can write our solution as

$$y(t) = \text{Re}[-C_1 \sin \beta L e^{j\omega t}]. \quad (f)$$

Part b

The transducer is itself made from solid materials having characteristics that do not differ greatly from those of the rod. Thus, there is the question of whether the elastic response of the transducer materials is of importance. Under the assumption that the rod and transducer are constructed from materials having essentially the same elastic properties, the assumption that the yoke and plunger are rigid, but that the rod supports acoustic waves, is justified provided the rod is long compared to the largest dimension of the transducer, and that an acoustic wavelength is long compared to the largest transducer dimension. (See Sec. 9.1.3).

PROBLEM 9.10

Part a

At the outset, we can write the equation of motion for the massless plate:

$$-aT(\ell, t) + f^e(t) = M \frac{\partial^2 \delta}{\partial t^2}(\ell, t) \approx 0 \quad (a)$$

Using the Maxwell stress tensor we find the force of electrical origin $f^e(t)$ to be

$$f^e(t) = \frac{\epsilon_0 A}{2} \left[\frac{(V_0 + v(t))^2}{(d - \delta(\ell, t))^2} - \frac{(V_0 - v(t))^2}{(d + \delta(\ell, t))^2} \right] \quad (b)$$

Since $v(t) \ll V_0$ and $\delta(\ell, t) \ll d$, we can linearize $f^e(t)$:

$$f^e(t) \approx \left[\frac{2\epsilon_0 A V_0^2}{d^3} \right] \delta(\ell, t) + \left[\frac{2\epsilon_0 A V_0}{d^2} \right] v(t) \quad (c)$$

Recognizing that $T(\ell, t) = E \frac{\partial \delta}{\partial x}(\ell, t)$ we can write our boundary condition at $x = \ell$ in the desired form:

$$aE \frac{\partial \delta}{\partial x}(\ell, t) = \frac{2\epsilon_0 A V_0^2}{d^3} \delta(\ell, t) + \frac{2\epsilon_0 A V_0}{d^2} v(t) \quad (d)$$

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PROBLEM 9.10 (Continued)

Longitudinal displacements in the rod obey the wave equation and for an assumed form of $\delta(x,t) = \text{Re}[\hat{\delta}(x)e^{j\omega t}]$ we can write $\hat{\delta}(x) = C_1 \sin \beta x + C_2 \cos \beta x$, where $\beta = \omega \sqrt{\rho/E}$. At $x = 0$ we have a fixed end, thus $\hat{\delta}(x=0) = 0$ and $C_2 = 0$. From part (a) and assuming sinusoidal time dependence, we can write our boundary condition at $x = l$ as

$$aE \frac{d\hat{\delta}}{dx}(l) = \frac{2\epsilon_o AV_o^2}{d^3} \hat{\delta}(l) + \frac{2\epsilon_o AV_o}{d^2} \hat{v} \quad (e)$$

Solving

$$C_1 = \frac{2\epsilon_o AV_o \hat{v}}{aEd^2 \beta \cos \beta l - \frac{2\epsilon_o AV_o^2}{d} \sin \beta l} \quad (f)$$

Finally, we can write our solution as

$$\delta(x,t) = \left[\frac{2\epsilon_o AV_o \sin \beta x}{aEd^2 \beta \cos \beta l - \frac{2\epsilon_o AV_o^2}{d} \sin \beta l} \right] \text{Re}[\hat{v} e^{j\omega t}] \quad (g) \quad *$$

PROBLEM 9.11

Part a

For no elastic wave reflection at the right-hand boundary we must have a boundary condition of the form

$$v(0,t) + \frac{1}{\sqrt{\rho E}} T(0,t) = 0 \quad (a)$$

(from Sec. 9.1.1b). Since $v(0,t) = \frac{\partial \delta}{\partial t}(0,t)$, we can write

$$-\sqrt{\rho E} \frac{\partial \delta}{\partial t}(0,t) = T(0,t) \quad (b)$$

If we write the boundary condition at $x = 0$ for our example we obtain

$$0 = -ST(0,t) + f_x^e(t), \quad (c)$$

or for perturbations

$$0 = -ST(0,t) + f_{a.c.}^e(t) \quad (d)$$

Combining (b) and (d)

$$f_{a.c.}^e(t) = -S\sqrt{\rho E} \frac{\partial \delta}{\partial t}(0,t) \quad (e)$$

SIMPLE ELASTIC CONTINUA

PROBLEM 9.11 (Continued)

and since $\partial\delta/\partial t (0,t) = dy_s/dt$,

$$f_{a.c.}^e(t) = -S\sqrt{\rho E} \frac{dy_s}{dt} \quad (f)$$

The perturbation electric force can be found using the Maxwell stress tensor (using a surface of integration similar to that illustrated by Prob. 8.10):

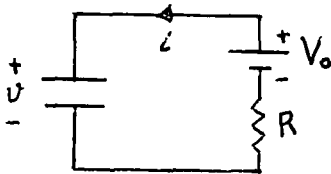
$$f_x^e(t) = \frac{\epsilon_o v^2 D}{a} = \frac{\epsilon_o V_o^2 D}{a} + \frac{2\epsilon_o V_o D v_s}{a} \quad (g)$$

where we associate $f_{a.c.}^e(t) = \frac{2\epsilon_o V_o D v_s}{a}$.

Equation (f) now becomes

$$\frac{2\epsilon_o V_o D v_s}{a} = -S\sqrt{\rho E} \frac{dy_s}{dt} \quad (h)$$

Now that we have dealt with the force balance we can write the circuit equations.



The capacitance of the device is found to be

$$C = \frac{2\epsilon_o y D}{a}$$

Note that $q = Cv$ and $i = \frac{dq}{dt}$. The basic circuit equation is

$$v + iR = V_o = v + R \frac{dq}{dt} = v + R \left[C \frac{dv}{dt} + v \frac{dC}{dt} \right] \quad (i)$$

Substituting, we obtain

$$V_o = v + RC \frac{dv}{dt} + \frac{2\epsilon_o DR}{a} \frac{dy}{dt} \quad (j)$$

and for perturbation quantities,

$$0 = v_s + RC_o \frac{dv_s}{dt} + \frac{2\epsilon_o DV_o R}{a} \frac{dy_s}{dt} \quad (k)$$

Since $\omega \ll \frac{1}{RC_o}$, $v_s \gg RC_o dv_s/dt$ and now we have

$$0 = v_s + \frac{2\epsilon_o DV_o R}{a} \frac{dy_s}{dt} \quad (l)$$

SIMPLE ELASTIC CONTINUA

PROBLEM 9.11 (Continued)

Equations (h) and (l) must be satisfied simultaneously and this can occur only if

$$\frac{2\epsilon_o DV_o R}{a} = \frac{aS\sqrt{\rho E}}{2\epsilon_o V_o D} \quad (m)$$

Finally from (m) we have the condition on the d.c. voltage,

$$V_o = \frac{a}{2\epsilon_o D} \left[\frac{S\sqrt{\rho E}}{R} \right]^{1/2} \quad (n)$$

PROBLEM 9.12

Part a

Note that there is no mutual capacitance between the two pairs. We can find the capacitance of the left-hand pair of plates to be

$$C_2 = \frac{\epsilon d(\frac{w}{2} - y_2)}{h} + \frac{\epsilon_o d(\frac{w}{2} + y_2)}{h} \quad (a)$$

The current i_2 can be found from $i_2 = dq_2/dt = d(V_o C_2)/dt = V_o dC_2/dt$, and upon substitution of C_2 we obtain

$$i_2 = \left[\frac{-(\epsilon - \epsilon_o)V_o d}{h} \right] \frac{dy_2}{dt} \quad (b)$$

If we solve for y_2 in terms of v_s our job will be done.

Define the y -axis from left to right with $y = 0$ at $y_1 = 0$. Assume all constant forces (with $v_s = 0$) to be balanced and consider only the perturbations. If we assume for the rod $\delta(y,t) = \text{Re}[\hat{\delta}(y)e^{j\omega t}]$ then we can write

$$\hat{\delta}(y) = C_1 \sin \beta y + C_2 \cos \beta y \quad (c)$$

where $\beta = \omega\sqrt{\rho/E}$. (We have assumed that the electrical forces act only on the surfaces of the rod. This is evident from the form of the force density, Eq. 8.5.45, if the effect of electrostriction can be ignored.) At $y = 0$ there is no perturbation force and for a.c. deflections we have a free end condition:

$$0 = T(0,t) \Rightarrow E \frac{d\hat{\delta}}{dy} (y = 0) = 0 \quad (d)$$

This forces C_1 to be zero. At $y = l$ we can write the boundary condition as

$$0 = -hdT(l,t) + f_{a.c.}^e(t)$$

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PROBLEM 9.12 (Continued)

Using the Maxwell stress tensor (or energy methods, as in Sec. 8.54)

$$f_1^e(t) = \frac{(\epsilon - \epsilon_o)d}{2h} (v_o + v_s)^2 \quad (e)$$

Linearizing and ignoring the d.c. term we have

$$f_{la.c.}^e(t) = \frac{(\epsilon - \epsilon_o)v_o d}{h} v_s.$$

From the boundary condition for complex amplitudes we obtain

$$0 = -hdE \frac{d\hat{\delta}}{dy}(l) + \frac{(\epsilon - \epsilon_o)v_o d}{h} \hat{v}_s \quad (f)$$

Substituting and solving for C_2 ;

$$C_2 = \frac{-(\epsilon - \epsilon_o)v_o}{h^2 E \beta \sin \beta l} \hat{v}_s. \quad (g)$$

Recognizing that $y_2(t) = \delta(0,t)$, we can now write

$$y_2(t) = \text{Re} \left[\frac{-(\epsilon - \epsilon_o)v_o \hat{v}_s}{h^2 E \beta \sin \beta l} e^{j\omega t} \right] \quad (h)$$

Since $i_2 = \frac{-(\epsilon - \epsilon_o)v_o d}{h} \frac{dy_2}{dt}$, we have

$$i_2 = \text{Re} \left[\frac{j\omega(\epsilon - \epsilon_o)^2 v_o^2 d}{h^3 E \beta \sin \beta l} \hat{v}_s e^{j\omega t} \right] \quad (i)$$

Finally, we can write

$$Y(j\omega) = \frac{\hat{i}_2}{\hat{v}_s} = \frac{j\omega(\epsilon - \epsilon_o)^2 v_o^2 d}{h^3 E \beta \sin \beta l} \quad (j)$$

Part b

The poles can be found from

$$h^3 E \beta \sin \beta l = 0 \quad (k)$$

where $\beta = \omega \sqrt{\rho/E}$. The lowest nonzero frequency can be found from

$\sin(\omega l \sqrt{\rho/E}) = 0$ to be

$$\omega = \frac{\pi}{l \sqrt{\rho/E}} \quad (l)$$

Note that the $\omega = 0$ is a pole because the rod is free to translate slowly between the plates.

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PROBLEM 9.13

Part a

The flux for the left-hand transducer is

$$\lambda_l = \frac{\mu_o N^2}{g} 2\pi R(a - \delta(0, t)) i_l \quad (a)$$

and for the right-hand one,

$$\lambda_r = \frac{\mu_o N^2}{g} 2\pi R(a + \delta(L, t)) i_r \quad (b)$$

For this electrically linear situation we have $W'_m = \frac{1}{2} Li^2 = \frac{1}{2} \lambda i$ and $f = \frac{\partial W'_m}{\partial \delta}$.
Hence we find, to linear terms

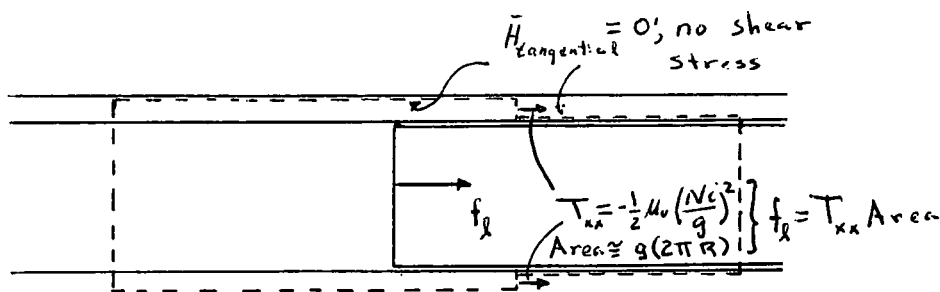
$$f_l = - \frac{\mu_o N^2}{g} \pi R(I_o^2 + 2I_o i_l) \quad (c)$$

and, because $i_r = I_o - Gv_{out}$

$$f_r = \frac{\mu_o N^2}{g} \pi R(I_o^2 - 2I_o Gv_{out}) \quad (d)$$

Part b

For the left-hand transducer, an acceptable stress-tensor surface is shown below,



and the mirror-image is acceptable for the right-hand transducer. Application of $f_x = \oint_S T_{xj} n_j da$ to the two surfaces yields the same result as in part (a).

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PROBLEM 9.13 (Continued)

Part c

The wave equation holds in the rod for $\delta(x,t)$. Assuming $\delta = \text{Re}[\hat{\delta}(x)e^{j\omega t}]$, we have $\hat{\delta}(x) = C_1 \sin \beta x + C_2 \cos \beta x$, where $\beta = \omega\sqrt{\rho/E}$. At $x = 0$, $f_\ell = -T(0,t)(\pi R^2)$ which yields

$$\hat{T}(0) = \frac{2\mu_o N^2 I_o \hat{I}}{Rg} = C_1 \hat{I},$$

which in turn implies $C_1 = \frac{C_I \hat{I}}{E\beta}$. At $x = \ell$, $f_r = T(\ell,t)(\pi R^2)$, which will yield C_2 .

The only other relation we need is the electrical circuit equation, which we can find from

$$\hat{v}_{\text{out}} = \frac{d\lambda_r}{dt} \quad C_2 = \frac{C_I}{E\beta \sin \beta \ell} (\hat{I} \cos \beta \ell + G \hat{v}_{\text{out}})$$

to be

$$\hat{v}_{\text{out}} = \frac{I_o L_1 j\omega \hat{\delta}(\ell)}{a(1+jGL_1\omega)} \quad (e)$$

where $L_1 = \mu_o N^2 (2\pi R a) / g$.

Finally we can write $G(\omega)$ as

$$G(\omega) = \frac{\hat{v}_{\text{out}}}{\hat{I}} = \frac{j\omega I_o L_1 C_I}{aE\beta \sin \beta \ell (1+jGL_1\omega) - j\omega G C_I I_o L_1 \cos \beta \ell} \quad (f)$$

Part d

If $G \ll \frac{1}{L_1\omega}$ so that the self inductance of the output transducer is negligible and the system is matched so that $a\sqrt{E\rho} = G C_I I_o L_1$ we have

$$\frac{\hat{v}_{\text{out}}}{\hat{I}} = \frac{j I_o L_1 C_I}{a\sqrt{E\rho} [\sin \beta \ell - j \cos \beta \ell]} \quad (g)$$

and

$$\left| \frac{\hat{v}_{\text{out}}}{\hat{I}} \right| = \frac{I_o L_1 C_I}{a\sqrt{E\rho}} \quad (h)$$

PROBLEM 9.14

Part a

With no perturbations and no volume force in the rod we know that the stress, $T(x_1)$, will be constant. At $x_1 = 0$,

$$0 = -AT(x_1 = 0) + f^e \quad (a)$$

where, using the Maxwell stress tensor, $f^e = \frac{\epsilon_o v_o^2 A_1}{2d^2}$. Hence,

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PROBLEM 9.14 (Continued)

$$T(x_1) = \frac{\epsilon_o v_o^2 A_1}{2Ad^2} . \quad (b)$$

Part b

The velocity of the wave will be $v_p = \sqrt{E/\rho}$ and the transit time will be $t_d = L/v_p$.

Using Table 9.1 we have

$$t_d = \frac{1}{5100} = 1.96 \times 10^{-4} \text{ sec.}$$

Part c

This part is similar to Prob. 9.11, where our condition for no reflection is

$$f_{a.c.}^e(t) = -A\sqrt{\rho E} \frac{\partial \delta}{\partial t}(0,t) \quad (c)$$

Using the Maxwell stress tensor

$$f^e = \frac{\epsilon_o A_1 v^2}{2d^2} \approx \frac{\epsilon_o A_1 v_o^2}{2d^2} + \frac{\epsilon_o A_1 v_o}{d^2} v'$$

where $v = v' + v_o$. Here, we ignore the effect on f^e of the change in d resulting from the motion of the plate.

Writing the circuit equation we have

$$iR + v = v_o = R \frac{dq}{dt} + v = R \left(C \frac{dv}{dt} + v \frac{dC}{dt} \right) \quad (d)$$

The capacitance C is

$$\frac{\epsilon_o A_1}{d - \delta(0,t)} \approx \frac{\epsilon_o A_1}{d} + \frac{\epsilon_o A_1}{d^2} \delta(0,t) \quad (e)$$

Our equation becomes

$$0 = v' + R \frac{\epsilon_o A_1}{d} \frac{dv'}{dt} + R v_o \frac{\epsilon_o A_1}{d^2} \frac{\partial \delta}{\partial t}(0,t) \quad (f)$$

and since

$$v' \gg \frac{\epsilon_o A_1 R}{d} \frac{dv'}{dt} ,$$

we have

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PROBLEM 9.14 (continued)

$$v' = - \frac{RV_o \epsilon_o A_1}{d^2} \frac{\partial \delta}{\partial t} (0, t) \quad (g)$$

Now we can use this result to write $f_{a.c.}^e = \epsilon_o A_1 V_o v' / d^2$, and the condition that this force take the form of (c) requires

$$A \sqrt{\rho E} d^4 = RV_o^2 \epsilon_o^2 A_1^2, \quad (h)$$

or equivalently

$$R = \frac{A \sqrt{\rho E} d^4}{\epsilon_o^2 A_1^2 V_o^2} \quad (i)$$

PROBLEM 9.15

Part a

We have from the problem statement

$$\psi(z + \Delta z) - \psi(z) = \beta \tau \Delta z.$$

If we take the limit $\Delta z \rightarrow 0$, then we obtain

$$\tau = \frac{1}{\beta} \frac{\partial \psi}{\partial z}.$$

Part b

We can write the equation of motion directly as

$$(J \Delta z) \frac{\partial^2 \psi}{\partial t^2} = \tau(z + \Delta z, t) - \tau(z, t).$$

Dividing by Δz we have

$$J \frac{\partial^2 \psi}{\partial t^2} = \frac{\tau(z + \Delta z, t) - \tau(z, t)}{\Delta z}$$

Taking the limit $\Delta z \rightarrow 0$ we obtain

$$J \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial \tau}{\partial z}$$

Part c

Substituting the result of part (a) into the result of part (b) we get

$$J \beta \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial z^2}.$$

SIMPLE ELASTIC CONTINUA

PROBLEM 9.16

Part a

We seek to write Newton's law for motions in the z direction of a slice of the material having x thickness dx . In our situation the mass is $\rho a dx dz$, where the acceleration is $\partial^2 \delta_z / \partial t^2$. The net force due to the stress is

$$F_z = [T_{zx}(x+dx) - T_{zx}(x)] a dz \quad (a)$$

and

$$\rho \frac{\partial^2 \delta_z}{\partial t^2} dx a dz = [T_{zx}(x+dx) - T_{zx}(x)] a dz \quad (b)$$

Finally, in the limit $dx \rightarrow 0$ we have

$$\rho \frac{\partial^2 \delta_z}{\partial t^2} = \frac{\partial T_{zx}}{\partial x} \quad (c)$$

Part b

The shear strain, e_{zx} , is defined so that it is proportional to $\delta_z(x+dx) - \delta_z(x)$ normalized to the distance between points dx . If $T_{zx} = 2G e_{zx}$ then in the limit $dx \rightarrow 0$ $T_{zx} = G \partial \delta_z / \partial x$ if we define

$$e_{zx} = \frac{1}{2} \frac{\partial \delta_z}{\partial x} \quad (d)$$

The $1/2$ is included to subtract out rigid body rotation, a point that is important in dealing with three-dimensional motions (see Chap. 11, Sec. 11.2.1a).

Part c

From part (a),

$$\rho \frac{\partial^2 \delta_z}{\partial t^2} = \frac{\partial T_{zx}}{\partial x} \quad (e)$$

Using the result of part (b) we have

$$\rho \frac{\partial^2 \delta_z}{\partial t^2} = G \frac{\partial^2 \delta_z}{\partial x^2} ; \quad (f)$$

the wave equation for shear waves with the propagational velocity

$$v_p = \sqrt{\frac{G}{\rho}} \quad .$$

SIMPLE ELASTIC CONTINUA

PROBLEM 9.17

Part a

Conservation of mass implies: net mass out per unit time = time rate of decrease of stored mass

$$[\rho v + \frac{\partial(\rho v)}{\partial x} \Delta x]A - (\rho v)A = - \frac{\partial}{\partial t} [\rho(\Delta x)A] \quad (a)$$

As $\Delta x \rightarrow 0$, we have

$$\frac{\partial}{\partial x} (\rho v) + \frac{\partial \rho}{\partial t} = 0 \quad (b)$$

If we write $\rho = \rho_0 + \rho'(x, t)$ and $v = v(x, t)$ then we obtain by substitution

$$\rho_0 \frac{\partial v}{\partial x} + \frac{\partial(\rho' v)}{\partial x} = - \frac{\partial \rho'}{\partial t} \quad (c)$$

Retaining only first-order terms we have

$$\rho_0 \frac{\partial v}{\partial x} = - \frac{\partial \rho'}{\partial t} \quad (d)$$

as desired.

Part b

Conservation of momentum implies:

time rate of increase of stored momentum = net momentum in
per unit time + externally applied force

$$\frac{\partial}{\partial t} (\rho v \Delta x A) = - [\rho v^2 + \frac{\partial(\rho v^2)}{\partial x} \Delta x]A + (\rho v^2)A + pA - (p + \frac{\partial p}{\partial x} \Delta x)A \quad (e)$$

as $\Delta x \rightarrow 0$, we have

$$\frac{\partial(\rho v)}{\partial t} = - \frac{\partial(\rho v^2)}{\partial x} - \frac{\partial p}{\partial x} \quad (f)$$

Expanding we have

$$\rho \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) + v \left(\frac{\partial(\rho v)}{\partial x} + \frac{\partial \rho}{\partial t} \right) = - \frac{\partial p}{\partial x} \quad (g)$$

this term is zero
by conservation of
mass

SIMPLE ELASTIC CONTINUA

PROBLEM 9.17 (continued)

Finally we have

$$\rho \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) = - \frac{\partial p}{\partial x} \quad (h)$$

Substituting the perturbation quantities and retaining only the first order terms we obtain

$$\rho_o \frac{\partial v}{\partial t} = - \frac{\partial p}{\partial x} \quad (i)$$

Part c

In terms of perturbation quantities we can write

$$p' = a^2 \rho' \quad (j)$$

where

$$a^2 \equiv \left(\frac{\partial p}{\partial \rho} \right)_{\rho_o}$$

Substitution for p' yields the two equations

$$\rho_o \frac{\partial v}{\partial x} = - \frac{\partial \rho'}{\partial t} \quad (k)$$

and

$$-a^2 \frac{\partial \rho'}{\partial x} = \rho_o \frac{\partial v}{\partial t} \quad (l)$$

Combining we obtain

$$\frac{\partial^2 v}{\partial t^2} = a^2 \frac{\partial^2 v}{\partial x^2} \quad (\text{scalar wave equation}) \quad (m)$$

Part d

If we substitute $v = \text{Re}[\hat{v}(x)e^{j\omega t}]$ in the above equation we obtain

$$\frac{d^2 \hat{v}(x)}{dx^2} + \frac{\omega^2}{a^2} \hat{v}(x) = 0 \quad (n)$$

which has solutions of the form

$$\hat{v}(x) = C_1 \sin\left(\frac{\omega}{a} x\right) + C_2 \cos\left(\frac{\omega}{a} x\right). \quad (o)$$

SIMPLE ELASTIC CONTINUA

PROBLEM 9.17 (continued)

A rigid wall at $x = 0$ implies that $\hat{v}(x=0) = 0$. The drive at $x = \ell$ and the equations of part (c) imply that

$$\frac{d\hat{v}}{dx} = - \frac{j\omega \hat{p}_0}{a^2 \rho_0} \quad (p)$$

at $x = \ell$.

The solution for $\hat{v}$ is

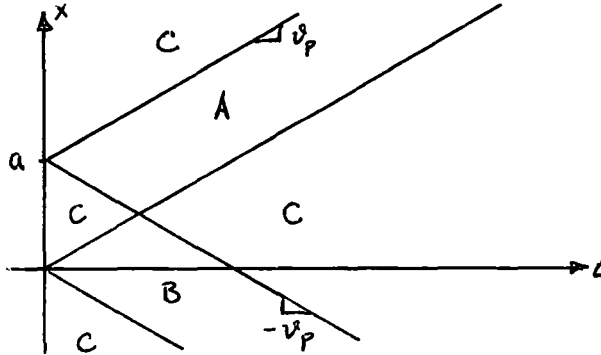
$$\hat{v}(x) = - \frac{j\hat{p}_0}{a\rho_0} \frac{\sin(\frac{\omega}{a} x)}{\cos(\frac{\omega}{a} \ell)} \quad (q)$$

and we can now obtain $v(x,t)$: for $\hat{p}_0$ real

$$v(x,t) = \frac{\hat{p}_0}{a\rho_0} \frac{\sin(\frac{\omega}{a} x)}{\cos(\frac{\omega}{a} \ell)} \sin \omega t. \quad (r)$$

PROBLEM 9.18

We can calculate the values of $d\delta+/d\alpha$ and $d\delta-/d\beta$ for three regions of the x - t plane as defined below.



Referring to equations from text, 9.1.23 and 9.1.24, 9.1.27 and 9.1.28:

Region A:

$$\frac{d\delta+}{d\alpha} = - \frac{1}{2} \frac{v_m}{v_p}, \quad \frac{d\delta-}{d\beta} = 0 \quad (a)$$

and

$$T = - \frac{E}{2} \frac{v_m}{v_p} \quad (b)$$

SIMPLE ELASTIC CONTINUA

PROBLEM 9.18 (continued)

Region B:

$$\frac{d\delta+}{d\alpha} = 0, \quad \frac{d\delta-}{d\beta} = \frac{1}{2} \frac{v_m}{v_p} \quad (c)$$

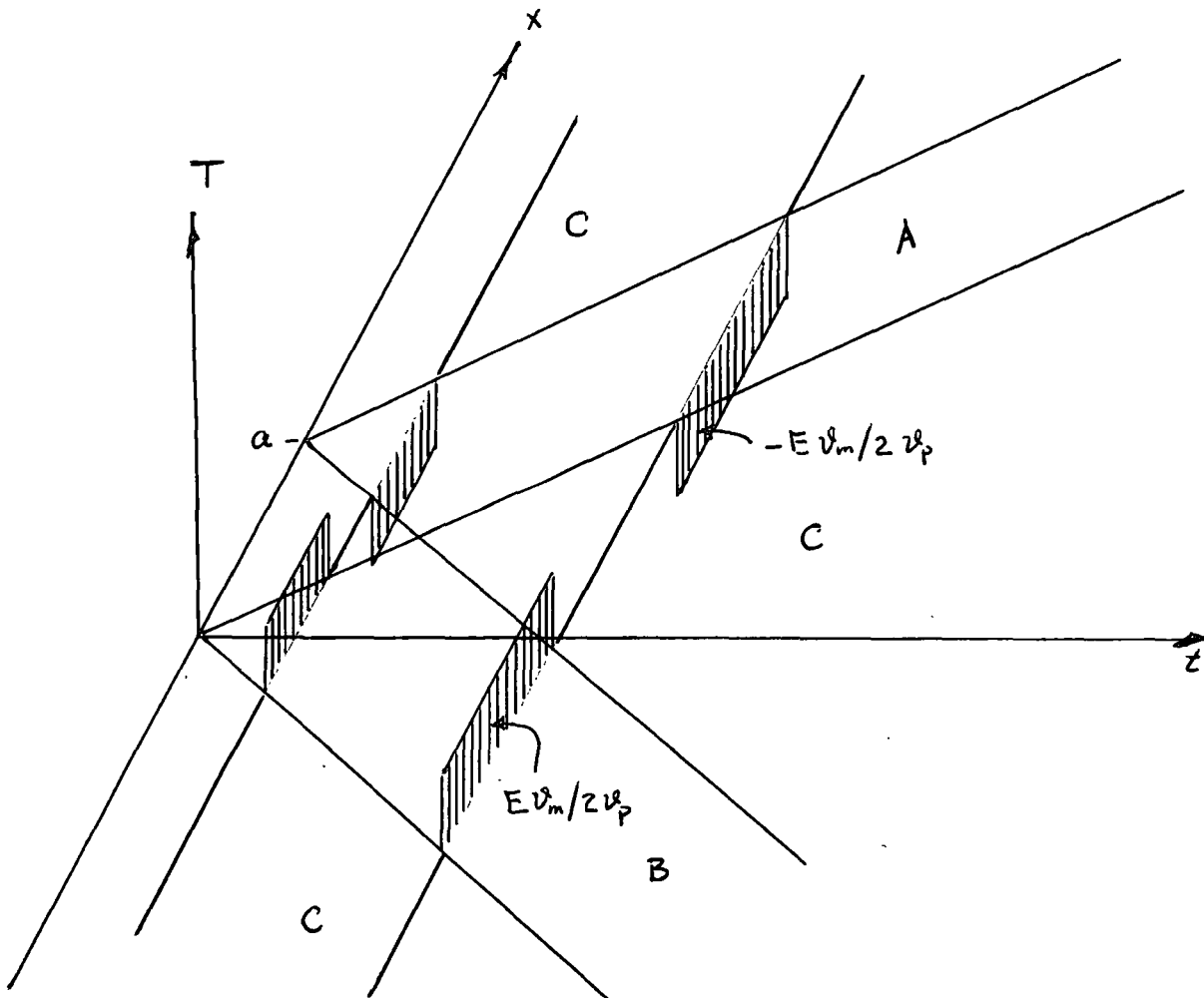
and

$$T = \frac{E}{2} \frac{v_m}{v_p} \quad (d)$$

Region C:

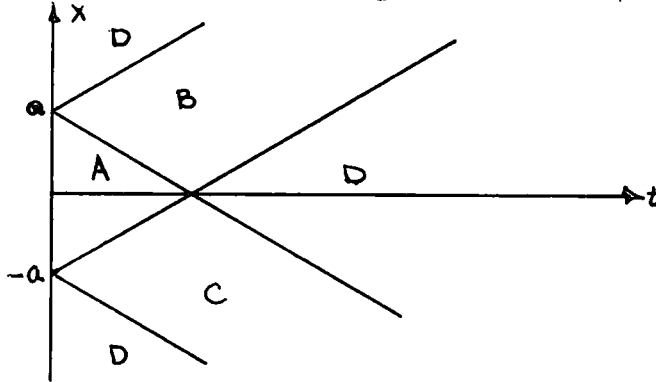
$$\frac{d\delta+}{d\alpha} = \frac{d\delta-}{d\beta} = 0 \text{ and } T = 0. \quad (e)$$

Plotting $T(x,t)$ in the x - t plane we have



PROBLEM 9.19

We can find $d\delta+/d\alpha$ and $d\delta-/d\beta$ for four regions of the x - t plane:



Referring to equations from the text 9.1.23, 9.1.24 and 9.1.27, 9.1.28 we have,

Region A:

$$\frac{d\delta+}{d\alpha} = \frac{1}{2} \frac{T(\alpha)}{E}, \quad \frac{d\delta-}{d\beta} = \frac{1}{2} \frac{T(\beta)}{E} \quad (a)$$

Region B:

$$\frac{d\delta+}{d\alpha} = \frac{1}{2} \frac{T(\alpha)}{E}, \quad \frac{d\delta-}{d\beta} = 0 \quad (b)$$

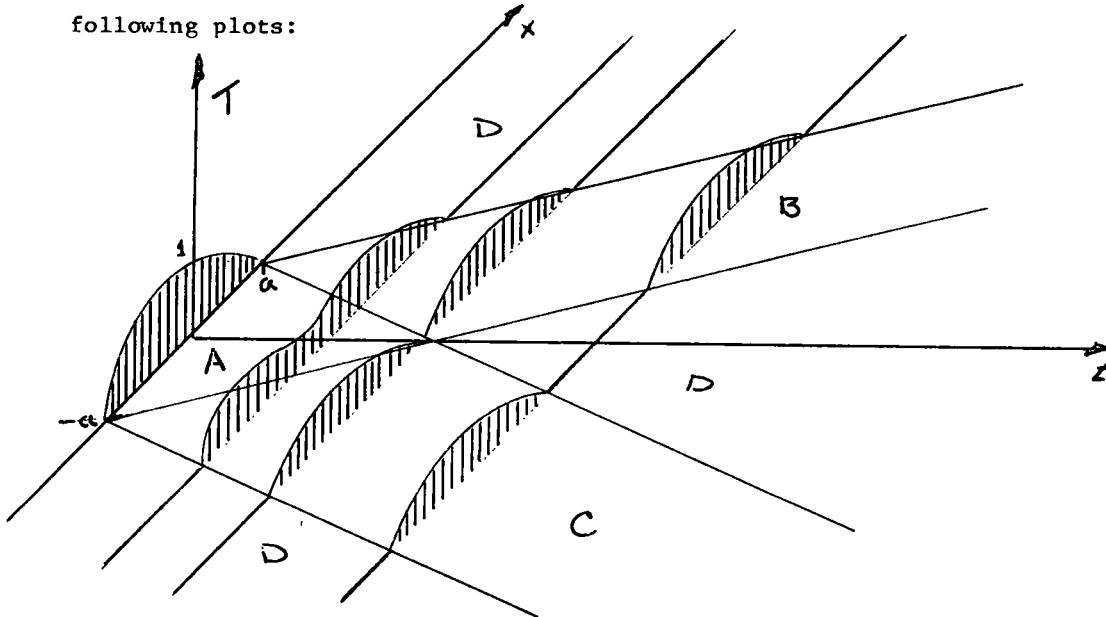
Region C:

$$\frac{d\delta+}{d\alpha} = 0, \quad \frac{d\delta-}{d\beta} = \frac{1}{2} \frac{T(\alpha)}{E} \quad (c)$$

Region D:

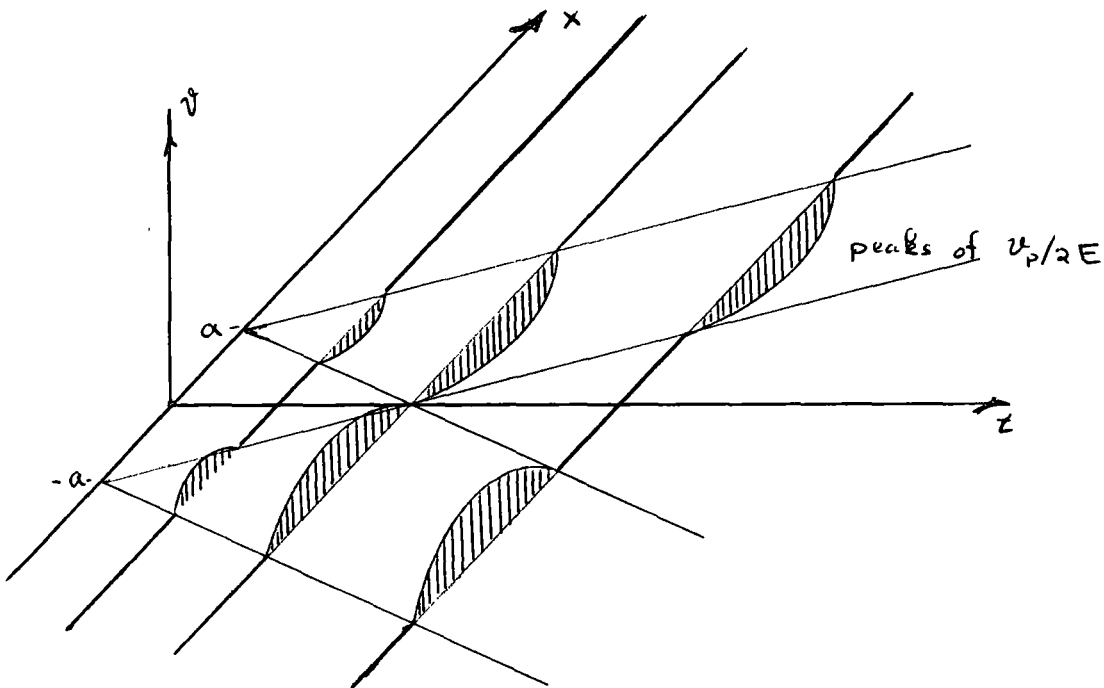
$$\frac{d\delta+}{d\alpha} = \frac{d\delta-}{d\beta} = 0 \quad (d)$$

We can use these values in equation 9.1.23 and 9.1.24 from text and make the following plots:



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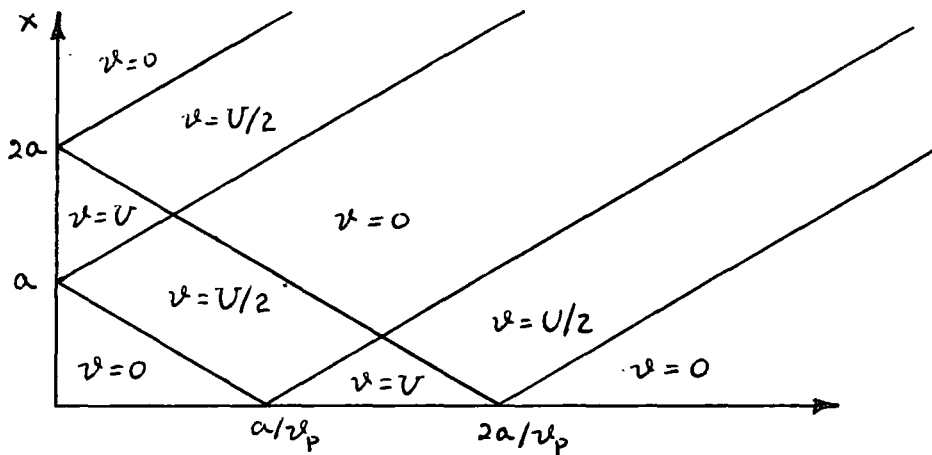
PROBLEM 9.19 (continued)



PROBLEM 9.20

Part a

The free end at $x = 0$ implies that $T(0, t) = 0$ and using equations 9.1.23 through 9.1.26 we can easily find that velocity pulses "bounce off" $x = 0$ boundary with the same sign and magnitude. For the x - t plane we can indicate the values for $v(x, t)$:

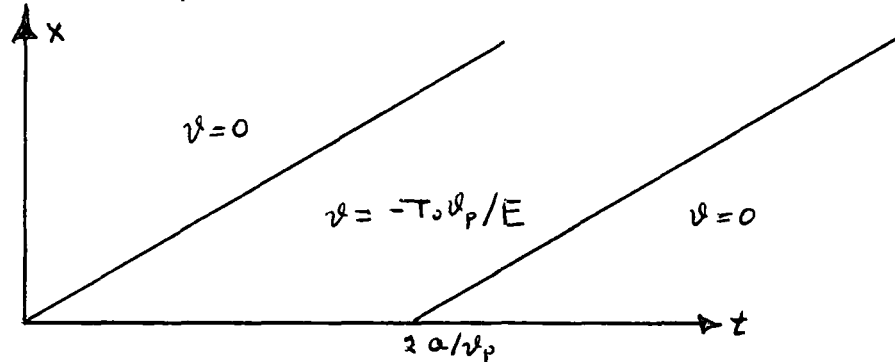


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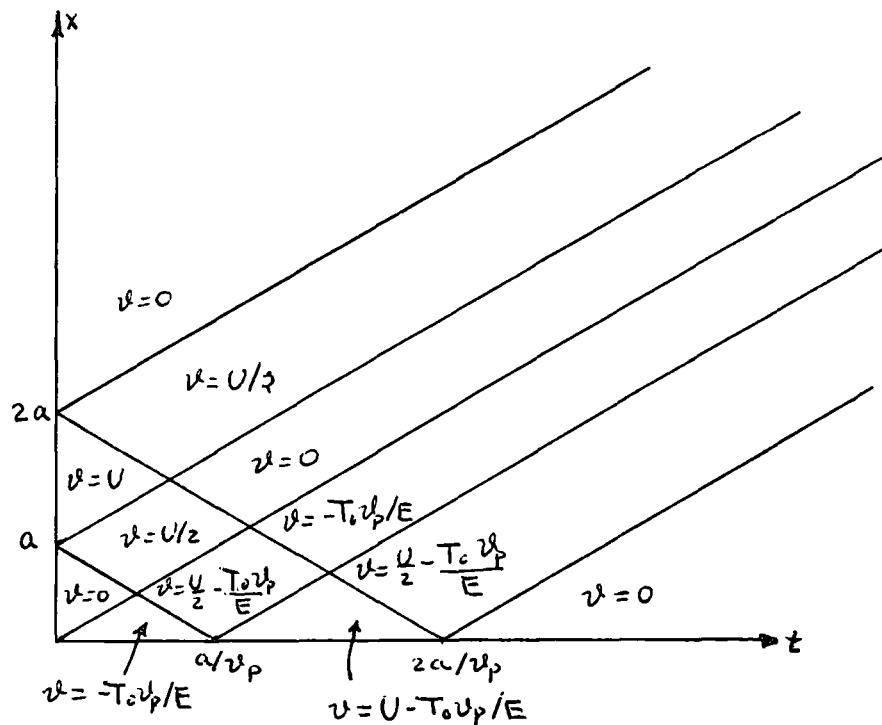
PROBLEM 9.20 (continued)

Part b

We can make use of part (a) if we use superposition. Consider the superposition of boundary and initial conditions; a free end, $T(0,t) = 0$ with the initial conditions in part (a) and the $T(0,t)$ as shown in Fig. 9.P20b with initial conditions on T and v zero. Since the system is linear, we can add the velocities that result from the two situations and thus have the net velocity. For the response to the second set of conditions we have



Add this velocity set to the set in part (a) and we obtain:

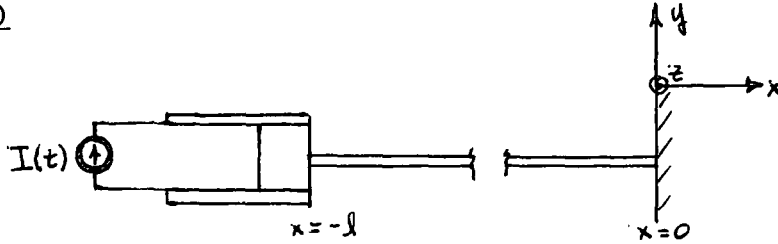


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PROBLEM 9.21

Part a

With the current returned on the inside surface the field in the air gap is $H_z = -\frac{I(t)}{D}$



and the force per unit area acting on the inside surface is

$$T_x = -\frac{1}{2} \mu_0 \frac{I^2}{D^2} \quad (a)$$

The force is $f_x = -T_x$ and $\frac{1}{2} \frac{\mu_0 a}{D} I^2(t)$ and the boundary condition at $x = -l$ is

$$M \frac{\partial^2 \delta}{\partial t^2} (-l, t) = \frac{1}{2} \frac{\mu_0 a}{D} I^2(t) + AT(-l, t) \quad (b)$$

Part b

The current will flow on the surface when the time τ is much shorter than the characteristic diffusion time τ_d over the length b :

$$\tau_d \gg \tau \quad \text{or} \quad \mu_0 \sigma b^2 \gg \tau \quad (c)$$

Part c

In order to ignore the mass M , the inertial term must be small compared to $AT(-l, t)$. For $t < \tau$, $\delta_- = 0$ on the rod, and from Eqs. 9.1.23 and 9.1.24,

$$T(-l, t) = -\frac{E}{v_p} \frac{\partial \delta}{\partial t} (-l, t) \quad (d)$$

Thus

$$\left| M \frac{\partial^2 \delta}{\partial t^2} (-l, t) \right| \ll \left| \frac{AE}{v_p} \frac{\partial \delta}{\partial t} \right| \quad (e)$$

or

$$M \ll AE \tau / v_p \quad (f)$$

Our boundary condition in part (a) now becomes:

$$0 = \frac{1}{2} \frac{\mu_0 a}{D} I^2(t) + AT(-l, t) \quad (g)$$

Since there is a fixed end at $x = 0$ we know that a stress wave traveling in the $+x$ direction will reflect at $x = 0$ with the same wave returning in the $-x$ direction. To satisfy the condition $v(0, t) = 0$, Eq. 9.1.23 shows that $d\delta_+/d\alpha = d\delta_-/d\beta$ at $x = 0$. Thus, from Eq. 9.1.24, the stress is twice that

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PROBLEM 9.21 (continued)

initiated at the left end

$$T_r = \frac{\mu_o a}{DA} I_o^2 \quad (h)$$

PROBLEM 9.22

Part a

We have $W = W'$ and $U = C + U'$ where W' and U' are perturbations from equilibrium. Rewriting the equations we have

$$\frac{\partial W'}{\partial t} + (W') \frac{\partial W'}{\partial x} + \frac{\partial U'}{\partial x} + \frac{K}{(C+U')^3} \frac{\partial U'}{\partial x} = 0 \quad (a)$$

and

$$\frac{\partial U'}{\partial t} + (C+U') \frac{\partial W'}{\partial x} + (W') \frac{\partial U'}{\partial x} = 0 \quad (b)$$

Neglecting all second-order perturbation terms we have

$$\frac{\partial W'}{\partial t} + (1 + \frac{K}{C^3}) \frac{\partial U'}{\partial x} = 0 \quad (c)$$

$$\frac{\partial U'}{\partial t} + (C) \frac{\partial W'}{\partial x} = 0 \quad (d)$$

Part b

Multiplying the above two equations by $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$, respectively, we have

$$\frac{\partial^2 W'}{\partial t^2} + (1 + \frac{K}{C^3}) \frac{\partial^2 U'}{\partial x \partial t} = 0 \quad (e)$$

and

$$\frac{\partial^2 U'}{\partial x \partial t} + (C) \frac{\partial^2 W'}{\partial x^2} = 0 \quad (f)$$

Eliminating U' we obtain

$$\frac{\partial^2 W'}{\partial t^2} = C(1 + \frac{K}{C^3}) \frac{\partial^2 W'}{\partial x^2} \quad (g)$$

which is the familiar wave equation with wave velocity $v_p = \sqrt{C(1 + \frac{K}{C^3})}$.

We can write the solution as $W' = \text{Re}[\hat{W}(x)e^{j\omega t}]$ where

$$\hat{W}(x) = C_1 \sin \beta x + C_2 \cos \beta x \quad (h)$$

with $\beta = \omega/v_p$.

At $x = 0$, $W = W' = 0$ and hence $C_2 = 0$. At $x = -L$, $W = W' = W_0 \cos \omega t$, or equivalently $\hat{W}(-L) = W_0$, hence $C_1 = -W_0/\sin \beta L$. Upon substitution we find that

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PROBLEM 9.22 (continued)

the solution is

$$W = W' = - \frac{W_o \sin \beta x}{\sin \beta L} \cos \omega t. \quad (1)$$

PROBLEM 9.23

Part a

This part is similar to Prob. 9.24 with two simplifications:

$$V_o = 0 \text{ and}$$

the mass is $M/\text{unit width } (M_w)$ instead of $2M$. The two separate relations yielding the natural frequencies are

$$\sin \left(\omega L \sqrt{\frac{\sigma_m}{S}} \right) = 0 \quad (a)$$

and

$$\frac{2\sigma_m}{M\omega \sqrt{\frac{\sigma_m}{S}}} = \tan \left(\omega L \sqrt{\frac{\sigma_m}{S}} \right) \quad (b)$$

(a) yields $\omega L \sqrt{\sigma_m/S} = n\pi$ where $n = 1, 2, \dots$ and corresponds to solutions which are "odd", or $\xi(x) = -\xi(-x)$. (b) can be solved graphically and corresponds to solutions which are "even", or $\xi(x) = \xi(-x)$.

Part b

The effect of raising M is to reduce the eigenfrequencies of the "even" modes. The "odd" solutions predicted by (a) are independent of the mass M . This is physically reasonable since there is a node at the mass, and since the mass doesn't move there is no inertial force. For the "even" solutions predicted by (b), we notice that if $M = 0$ we have essentially the natural frequencies of a membrane of length $2L$. As $M \rightarrow \infty$, the system responds like two different membranes of length L . The infinite mass acts like a rigid boundary.

PROBLEM 9.24

Part a

We can use the Maxwell Stress Tensor to find the forces of electric origin. If f_u^e corresponds to the force due to the upper electrode and f_l^e corresponds to the force due to the lower electrode, then we have:

$$\vec{f}_u^e(t) = \frac{\epsilon_o V_o^2 A}{2[d - \xi(0, t)]^2} \vec{i}_y \quad (a)$$

PROBLEM 9.24 (Continued)

$$\hat{f}_l^e(t) = - \frac{\epsilon_o v_o^2 A}{2[d + \xi(0, t)]^2} \hat{1}_y \quad (b)$$

Our equation for the membranes is $\sigma_m \partial^2 \xi / \partial t^2 = S \frac{\partial^2 \xi}{\partial x^2}$ and if we assume $\xi = \text{Re}[\hat{\xi}(x)e^{j\omega t}]$, then we can write

$$\hat{\xi}(x) = C_1 \sin \beta x + C_2 \cos \beta x \quad (c)$$

for $x > 0$ and

$$\hat{\xi}(x) = C_3 \sin \beta x + C_4 \cos \beta x \quad (d)$$

for $x < 0$ where $\beta = \omega \sqrt{\sigma_m / S}$.

Our boundary condition will yield the four constants. We have

$$\begin{aligned} \hat{\xi}(x = -L) &= 0 \\ \hat{\xi}(x = L) &= 0 \\ \hat{\xi}(x = 0^+) &= \hat{\xi}(x = 0^-) \end{aligned} \quad (e)$$

and

$$2M \frac{\partial^2 \xi}{\partial t^2}(0, t) = S \omega \left[\frac{\partial \hat{\xi}}{\partial x}(0^+) - \frac{\partial \hat{\xi}}{\partial x}(0^-) \right] + f_u^e(t) + f_l^e(t) \quad (f)$$

which reduces to

$$-2M\omega^2 \hat{\xi}(0) = S \omega \left[\frac{d\hat{\xi}}{dx}(0^+) - \frac{d\hat{\xi}}{dx}(0^-) \right] + \frac{2\epsilon_o v_o^2 A}{d^3} \hat{\xi}(0) \quad (g)$$

after we linearize $[f_u^e(t) + f_l^e(t)]$. Substituting, we immediately find $C_2 = C_4$. Writing the remaining equations we have

$$0 = -C_3 \sin \beta L + C_2 \cos \beta L \quad (h)$$

$$0 = C_1 \sin \beta L + C_2 \cos \beta L \quad (i)$$

$$0 = S \omega \beta C_1 + \left[\frac{2\epsilon_o v_o^2 A}{d^3} + 2M\omega^2 \right] C_2 - S \omega \beta C_3 \quad (j)$$

If we eliminate the constants by setting the determinant of the coefficients C_1 , C_2 , and C_3 equal to zero, we obtain two separate relations:

$$\sin \beta L = 0 \text{ and } \frac{S \omega \beta}{\frac{\epsilon_o v_o^2 A}{d^3} + M \omega^2} = \tan \beta L. \quad (k)$$

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PROBLEM 9.24 (continued)

Substituting for β we have

$$\sin(\omega L \sqrt{\frac{\sigma_m}{S}}) = 0 \text{ and } \frac{S \omega \omega \sqrt{\frac{\sigma_m}{S}}}{\frac{\epsilon_o V_o^2 A}{d^3} + M \omega^2} = \tan \omega L \sqrt{\frac{\sigma_m}{S}} \quad (l)$$

The first relation implies that $\omega L \sqrt{\frac{\sigma_m}{S}} = n\pi$ where $n = 1, 2, \dots$. The second relation can be solved graphically.

Part b

As V_o is increased from $V_o = 0$, the lowest natural frequency decreases.

When V_o approaches the value

$$\sqrt{\frac{S \omega d^3}{\epsilon_o A L}},$$

the lowest natural frequency approaches zero; as V_o is further increased, there will be an imaginary solution for ω and the system will be unstable.

PROBLEM 9.25

Part a

The force of the lower plunger is $f_\ell^m = \frac{\partial W'}{\partial z} = \frac{L_o}{2a^2} i_\ell^2$. By symmetry the upper plunger has a force $f_u^m = -(L_o/a^2) i_u^2$. From the circuit $i_\ell^2 = (I_o + i_1)^2 \approx I_o^2 + 2I_o i_1$ and $i_u^2 = (I_o - i_1)^2 \approx I_o^2 - 2I_o i_1$. Hence the total magnetic force is

$$f^m = \frac{2L_o I_o i_1}{a} = \frac{2L_o I_o G}{a} \frac{\partial \xi}{\partial x}(0, t) \quad (a)$$

Writing the force balance on the tip of the wire at $x = -\ell$ we have

$$f \frac{\partial \xi}{\partial x}(-\ell, t) + \frac{2L_o I_o G}{a} \frac{\partial \xi}{\partial x}(0, t) = 0 \quad (b)$$

Part b

Away from the ends

$$M \frac{\partial^2 \xi}{\partial t^2} = f \frac{\partial^2 \xi}{\partial x^2} \quad (c)$$

and if $\xi = \text{Re}[\hat{\xi}(x)e^{j\omega t}]$ then

$$\hat{\xi}(x) = C_1 \sin \beta x + C_2 \cos \beta x \quad (d)$$

where $\beta = \omega \sqrt{m/f}$. $\xi(0, t) = 0$ implies that $C_2 = 0$. From part (a) we have

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PROBLEM 9.25 (continued)

$$f \frac{d\hat{\xi}}{dx}(-l) + \frac{2L_o I_o G}{a} \frac{d\hat{\xi}}{dx}(0) = 0. \quad (e)$$

Upon substitution we obtain

$$f\beta C_1 \cos \beta l + \frac{2L_o I_o G}{a} G\beta C_1 = 0 \quad (f)$$

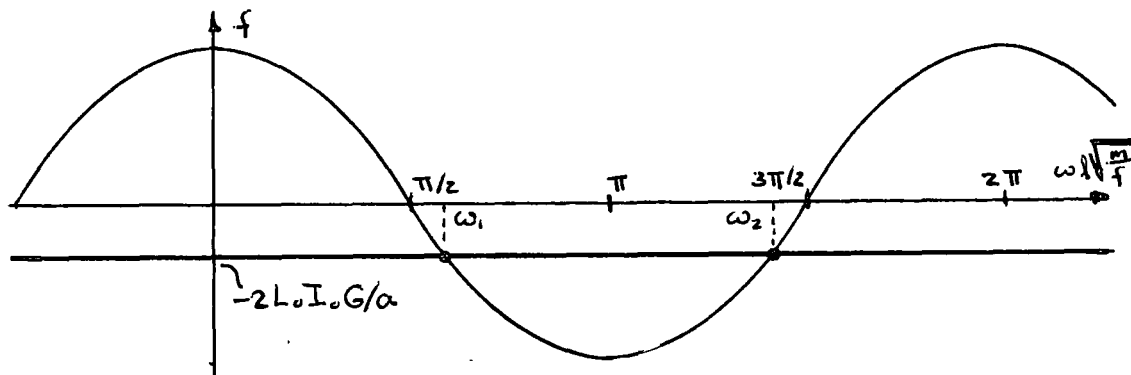
Since C_1 must be finite for a finite response, we have

$$f\beta \cos \beta l + \frac{2L_o I_o G}{a} G\beta = 0, \quad (g)$$

or

$$f \cos \omega l \sqrt{\frac{m}{f}} + \frac{2L_o I_o G}{a} = 0 \quad (h)$$

(We have ruled out one solution, because it is trivial.) A graphical solution of (h) is shown in the figure.



Part c

If $G = 0$, then

$$\omega l \sqrt{\frac{m}{f}} = \left(\frac{2n+1}{2}\right)\pi \quad (i)$$

with $n = 0, 1, 2, \dots$

Part d

From the figure, ω_1 increases toward $\omega_1 l \sqrt{m/f} = \pi$ and ω_2 decreases toward the same value. They come together at $G = af/2L_o I_o$ and seemingly disappear if $G > \frac{af}{2L_o I_o}$.

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PROBLEM 9.25 (continued)

Part e

If $|G| > \frac{af}{2L_o I_o}$, then (h) has imaginary solutions for ω , hence the system will be unstable.

PROBLEM 9.26

Part a

First of all we notice that $y(t) = \xi(-L, t)$. For the membrane

$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \frac{\partial^2 \xi}{\partial x^2}$ and if $\xi = \text{Re}[\hat{\xi}(x)e^{j\omega t}]$ then $\hat{\xi}(x) = C_1 \sin \beta x + C_2 \cos \beta x$ where

$\beta = \omega \sqrt{\sigma_m / S}$. At $x = 0$, $\hat{\xi}(x=0) = 0$ and therefore $C_2 = 0$. At $x = -L$, we can write the boundary condition

$$M \frac{\partial^2 \xi}{\partial t^2} (-L, t) = SD \frac{\partial \xi}{\partial x} (-L, t) + f_y^m(t) \quad (a)$$

We can find $f_y^e(t)$ using Ampere's Law and the Maxwell stress tensor

$$f_y^e(t) = \frac{\mu_o AN^2}{2} \left[\frac{(I_o + I(t))^2}{(d-D-\xi(-L, t))^2} - \frac{(I_o - I(t))^2}{(d-D+\xi(-L, t))^2} \right] \quad (b)$$

Since $I_o \gg I(t)$ and $(d-D) \gg \xi(-L, t)$ then we can linearize:

$$f_y^e(t) \approx 2N^2 A \mu_o \left[\frac{I_o}{(d-D)^2} I(t) + \frac{I_o^2}{(d-D)^3} \xi(-L, t) \right] \quad (c)$$

Substitution of (c) into (a) and definition of $C_I \equiv \frac{2N^2 A \mu_o I_o}{(d-D)^2}$ and

$$C_y \equiv \frac{2N^2 A \mu_o I_o^2}{(d-D)^3} \text{ gives}$$

$$M \frac{\partial^2 \xi}{\partial t^2} (-L, t) = SD \frac{\partial \xi}{\partial x} (-L, t) + C_I I(t) + C_y \xi(-L, t) \quad (d)$$

or in complex form,

$$-M\omega^2 \hat{\xi}(-L) = SD \frac{\partial \hat{\xi}}{\partial x} (-L) + C_I \hat{I} + C_y \hat{\xi}(-L) \quad (e)$$

After solving for C_1 , we can write

$$\hat{\xi}(x) = \frac{C_I \sin \beta x \hat{I}}{(M\omega^2 + C_y) \sin \beta L - SD \beta \cos \beta L} \quad (f)$$

or finally

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PROBLEM 9.26 (continued)

$$\hat{y} = \frac{c_I \sin \beta L \hat{I}}{SD\beta \cos \beta L - (M\omega^2 + C_y) \sin \beta L} \quad (g)$$

where $y(t) = \text{Re}[\hat{y} e^{j\omega t}]$.

Part b

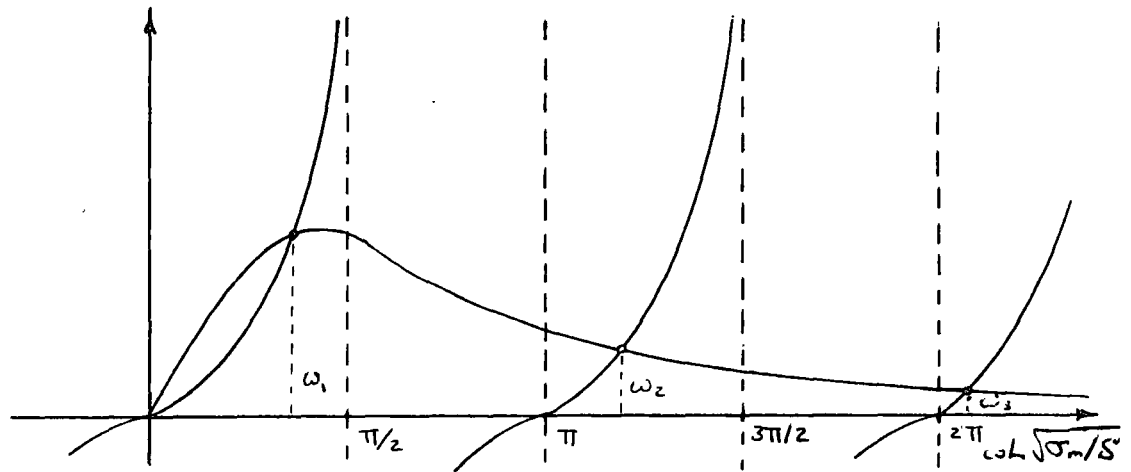
To find the resonance frequencies we look at the poles of $\hat{y}/\hat{I}$. This amounts to finding the zeros of the denominator of $\hat{y}/\hat{I}$. We have

$$SD\omega \sqrt{\frac{\sigma_m}{S}} \cos \omega L \sqrt{\frac{\sigma_m}{S}} = [M\omega^2 + C_y] \sin \omega L \sqrt{\frac{\sigma_m}{S}} \quad (h)$$

or

$$\frac{SD\omega \sqrt{\frac{\sigma_m}{S}}}{M\omega^2 + C_y} = \tan(\omega L \sqrt{\frac{\sigma_m}{S}}) \quad (i)$$

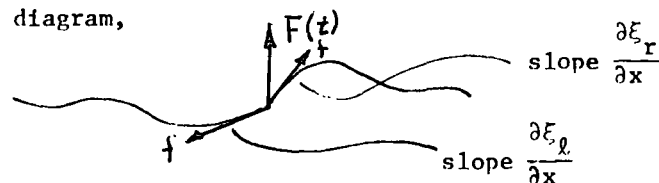
We can represent the solution graphically:



PROBLEM 9.27

Part a

The boundary condition may be obtained by applying force equilibrium using the following diagram,



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PROBLEM 9.27 (continued)

thus

$$F(t) = f \left[\frac{\partial \xi_l}{\partial x} (0^-) - \frac{\partial \xi_r}{\partial x} (0^+) \right] \quad (a)$$

Part b

For the odd solution, $\xi_l(x,t) = -\xi_r(-x,t)$ and it follows that $\frac{\partial \xi_l}{\partial x} - \frac{\partial \xi_r}{\partial x} = 0$. This implies that the odd solution is not excited by the force $F(t)$.

Part c

For the even solution, $\xi_l(x,t) = \xi_r(-x,t)$, we have $\frac{\partial \xi_l}{\partial x} = -\frac{\partial \xi_r}{\partial x}$ and the boundary condition (a) from part (a) becomes

$$F(t) = -2f \frac{\partial \xi_r}{\partial x} \text{ at } x = 0 \quad (b)$$

For $0 < x < l$ we have

$$m \frac{\partial^2 \xi}{\partial t^2} = f \frac{\partial^2 \xi}{\partial x^2} \quad (c)$$

with $\xi(x,t) = 0$ at $x = l$.

For $t < 0$, this reduces to

$$\frac{\partial^2 \xi}{\partial x^2} = 0 \quad (d)$$

and we obtain

$$\xi(x) = \frac{F_0 l}{2f} \left(1 - \frac{x}{l} \right) \text{ for } 0 < x < l \quad (e)$$

Part d

We now have a combined transient and driven response, as discussed in Sec. 9.2.1. By contrast with the developments of that section, we now have a boundary condition at $x = 0$ on the slope $\partial \xi_r / \partial x$ (see (b) of part (c)). Our program is: ($\xi \equiv \xi_r$ in the following)

- i. Find the driven sinusoidal steady-state response. This satisfies the boundary conditions:

$$F_0 \cos \omega t = -2f \frac{\partial \xi}{\partial x} (0,t) \quad (f)$$

$$\xi(l,t) = 0 \quad (g)$$

- ii. Find normal modes, which satisfy homogeneous boundary conditions;

$$\frac{\partial \xi}{\partial x} (0,t) = 0 \quad (h)$$

$$\xi(l,t) = 0 \quad (i)$$

The sum of these modes takes the form of a Fourier series.

PROBLEM 9.27 (continued)

iii. Superimpose (i) and (ii) and use the initial conditions found in parts (a)-(c) to evaluate the arbitrary coefficients.

The driven response is of the form

$$\xi = \text{Re}(C_1 \sin \beta x + C_2 \cos \beta x) e^{j\omega t}; \quad \beta = \omega \sqrt{\frac{m}{f}} \quad (j)$$

a linear combination which satisfies (g)

$$\xi = \text{Re} C_3 \sin \beta(x-l) e^{j\omega t} \quad (k)$$

while (f) evaluates C_3 and the driven response is

$$\xi = - \text{Re} \frac{F_0 \sin \beta(x-l) e^{j\omega t}}{2f\beta \cos \beta l} \quad (l)$$

The normal modes are in this lossless case the resonances of the driven response and occur as $\cos \beta l = 0$. Thus

$$\omega_n l \sqrt{\frac{m}{f}} = \left(\frac{2n+1}{2}\right)\pi, \quad n = 0, 1, 2, 3 \dots \quad (m)$$

and the total solution for $0 < x < l$ is

$$\xi = - \frac{F_0 \sin \beta(x-l)}{2f\beta \cos \beta l} \cos \omega t + \sum_{n=0}^{\infty} [A_n^+ e^{j\omega_n t} + A_n^- e^{-j\omega_n t}] \sin\left[\left(\frac{2n+1}{2}\right)\frac{\pi}{l}(x-l)\right] \quad (n)$$

The coefficients A_n^+ and A_n^- are evaluated by requiring that

$$\xi(x, 0) = \frac{F_0 l}{2f} \left(1 - \frac{x}{l}\right) = - \frac{F_0 \sin \beta(x-l)}{2f\beta \cos \beta l} + \sum_{n=0}^{\infty} (A_n^+ + A_n^-) \sin\left[\left(\frac{2n+1}{2}\right)\frac{\pi}{l}(x-l)\right] \quad (o)$$

and

$$\frac{\partial \xi}{\partial t}(x, 0) = 0 = \sum_{n=0}^{\infty} [j\omega_n A_n^+ - j\omega_n A_n^-] \sin\left[\left(\frac{2n+1}{2}\right)\frac{\pi}{l}(x-l)\right] \quad (p)$$

This last condition is satisfied if $A_n^+ = A_n^-$. The A_n^+ 's follow from (o) by using the orthogonality of the functions $\sin\left[\left(\frac{2n+1}{2}\right)\frac{\pi}{l}(x-l)\right]$ and $\sin\left[\left(\frac{2m+1}{2}\right)\frac{\pi}{l}(x-l)\right]$, $m \neq n$, over the interval l .

PROBLEM 10.1Part a

At $x = 0$, the net force on an incremental length of the string has to be zero.

$$-2B \frac{\partial \xi}{\partial t} - f \frac{\partial \xi}{\partial x} = 0$$

This is the required boundary condition at $x = 0$.

Part b

The power absorbed by the dashpots is the product of force $2B \partial \xi / \partial t$ and the velocity $\partial \xi / \partial t$. Thus

$$P = 2B \left(\frac{\partial \xi}{\partial t} \right)^2$$

If we solve Eq. 10.1.6 for

$$\xi(x, t) = \text{Re} [\hat{\xi} e^{j(\omega t - kx)}]$$

and assume that $\omega < \omega_c$ we get

$$\xi(x, t) = \text{Re} \{ [A_1 \sinh |k|x + A_2 \cosh |k|x] e^{j\omega t} \}$$

where

$$k = \left[\frac{\omega^2 - \omega_c^2}{v_s^2} \right]^{1/2}$$

We can calculate A_1 and A_2 using the boundary condition of part (a) and the boundary condition at $x = \ell$

$$\xi(-\ell, t) = \text{Re} \xi_0 e^{j\omega t}$$

We then get

$$A_1 = - \frac{j \xi_0 2B\omega}{[f|k| \cosh |k|\ell + j\omega 2B \sinh |k|\ell]}$$

$$A_2 = \frac{\xi_0 f|k|}{[f|k| \cosh |k|\ell + j\omega 2B \sinh |k|\ell]}$$

If we plug these values into the expression for power, and then time average, we have

$$\langle P \rangle = \frac{B(f|k| \xi_0 \omega)^2}{[(f|k| \cosh |k|\ell)^2 + (2B\omega \sinh |k|\ell)^2]}$$

where it is convenient to use the identity

$$\langle \text{Re } A e^{j\omega t} \text{ Re } B e^{j\omega t} \rangle = \frac{1}{2} AB^*$$

PROBLEM 10.2Part a

We use Eq. (10.1.6)

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2} - \omega_c^2 \xi, \quad \omega_c^2 = \frac{Ib}{m}$$

Assume solutions $\xi = \text{Re}[(Ae^{-jkx} + Be^{jkx})e^{j\omega t}]$. The dispersion equation is:

$$k^2 = \frac{\omega_d^2 - \omega_c^2}{v_s^2}$$

Now use the boundary conditions, which require

$$A e^{jkl} + B e^{-jkl} = \xi_d$$

$$-jk[A - B] = 0$$

(i) $\omega_d < \omega_c$ (below cutoff)

$$\xi = \frac{\xi_d \cosh \alpha x}{\cosh \alpha l} \cos \omega_d t$$

$$\alpha = \sqrt{\omega_c^2 - \omega_d^2} / v_s$$

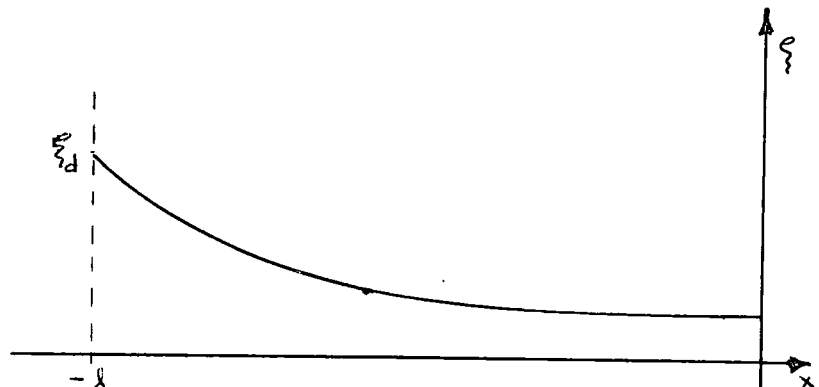
(ii) $\omega_d > \omega_c$ (above cutoff)

$$\xi(x, t) = \frac{\xi_d \cos \beta x}{\cos \beta l} \cos \omega_d t$$

$$\beta = \sqrt{\omega_d^2 - \omega_c^2} / v_s$$

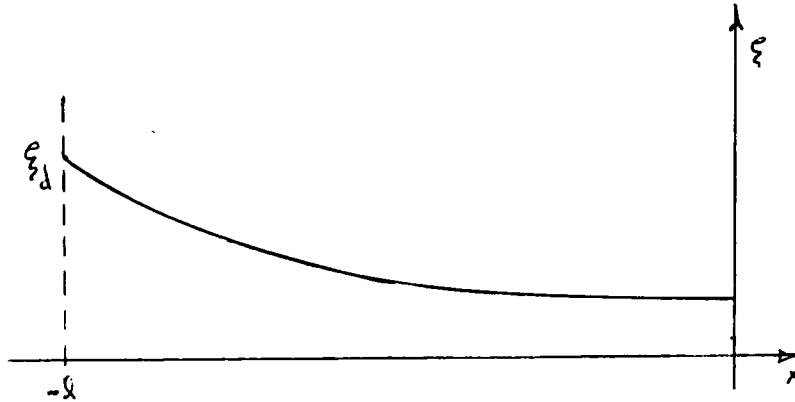
Part b

$$\omega_d = 0$$

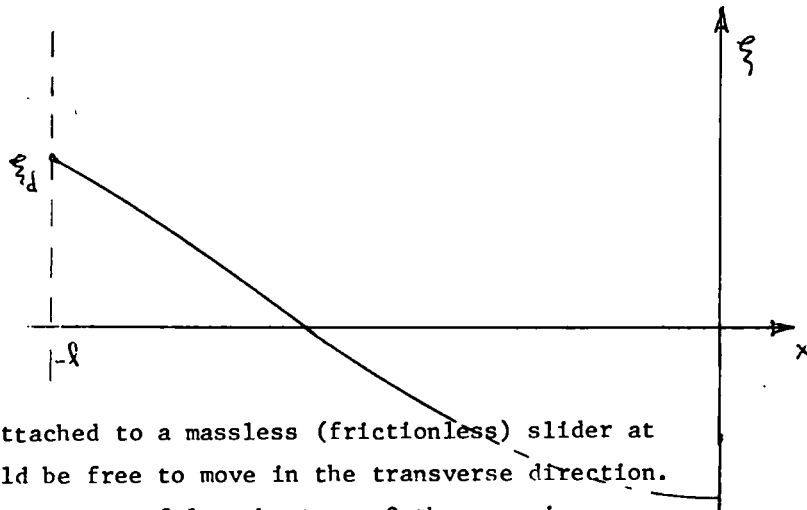


PROBLEM 10.2 (continued)

$$\omega_d < \omega_c$$



$$\omega_d > \omega_c$$



Part c

The string might be attached to a massless (frictionless) slider at $x = 0$, so that the end would be free to move in the transverse direction. Force equilibrium for the increment of length at $x = 0$ then requires $\partial \xi / \partial x = 0$ at $x = 0$.

PROBLEM 10.3

Part a

From Eq. 10.1.10 we have

$$k = \left[\frac{\omega^2 - \omega_c^2}{v_s^2} \right]^{1/2}$$

with our solution of the form

$$\xi(x, t) = \text{Re} \left(A_1 e^{j(\omega t - kx)} + A_2 e^{j(\omega t + kx)} \right)$$

We have the boundary conditions

PROBLEM 10.3(continued)

$$\frac{\partial \xi}{\partial x}(0, t) = 0$$

and

$$\frac{\partial \xi}{\partial x}(-\ell, t) = 0.$$

From the first boundary condition, we obtain

$$A_1 = A_2 ; \xi(x, t) = \text{Re } A_3 \cos kx e^{j\omega t}$$

From the second boundary condition, we obtain

$$\sin k\ell = 0$$

This implies that

$$k = \frac{n\pi}{\ell} ; n = 0, 1, 2, 3, \dots$$

Note that by contrast with the case where the ends are fixed, $n = 0$ is a valid (nontrivial) and crucial solution. It corresponds to an eigenmode which is simply a rigid body translation.

From Eq. 10.1.7

$$\omega^2 = k^2 v_s^2 + \omega_c^2$$

Therefore, the eigenfrequencies are

$$\omega = \pm \left[\omega_c^2 + \left(\frac{n\pi}{\ell} v_s \right)^2 \right]^{1/2}$$

For the $n = 0$ mode, $\omega = \pm \omega_c$.

Part b

With I as in Fig. 10.1.9, we have the same equations as in part (a) if we replace ω_c^2 by $-\omega_c^2$. Therefore, for this case, the eigenfrequencies are

$$\omega = \left[\left(\frac{n\pi}{\ell} v_s \right)^2 - \omega_c^2 \right]^{1/2}$$

Part c

With I as in Fig. 10.1.9, the $\bar{I}x\bar{B}$ force is destabilizing, as a small perturbation from $x = 0$ tends to increase this force. If ω in part (b) became imaginary, the equilibrium $\xi = 0$ would become unstable as the solutions are unbounded in time. This will happen as

$$\left(\frac{n\pi}{\ell} v_s \right)^2 - \omega_c^2 < 0$$

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PROBLEM 10.3 (continued)

or in terms of the current

$$I > \frac{n}{b} \left(\frac{\pi}{\ell} v_s \right)^2$$

Note that any finite current makes the $n = 0$ mode unstable, since for this mode there is no elastic restoring force.

PROBLEM 10.4

Multiply the system equation by $\frac{\partial \xi}{\partial t}$,

$$m \frac{\partial \xi}{\partial t} \cdot \frac{\partial^2 \xi}{\partial t^2} = f \frac{\partial \xi}{\partial t} \cdot \frac{\partial^2 \xi}{\partial x^2} - I b \frac{\partial \xi}{\partial t} \cdot \xi + \frac{\partial \xi}{\partial t} \cdot F(x, t)$$

Proper substitution of partial differential identities yields:

$$\frac{\partial}{\partial t} \left[\frac{m}{2} \left(\frac{\partial \xi}{\partial t} \right)^2 + \frac{f}{2} \left(\frac{\partial \xi}{\partial x} \right)^2 + \frac{I b}{2} \xi^2 \right] - \frac{\partial}{\partial x} \left[f \frac{\partial \xi}{\partial x} \cdot \frac{\partial \xi}{\partial t} \right] = \frac{\partial \xi}{\partial t} \cdot F(x, t)$$

PROBLEM 10.5

We have that

$$\xi(x, t) = \text{Re} \left[\hat{\xi}_+ e^{j(\omega t - kx)} + \hat{\xi}_- e^{j(\omega t + kx)} \right]$$

Part a

For k real, we might write this in the form

$$\xi(x, t) = \frac{1}{2} \left[\hat{\xi}_+ e^{j(\omega t - kx)} + \hat{\xi}_+^* e^{-j(\omega t - kx)} + \hat{\xi}_- e^{j(\omega t + kx)} + \hat{\xi}_-^* e^{-j(\omega t + kx)} \right]$$

From Prob. 10.4 we have that the power carried by the string is

$$P = -f \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial t}.$$

If we do the indicated differentiations, then substitute into this expression, and then time average, we will obtain

$$\langle P \rangle = \frac{f \omega k}{2} [\hat{\xi}_+ \hat{\xi}_+^* - \hat{\xi}_- \hat{\xi}_-^*]$$

Part b

For k purely imaginary

$$k = j\beta$$

with β real, we can write $\xi(x, t)$ in the form

PROBLEM 10.5 (continued)

$$\xi(x,t) = \frac{1}{2} \left[\hat{\xi}_+ e^{j\omega t + \beta x} + \hat{\xi}_+^* e^{-j\omega t + \beta x} + \hat{\xi}_- e^{j\omega t - \beta x} + \hat{\xi}_-^* e^{-j\omega t - \beta x} \right]$$

If we again substitute into our expression for power and average over time we obtain

$$\langle P \rangle = - \frac{j\omega\beta f}{2} [\hat{\xi}_+^* \hat{\xi}_- - \hat{\xi}_+ \hat{\xi}_-^*]$$

From (b), we see that it is possible to have a net power flow from two evanescent waves, but not from a single evanescent wave. Suppose that a single evanescent wave did carry power away from the driving source. This would correspond physically to a string driven at the left and infinite to the right. With $\omega_d < \omega_c$, the response as $x \rightarrow \infty$ becomes vanishingly small; clearly there can be no power flow at $x \rightarrow \infty$. Yet, there is no mechanism for power absorption by the string and so there can be no power flow into the string from the drive. With a dissipative load, a second evanescent wave is established, decaying to the left, and the conditions for power flow are met.

PROBLEM 10.6

From the dispersion relation, we calculate:

$$v_g \equiv \frac{\partial \omega}{\partial k} = v_s \left[1 - \frac{\omega_c^2}{\omega^2} \right]^{1/2}$$

Now, assuming a single forward traveling wave:

$$\xi = \xi_+ \cos[\omega t - k(\omega)x]$$

Then:

$$\langle W \rangle = \left(\frac{m\omega^2}{4} + \frac{fk^2}{4} + \frac{Ib}{4} \right) \xi_+^2$$

$$\langle P \rangle = \left(\frac{fk\omega}{2} \right) \xi_+^2$$

Thus, substitution gives

$$\begin{aligned} \frac{\langle P \rangle}{\langle W \rangle} &= \frac{fk\omega/2}{\left(\frac{m\omega^2}{4} + \frac{fk^2}{4} + \frac{Ib}{4} \right)} \\ &= v_s \left[1 - \frac{\omega_c^2}{\omega^2} \right]^{1/2} = v_g \end{aligned}$$

which is the desired relation. This result is of some general significance, but has been shown here for a particular case.

PROBLEM 10.7

Part a

The equations of motion for the membranes are

$$\sigma_m \frac{\partial^2 \xi_1}{\partial t^2} = S \frac{\partial^2 \xi_1}{\partial x^2} + T_1$$

$$\sigma_m \frac{\partial^2 \xi_2}{\partial t^2} = S \frac{\partial^2 \xi_2}{\partial x^2} + T_2$$

where T_1 and T_2 are the transverse magnetic forces/area. If the membranes extend a distance w into the paper, and if we define regions 1, 2, and 3 as the top, middle, and bottom regions respectively in Fig. 10P.7, the flux in each region is

$$\lambda_1 = \mu_o H_1 w(d - \xi_1)$$

$$\lambda_2 = \mu_o H_2 w(d + \xi_1 - \xi_2)$$

$$\lambda_3 = \mu_o H_3 w(d + \xi_2)$$

where H_1 , H_2 , and H_3 are the magnetic field intensities within each region. Since the flux is conserved, when $\xi_1 = \xi_2 = 0$ we have

$$\lambda_1 = \lambda_3 = -\mu_o H_o w d \quad \lambda_2 = +\mu_o H_o w d$$

Therefore,

$$\bar{H}_1 = -\frac{H_o d}{d - \xi_1} \bar{i}_x$$

and

$$\bar{H}_2 = +\frac{H_o d}{d + \xi_1 - \xi_2} \bar{i}_x$$

and

$$\bar{H}_3 = -\frac{H_o d}{d + \xi_2} \bar{i}_x.$$

We will use the Maxwell stress tensor to calculate T_1 and T_2 , using a pill-box volume enclosing a section of surface on each membrane.

We then obtain

$$T_1 \cong -\frac{\mu_o}{2} [H_1^2 - H_2^2]$$

and

$$T_2 \cong -\frac{\mu_o}{2} [H_2^2 - H_3^2]$$

Substituting the expression for the $\bar{H}$ fields, and realizing that $\xi_1 \ll d$ and $\xi_2 \ll d$, we finally obtain for the forces

PROBLEM 10.7 (continued)

$$T_1 = - \frac{\mu_o H_o^2 (2\xi_1 - \xi_2)}{d}$$

and

$$T_2 = - \frac{\mu_o H_o^2 (2\xi_2 - \xi_1)}{d}$$

Our equations of motion are then

$$\sigma_m \frac{\partial^2 \xi_1}{\partial t^2} = S \frac{\partial^2 \xi_1}{\partial x^2} - \frac{\mu_o H_o^2}{d} (2\xi_1 - \xi_2)$$

and

$$\sigma_m \frac{\partial^2 \xi_2}{\partial t^2} = S \frac{\partial^2 \xi_2}{\partial x^2} - \frac{\mu_o H_o^2}{d} (2\xi_2 - \xi_1)$$

Part b

We assume that

$$\xi_1 = \text{Re } \hat{\xi}_1 e^{j(\omega t - kx)}$$

and

$$\xi_2 = \text{Re } \hat{\xi}_2 e^{j(\omega t - kx)}$$

We can substitute these functions into the equations of motion from part (a), and solve for the relation between ω and k such that the 2 equations of motion are consistent. This dispersion relation is

$$-\sigma_m \omega^2 + Sk^2 + \frac{2\mu_o H_o^2}{d} = \pm \frac{\mu_o H_o^2}{d}$$

We see that the dispersion equation factors into two dispersion relations. If we substitute this relation back into the equations of motion from part (a), we see that we obtain even and odd solutions.

The dispersion relation

$$\omega^2 = \frac{Sk^2}{\sigma_m} + \frac{\mu_o H_o^2}{\sigma_m d}$$

yields

$$\xi_1 = \xi_2.$$

The dispersion relation

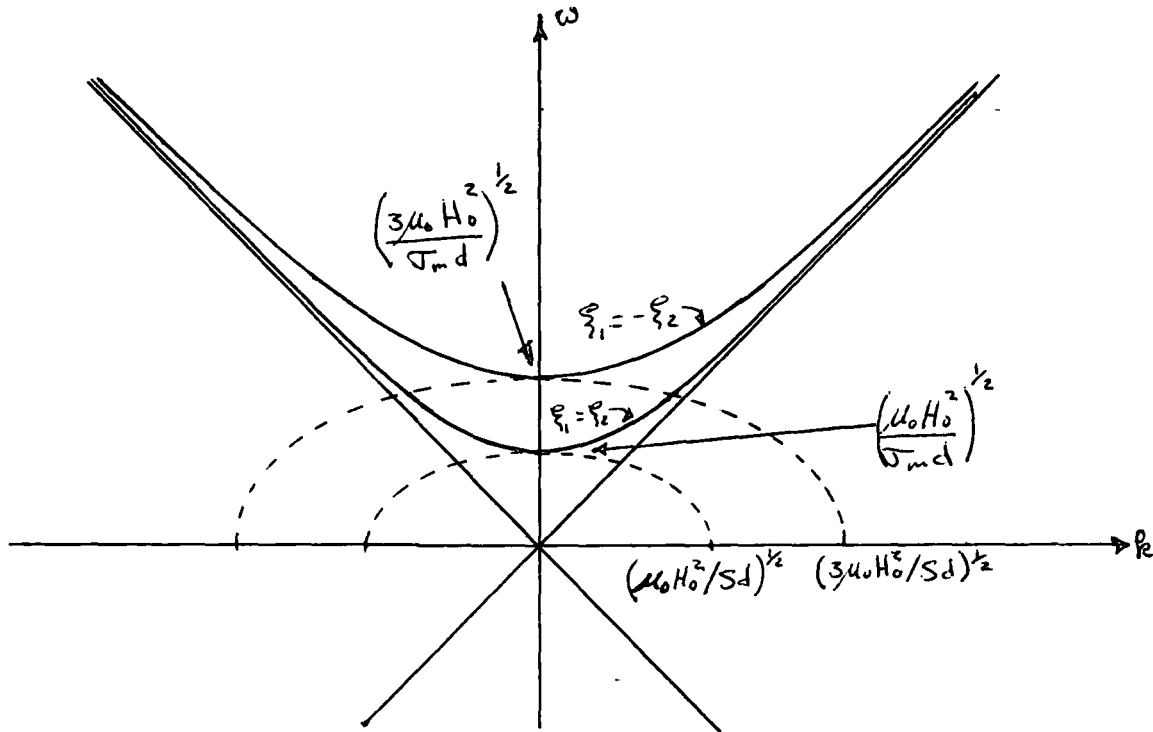
$$\omega^2 = \frac{Sk^2}{\sigma_m} + \frac{3\mu_o H_o^2}{\sigma_m d}$$

yields $\xi_1 = -\xi_2$.

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PROBLEM 10.7 (continued)

Plotting ω versus k , we obtain



_____ k real
 - - - - k imaginary

From the plot we see that the lowest frequency for which we have propagation (k real) for the even mode is

$$\omega_{ce} = \left(\frac{\mu_o H_o^2}{\sigma_m d} \right)^{1/2}$$

For the odd mode, the cut off frequency is

$$\omega_{co} = \left(\frac{3\mu_o H_o^2}{\sigma_m d} \right)^{1/2}$$

Part d

We are given the boundary conditions that at $x = 0$

$$\xi_1 = 0 \quad \xi_2 = 0$$

and at $x = -l$

$$\xi_1 = -\xi_2 = \text{Re } \xi_o e^{j\omega t}.$$

PROBLEM 10.7 (continued)

From the boundary condition, we see that our solution is purely odd. Therefore

$$k = \left[\frac{\omega^2 \sigma_m}{S} - \frac{3\mu_o H_o^2}{Sd} \right]^{1/2}$$

We assume a solution of the form

$$\xi_1(x,t) = -\xi_2(x,t) = \text{Re} \{ A_1 e^{j(\omega t - kx)} + A_2 e^{j(\omega t + kx)} \}$$

Evaluating A_1 and A_2 through the boundary conditions, we obtain

$$A_1 = -A_2 = \frac{\xi_o}{e^{jkl} - e^{-jkl}}$$

Therefore

$$\xi_1(x,t) = -\xi_2(x,t) = \text{Re} \frac{\xi_o [e^{-jkx} - e^{+jkx}] e^{j\omega t}}{[e^{jkl} - e^{-jkl}]}$$

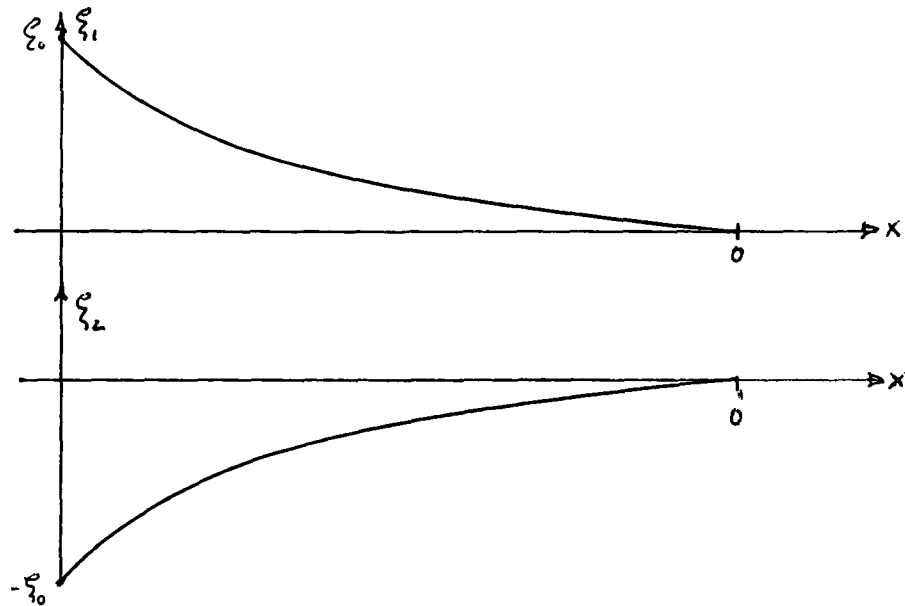
For $\omega = 0$, k is pure imaginary. We define $k = j\beta$, with β real with value

$$\beta = \left(\frac{3\mu_o H_o^2}{Sd} \right)^{1/2}$$

Therefore

$$\xi_1(x,t) = -\frac{\xi_o \sinh \beta x}{\sinh \beta l} \cos \omega t$$

A sketch appears below.



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PROBLEM 10.8

Part a

The given equations follow by writing out Maxwell's equations and assuming $\vec{E}$ and $\vec{H}$ have the given directions and dependences.

Part b

The force equation for an incremental volume element is

$$\vec{F} = \vec{i}_x m n_e \frac{\partial v_x}{\partial t} \quad (a)$$

where $\vec{F}$ is the force density due to electrical forces on the electrons

$$\vec{F} = - \vec{i}_x e n_e E_x \quad (b)$$

Thus,

$$-e n_e E_x = m n_e \frac{\partial v_x}{\partial t} \quad (c)$$

Part c

As the electrons move, they give rise to the current density

$$J_x \cong - e n_e v_x \text{ (linearized)} \quad (d)$$

Part d

Assume $e^{j(\omega t - kx)}$ dependence and (c) and (d) require

$$\hat{J}_x = - j \frac{e^2 n_e}{\omega m} \hat{E}_x \quad (e)$$

$$= - j \omega \epsilon_0 \left[\frac{\omega_p^2}{\omega^2} \right] \hat{E}_x \quad (f)$$

where $\omega_p = \sqrt{e^2 n_e / m \epsilon_0}$ is called the plasma frequency. (See page 600)

Combining this with Maxwell's equations:

$$k^2 = \frac{\omega^2}{c^2} \left[1 - \frac{\omega_p^2}{\omega^2} \right]; \quad c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \quad (g)$$

Part e

We have a dispersion which yields evanescent waves below the plasma (cutoff) frequency. Below this frequency, the electrons respond to the electric field associated with the wave in such a way as to reflect rather than transmit an incident electromagnetic wave.

Part f

Waves impinging upon a boundary between free space and plasma will be totally reflected if the wave frequency $\omega < \omega_p$. The plasma frequency for the ionosphere

PROBLEM 10.8 (continued)

is typically

$$f_p \sim 10 \text{ MHz}$$

This result explains why AM broadcasts ($500 \text{ kHz} < f < 1500 \text{ kHz}$) can commonly be monitored all over the world, whereas FM ($88 \text{ MHz} < f < 108 \text{ MHz}$) has a range limited to "line-of-sight".

PROBLEM 10.9

In the regions

$$x < -\ell \text{ and } x > 0$$

the equation of motion for the string is

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2}$$

In the region $-\ell \leq x \leq 0$, this equation is modified due to the magnetic force to

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2} - \omega_c^2 \xi$$

If we assume

$$\xi(x, t) = \text{Re} \{ \hat{\xi} e^{j(\omega t - kx)} \}$$

and substitute back into the equations of motion we obtain the dispersion relations

$$k = \pm \left[\frac{\omega^2 - \omega_c^2}{v_s^2} \right]^{1/2} \quad -\ell \leq x \leq 0$$

$$k = \pm \frac{\omega}{v_s} \quad x < -\ell, x > 0$$

The boundary conditions are

$$\text{at } x = -\ell \quad \hat{\xi} = \hat{\xi}_0$$

$$\text{at } x = 0 \quad \xi \text{ and } \frac{\partial \xi}{\partial x} \text{ must be continuous.}$$

We assume that

$$\xi(x, t) = \text{Re} \left\{ \left[A e^{-\beta x} + B e^{+\beta x} \right] e^{j\omega t} \right\} \quad \text{for } -\ell < x < 0$$

PROBLEM 10.9 (continued)

where

$$\beta = \left[\frac{\omega_c^2 - \omega^2}{v_s^2} \right]^{1/2} \quad \text{for } \omega < \omega_c$$

and

$$\xi(x, t) = \text{Re} \{ \hat{\xi}_b e^{-jk_b x} e^{j\omega t} \} \quad \text{for } x > 0$$

where

$$k_b = \frac{\omega}{v_s}$$

Using the above boundary conditions, we obtain

$$A = \frac{\xi_o (\beta + jk_b)}{2(\beta \cosh \beta l + jk_b \sinh \beta l)}$$

$$B = \frac{\xi_o (\beta - jk_b)}{2(\beta \cosh \beta l + jk_b \sinh \beta l)}$$

$$\text{But } \hat{\xi}_b = A + B$$

Therefore

$$\frac{\hat{\xi}_b}{\xi_o} = \frac{1}{\left[\cosh \beta l + \frac{jk_b}{\beta} \sinh \beta l \right]}$$

Part b

$$\text{As } l \rightarrow 0$$

$$\frac{\hat{\xi}_b}{\xi_o} \rightarrow 1$$

$$\text{As } l \rightarrow \infty$$

$$\frac{\hat{\xi}_b}{\xi_o} \rightarrow 0$$

PROBLEM 10.10Part a

The equation of motion for the string is

$$m \frac{\partial^2 \xi}{\partial t^2} = f \frac{\partial^2 \xi}{\partial x^2} + S - mg \quad (a)$$

PROBLEM 10.10 (continued)

where, for small deflections ξ in the "1/r" field from Q,

$$S \approx \frac{qQ}{2\pi\epsilon_0 d} \left[1 + \frac{\xi}{d} \right]$$

In static equilibrium, $\xi = 0$ and from (a)

$$qQ = 2\pi d \epsilon_0 \cdot mg \quad (b)$$

Part b

The perturbation equation of motion remains;

$$m \frac{\partial^2 \xi}{\partial t^2} = f \frac{\partial^2 \xi}{\partial x^2} + \left(\frac{qQ}{2\pi d^2 \epsilon_0} \right) \xi \quad (c)$$

Assume $e^{j(\omega t - kx)}$ dependence and (c) requires ($v_s = \sqrt{f/m}$)

$$\omega^2 = v_s^2 k^2 - \frac{qQ}{2\pi d^2 \epsilon_0 m} \quad (d)$$

or from (b),

$$\omega^2 = v_s^2 k^2 - \frac{g}{d}$$

The boundary conditions require $k = n\pi/\ell$, and for stability the most critical mode is $n = 1$; thus

$$v_s^2 \left(\frac{\pi}{\ell} \right)^2 > \frac{g}{d} \quad (e)$$

$$m < \frac{fd}{g} \left(\frac{\pi}{\ell} \right)^2 \quad (f)$$

Part c

Increase f , d , or decrease ℓ .

PROBLEM 10.11

$$m \frac{\partial^2 \xi}{\partial t^2} = f \frac{\partial^2 \xi}{\partial x^2} + S - mg$$

where $S = (\vec{I} \times \vec{B})_{r=\xi}$ and $|B| = \frac{\mu_0 I_0}{2r}$, r the radial distance from the fixed wire.

Therefore $S = \frac{\mu_0 I_0^2}{2\pi r}$.

For static equilibrium

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PROBLEM 10.11 (continued)

$$S = mg = \frac{\mu_o I_o I}{2\pi \xi_o}$$

Therefore

$$I = \frac{2\pi mg \xi_o}{\mu_o I_o}$$

Note that $I_o I > 0$ for the required equilibrium.

Part b

The force per unit length is linearized to obtain the perturbation equation.

$$S = \frac{\mu_o I_o I}{2\pi(\xi_o + \xi)} \approx \frac{\mu_o I_o I}{2\pi} \left[\frac{1}{\xi_o} - \frac{\xi}{\xi_o^2} \right]$$

Therefore

$$m \frac{\partial^2 \xi}{\partial t^2} = f \frac{\partial^2 \xi}{\partial x^2} - \frac{\mu_o I_o I}{2\pi \xi_o^2} \xi$$

Part c

Assuming $e^{j(\omega t - kx)}$ solutions, the dispersion relation is

$$-m\omega^2 = -f k^2 - \frac{\mu_o I_o I}{2\pi \xi_o^2}$$

Solving for ω , we obtain

$$\omega = \left[k^2 \frac{f}{m} + \frac{\mu_o I_o I}{2\pi m \xi_o^2} \right]^{1/2}$$

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As long as $I_o I > 0$ the equilibrium will always be stable as ω will always be real. Note that this condition is required for the desired static equilibrium to exist.

PROBLEM 10.12

The equation of motion is given as

$$m \frac{\partial^2 \xi}{\partial t^2} = f \frac{\partial^2 \xi}{\partial x^2} + P \xi \quad (a)$$

Part a

Boundary conditions follow from force equilibrium for the ends of the wire

PROBLEM 10.12 (continued)

$$(i) -2K\xi(0,t) + f \frac{\partial \xi(0,t)}{\partial x} = 0 \quad (b)$$

$$(ii) 2K\xi(l,t) + f \frac{\partial \xi(l,t)}{\partial x} = 0 \quad (c)$$

Part b

The dispersion relation follows from (a) as

$$\omega^2 = v_s^2 k^2 - \frac{P}{m}; \quad v_s = \sqrt{f/m} \quad (d)$$

where solutions have been assumed of the following form:

$$\xi = \text{Re}[(A \sin kx + B \cos kx)e^{j\omega t}] \quad (e)$$

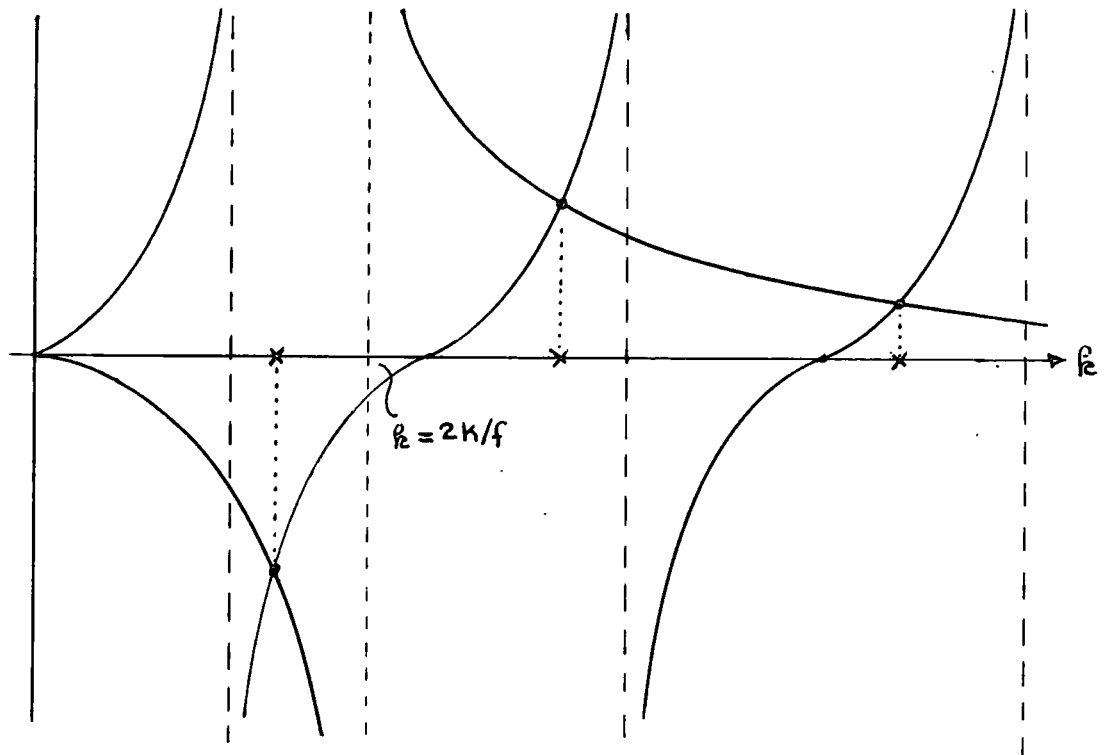
Application of the boundary conditions yields a transcendental equation for k :

$$\tan kl = \frac{4Kf k}{f^2 k^2 - 4K^2} \quad (f)$$

where, from (d),

$$k = \frac{1}{v_s} \sqrt{\omega^2 + P/m} \quad (g)$$

Thus, (f) is the desired equation for the natural frequencies.



PROBLEM 10.12 (continued)

Part c

As $K \rightarrow 0$, the lowest root of graphic solution goes to $k \rightarrow 0$, for which stability criterion is:

$$0 > \frac{P}{m}$$

PROBLEM 10.13

Part a

This problem is very similar to that of problem 10.7. Using the same reasoning as in that problem, we obtain

$$\sigma_m \frac{\partial^2 \xi_1}{\partial t^2} = S \frac{\partial^2 \xi_1}{\partial x^2} + \frac{\epsilon_o V_o^2}{d^3} (2\xi_1 - \xi_2)$$

$$\sigma_m \frac{\partial^2 \xi_2}{\partial t^2} = S \frac{\partial^2 \xi_2}{\partial x^2} + \frac{\epsilon_o V_o^2}{d^3} (2\xi_2 - \xi_1)$$

Part b

Assuming sinusoidal solutions in time and space, the dispersion relation is

$$-\sigma_m \omega^2 + Sk^2 - \frac{2\epsilon_o V_o^2}{d^3} = \pm \frac{\epsilon_o V_o^2}{d^3}$$

We have a dispersion relation that factors into two parts. The odd mode,

$\xi_1 = -\xi_2$ has the dispersion relation

$$\omega = \left[\frac{Sk^2}{\sigma_m} - \frac{3\epsilon_o V_o^2}{\sigma_m d^3} \right]^{1/2}$$

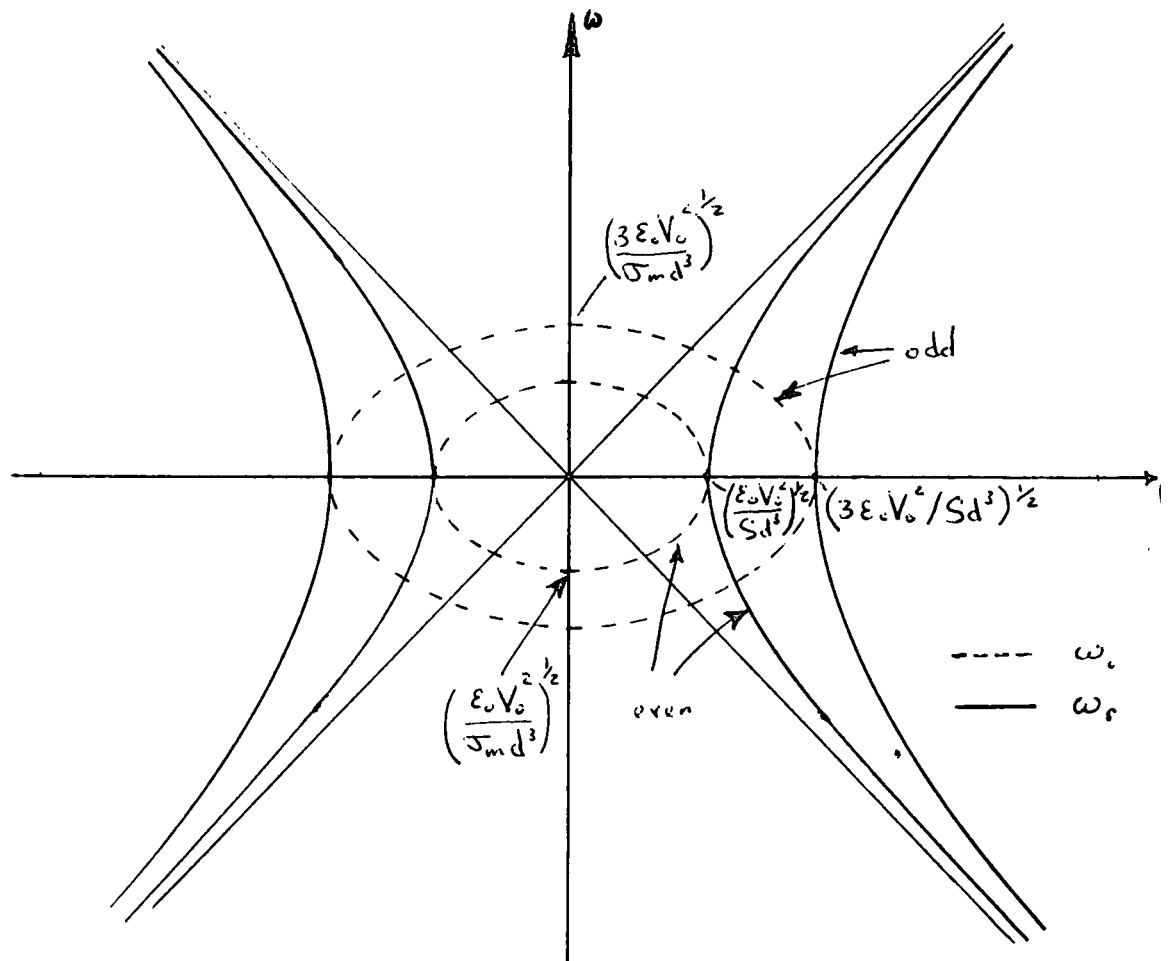
The even mode, $\xi_1 = \xi_2$ has the dispersion relation

$$\omega = \left[\frac{Sk^2}{\sigma_m} - \frac{\epsilon_o V_o^2}{\sigma_m d^3} \right]^{1/2}$$

Part c

A plot of the dispersion relation appears below.

PROBLEM 10.13 (continued)



Part d

The lowest allowed value of k is $k = \frac{\pi}{L}$ since the membranes are fixed at $x = 0$ and $x = L$. Therefore the first mode to go unstable is the even mode. This happens as

$$\left(\frac{3\epsilon_0 V_0^2}{Sd^3} \right) = \frac{\pi^2}{L^2}$$

↗
000

or

$$V_0 = \left| \frac{\pi^2}{L^2} \frac{Sd^3}{\epsilon_0 3} \right|^{1/2}$$

PROBLEM 10.14

The equation of motion is

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2} - v \frac{\partial \xi}{\partial t} \quad (a)$$

Part a

The dispersion for this system is:

$$\omega^2 - jv\omega - v_s^2 k^2 = 0 \quad (b)$$

We may solve for ω ,

$$\begin{aligned} \omega &= j \left[\frac{v}{2} \pm \sqrt{\left(\frac{v}{2}\right)^2 - v_s^2 k^2} \right] \\ &\equiv j[\alpha \pm \gamma] \end{aligned} \quad (c)$$

We assume solutions of the form:

$$\xi(x, t) = \text{Re} \left\{ \sum_{n \text{ odd}} \left[A_n e^{-(\alpha+\gamma_n)t} + B_n e^{-(\alpha-\gamma_n)t} \right] \sin \frac{n\pi x}{\ell} \right\} \quad (d)$$

Now, we may use the initial condition on $\frac{\partial \xi}{\partial t}$ to relate A_n and B_n . Thus we obtain:

$$\xi(x, t) = \text{Re} \left\{ \sum_{n \text{ odd}} A_n \left[e^{-\gamma_n t} - \left(\frac{\alpha+\gamma_n}{\alpha-\gamma_n} \right) e^{\gamma_n t} \right] e^{-\alpha t} \sin \frac{n\pi x}{\ell} \right\} \quad (e)$$

Now, we apply the initial condition on $\xi(x, t = 0)$ to determine A_n .

$$\begin{aligned} \xi(x, 0) &= \sum_{n \text{ odd}} A_n \left[\frac{2\gamma_n}{\gamma_n - \alpha} \right] \sin \frac{n\pi x}{\ell} \\ &\equiv \sum_{n \text{ odd}} A'_n \sin \frac{n\pi x}{\ell} \end{aligned} \quad (f)$$

The coefficient A'_n is determined from a Fourier analysis of the displacement:

$$A'_n = \frac{4\xi_0}{n\pi}, \quad (g)$$

So that:

$$A_n = \left(\frac{\gamma_n - \alpha}{2\gamma_n} \right) \left(\frac{4\xi_0}{n\pi} \right) \quad (h)$$

Part b

There is one important difference between this problem and the magnetic diffusion problems of Chap. VII. While magnetic diffusion is "true diffusion" and satisfies the normal diffusion equation, the string equation is basically a wave equation modified by viscosity. Hence, we note (c) that especially the

PROBLEM 10.14 (continued)

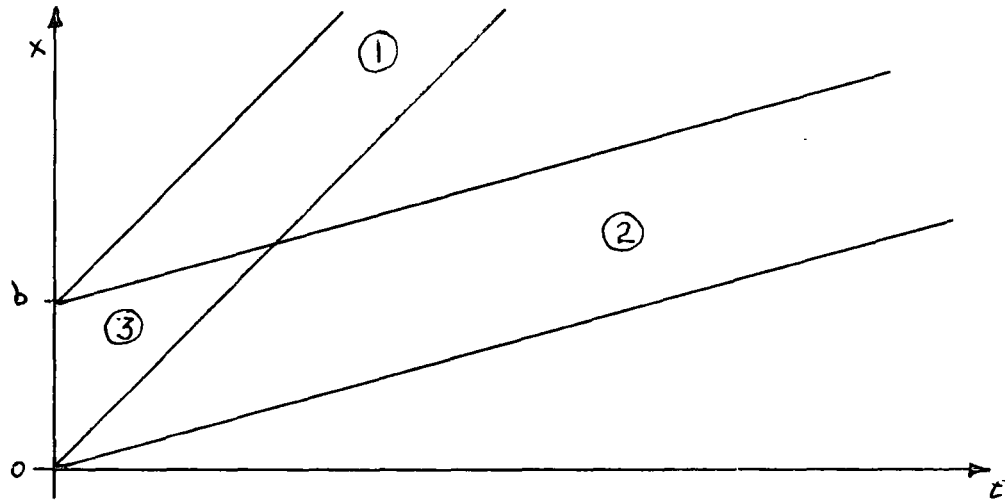
higher modes in the solution to this problem have sinusoidal time dependence as well as decay. Magnetic diffusion as discussed in Chap. 7 exhibits no such oscillation, because there is no mathematical analog to the inertia of the string. If we had included the effects of electromagnetic wave propagation (displacement current) the analogy would be more complete.

PROBLEM 10.15

From Chap. 10, page 588, Eqs. (e) and (f) we have

$$\frac{d\xi_+}{d\alpha} = \frac{(v_s - U)}{2v_s} \frac{\partial \xi}{\partial x} - \frac{1}{2v_s} \frac{\partial \xi}{\partial t}$$

$$\frac{d\xi_-}{d\beta} = \frac{(v_s + U)}{2v_s} \frac{\partial \xi}{\partial x} + \frac{1}{2v_s} \frac{\partial \xi}{\partial t}$$



Since $\frac{\partial \xi}{\partial x}(x=0) = 0$, we have the following relations in the three regions.

Region 1

$$\frac{d\xi_+}{d\alpha} = -\frac{v_o}{2v_s} ; \frac{d\xi_-}{d\beta} = 0$$

Region 2

$$\frac{d\xi_+}{d\alpha} = 0 ; \frac{d\xi_-}{d\beta} = \frac{v_o}{2v_s}$$

Region 3

$$\frac{d\xi_+}{d\alpha} = -\frac{v_o}{2v_s} ; \frac{d\xi_-}{d\beta} = \frac{v_o}{2v_s}$$

PROBLEM 10.15 (continued)

In the other regions, the derivatives are zero.

From Eq. 10.2.10 on page 586,

$$\frac{\partial \xi}{\partial x} = \frac{d\xi_+}{d\alpha} + \frac{d\xi_-}{d\beta}$$

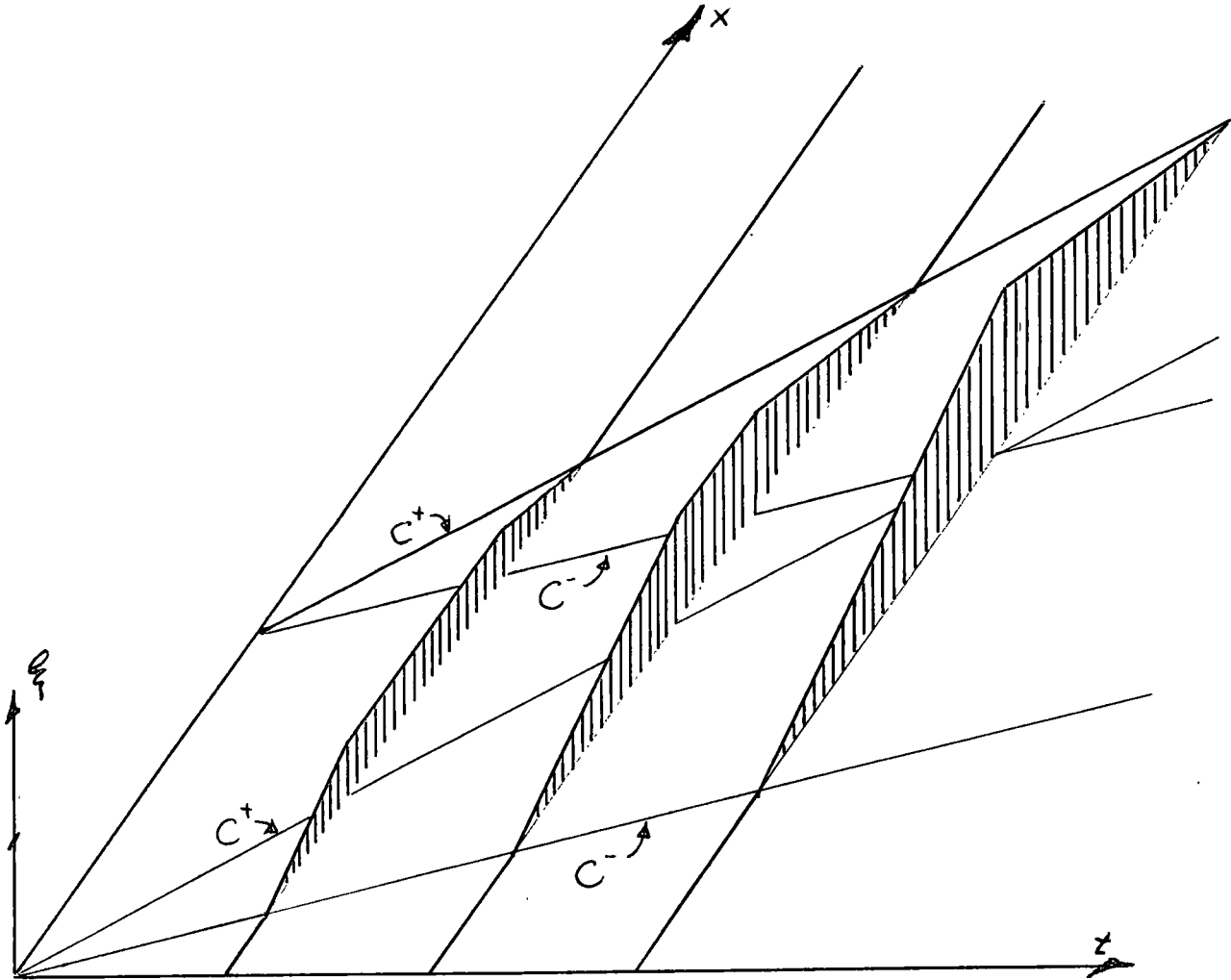
we have

$$\frac{\partial \xi}{\partial x} = \frac{v_0}{2v_s} [u_{-1}(\beta) - u_{-1}(\beta-b) - u_{-1}(\alpha) + u_{-1}(\alpha-b)]$$

Integrating with respect to x , we obtain

$$\xi(x,t) = \frac{v_0}{2v_s} [u_{-2}(\beta) - u_{-2}(\beta-b) - u_{-2}(\alpha) + u_{-2}(\alpha-b)]$$

A sketch of this deflection is shown in the figure.



PROBLEM 10.16

Part a

The equation of motion is simply

$$m \frac{\partial^2 \xi}{\partial t^2} = f \frac{\partial^2 \xi}{\partial x^2}$$

The dispersion equation follows as:

$$(\omega - kU)^2 = v_s^2 k^2$$

or

$$k_{\pm} = \frac{\omega}{U \pm v_s} = \frac{\omega(U \mp v_s)}{U^2 - v_s^2} \equiv \alpha \mp \beta$$

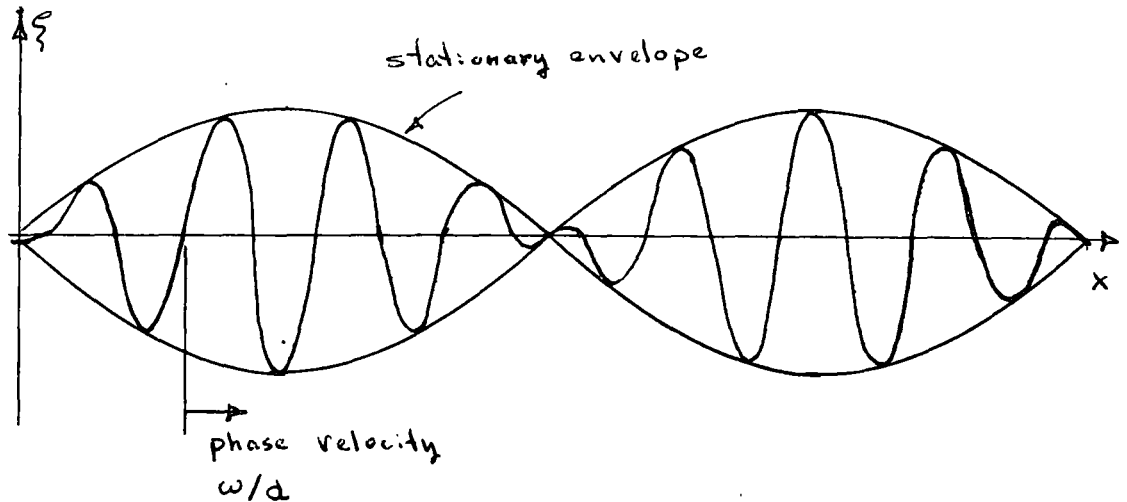
Where solutions are assumed of the form:

$$\xi(x, t) = \text{Re}\{(\xi_+ e^{j\beta x} + \xi_- e^{-j\beta x}) e^{j(\omega t - \alpha x)}\}$$

The boundary conditions are both applied at $x = 0$, because string is moving at a "supersonic" velocity.

$$\xi(x, t) = \xi_0 \left\{ \cos \beta x \cos[\omega t - \alpha x] - \frac{U}{v_s} \sin \beta x \sin[\omega t - \alpha x] \right\}$$

Part b



PROBLEM 10.17

We use Eq. 10.2.9

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right)^2 \xi = v_s^2 \frac{\partial^2 \xi}{\partial x^2}$$

Assuming sinusoidal solutions in time and space we obtain the dispersion relation

$$(\omega - kU)^2 = k^2 v_s^2$$

Thus

$$k = \frac{\omega}{U - v_s} = \frac{\omega(U + v_s)}{U^2 - v_s^2}$$

We let

$$\alpha = \frac{\omega U}{U^2 - v_s^2}$$

$$\beta = \frac{\omega v_s}{U^2 - v_s^2}$$

Therefore, $k = \alpha + \beta$ and

$$\xi(x, t) = \text{Re}[A e^{-j(\alpha - \beta)x} + B e^{-j(\alpha + \beta)x}] e^{j\omega t}$$

The boundary conditions are

$$\xi(x = 0) = 0 \text{ which implies } A = -B$$

$$\xi(x = -l) = \xi_0$$

Therefore

$$\begin{aligned} \xi(x, t) &= \text{Re} [A [e^{-j(\alpha - \beta)x} - e^{-j(\alpha + \beta)x}] e^{j\omega t}] \\ &= \text{Re} [A 2j \sin \beta x e^{j(\omega t - \alpha x)}] \end{aligned}$$

However,

$$\xi(-l, t) = \text{Re} [\xi_0 e^{j\omega t}]$$

Therefore

$$\xi(x, t) = -\frac{\xi_0}{\sin \beta l} \sin \beta x \cos[\omega t - \alpha(x + l)]$$

Part b

For $\xi = 0$ at $x = 0$ and at $x = -l$ we must have $\beta = n\pi/l$

$$\frac{\omega v_s}{U^2 - v_s^2} = \frac{n\pi}{l}$$

or

PROBLEM 10.17 (continued)

$$\omega = \frac{n\pi}{\ell} \frac{(U^2 - v_s^2)}{v_s}$$

These are the natural frequencies of the wire.

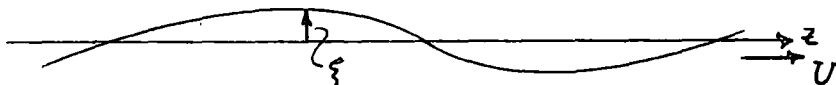
Part c

The results are meaningful only for $|U| < |v_s|$. If this inequality were not true, we would not be able to use a downstream boundary condition to determine upstream behavior and arrive at a result that would be obtained by "turning the drive on". That is, if $U > v_s$ the predictions are not consistent with causality.

PROBLEM 10.18

Part a

In the limit of wavelength short compared to the radius, we may "unwrap" the system:



$$m \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial z} \right)^2 \xi = f \frac{\partial^2 \xi}{\partial z^2} \quad (a)$$

Now let $z \rightarrow R\theta$, $U \rightarrow R\Omega$. Then, it follows that

$$m \left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \theta} \right)^2 \xi = \frac{f}{R^2} \frac{\partial^2 \xi}{\partial \theta^2} \quad (b)$$

where $\Omega_s = \sqrt{f/(m R^2)}$

Part b

The initial conditions are

$$\partial \xi / \partial t (\theta, t = 0) = 0 \quad (c)$$

$$\xi(\theta, t = 0) = \begin{cases} \xi_0 & 0 \leq \theta \leq \pi/4 \\ 0, & \text{elsewhere} \end{cases} \quad (d)$$

Solutions take the form

$$\xi = \xi_+(\alpha) + \xi_-(\beta) \quad (e)$$

where

PROBLEM 10.18 (continued)

$$\alpha = \theta - \Omega_s t$$

$$\beta = \theta - 3\Omega_s t$$

Because $\partial \xi / \partial t (t = 0) = 0$,

$$= -\Omega_s \frac{d\xi_+}{d\alpha} - 3\Omega_s \frac{d\xi_-}{d\beta} \quad (f)$$

Also,

$$\frac{\partial \xi}{\partial \theta} (t=0) = \frac{d\xi_+}{d\alpha} + \frac{d\xi_-}{d\beta} = \xi_o [u_o(0) - u_o(\pi/4)] \quad (g)$$

Thus, from (f) and (g).

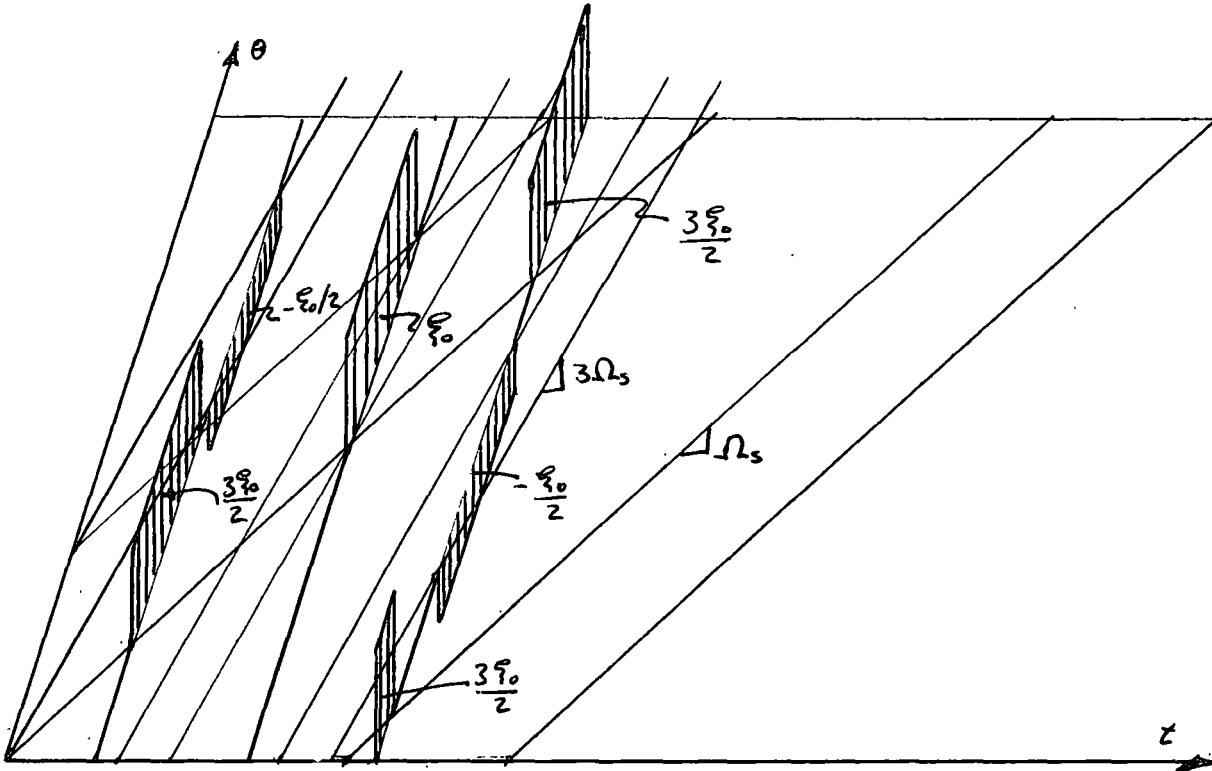
$$\frac{d\xi_+}{d\alpha} = \frac{3}{2} \xi_o [u_o(0) - u_o(\pi/4)]; \text{ on } \alpha \quad (h)$$

$$\frac{d\xi_-}{d\beta} = -\frac{1}{2} \xi_o [u_o(0) - u_o(\pi/4)]; \text{ on } \beta$$

The solution in the θ - t plane follows from

$$\frac{\partial \xi}{\partial \theta} = \frac{d\xi_+}{d\alpha} + \frac{d\xi_-}{d\beta} \quad (j)$$

and an integration at constant t on θ . The result is shown in the figure. Note that the characteristics that leave the interval $0 < \theta < 2\pi$ at $\theta = 2\pi$ reappear at $\theta = 0$ to account for the reentrant nature of the rotating wire.



PROBLEM 10.19

In the moving frame we can write

$$m \frac{\partial^2 \xi}{\partial t'^2} = f \frac{\partial^2 \xi}{\partial x'^2} + F(x', t') \quad (a)$$

and so from Prob. 10.4, we can write

$$P'_{in} = \frac{\partial W'}{\partial t'} + \frac{\partial P'}{\partial x'} \quad (b)$$

where

$$P'_{in} = F \frac{\partial \xi}{\partial t'} \quad (c)$$

$$W' = \frac{1}{2} m \left(\frac{\partial \xi}{\partial t'} \right)^2 + \frac{1}{2} f \left(\frac{\partial \xi}{\partial x'} \right)^2 \quad (d)$$

$$P' = - f \frac{\partial \xi}{\partial x'} \frac{\partial \xi}{\partial t'} \quad (e)$$

$$\text{But } \frac{\partial}{\partial x'} = \frac{\partial}{\partial x} \text{ and } \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + U \frac{\partial}{\partial x}$$

Therefore (c)-(e) become

$$P'_{in} = F \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \xi \quad (f)$$

$$W' = \frac{1}{2} m \left(\frac{\partial \xi}{\partial t} + U \frac{\partial \xi}{\partial x} \right)^2 + \frac{1}{2} f \left(\frac{\partial \xi}{\partial x} \right)^2 \quad (g)$$

$$P' = - f \frac{\partial \xi}{\partial x} \left(\frac{\partial \xi}{\partial t} + U \frac{\partial \xi}{\partial x} \right) \quad (h)$$

The conservation of energy equation, in terms of fixed frame coordinates, becomes

$$P'_{in} = \frac{\partial W'}{\partial t} + U \frac{\partial W'}{\partial x} + \frac{\partial P'}{\partial x} \quad (i)$$

$$= \frac{\partial W'}{\partial t} + \frac{\partial}{\partial x} (P' + W'U) \quad (j)$$

If we let

$$P_{in} = P'_{in}$$

$$W = W' \quad (k)$$

$$P = P' + W'U$$

we can write

$$P_{in} = \frac{\partial W}{\partial t} + \frac{\partial P}{\partial x} \quad (l)$$

which is the required form.

PROBLEM 10.20

The equation of motion is given by Eq. 10.2.33, and hence the dispersion equation is 10.2.36;

$$k = \eta \pm j\gamma \quad (a)$$

where

$$\eta = \omega_d U / (U^2 - v_s^2)$$

$$\gamma = v_s \sqrt{(U^2 - v_s^2) k_c^2 - \omega_d^2} / (U^2 - v_s^2)$$

Solutions are assumed of the form

$$\xi = \text{Re}[A \sinh \gamma x + B \cosh \gamma x] e^{j(\omega t - \eta x)} \quad (b)$$

Boundary conditions require;

$$B = \xi_0 \quad (c)$$

$$A = j\eta \xi_0 / \gamma \quad (d)$$

Thus

$$\xi = \text{Re} \xi_0 \left[\frac{j\eta}{\gamma} \sinh \gamma x + \cosh \gamma x \right] e^{j(\omega t - \eta x)} \quad (e)$$

The deflection has an envelope with an essentially exponentially increasing dependence on x , with the instantaneous deflection traveling in the $+x$ direction.

PROBLEM 10.21 *"m" SHOULD BE "ε_m" THROUGHOUT PROBLEM*
Part a

The equation of motion is

$$\sigma_m \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 \xi = S \frac{\partial^2 \xi}{\partial x^2} - mg + T \quad (a)$$

$$\text{with } T = \frac{\epsilon_0}{2} \frac{v_o^2}{(d-\xi)^2} \approx \frac{\epsilon_0}{2} v_o^2 \left[\frac{1}{d^2} + \frac{2\xi}{d^3} \right]$$

For equilibrium, $\xi = 0$ and from (a)

$$\frac{\epsilon_0 v_o^2}{2d^2} = mg \quad (b)$$

or

$$v_o = \left[\frac{2mg d^2}{\epsilon_0} \right]^{1/2} \quad (c)$$

PROBLEM 10.21 (continued)
Part b

With solutions of the form $e^{j(\omega t - kx)}$ the dispersion relation is

$$(\omega - kU)^2 = \frac{S}{\sigma_m} k^2 - \frac{\epsilon_o v_o^2}{\sigma_m d^3} \quad (d)$$

Solving for k , we obtain

$$k = \omega U \pm \sqrt{\frac{S}{\sigma_m} \omega^2 - \left(U^2 - \frac{S}{\sigma_m}\right) \left(\frac{\epsilon_o v_o^2}{\sigma_m d^3}\right)} \quad (e)$$

$$\left(U^2 - \frac{S}{\sigma_m}\right)$$

For $U > \sqrt{S/\sigma_m}$, and not to have spatially growing waves

$$\frac{S}{\sigma_m} \omega^2 - \left(U^2 - \frac{S}{\sigma_m}\right) \left(\frac{\epsilon_o v_o^2}{\sigma_m d^3}\right) > 0 \quad (f)$$

or

$$\omega^2 > \left[\left(U^2 - \frac{S}{\sigma_m}\right) \frac{\epsilon_o v_o^2}{S d^3} \right] \quad (g)$$

PROBLEM 10.22
Part a

Neglecting the curvature of the system, as in Prob. 10.18, we write:

$$\sigma_m \left(\frac{\partial}{\partial t} + \frac{\Omega R}{R} \frac{\partial}{\partial \theta} \right)^2 \xi = \frac{S}{R^2} \cdot \frac{\partial^2 \xi}{\partial \theta^2} + T_r \quad (a)$$

where the linearized perturbation force/unit area is

$$T_r = \left(\frac{2\epsilon_o v_o^2}{3a} \right) \xi \quad (b)$$

Therefore, the equation of motion is

$$\left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \theta} \right)^2 \xi = \Omega_s^2 \left(\frac{\partial^2 \xi}{\partial \theta^2} + m_c^2 \xi \right) \quad (c)$$

$$\Omega_s^2 = \frac{S}{\sigma_m R^2}$$

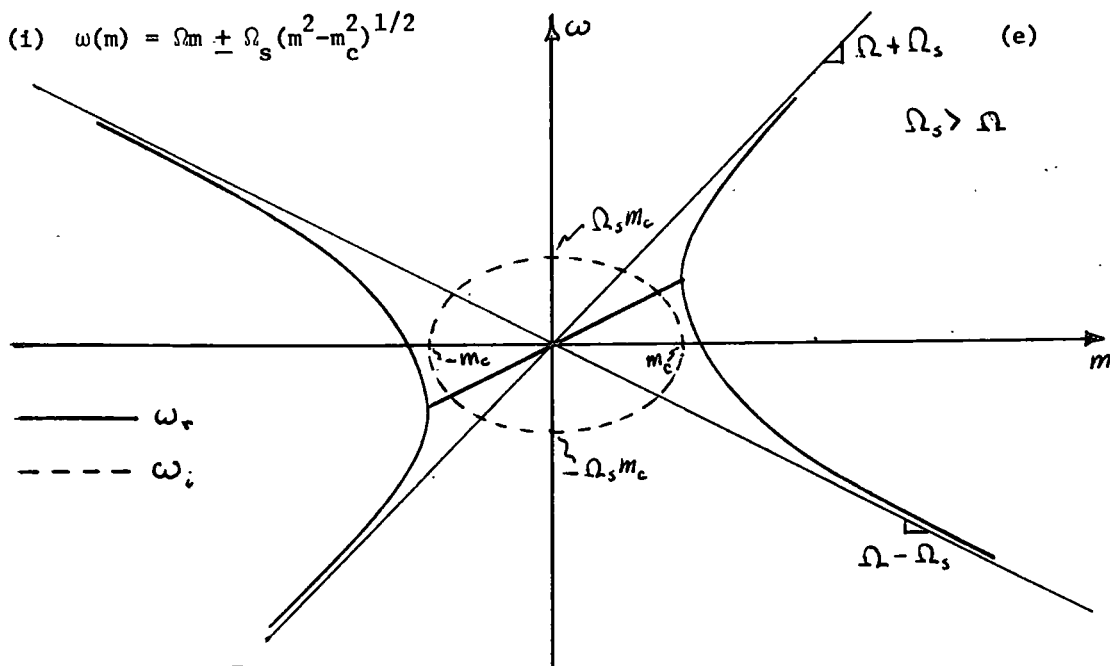
$$m_c^2 = \frac{2\epsilon_o v_o^2}{3a} \frac{R^2}{S}$$

PROBLEM 10.22

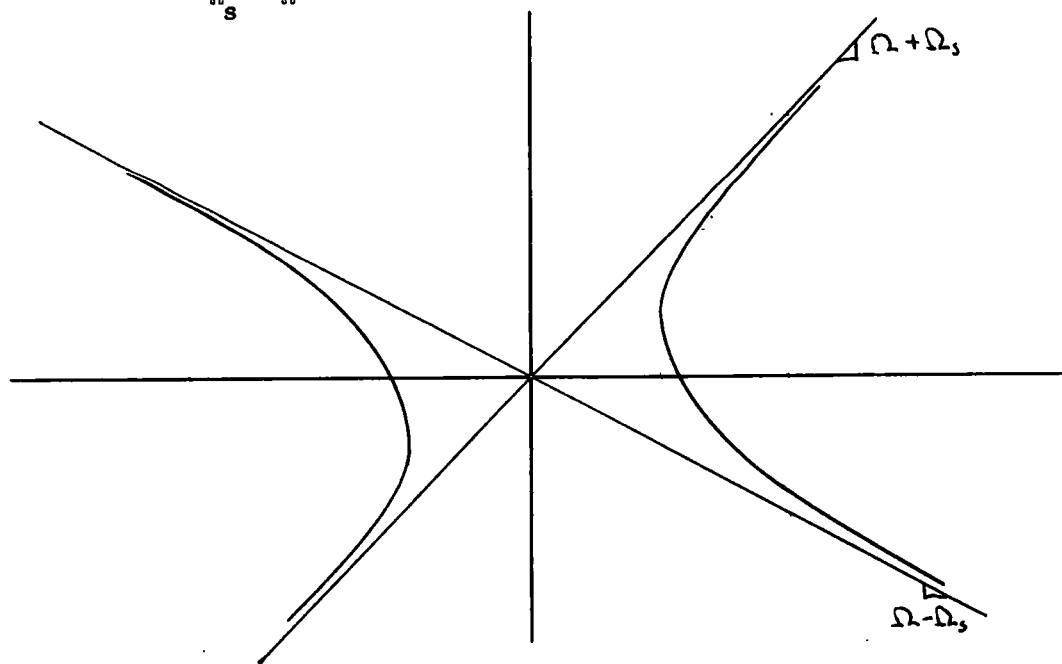
Part b

$$(\omega - m\Omega)^2 = \Omega_s^2 (m^2 - m_c^2) \quad (d)$$

$$(i) \quad \omega(m) = \Omega m \pm \Omega_s (m^2 - m_c^2)^{1/2} \quad (e)$$



$$(ii) \quad m = \frac{\Omega \omega \pm \sqrt{\Omega_s^2 \omega^2 + (\Omega_s^2 - \Omega^2)(\Omega_s^2 m_c^2)}}{\Omega_s^2 - \Omega^2} \quad (f)$$



DYNAMICS OF ELECTROMECHANICAL CONTINUA

PROBLEM 10.22

Part c

Because the membrane closes on itself it can be absolutely unstable regardless of Ω relative to Ω_g . Allowed values of m are determined by the requirement that the deflections be periodic in θ ; $m = 0, 1, 2, 3, \dots$. Thus, from (e) any finite m_c will lead to instability in the $m = 0$ mode. Note however that this mode does not meet the requirement that wavelengths be short compared to R .

PROBLEM 10.23

We may take the results of Prob. 10.13, replacing $\frac{\partial}{\partial t}$ by $\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}$ and replacing ω by $\omega - kU$.

Part a

The equations of motion are

$$\sigma_m \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 \xi_1 = S \frac{\partial^2 \xi_1}{\partial x^2} + \frac{\epsilon_o V_o^2}{d^3} (2\xi_1 - \xi_2) \quad (a)$$

and

$$\sigma_m \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 \xi_2 = S \frac{\partial^2 \xi_2}{\partial x^2} + \frac{\epsilon_o V_o^2}{d^3} (2\xi_2 - \xi_1) \quad (b)$$

Part b

The dispersion relation is biquadratic, and factors into

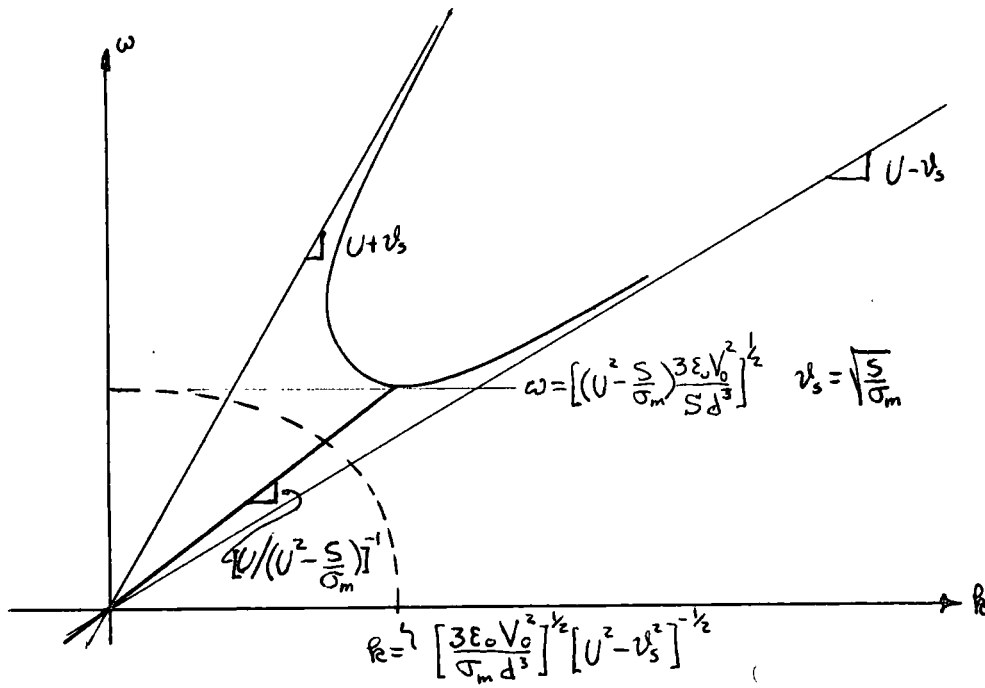
$$-\sigma_m (\omega - kU)^2 + Sk^2 - \frac{2\epsilon_o V_o^2}{d^3} = \pm \frac{\epsilon_o V_o^2}{d^3} \quad (c)$$

The (+) signs correspond to the cases $\xi_1 = -\xi_2$ and $\xi_1 = \xi_2$ respectively, as will be seen in part (d).

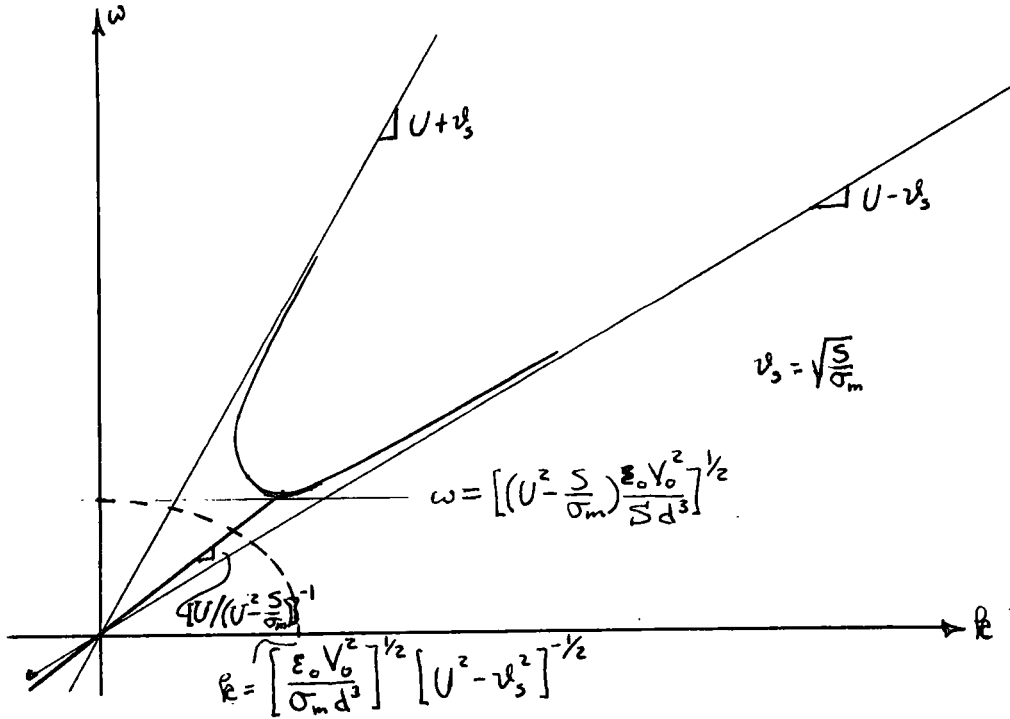
Part c

The dispersion relations are plotted in the figure for $U > \sqrt{S/\sigma_m}$.

PROBLEM 10.23 (continued)



and



PROBLEM 10.23 (continued)

Part d

Let $\xi_1 = \xi_2$. Then (a) and (b) become

$$\sigma_m \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 \xi_1 = S \frac{\partial^2 \xi_1}{\partial x^2} + \frac{\epsilon_o v_o^2}{d^3} \xi_1 \quad (d)$$

and

$$\sigma_m \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 \xi_2 = S \frac{\partial^2 \xi_2}{\partial x^2} + \frac{\epsilon_o v_o^2}{d^3} \xi_2 \quad (e)$$

These equations are identical for $\xi_1 = \xi_2$; the dispersion equation is (c) with the minus sign. Now let $\xi_1 = -\xi_2$ and (a) and (b) require

$$\sigma_m \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 \xi_1 = S \frac{\partial^2 \xi_1}{\partial x^2} + \frac{3\epsilon_o v_o^2}{d^3} \xi_1 \quad (f)$$

and

$$\sigma_m \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 \xi_2 = S \frac{\partial^2 \xi_2}{\partial x^2} + \frac{3\epsilon_o v_o^2}{d^3} \xi_2 \quad (g)$$

These equations are identical for $\xi_1 = -\xi_2$; the dispersion equation is (c) with the + sign.

Part e

$$\xi_1(0, t) = \text{Re } \hat{\xi} e^{j\omega t} = -\xi_2(0, t) \quad (h)$$

$$\frac{\partial \xi_1}{\partial x} = \frac{\partial \xi_2}{\partial x} = 0 \text{ at } x = 0 \quad (i)$$

The odd mode is excited. Hence, we use the + sign in (c)

$$-\sigma_m (\omega - kU)^2 + Sk^2 - \frac{3\epsilon_o v_o^2}{d^3} = 0 \quad (j)$$

$$k^2 (S - \sigma_m U^2) + 2\sigma_m \omega kU - \sigma_m \omega^2 - \frac{3\epsilon_o v_o^2}{d^3} = 0 \quad (k)$$

Solving for k, we obtain

$$k_{\pm} = \alpha \pm \beta \quad (l)$$

$$\text{where } \alpha = \frac{\omega U}{U^2 - v_s^2} \quad (m)$$

$$\beta = \frac{[\omega^2 v_s^2 - \frac{3\epsilon_o v_o^2 (U^2 - v_s^2)}{\sigma_m d^3}]^{1/2}}{U^2 - v_s^2} \quad (n)$$

PROBLEM 10.23 (continued)

with $v_s^2 = S/\sigma_m$.

Therefore

$$\xi_1 = \text{Re} \{ [A e^{-j(\alpha+\beta)x} + B e^{-j(\alpha-\beta)x}] e^{j\omega t} \} \quad (o)$$

Applying the boundary conditions, we obtain

$$A = \hat{\xi} \frac{(\beta - \alpha)}{2\beta} \quad (p)$$

$$B = \frac{(\alpha + \beta) \hat{\xi}}{2\beta} \quad (q)$$

Therefore, if $\hat{\xi}$ is real

$$\xi_1(x, t) = -\xi_2(x, t) = \hat{\xi} \cos \beta x \cos(\omega t - \alpha x) - \frac{\alpha}{\beta} \hat{\xi} \sin \beta x \sin(\omega t - \alpha x) \quad (r)$$

Part f

We can see that β can be imaginary, for which we will have spatially growing curves. This can happen when

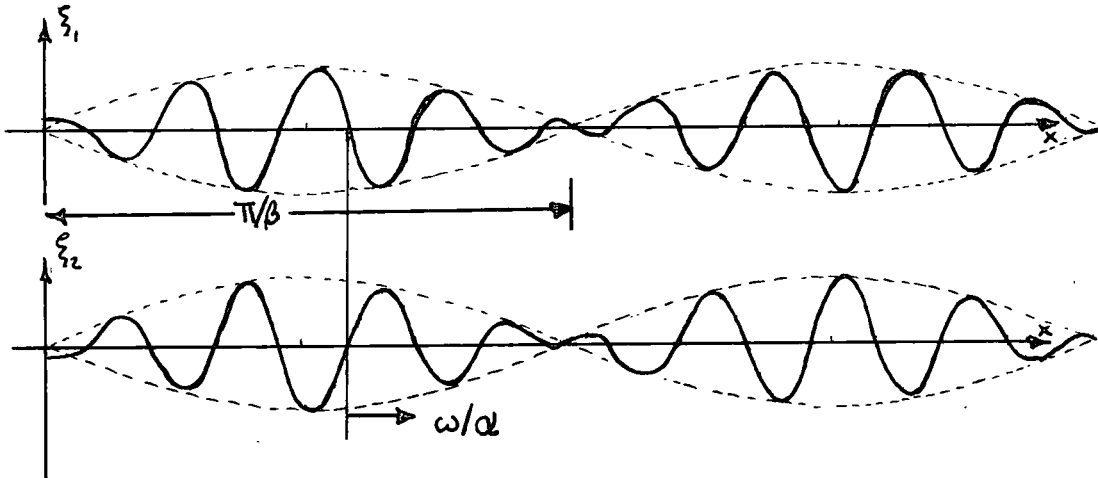
$$\omega^2 v_s^2 - \frac{3\epsilon_o v_o^2}{\sigma_m d^3} (U^2 - v_s^2) < 0 \quad (s)$$

or

$$v_o^2 > \frac{\sigma_m d^3 \omega^2 v_s^2}{3\epsilon_o (U^2 - v_s^2)} \quad (t)$$

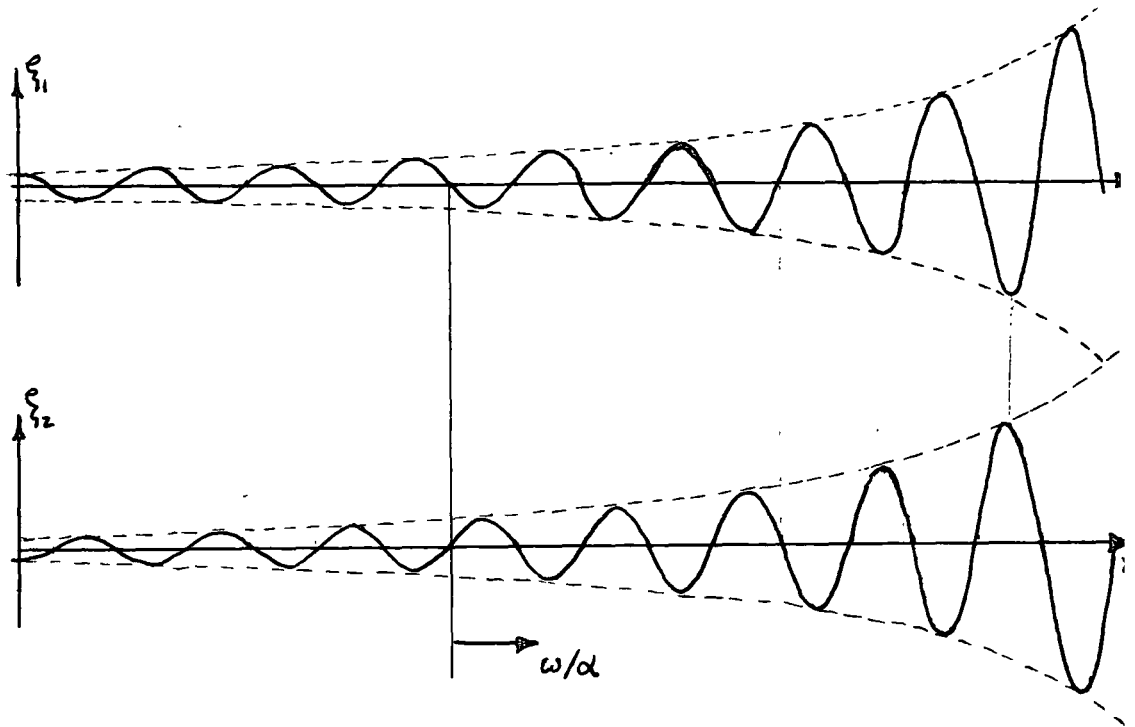
Part g

With $V_o = 0$ and $v > v_s$;



PROBLEM 10.23 (continued)

Amplifying waves are obtained as (t) is satisfied;



PROBLEM 10.24

Part a

The equation of motion for the membrane is:

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \left[\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right] + T_z \quad (a)$$

where

$$T_z = T_{zz} = 2\epsilon_o V_o^2 \xi / s^3 \quad (b)$$

The equation may be rewritten as follows:

PROBLEM 10.24 (continued)

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \left[\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + k_c^2 \xi \right] \quad (c)$$

where

$$k_c^2 \equiv \frac{2\epsilon_o v_o^2}{S s^3}$$

Assume solutions of the following form:

$$\xi(x,y,t) = \text{Re}[\hat{\xi} e^{j(\omega t - k_x x - k_y y)}] \quad (d)$$

The dispersion is:

$$\omega^2 = v_s^2 [k_x^2 + k_y^2 - k_c^2] \quad (e)$$

The mode which goes unstable first is the lowest spatial mode:

$$k_x = \frac{\pi}{a}, \quad k_y = \frac{\pi}{b} \quad (f)$$

Instability occurs at

$$k_c^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 \quad (g)$$

or,

$$v_o = \left\{ \frac{S s^3}{2\epsilon_o} \left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 \right] \right\}^{1/2} \quad (h)$$

Part b

The natural frequencies follow from Eq. (e) as

$$\omega_{mn} = v_s \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k_c^2 \right]^{1/2} \quad (i)$$

Part c

We superimpose eigensolutions to obtain the membrane motion for $t > 0$.

The solution that already satisfies the initial condition on velocity is

$$\xi(x,y,t) = \sum_m \sum_n \xi_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos \omega_{mn} t \quad (j)$$

where m and n are odd only, since the initial condition on $\xi(x,y,t=0)$ requires no even modes. Now use the principle of orthogonality of modes. Multiply (j) by $\sin(p\pi x/a) \sin(q\pi y/b)$ and integrate over the area of the membrane. The left hand side becomes

PROBLEM 10.24 (continued)

$$\begin{aligned} \int_0^b \int_0^a \left[\xi(x, y, t=0) \sin \frac{p\pi x}{a} \sin \frac{q\pi y}{b} \right] dx dy \\ = \int_0^b \int_0^a J_0 u_0 \left(x - \frac{a}{2}\right) u_0 \left(y - \frac{b}{2}\right) \sin \frac{p\pi x}{a} \sin \frac{q\pi y}{b} dx dy \end{aligned} \quad (k)$$

Thus, (j) reduces to

$$J_0 = \xi_{pq} \left(\frac{a}{2}\right) \left(\frac{b}{2}\right) \quad (l)$$

which makes it possible to evaluate the Fourier amplitudes

$$\xi_{mn} = \frac{4J_0}{ab} \quad (m)$$

The desired response is (j) with ξ_{mn} given by (m).

$$\xi(x, y, t) = \sum_{\substack{m \\ \text{(odd)}}} \sum_{\substack{n \\ \text{(odd)}}} \left(\frac{4J_0}{ab}\right) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos \omega_{mn} t$$

Note that the analysis is valid even if the lowest mode(s) is (are) unstable, for which case:

$$\cos \omega_{pq} t \rightarrow \cosh \alpha_{pq} t$$

PROBLEM 10.25

The equation of motion is (see Table 9.2, page 535):

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right) \quad (a)$$

With solutions of the form $\xi = \text{Re } \hat{\xi} e^{j(\omega t - k_x x - k_y y)}$, the dispersion equation is

$$\omega = \pm v_s \sqrt{k_x^2 + k_y^2} \quad (b)$$

A particular superposition of these solutions that satisfies the boundary conditions along three of the four edges is

$$\xi = A \sin \frac{n\pi y}{a} \sin k_x (x-b) \cos \omega_0 t \quad (c)$$

where in view of (b),

$$\omega_0^2 = v_s^2 \left[k_x^2 + \left(\frac{n\pi}{a}\right)^2 \right] \quad (d)$$

Thus, there is a solution for each value of n , and

PROBLEM 10.25 (continued)

$$\xi = \sum_{n=1}^{\infty} A_n \sin k_n (x-b) \cos \omega_0 t \sin \frac{n\pi y}{a} \quad (e)$$

where, from (d)

$$k_n = \left[\left(\frac{\omega_0}{v_s} \right)^2 - \left(\frac{n\pi}{a} \right)^2 \right]^{1/2} \quad (f)$$

At $x = 0$, (e) takes the form of a Fourier series

$$\xi(y=0) = \sum_{n=1}^{\infty} -A_n \sin k_n b \cos \omega_0 t \sin \frac{n\pi y}{a} \quad (g)$$

This function of (y,t) has the correct dependence on t . The dependence on y is made that of Fig. 10P.25 by adjusting the coefficients A_n as is usual in a Fourier series. Note that because of the symmetry of the excitation about $y = a/2$, only odd values of n give finite A_n . Thus

$$\begin{aligned} \int_0^{a/2} \frac{2\xi_0}{a} y \sin \frac{m\pi}{a} y \, dy + \int_{a/2}^a \frac{2\xi_0}{a} (a-y) \sin \frac{m\pi}{a} y \, dy \\ = - \int_0^a A_n \sin k_n b \sin \frac{n\pi y}{a} \sin \frac{m\pi y}{a} \, dy \end{aligned} \quad (h)$$

Evaluation of the integrals gives

$$\frac{4\xi_0 a}{(m\pi)^2} \sin \left(\frac{m\pi}{2} \right) = - \frac{A_n a \sin k_n b}{2} \quad (i)$$

Hence, the required function is (e) with k_n given by (f) and A_n given by solving (i)

$$A_n = - \frac{8\xi_0}{(m\pi)^2} \sin \left(\frac{m\pi}{2} \right) / \sin k_n b \quad (j)$$

PROBLEM 10.26

The force per unit length is $\mu_0 \bar{I} \times \bar{H}$, where $\bar{H}$ is the magnetic field intensity evaluated at the position of the wire. That is,

$$\bar{S} = \mu_0 I [H_x \bar{i}_y - H_y \bar{i}_x] \quad (a)$$

To evaluate H_x and H_y at $u\bar{i}_x + v\bar{i}_y$, note that $\bar{H}(0,0) = 0$. By symmetry $H_y(0,y) = 0$ and therefore $\partial H_y / \partial y (0,0) = 0$. Then, $\nabla \cdot \bar{B} = 0$ requires that $\partial H_x / \partial x (0,0) = 0$. Thus, an expansion of (a) about the origin gives

PROBLEM 10.26 (continued)

$$\bar{S} \cong \mu_o I \left[\frac{\partial H_x}{\partial y} v \bar{i}_y - \frac{\partial H_y}{\partial x} u \bar{i}_x \right] \quad (b)$$

Note that because $\nabla \times \bar{H} = 0$ at the origin, $\partial H_x / \partial y = \partial H_y / \partial x$. Thus, (b) becomes

$$\bar{S} = \mu_o I \frac{\partial H_y}{\partial x} [-u \bar{i}_x + v \bar{i}_y] \quad (c)$$

and $\partial H_y / \partial x (0,0)$ is easily computed because

$$H_y(x,0) = \frac{I_o}{2\pi} \left[\frac{1}{a-x} - \frac{1}{a+x} \right] \cong \frac{I_o}{2\pi} \left[\frac{2x}{a^2} \right] \quad (d)$$

Thus,

$$\bar{S} = \frac{\mu_o I I_o}{\pi a^2} [-u \bar{i}_x + v \bar{i}_y] \quad (e)$$

It is the fact that $\nabla \times \bar{H} = 0$ in the neighborhood of the origin that requires that the contributions to (e) be negatives.

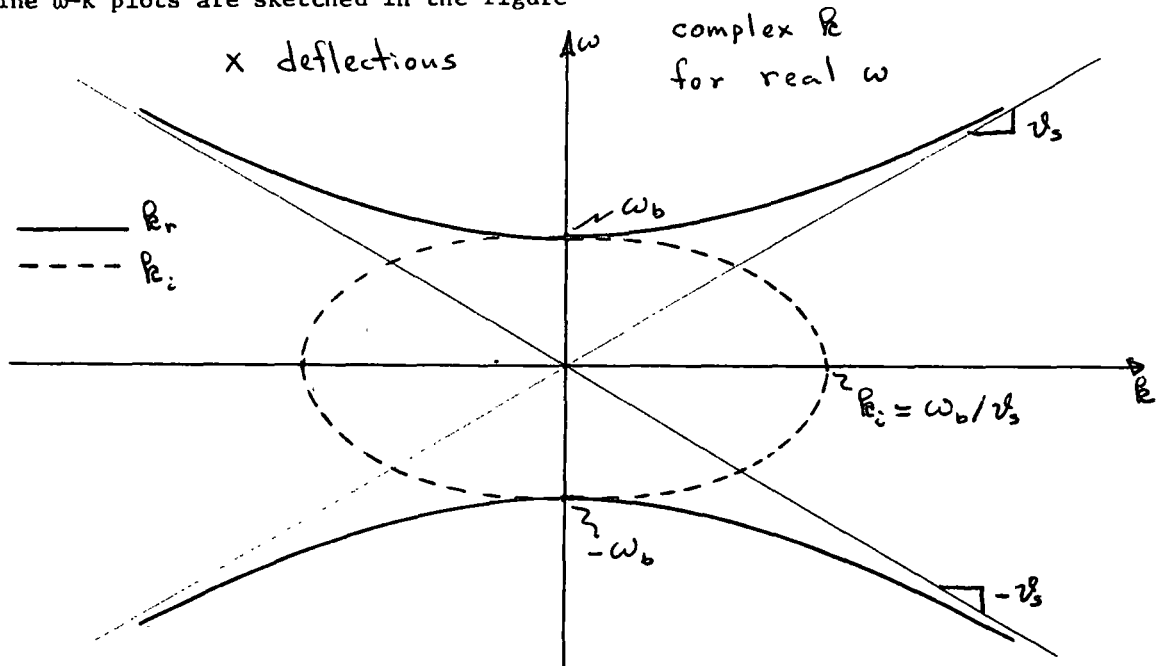
Part b

$$(i) \text{ Assume } u = \text{Re}[\hat{u} e^{j(\omega t - kz)}] \quad (f)$$

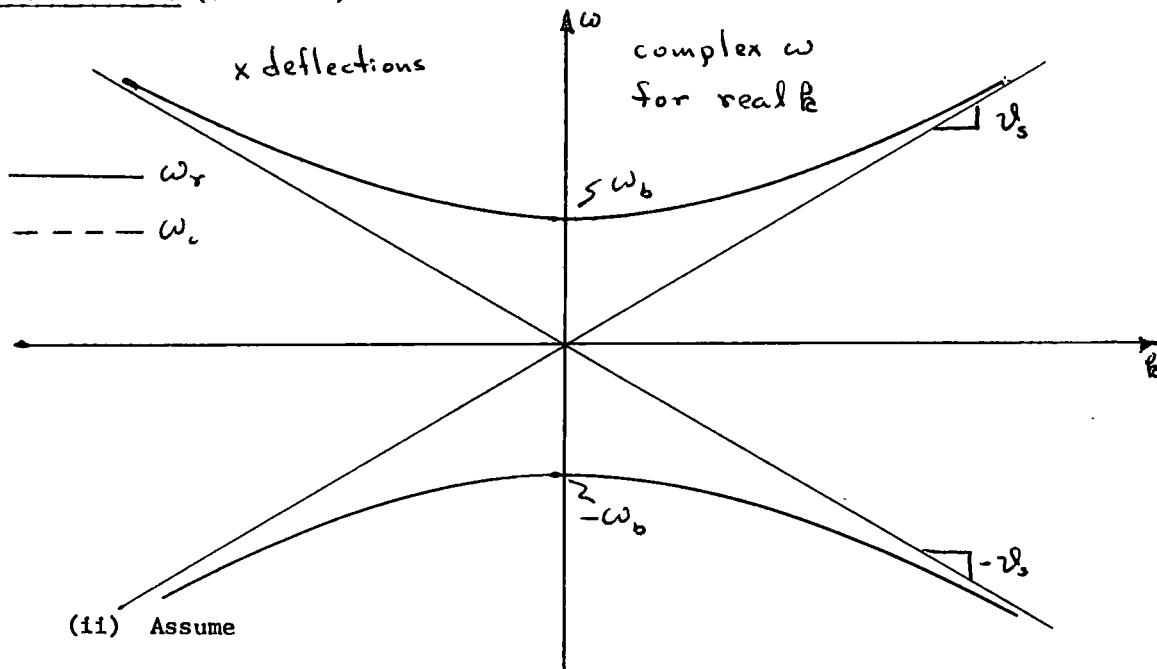
Then

$$\omega^2 = v_s^2 k^2 + \omega_b^2, \quad v_s^2 = \frac{f}{m}; \quad \omega_b^2 = \frac{Ib}{m} \quad (g)$$

The ω - k plots are sketched in the figure



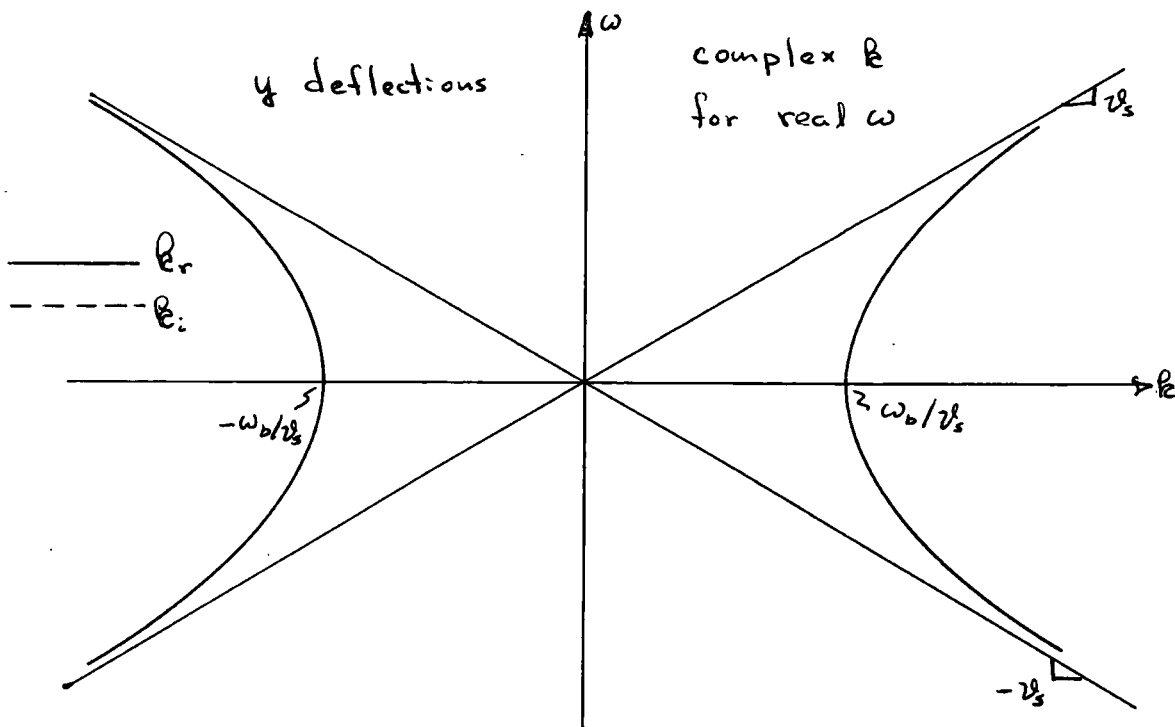
PROBLEM 10.26 (continued)



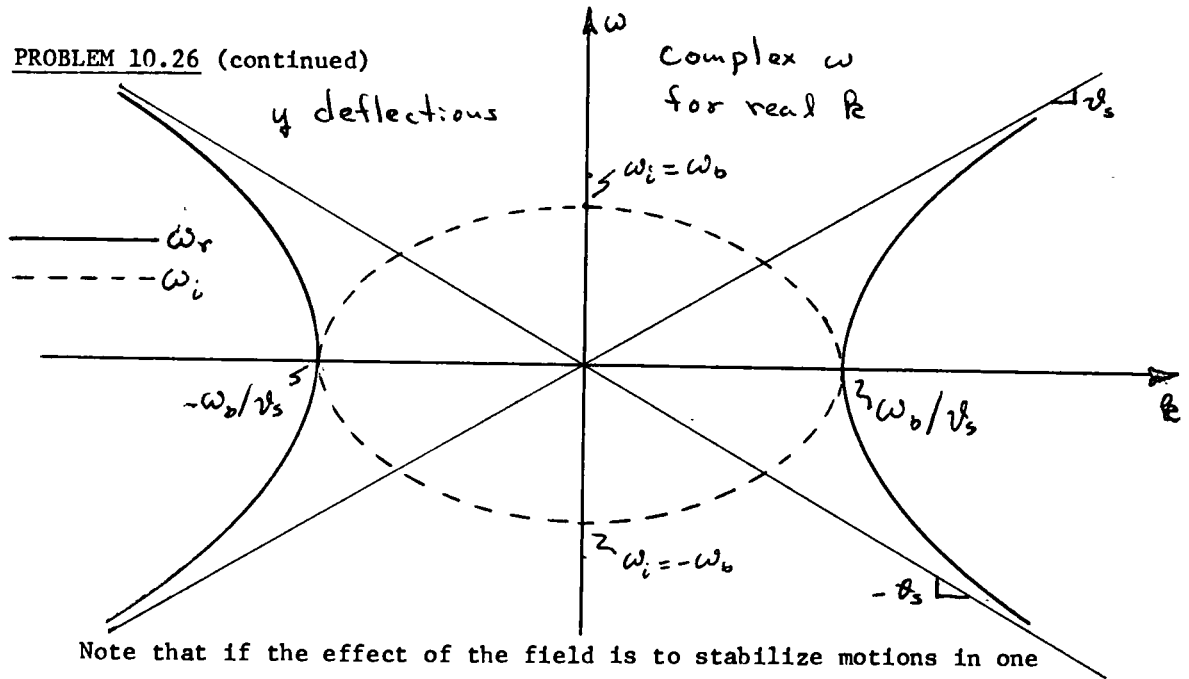
$$v = \text{Re}[\hat{v} e^{j(\omega t - kz)}] \quad (h)$$

Then

$$\omega^2 = v_s^2 k^2 - \omega_b^2 \quad (i)$$

 and the ω - k plots are as shown


PROBLEM 10.26 (continued)



Part c

Driven response is found in a manner similar to that for Prob. 10.2.

Thus for

$$\omega < \omega_b \quad (\text{cutoff})$$

$$u(z, t) = - \frac{u_o \sinh \alpha_u x}{\sinh \alpha l} \cos \omega_o t \quad (j)$$

$$v(z, t) = - \frac{v_o \sin k_r x}{\sin k_v l} \sin \omega_o t \quad (k)$$

$$\omega > \omega_b$$

$$u(z, t) = - \frac{u_o \sin k_u x}{\sin k_u l} \cos \omega_o t \quad (l)$$

$$v(z, t) = - \frac{v_o \sin k_v x}{\sin k_v l} \sin \omega_o t \quad (m)$$

where

$$\alpha_u = \left[\frac{\omega_b^2 - \omega_o^2}{v_s^2} \right]^{1/2}$$

$$k_u = \left[\frac{\omega_o^2 - \omega_b^2}{v_s^2} \right]^{1/2}$$

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PROBLEM 10.26 (continued)

$$k_v = \left[\frac{\omega_o^2 + \omega_b^2}{v_s^2} \right]^{1/2}$$

Part d

We must suppress instability of lowest natural mode in v.

$$v_s^2 \left(\frac{\pi}{\ell} \right)^2 > \omega_b^2 \quad (n)$$

or

$$I I_o < \frac{f \pi a^2}{\mu_o} \left(\frac{\pi}{\ell} \right)^2 \quad (o)$$

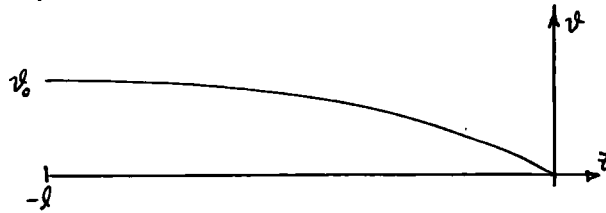
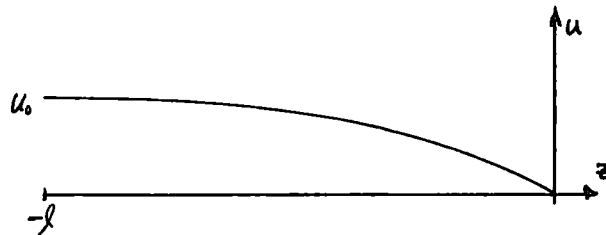
for evanescent waves

$$\omega_o^2 < \omega_b^2 \quad (p)$$

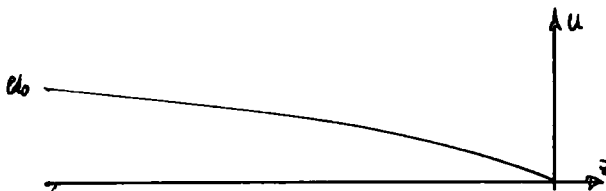
Thus, from (n) and (p), $\omega_o^2 < v_s^2 (\pi/\ell)^2$.

Part e

$$I = 0$$

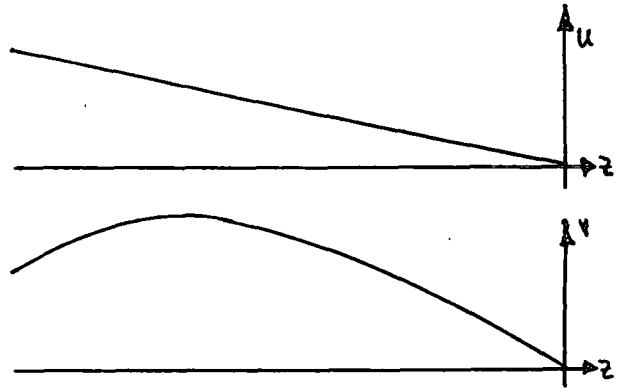


$$I < \omega_o^2 / (b/m)$$



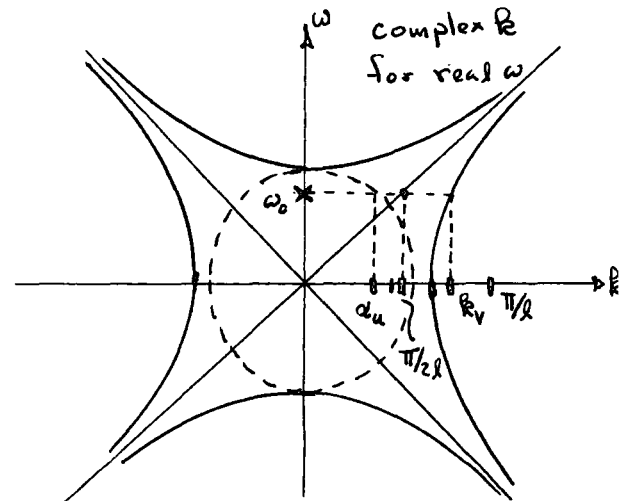
PROBLEM 10.26 (continued)

$$I = \frac{\omega_o^2}{b/m}$$



The effect of raising the current is summarized by the ω - k plot, with complex k plotted for real ω .

As I is raised the hyperbola moves outward. Thus, k_v increases and k_u decreases to zero and becomes imaginary. Thus, wavelengths for the v deflection shorten while those for u lengthen to infinity and then deflections decay. Note that v waves shorter than $\lambda=2\ell$ will not be observed because of instability.



PROBLEM 10.27

Part a

We may take the results of problem 10.26 and replace $\frac{\partial}{\partial t}$ by $\frac{\partial}{\partial t} + U \frac{\partial}{\partial z}$ in the differential equations, and ω by $\omega - kU$ in the dispersion equations.

Therefore, the equations of motion are

$$m \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial z} \right)^2 u = f \frac{\partial^2 u}{\partial z^2} - I b u \quad (a)$$

$$m \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial z} \right)^2 v = f \frac{\partial^2 v}{\partial z^2} + I b v \quad (b)$$

Part b

For the x motions, the dispersion relation is

$$-m(\omega - kU)^2 = -fk^2 - Ib \quad (c)$$

We let

PROBLEM 10.27 (continued)

$$\frac{Ib}{m} = \omega_b^2$$

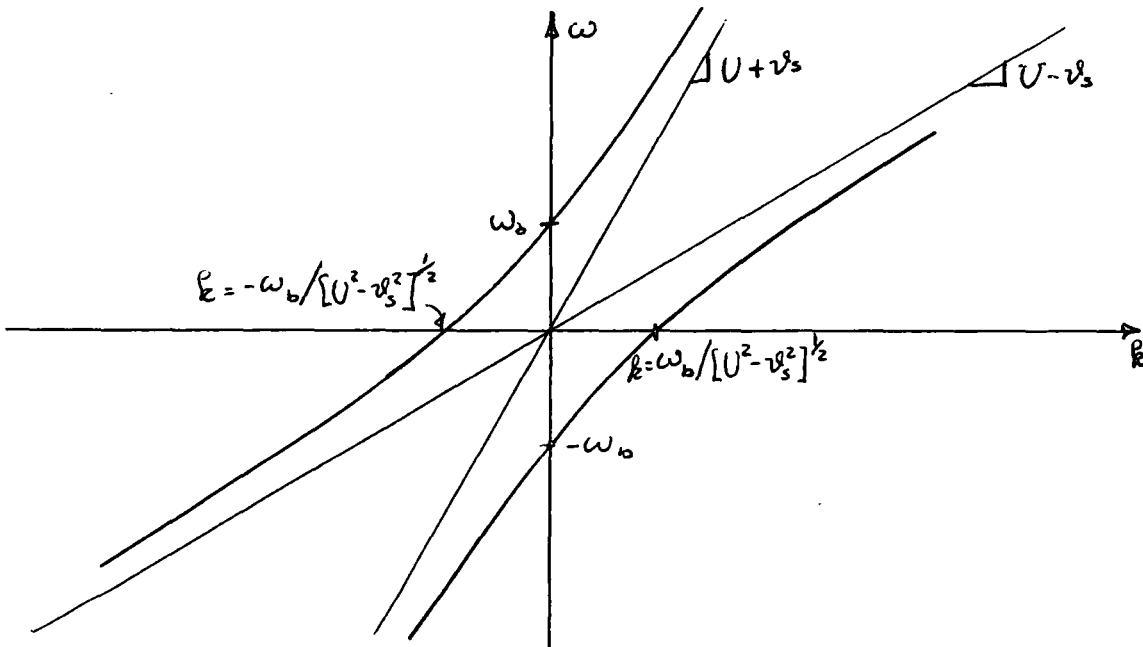
$$\frac{f}{m} = v_s^2$$

$$\text{Therefore } \omega = kU \pm \sqrt{k^2 v_s^2 + \omega_b^2} \quad (d)$$

or solving for k

$$k = \frac{\omega U \pm \sqrt{\omega^2 v_s^2 + \omega_b^2 (U^2 - v_s^2)}}{(U^2 - v_s^2)} \quad (e)$$

The ω - k plot for x motions is sketched as



For real ω , we have only k real. For real k , we have only real ω .

For the y motions, we obtain

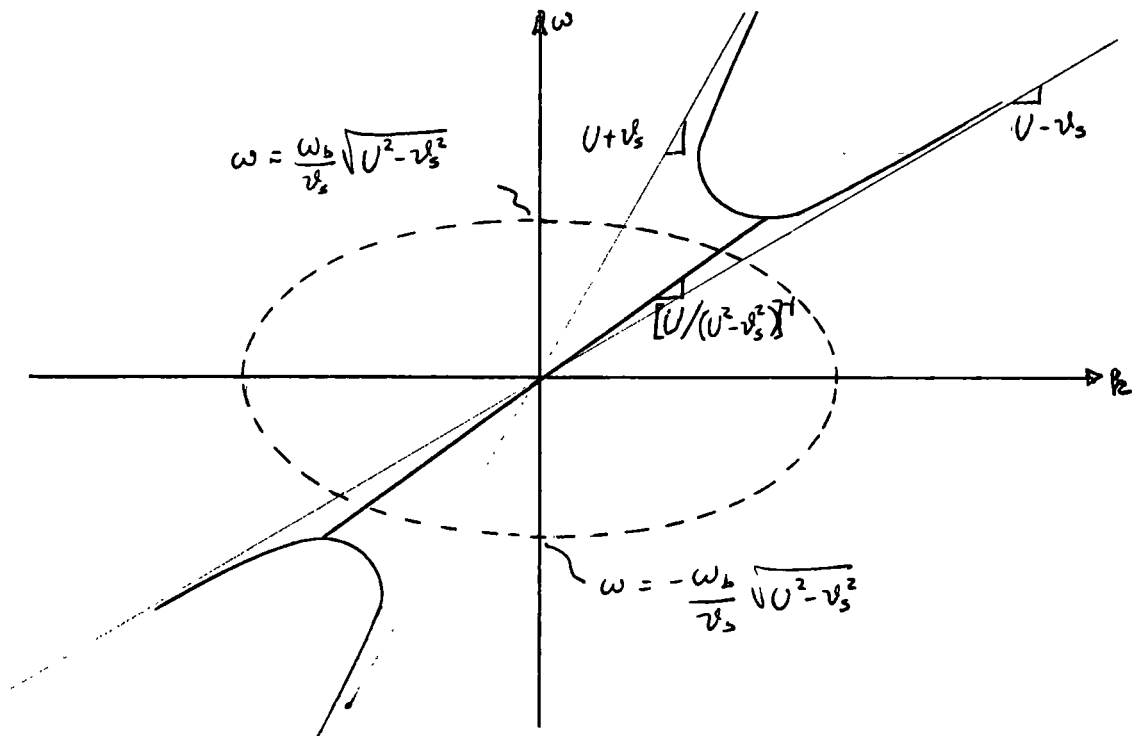
$$\omega = kU \pm \sqrt{v_s^2 k^2 - \omega_b^2} \quad (f)$$

$$k = \frac{\omega U \pm \sqrt{\omega^2 v_s^2 - \omega_b^2 (U^2 - v_s^2)}}{(U^2 - v_s^2)} \quad (g)$$

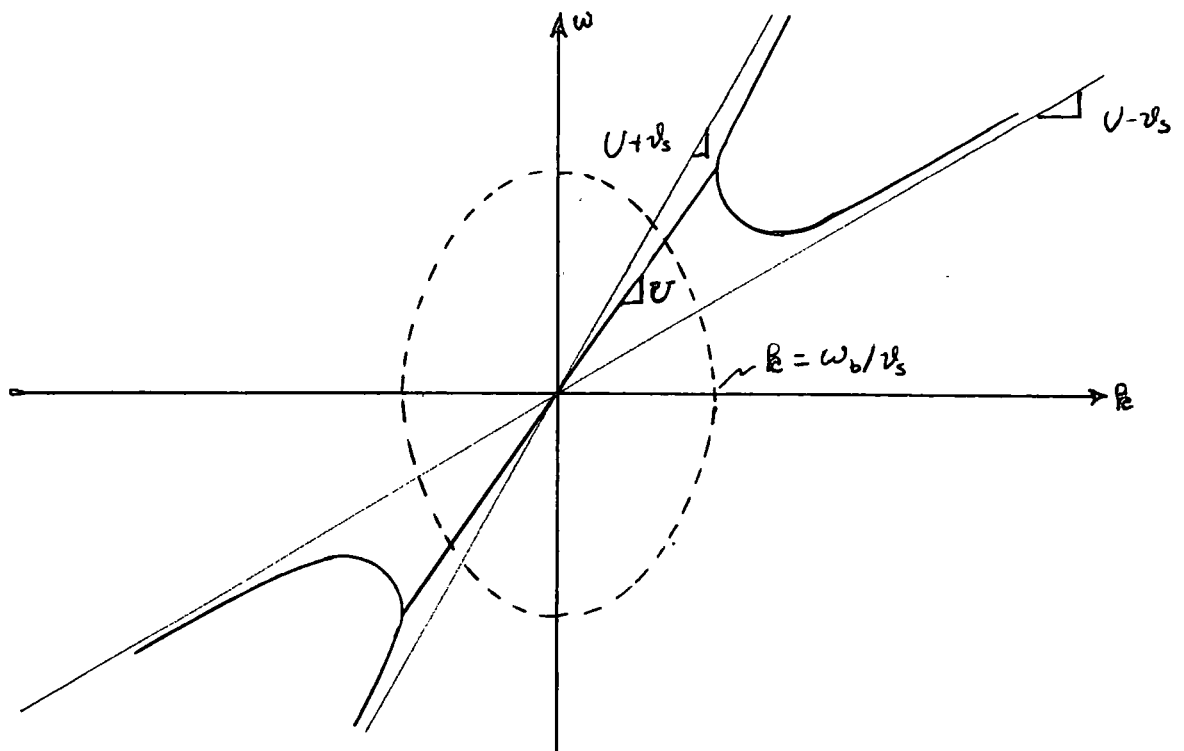
Thus for real ω , the sketch is

DYNAMICS OF ELECTROMECHANICAL CONTINUA

PROBLEM 10.27 (continued)



while for real k , the ω - k plot is



PROBLEM 10.27 (continued)

Part c

Since the wire is traveling at a "supersonic" velocity, we cannot impose a downstream boundary condition to determine upstream behavior.

We are given

$$u(0,t)\bar{i}_x + v(0,t)\bar{i}_y = u_o \cos \omega_o t \bar{i}_x + v_o \sin \omega_o t \bar{i}_y \quad (h)$$

and the boundary conditions

$$\frac{\partial u}{\partial z}(0,t) = 0; \frac{\partial v}{\partial z}(0,t) = 0 \quad (i)$$

We let

$$\alpha = \frac{\omega U}{U^2 - v_s^2}; \beta = \sqrt{\frac{\omega^2 v_s^2 + \omega_b^2 (U^2 - v_s^2)}{(U^2 - v_s^2)}} \quad (j)$$

$$\gamma = \sqrt{\frac{\omega^2 v_s^2 - \omega_b^2 (U^2 - v_s^2)}{(U^2 - v_s^2)}}$$

For the x motions, the allowed values of k are

$$k_1 = \alpha + \beta \quad \text{with } \omega = \omega_o$$

$$k_2 = \alpha - \beta \quad (k)$$

Therefore

$$u = \text{Re} \left\{ \left[A_1 e^{-jk_1 z} + A_2 e^{-jk_2 z} \right] e^{j\omega_o t} \right\} \quad (l)$$

Applying the boundary conditions and simplifying, we obtain

$$u = u_o \text{Re} \left[\left(\frac{j\alpha}{\beta} \sin \beta z + \cos \beta z \right) e^{j(\omega t - \alpha z)} \right] \quad (m)$$

For the y motions, the allowed values of k are

$$k_3 = \alpha + \gamma$$

$$k_4 = \alpha - \gamma \quad (n)$$

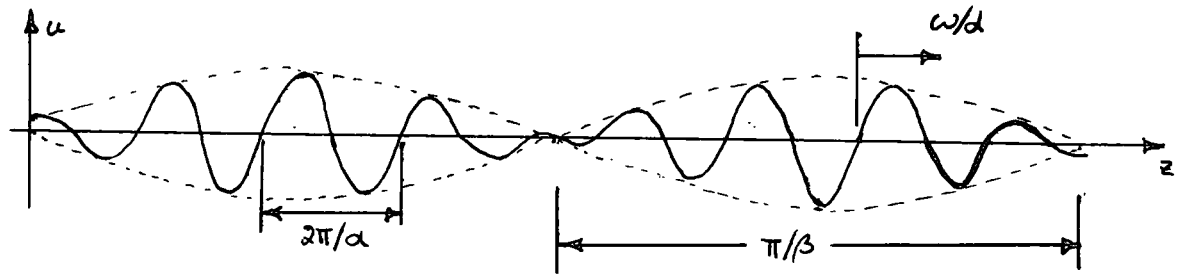
Therefore

$$v = -v_o \text{Re} \left[\left(-\frac{\alpha}{\gamma} \sin \gamma z + j \cos \gamma z \right) e^{j(\omega t - \alpha z)} \right] \quad (o)$$

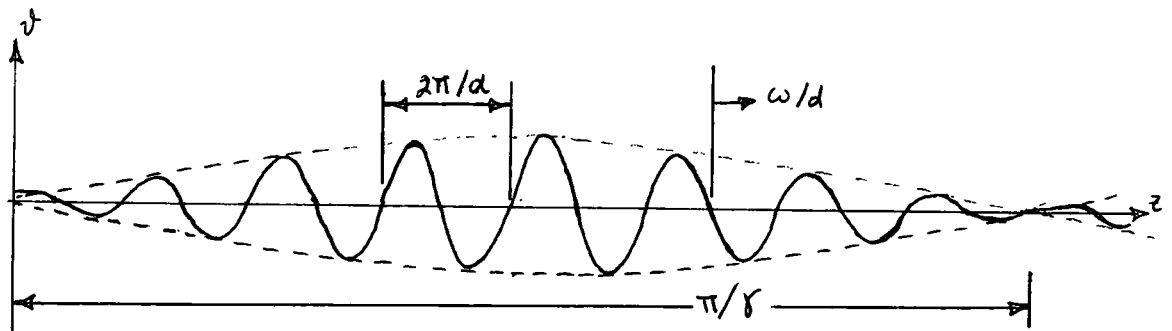
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PROBLEM 10.27 (continued)

Part d



As long as $U > v_s$ this is the form of u , no matter ^{what} the value of I (as long as $I > 0$). As the magnitude of I increases, β increases but α remains unchanged.

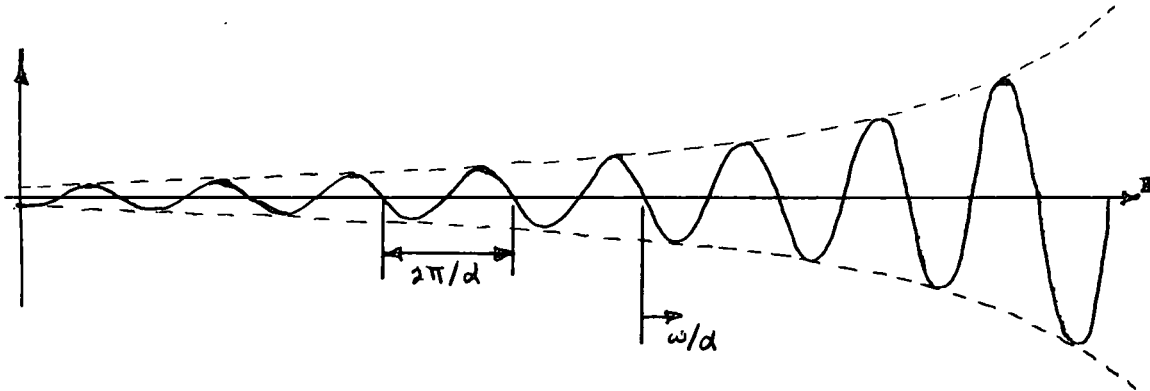


PROBLEM 10.27 (continued)

This is the form of v , as long as

$$\omega^2 v_s^2 - \omega_b^2 (U^2 - v_s^2) > 0 \quad (p)$$

As I is increased, we reach a value whereby this inequality no longer holds. At this point γ becomes imaginary and we have spatial growth.



As I is increased beyond the critical value, v will begin to grow exponentially with z .

Part e

To simulate the moving wire, we could use a moving stream of a conducting liquid such as mercury. We would introduce current onto the stream at the nozzle and complete the circuit by having the stream strike a metal plate at some downstream position.

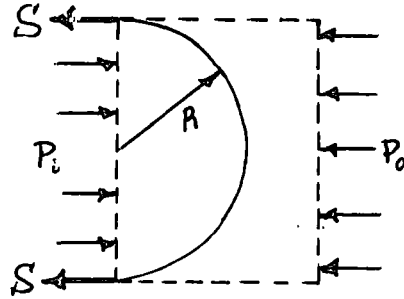
PROBLEM 10.28

Part a

A simple static argument establishes the required pressure difference. The pressure, as a mechanical stress that occurs in a fluid, always acts on a surface in the normal direction. The figure shows a section of length Δz from the membrane. Since the volume which encloses this section must be in force equilibrium, we can write

PROBLEM 10.28 (continued)

$$2R(\Delta z)[p_i - p_o] = 2S(\Delta z) \quad (a)$$



where we have summed the forces acting on the surfaces. It follows that the required pressure difference is

$$p_i - p_o = \frac{S}{R} \quad (b)$$

Part b

To answer this question, and other questions concerning the dynamics of the circular membrane, we must include in our description a perturbation displacement from the equilibrium at $r = R$. Hence, we define the membrane surface by the relation

$$r = R + \xi(\theta, z, t) \quad (c)$$

The pressure difference $p_i - p_o$ is a force per unit area acting on the membrane in the normal direction. It is the surface force density necessary to counteract a mechanical force per unit area T_m

$$T_m = -\frac{S}{R} \quad (d)$$

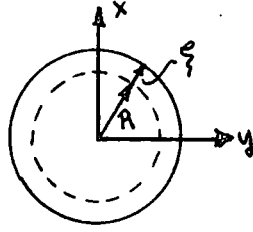
which acts on each section of the membrane in the radial direction. We wish now to determine the mechanical force acting on each section, when the surface is perturbed to the position given by (c). We can do this in steps. First, consider the case where ξ is independent of θ and z , as shown in the figure. Then from (d)

$$T_m = -\frac{S}{R + \xi} = -S\left[\frac{1}{R} - \frac{\xi}{R^2}\right] \quad (e)$$

where we have kept only the linear term in the expansion of T_m about $r = R$.

When the perturbation ξ depends on θ , the surface has a tilt, as shown. We can sum the components to S acting on the section in the radial direction as

PROBLEM 10.28 (continued)



$$\lim_{\Delta\theta \rightarrow 0} \frac{S}{R\Delta\theta} \left[\left. \frac{1}{R} \frac{\partial \xi}{\partial \theta} \right|_{\theta + \frac{\Delta\theta}{2}} - \left. \frac{1}{R} \frac{\partial \xi}{\partial \theta} \right|_{\theta - \frac{\Delta\theta}{2}} \right] = \frac{S}{R} \frac{\partial^2 \xi}{\partial \theta^2} \quad (f)$$

Similarly, a dependence on z gives rise to a radial force on the section due to the mechanical tension S ,

$$\lim_{\Delta z \rightarrow 0} \frac{S}{\Delta z} \left[\left. \frac{\partial \xi}{\partial z} \right|_{z + \frac{\Delta z}{2}} - \left. \frac{\partial \xi}{\partial z} \right|_{z - \frac{\Delta z}{2}} \right] = S \frac{\partial^2 \xi}{\partial z^2} \quad (g)$$

In general, the force per unit area exerted on a small section of membrane under the constant tension S from the adjacent material is the sum of the forces given by (e), (f) and (g),

$$T_m = S \left(-\frac{1}{R} + \frac{\xi}{R^2} + \frac{1}{R} \frac{\partial^2 \xi}{\partial \theta^2} + \frac{\partial^2 \xi}{\partial z^2} \right) \quad (h)$$

It is now possible to write the dynamic force equation for radial motions. In addition to the pressure difference $p_i - p_o$ acting in the radial direction, we will include the inertial force density $\sigma_m / (\partial^2 \xi / \partial t^2)$ and a surface force density T_r due to electric or magnetic fields. Hence,

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \left(-\frac{1}{R} + \frac{\xi}{R^2} + \frac{1}{R} \frac{\partial^2 \xi}{\partial \theta^2} + \frac{\partial^2 \xi}{\partial z^2} \right) + T_r + p_i - p_o \quad (i)$$

Consider now the case where there is no electromechanical interaction.

Then $T_r = 0$, and static equilibrium requires that (b) hold. Hence, the constant terms in (i) cancel, leaving the perturbation equation

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \left(\frac{\xi}{R^2} + \frac{1}{R} \frac{\partial^2 \xi}{\partial \theta^2} + \frac{\partial^2 \xi}{\partial z^2} \right) \quad (j)$$

PROBLEM 10.28 (continued)

Parts c & d

This equation is formally the same as those that we have encountered previously (see Sec. 10.1.3). However, the cylindrical geometry imposes additional requirements on the solutions. That is, if we assume solutions having the form,

$$\xi = \text{Re } \hat{\xi}(z) e^{j(\omega t + m\theta)} \quad (k)$$

the assumed dependence on θ is a linear combination of $\sin m\theta$ and $\cos m\theta$.

If the displacement is to be single valued, m must have integer values. Otherwise we would not have $\xi(\theta, z, t) = \xi(\theta + 2\pi, z, t)$.

With the assumed dependence on θ and t , (j) becomes,

$$\frac{d^2 \hat{\xi}}{dz^2} + k^2 \hat{\xi} = 0 \quad (l)$$

where

$$k^2 = \frac{1}{R^2} (1 - m^2) + \frac{\omega^2 \sigma_m}{S}$$

The membrane is attached to the rigid tubes at $z = 0$ and $z = \ell$. The solution to (l) which satisfies this condition is

$$\hat{\xi} = A \sin k_n x \quad (m)$$

where

$$k_n = \frac{n\pi}{\ell}, \quad n = 1, 2, 3, \dots$$

The eigenvalue k_n determines the eigenfrequency, because of (l).

$$\omega_n^2 = \left[\left(\frac{n\pi}{\ell} \right)^2 + \frac{(m^2 - 1)}{R^2} \right] \frac{S}{\sigma_m} \quad (n)$$

To obtain a picture of how these modes appear, consider the case where A is real, and (m) and (k) become

$$\xi(\theta, z, t) = A \sin \frac{n\pi}{\ell} x \cos m\theta \cos \omega t \quad (o)$$

The instantaneous displacements for the first four modes are shown in the figure.

There is the possibility that the $m = 0$ mode is unstable, as can be seen from (n), where if

$$\left(\frac{n\pi}{\ell} \right)^2 < \frac{1}{R^2} \quad (p)$$

we find that the time dependence has the form $\exp \pm |\omega|t$. The first mode to meet this condition for instability is the $n = 1$ mode. Hence, it is not possible

DYNAMICS OF ELECTROMECHANICAL CONTINUA

PROBLEM 10.28 (continued)

$$m=0$$

$$n=1$$

$$m=1$$

$$n=1$$

$$m=2$$

$$n=1$$

$$m=0$$

$$n=2$$

PROBLEM 10.28 (continued)

to maintain the uniform cylindrical shape of the static equilibrium if

$$R\pi/\ell < 1 \quad (q)$$

This condition for static instability is easily understood if we remember that in the $m = 0$, $n = 1$ mode, there are two perturbation surface forces on a small section of the membrane surface. One of these is the perturbation part of (e) and arises because of the curvature in static equilibrium. This force acts in the same direction as the displacement, and hence tends to produce static instability. It is counteracted by a restoring force proportional to the second derivative in the z direction, as given by (g). Condition (q) is satisfied when the effect of the initial curvature predominates the stiffness from the boundaries.

Part e

With rotation, the dispersion becomes:

$$(\omega - m\Omega)^2 = \Omega_s^2 [m^2 - 1 - m_c^2]$$

with

$$\Omega_s^2 = \frac{S}{\sigma_m R^2}$$

$$m_c^2 = \frac{2\epsilon_o V_o^2}{a^3} \frac{R^2}{S}$$

Because there is no z dependence (no surface curvature in the z direction) the equilibrium is unstable in the $m = 0$ mode even in the absence of an applied voltage.

PROBLEM 10.29

The solution is of the form

$$\xi = \xi_+(\alpha) + \xi_-(\beta) \quad (a)$$

where

$$\alpha = x - y$$

$$\beta = x + y$$

We are given that at $x = 0$

$$\frac{\partial \xi}{\partial x} = \frac{d\xi_+}{d\alpha} + \frac{d\xi_-}{d\beta} = \Delta[u_{-1}(y) - u_{-1}(y-a)] \quad (b)$$

and that

PROBLEM 10.29 (continued)

$$\xi = 0$$

which implies that

$$\frac{\partial \xi}{\partial y} = 0 = -\frac{d\xi_+}{d\alpha} + \frac{d\xi_-}{d\beta} \quad (c)$$

We therefore have

$$\frac{d\xi_+}{d\alpha} = \frac{\Delta}{2} [u_{-1}(-\alpha) - u_{-1}(-\alpha-a)]$$

and

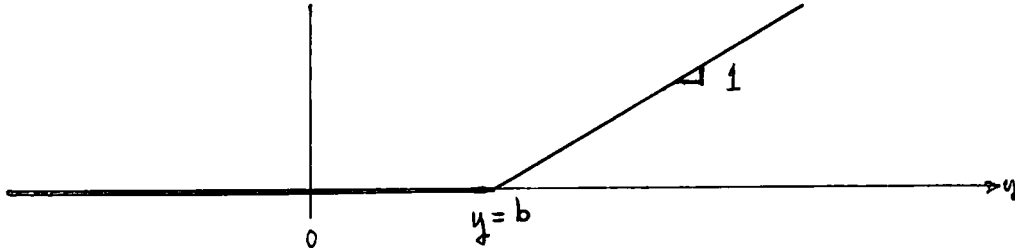
$$\frac{d\xi_-}{d\beta} = \frac{\Delta}{2} [u_{-1}(\beta) - u_{-1}(\beta-a)] \quad (e)$$

$$\text{Then } \frac{\partial \xi}{\partial y} = -\frac{d\xi_+}{d\alpha} + \frac{d\xi_-}{d\beta} = \frac{\Delta}{2} \{u_{-1}(y-x) - u_{-1}(y-x-a) + u_{-1}(x+y) - u_{-1}(x+y-a)\} \quad (f)$$

Integrating with respect to y , we obtain

$$\xi = \frac{\Delta}{2} \{-u_{-2}(y-x) + u_{-2}(y-x-a) + u_{-2}(y+x) - u_{-2}(y+x-a)\} \quad (g)$$

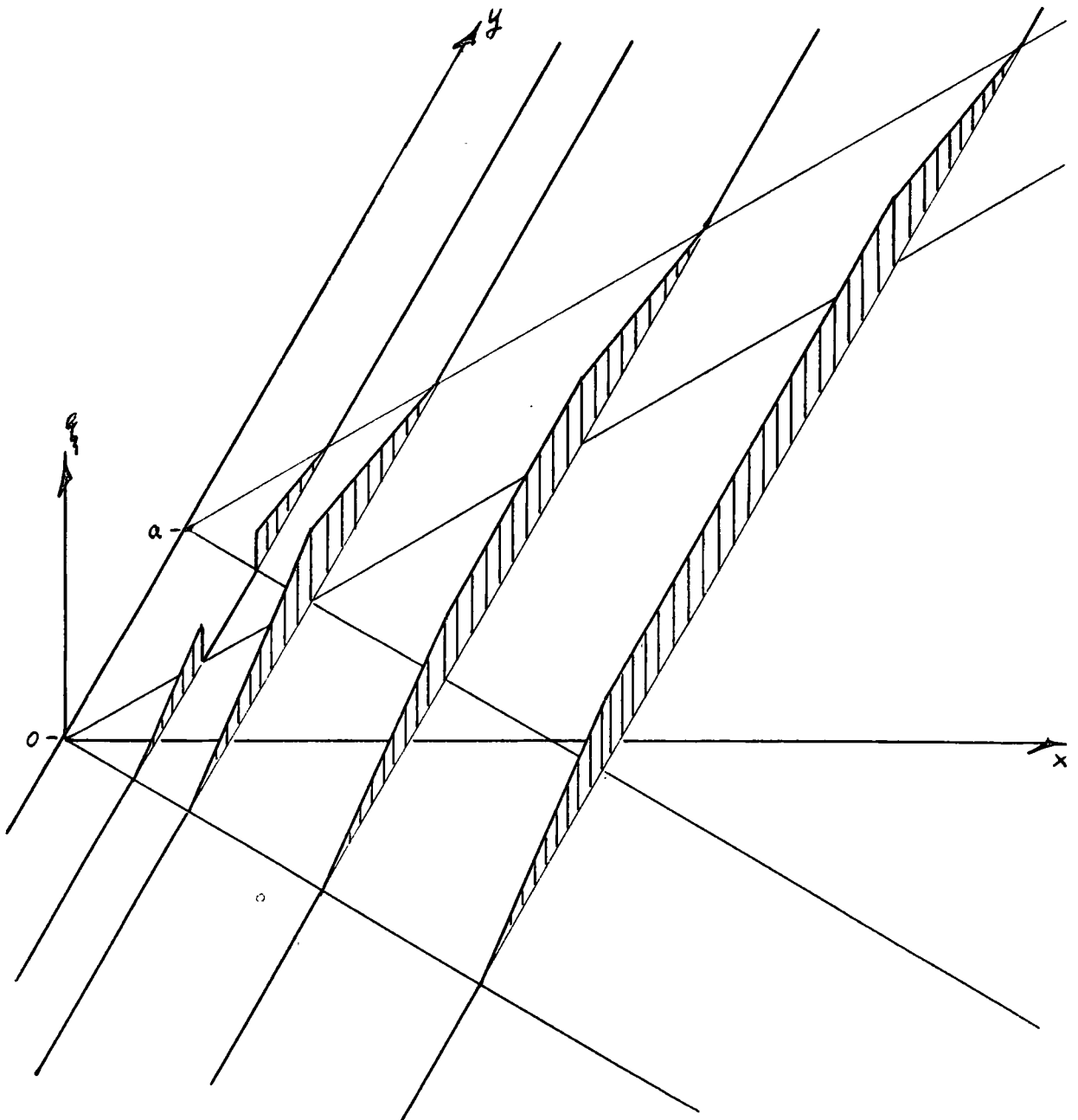
where u_{-2} is a ramp function; that is $u_{-2}(y-b)$ is defined as



Part b

The constraint represented by Fig. 10P.29 could be obtained by ejecting the membrane from a slit at $x = 0$ that is planar, but tilted over the range $0 < y < a$. Thus, the membrane would have no deflection ξ at $x = 0$, but would have the required constant slope Δ over the range $0 < y < a$, and zero slope elsewhere.

PROBLEM 10.29 (continued)



PROBLEM 10.30

For this situation, the governing equation is (10.4.15) of the text.

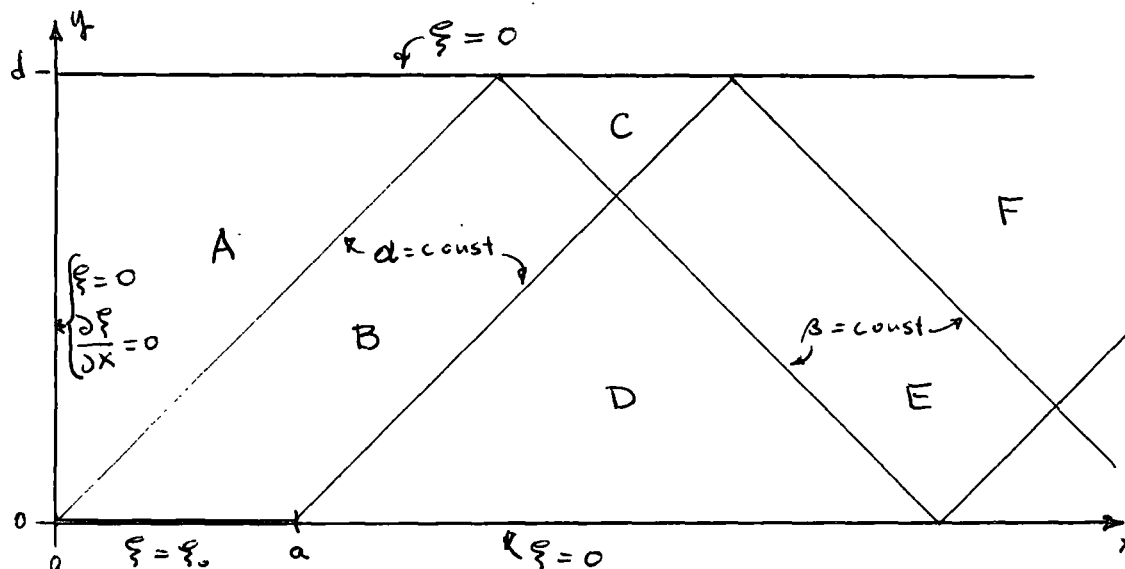
$$(M^2-1) \frac{\partial^2 \xi}{\partial x^2} = \frac{\partial^2 \xi}{\partial v^2} \quad (a)$$

Here $M^2 = 2$; so we have the equation:

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{\partial^2 \xi}{\partial v^2} \quad (b)$$

The characteristics are determined from equations (10.4.17) and (10.4.18) to be:

$$\begin{aligned}\alpha &= \mathbf{x} - \mathbf{y} \\ \beta &= \mathbf{x} + \mathbf{y}\end{aligned}\tag{c}$$



The x-y plane divides into regions A...F, as shown in the sketch. Tracing back on the characteristics from points in regions A, D, F... shows that in these regions $\xi = 0$; the characteristics originate on "zero" boundary conditions. At points in region B, only the C^+ characteristic originates on finite data; $\xi_+(\alpha) = \xi_0$, $\xi_-(\beta) = 0$ and hence

$$\xi = \xi_0 \text{ in region B}$$

PROBLEM 10.30 (continued)

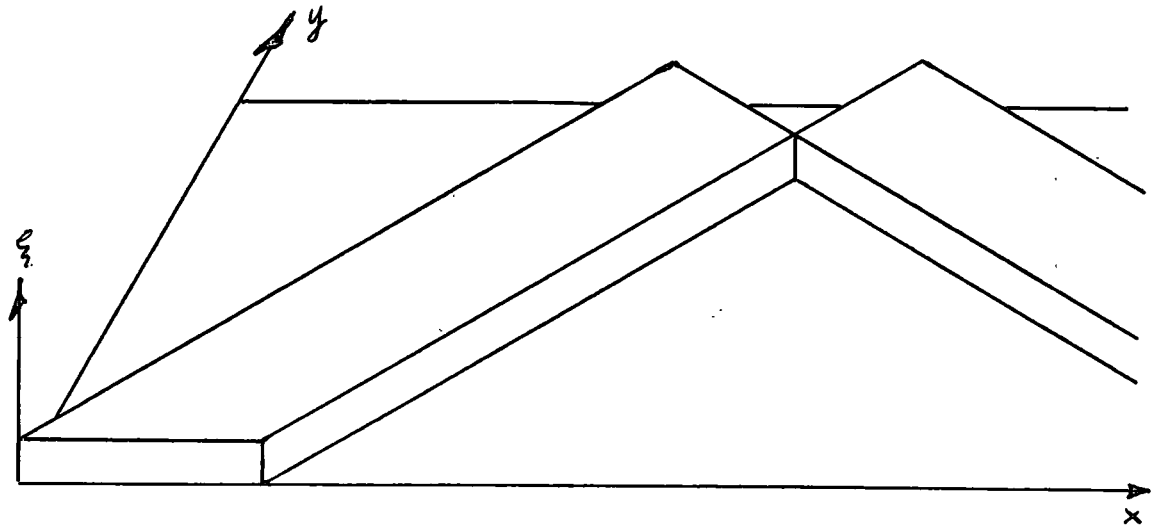
In C, deflections are determined by waves, both originating from the initial data. Hence $\xi_+(\alpha) = \xi_0$, but $\xi_-(\beta)$ is determined by the reflection of an incident wave on the boundary at $y = d$. Hence $\xi_-(\beta) = -\xi_0$ and

$$\xi = 0 \text{ in region C}$$

In region E only the $\xi_-(\beta)$ wave is finite because the $\xi_+(\alpha)$ wave originates on zero conditions and

$$\xi = -\xi_0 \text{ in region E}$$

The deflection has the stationary appearance shown in the figure.



PROBLEM 10.31

From equation 10.4.30, we have

$$\omega^2 = k^2 v_s^2 + \frac{k B_o I}{m} \quad (a)$$

We define

$$\alpha = + \frac{I B_o}{2 m v_s^2} \quad (b)$$

and

$$\beta = \sqrt{\left(\frac{B_o I}{2 m v_s^2} \right)^2 + \frac{\omega^2}{v_s^2}} \quad (c)$$

DYNAMICS OF ELECTROMECHANICAL CONTINUA

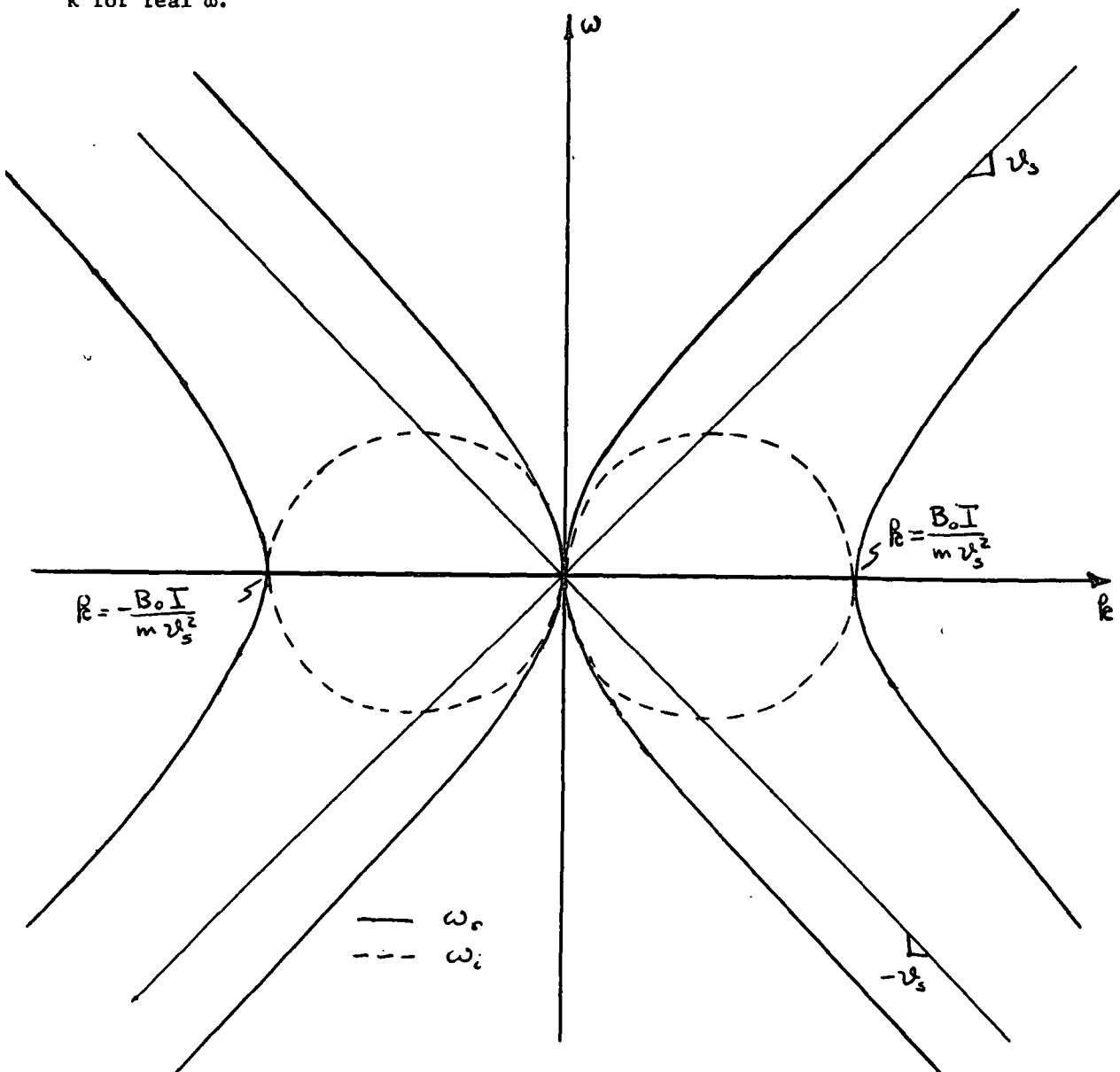
PROBLEM 10.31 (continued)

The four allowed branches of k as a function of ω are therefore $k = \pm k_1$, and $\pm k_2$, where

$$k_1 = \alpha + \beta \quad (d)$$

$$k_2 = -\alpha + \beta \quad (e)$$

The sketch shows complex ω for real k . Note however, that only real values of k are given if ω is real and hence the solid lines represent the plot of complex k for real ω .



PROBLEM 10.32

The effect of the longitudinal convection is accounted for by replacing ω in Eq. 10.4.3 by $\omega - kU$ (see for example page 594). Thus,

$$(\omega - kU)^2 = k^2 v_s^2 + \frac{k B_o I}{m} \quad (a)$$

This expression can be solved to give

$$k = \frac{(2\omega U + \frac{B_o I}{m}) \pm \sqrt{4v_s^2 \omega^2 + 4\omega U \left(\pm \frac{B_o I}{m} \right) + \left(\frac{B_o I}{m} \right)^2}}{2(U^2 - v_s^2)} \quad (b)$$

The sketch of complex k for real ω is made with the help of the following observations: Consider the modes that are represented by $-B_o$.

- 1) Asymptotes for branches are $k = \omega / (U \mp v_s)$ as $\omega \rightarrow \infty$.
- 2) As ω is lowered, the $(-B_o)$ branches become complex as

$$4v_s^2 \omega^2 + 4\omega U \left(\frac{-B_o I}{m} \right) + \left(\frac{B_o I}{m} \right)^2 = 0$$

or at the frequencies

$$\omega = \frac{B_o I}{2v_s^2 m} \left(U \pm \sqrt{U^2 - v_s^2} \right)$$

Thus, for $U > v_s$ there is a lower as well as an upper positive frequency at which k switches from real to complex values.

In this range of complex k , real k is

$$k = (2\omega U - \frac{B_o I}{m}) / 2(U^2 - v_s^2)$$

or a straight line intercepting the $k = 0$ axis at

$$\omega = \frac{B_o I}{2Um}$$

- 3) as $\omega \rightarrow 0$,

$$k \rightarrow 0 \text{ and } k \rightarrow \pm B_o I / m(U^2 - v_s^2)$$

where the $-$ sign goes with the unstable branches.

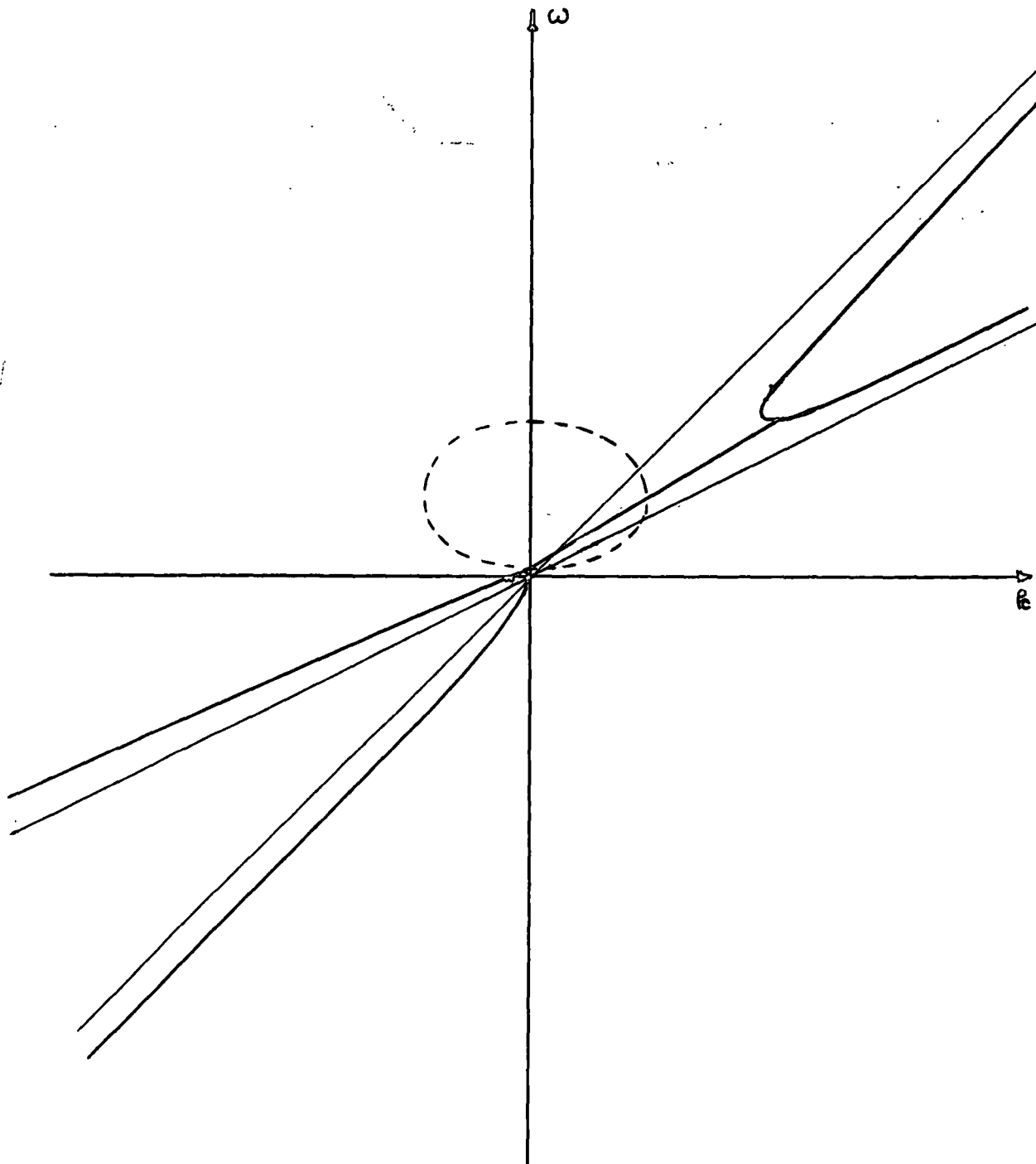
- 4) As $\omega \rightarrow -\infty$ the values of k are real and approach the asymptotes

$$k = \omega / (U \mp v_s).$$

DYNAMICS OF ELECTROMECHANICAL CONTINUA

PROBLEM 10.32 (continued)

Similar reasoning gives the modes represented in (b) by $+B_0$. Note that these modes have a plot obtained by replacing $\omega \rightarrow -\omega$ and $k \rightarrow -k$ in the figure.

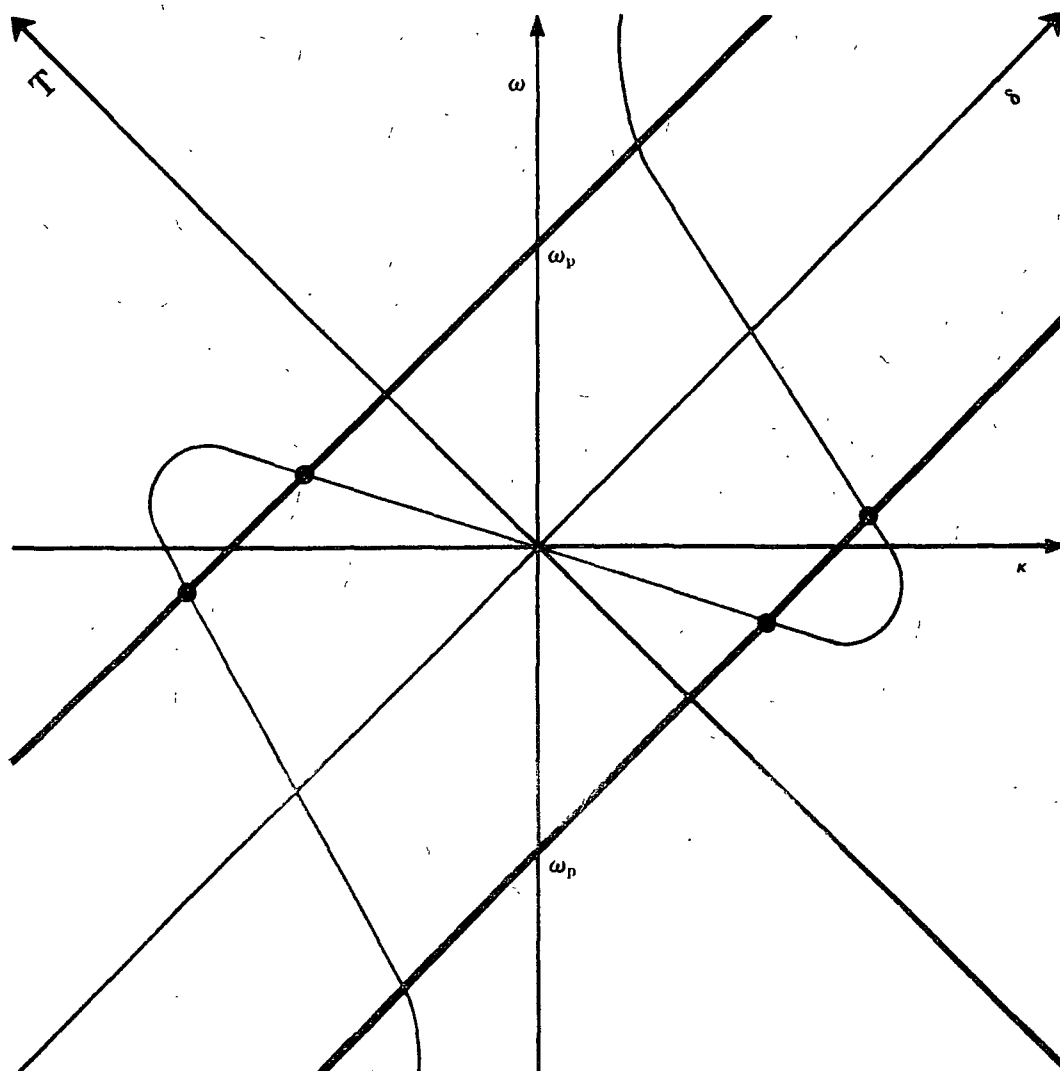


SOLUTIONS MANUAL FOR

ELECTROMECHANICAL DYNAMICS

PART III: Elastic and Fluid Media

HERBERT H. WOODSON JAMES R. MELCHER



Prepared by MARKUS ZAHN

JOHN WILEY & SONS, INC. NEW YORK • LONDON • SYDNEY • TORONTO

Mark Zahn

SOLUTIONS MANUAL FOR

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DYNAMICS

Part III: Elastic and Fluid Media

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PREFACE TO: SOLUTIONS MANUAL TO

ELECTROMECHANICAL DYNAMICS, PART III:

ELASTIC AND FLUID MEDIA

This manual presents, in an informal format, solutions to the problems found at the ends of chapters in Part III of the book, Electromechanical Dynamics. It is intended as an aid for instructors, and in special circumstances, for use by students. A sufficient amount of explanatory material is included such that the solutions, together with problem statements, are in themselves a teaching aid. They are substantially as found in the records for the undergraduate and graduate courses 6.06, 6.526, and 6.527, as taught at Massachusetts Institute of Technology over a period of several years.

It is difficult to give proper credit to all of those who contributed to these solutions, because the individuals involved range over teaching assistants, instructors, and faculty who have taught the material over a period of more than four years. However, special thanks are due the authors, Professor J. R. Melcher and Professor H. H. Woodson, who gave me the opportunity and incentive to write this manual. This work has greatly increased the value of my graduate education, in addition to giving me the pleasure of working with these two men.

The manuscript was typed by Mrs. Evelyn M. Holmes, whom I especially thank for her sense of humor, advice, patience and expertise which has made this work possible.

Of most value during the course of this work was the understanding of my girl friend, then fiancée, and now my wife, Linda, in spite of the competition for time.

Markus Zahn

Cambridge, Massachusetts
October, 1969

PROBLEM 11.1
Part a

We add up all the volume force densities on the elastic material, and with the help of equation 11.1.4, we write Newton's law as

$$\rho \frac{\partial^2 \delta_1}{\partial t^2} = \frac{\partial T_{11}}{\partial x_1} - \rho g \quad (a)$$

where we have taken $\frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_3} = 0$. Since this is a static problem, we let $\frac{\partial}{\partial t} = 0$. Thus,

$$\frac{\partial T_{11}}{\partial x_1} = \rho g. \quad (b)$$

From 11.2.32, we obtain

$$T_{11} = (2G + \lambda) \frac{\partial \delta_1}{\partial x_1} \quad (c)$$

Therefore

$$(2G + \lambda) \frac{\partial^2 \delta_1}{\partial x_1^2} = \rho g \quad (d)$$

Solving for δ_1 , we obtain

$$\delta_1 = \frac{\rho g}{2(2G + \lambda)} x_1^2 + C_1 x + C_2 \quad (e)$$

where C_1 and C_2 are arbitrary constants of integration, which can be evaluated by the boundary conditions

$$\delta_1(0) = 0 \quad (f)$$

and

$$T_{11}(L) = (2G + \lambda) \frac{\partial \delta_1}{\partial x_1}(L) = 0 \quad (g)$$

since $x_1 = L$ is a free surface. Therefore, the solution is

$$\delta_1 = \frac{\rho g x_1}{2(2G + \lambda)} [x_1 - 2L]. \quad (f)$$

Part b

Again applying 11.2.32

INTRODUCTION TO THE ELECTROMECHANICS OF ELASTIC MEDIA

PROBLEM 11.1 (Continued)

$$\begin{aligned}
 T_{11} &= (2G+\lambda) \frac{\partial \delta_1}{\partial x_1} = \rho g [x_1 - L] \\
 T_{12} &= T_{21} = 0 \\
 T_{13} &= T_{31} = 0 \\
 T_{22} &= \lambda \frac{\partial \delta_1}{\partial x_1} = \frac{\lambda \rho g}{(2G+\lambda)} [x_1 - L] \\
 T_{33} &= \lambda \frac{\partial \delta_1}{\partial x_1} = \frac{\lambda \rho g}{(2G+\lambda)} [x_1 - L] \\
 T_{32} &= T_{23} = 0
 \end{aligned} \tag{g}$$

$$\bar{\bar{T}} = \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix} \tag{h}$$

PROBLEM 11.2

Since the electric force only acts on the surface at $x_1 = -L$, the equation of motion for the elastic material ($-L \leq x_1 \leq 0$) is from Eqs. (11.1.4) and (11.2.32),

$$\rho \frac{\partial^2 \delta_1}{\partial t^2} = (2G+\lambda) \frac{\partial^2 \delta_1}{\partial x_1^2} \tag{a}$$

The boundary conditions are

$$\delta_1(0, t) = 0$$

and

$$M \frac{\partial^2 \delta_1(-L, t)}{\partial t^2} = aD(2G+\lambda) \frac{\partial \delta_1}{\partial x_1}(-L, t) + f^e \tag{b}$$

f^e is the electric force in the x_1 direction at $x_1 = -L$, and may be found by using the Maxwell Stress Tensor $T_{ij} = \epsilon E_i E_j - \frac{1}{2} \delta_{ij} \epsilon E_k E_k$ to be (see Appendix G for discussion of stress tensor),

$$f^e = - \frac{\epsilon}{2} E^2 aD$$

with

$$E = \frac{V_0 + V_1 \cos \omega t}{d + \delta_1(-L, t)} \tag{c}$$

PROBLEM 11.2 (continued)

Expanding f^e to linear terms only, we obtain

$$f^e = -\frac{\epsilon a D}{2} \left[\frac{V_o^2}{d^2} + \frac{2V_o V_1 \cos \omega t}{d^2} - \frac{2V_o^2}{d^3} \delta_1(-L, t) \right] \quad (d)$$

We have neglected all second order products of small quantities.

Because of the constant bias V_o , and the sinusoidal nature of the perturbations, we assume solutions of the form

$$\delta_1(x_1, t) = \delta_1(x_1) + \text{Re } \hat{\delta} e^{j(\omega t - kx_1)} \quad (e)$$

where

$$\hat{\delta} \ll \delta_1(x_1) \ll L$$

The relationship between ω and k is readily found by substituting (e) into (a), from which we obtain

$$k = \pm \frac{\omega}{v_p} \text{ with } v_p = \sqrt{\frac{2G+\lambda}{\rho}} \quad (f)$$

We first solve for the equilibrium configuration which is time independent.

Thus

$$\frac{\partial^2 \delta_1(x_1)}{\partial x_1^2} = 0 \quad (g)$$

This implies

$$\delta_1(x_1) = C_1 x_1 + C_2$$

Because $\delta_1(0) = 0$, $C_2 = 0$.

From the boundary condition at $x_1 = -L$ ((b) & (d))

$$aD(2G+\lambda)C_1 - \frac{\epsilon}{2} \frac{V_o^2}{d^2} aD = 0 \quad (h)$$

Therefore

$$\delta_1(x_1) = + \frac{\epsilon}{2} \frac{V_o^2}{d^2(2G+\lambda)} x_1 \quad (i)$$

Note that $\delta_1(x_1 = -L)$ is negative, as it should be.

For the time varying part of the solution, using (f) and the boundary condition

$$\delta(0, t) = 0$$

PROBLEM 11.2 (continued)

we can let the perturbation δ_1 be of the form

$$\delta_1(x_1, t) = \text{Re } \hat{\delta} \sin kx_1 e^{j\omega t} \quad (j)$$

Substituting this assumed solution into (b) and using (d), we obtain

$$+ M\omega^2 \hat{\delta} \sin kL = aD(2G+\lambda)k \hat{\delta} \cos kL - \frac{\epsilon a D V_o V_1}{d^2} - \frac{\epsilon a D V_o^2}{d^3} \hat{\delta} \sin kL \quad (k)$$

Solving for $\hat{\delta}$, we have

$$\hat{\delta} = - \frac{\epsilon a D V_o V_1}{d^2 \left[M\omega^2 \sin kL - aD(2G+\lambda)k \cos kL + \frac{\epsilon a D V_o^2}{d^3} \sin kL \right]}$$

Thus, because $\hat{\delta}$ has been shown to be real,

$$\delta_1(-L, t) = - \frac{\epsilon}{2} \frac{V_o^2 L}{d^2(2G+\lambda)} - \hat{\delta} \sin kL \cos \omega t \quad (m)$$

Part b

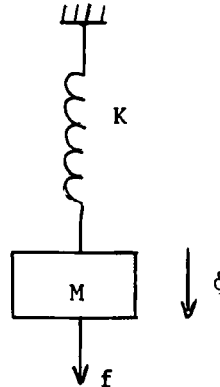
If $kL \ll 1$, we can approximate the sinusoidal part of (m) as

$$\delta_1(-L, t) = \frac{\epsilon a D V_o V_1 \cos \omega t}{d^2 \left[M\omega^2 - \frac{aD(2G+\lambda)}{L} + \frac{\epsilon a D V_o^2}{d^3} \right]} \quad (n)$$

We recognize this as a force-displacement relation for a mass on the end of a spring.

Part c

We thus can model (n) as



PROBLEM 11.2 (Continued)

where

$$f = - \frac{\epsilon a D V_o V_1 \cos \omega t}{d^2}$$

and

$$K = \frac{a D (2G + \lambda)}{L} - \frac{\epsilon a D V_o^2}{d^3}$$

We see that the electrical force acts like a negative spring constant.

PROBLEM 11.3

Part a

From (11.1.4), we have the equation of motion in the x_2 direction as

$$\rho \frac{\partial^2 \delta_2}{\partial t^2} = \frac{\partial T_{21}}{\partial x_1} \quad (a)$$

From (11.2.32),

$$T_{21} = G \left[\frac{\partial \delta_2}{\partial x_1} \right] \quad (b)$$

Therefore, substituting (b) into (a), we obtain an equation for δ_2

$$\rho \frac{\partial^2 \delta_2}{\partial t^2} = G \frac{\partial^2 \delta_2}{\partial x_1^2} \quad (c)$$

We assume solutions of the form

$$\delta_2 = \text{Re } \hat{\delta}_2 e^{j(\omega t - kx_1)} \quad (d)$$

where from (c) we obtain

$$k = \pm \frac{\omega}{v_p} \quad v_p^2 = \frac{G}{\rho}$$

Thus we let

$$\delta_2 = \text{Re } \left[\delta_a e^{j(\omega t - kx_1)} + \delta_b e^{j(\omega t + kx_1)} \right] \quad (e)$$

$$\text{with } k = \frac{\omega}{v_p}$$

The boundary conditions are

$$\delta_2(l, t) = \delta_o e^{j\omega t} \quad (f)$$

INTRODUCTION TO THE ELECTROMECHANICS OF ELASTIC MEDIA

PROBLEM 11.3 (continued)

and

$$\left. \frac{\partial \delta_2}{\partial x_1} \right|_{x_1=0} = 0 \quad (g)$$

since the surface at $x_1 = 0$ is free.

Therefore

$$\delta_a e^{-jk\ell} + \delta_b e^{jk\ell} = \delta_o \quad (h)$$

and

$$-jk \delta_a + jk \delta_b = 0 \quad (i)$$

Solving, we obtain

$$\delta_a = \delta_b = \frac{\delta_o}{2 \cos k\ell} \quad (j)$$

Therefore

$$\delta_2(x_1, t) = \text{Re} \left[\frac{\delta_o}{\cos k\ell} \cos kx_1 e^{j\omega t} \right] = \frac{\delta_o}{\cos k\ell} \cos kx_1 \cos \omega t \quad (k)$$

and

$$\begin{aligned} T_{21}(x_1, t) &= -\text{Re} \left[\frac{G\delta_o k}{\cos k\ell} \sin kx_1 e^{j\omega t} \right] \\ &= -\frac{G\delta_o k}{\cos k\ell} \sin kx_1 \cos \omega t \end{aligned} \quad (l)$$

Part b

In the limit as ω gets small

$$\delta_2(x_1, t) \rightarrow \text{Re}[\delta_o e^{j\omega t}] \quad (m)$$

In this limit, δ_2 varies everywhere in phase with the source. The slab of elastic material moves as a rigid body. Note from (l) that the force per unit area at $x_1 = \ell$ required to set the slab into motion is $T_{21}(\ell, t) = \rho \ell \frac{d^2}{dt^2}(\delta_o \cos \omega t)$ or the mass $/(x_2 - x_3)$ area times the rigid body acceleration.

Part c

The slab can resonate if we can have a finite displacement, even as $\delta_o \rightarrow 0$. This can happen if the denominator of (k) vanishes

$$\cos k\ell = 0 \quad (n)$$

or

$$\omega = \frac{(2n+1)\pi v}{2\ell} \quad n = 0, 1, 2, \dots \quad (o)$$

PROBLEM 11.3 (continued)

The lowest frequency is for $n = 0$

$$\text{or } \omega_{\text{low}} = \frac{\pi v}{2\ell} p \quad (p)$$

PROBLEM 11.4
Part a

We have that

$$\tau_i = T_{ij} n_j = \alpha \delta_{ij} n_j$$

It is given that the T_{ij} are known, thus the above equation may be written as three scalar equations $(T_{ij} - \alpha \delta_{ij}) n_j = 0$, or:

$$\begin{aligned} (T_{11} - \alpha) n_1 + T_{12} n_2 + T_{13} n_3 &= 0 \\ T_{21} n_1 + (T_{22} - \alpha) n_2 + T_{23} n_3 &= 0 \\ T_{31} n_1 + T_{32} n_2 + (T_{33} - \alpha) n_3 &= 0 \end{aligned} \quad (a)$$

Part b

The solution for these homogeneous equations requires that the determinant of the coefficients of the n_i 's equal zero.

Thus

$$\begin{aligned} (T_{11} - \alpha) [(T_{22} - \alpha)(T_{33} - \alpha) - (T_{23})^2] \\ - T_{12} [T_{12}(T_{33} - \alpha) - T_{13}T_{23}] \\ + T_{13} [T_{12}T_{23} - T_{13}(T_{22} - \alpha)] = 0 \end{aligned} \quad (b)$$

where we have used the fact that

$$T_{ij} = T_{ji}. \quad (c)$$

Since the T_{ij} are known, this equation can be solved for α .

Part c

Consider $T_{12} = T_{21} = T_0$, with all other components equal to zero. The determinant of coefficients then reduces to

$$-\alpha^3 + T_0^2 \alpha = 0 \quad (d)$$

$$\text{for which } \alpha = 0 \quad (e)$$

$$\text{or } \alpha = \pm T_0 \quad (f)$$

The $\alpha = 0$ solution indicates that with the normal in the x_3 direction, there is no normal stress. The $\alpha = \pm T_0$ solution implies that there are two surfaces where the net traction is purely normal with stresses $\pm T_0$, respectively, as

PROBLEM 11.4 (continued)

found in example 11.2.1. Note that the normal to the surface for which the shear stress is zero can be found from (a), since α is known, and it is known that $|\bar{n}| = 1$.

PROBLEM 11.5

From Eqs. 11.2.25 - 11.2.28, we have

$$e_{11} = \frac{1}{E} [T_{11} - \nu(T_{22} + T_{33})] \quad (a)$$

$$e_{22} = \frac{1}{E} [T_{22} - \nu(T_{33} + T_{11})] \quad (b)$$

$$e_{33} = \frac{1}{E} [T_{33} - \nu(T_{11} + T_{22})] \quad (c)$$

and

$$e_{ij} = \frac{T_{ij}}{2G} \quad i \neq j \quad (d)$$

These relations must still hold in a primed coordinate system, where we can use the transformations

$$T'_{ij} = a_{ik} a_{jl} T_{kl} \quad (e)$$

and

$$e'_{ij} = a_{ik} a_{jl} e_{kl} \quad (f)$$

For an example, we look at e'_{11}

$$e'_{11} = a_{1k} a_{1l} e_{kl} = \frac{1}{E} [T'_{11} - \nu(T'_{22} + T'_{33})] \quad (g)$$

This may be rewritten as

$$a_{1k} a_{1l} e_{kl} = \frac{1}{E} [(1 + \nu) a_{1k} a_{1l} T_{kl} - \nu \delta_{kl} T_{kl}] \quad (h)$$

where we have used the relation from Eq.(8.2.23), page G10 or 439.

$$a_{pr} a_{ps} = \delta_{ps} \quad (i)$$

Consider some values of k and l where $k \neq l$.

Then, from the stress-strain relation in the unprimed frame,

$$a_{1k} a_{1l} e_{kl} = a_{1k} a_{1l} \frac{T_{kl}}{2G} = \frac{a_{1k} a_{1l}}{E} (1 + \nu) T_{kl} \quad (j)$$

Thus

$$\frac{1}{2G} = \frac{1 + \nu}{E} \quad (k)$$

or $E = 2G(1 + \nu)$ which agrees with Eq. (g) of example 11.2.1.

PROBLEM 11.6

Part a

Following the analysis in Eqs. 11.4.16 - 11.4.26, the equation of motion for the bar is

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{Eb^2}{3\rho} \frac{\partial^4 \xi}{\partial x_1^4} = 0 \quad (a)$$

where ξ measures the bar displacement in the x_2 direction, T_2 in Eq. 11.4.26 = 0 as the surfaces at $x_2 = \pm b$ are free. The boundary conditions for this problem are that at $x_1 = 0$ and at $x_1 = L$

$$T_{21} = 0 \quad \text{and} \quad T_{11} = 0 \quad (b)$$

as the ends are free.

We assume solutions of the form

$$\xi = \text{Re } \hat{\xi}(x) e^{j\omega t} \quad (c)$$

As in example 11.4.4, the solutions for $\hat{\xi}(x)$ are

$$\hat{\xi}(x) = A \sin \alpha x_1 + B \cos \alpha x_1 + C \sinh \alpha x_1 + D \cosh \alpha x_1 \quad (d)$$

with

$$\alpha = \left[\omega^2 \left(\frac{3\rho}{Eb^2} \right) \right]^{1/4}$$

Now, from Eqs. 11.4.18 and 11.4.21,

$$T_{21} = \frac{(x_2^2 - b^2)E}{2} \frac{\partial^3 \xi}{\partial x_1^3} \quad (e)$$

which implies

$$\frac{\partial^3 \xi}{\partial x_1^3} = 0 \quad (f)$$

$$\text{at } x_1 = 0, x_1 = L$$

and

$$T_{11} = -x_2 E \frac{\partial^2 \xi}{\partial x_1^2} \quad (g)$$

which implies

$$\frac{\partial^2 \xi}{\partial x_1^2} = 0 \quad (h)$$

$$\text{at } x_1 = 0 \text{ and } x_1 = L$$

PROBLEM 11.6 (continued)

With these relations, the boundary conditions require that

$$\begin{aligned}
 -A & \qquad \qquad \qquad + C & = 0 \\
 -A \cos \alpha L + B \sin \alpha L + C \cosh \alpha L + D \sinh \alpha L & = 0 \\
 -B & \qquad \qquad \qquad + D & = 0 \\
 -A \sin \alpha L - B \cos \alpha L + C \sinh \alpha L + D \cosh \alpha L & = 0
 \end{aligned} \tag{i}$$

The solution to this set of homogeneous equations requires that the determinant of the coefficients of A, B, C, and D equal zero. Performing this operation, we obtain

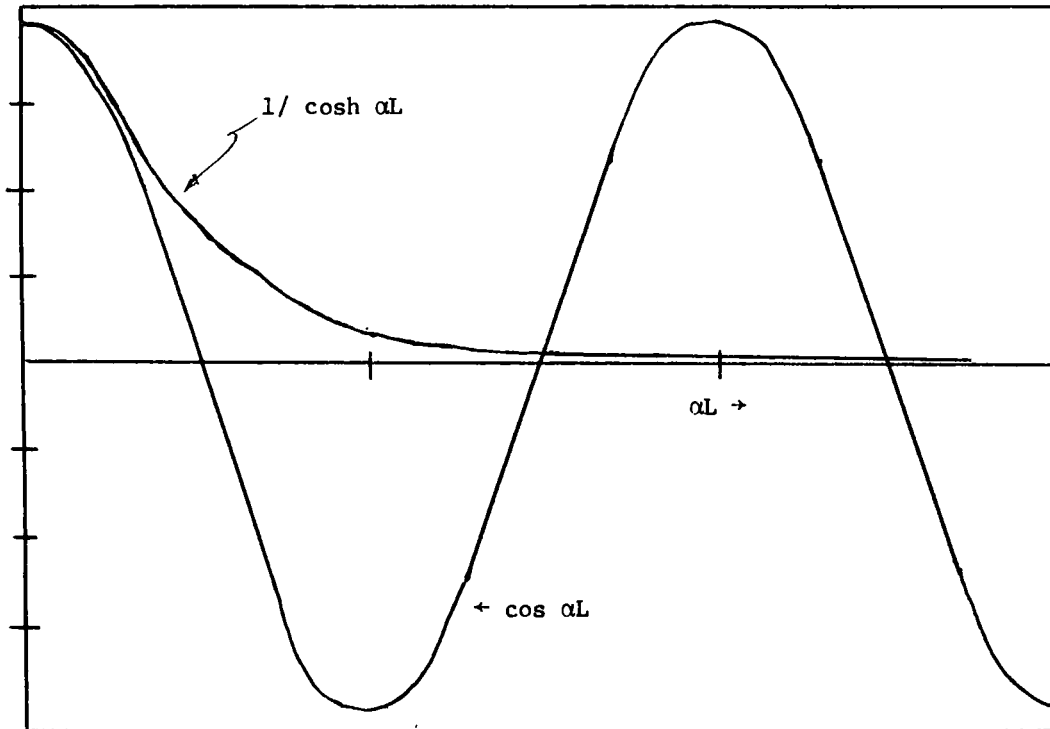
$$\cos \alpha L \cosh \alpha L = 1 \tag{j}$$

Thus,

$$\beta = \alpha L = \left[\omega^2 \left(\frac{3\rho}{Eb^2} \right) \right]^{1/4} L \tag{k}$$

Part b

The roots of $\cos \beta = \frac{1}{\cosh \beta}$ follow from the figure.



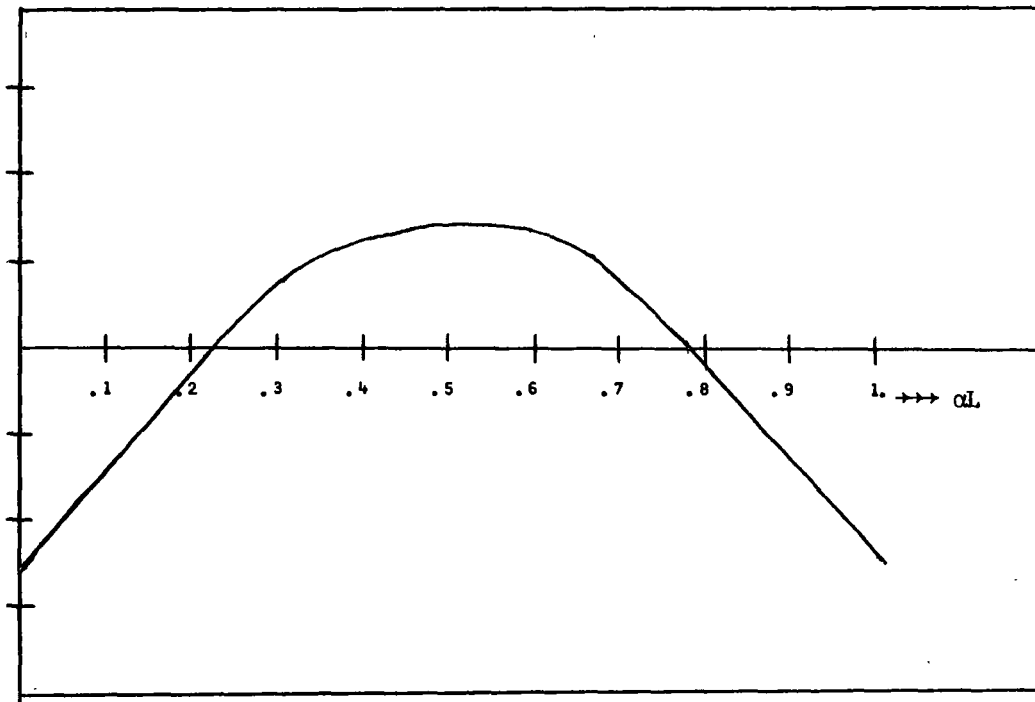
Note from the figure that the roots αL are essentially the roots $3\pi/2, 5\pi/2, \dots$ of $\cos \alpha L = 0$.

PROBLEM 11.6 (continued)Part c

It follows from (i) that the eigenfunction is

$$\begin{aligned} \hat{\xi} = A' [& (\sin \alpha x_1 + \sinh \alpha x_1)(\sin \alpha L + \sinh \alpha L) \\ & + (\cos \alpha L - \cosh \alpha L)(\cos \alpha x_1 + \cosh \alpha x_1) \end{aligned} \quad (2)$$

where A' is an arbitrary amplitude. This expression is found by taking one of the constants $A \dots D$ as known, and solving for the others. Then, (d) gives the required dependence on x_1 to within an arbitrary constant. A sketch of this function is shown in the figure.



INTRODUCTION TO THE ELECTROMECHANICS OF ELASTIC MEDIA

PROBLEM 11.7

As in problem 11.6, the equation of motion for the elastic beam is

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{Eb^2}{3\rho} \frac{\partial^4 \xi}{\partial x_1^4} = 0 \quad (a)$$

The four boundary conditions for this problem are:

$$\xi(x_1 = 0) = 0 \quad \xi(x_1 = L) = 0$$

$$\delta_1(0) = -x_2 \left. \frac{\partial \xi}{\partial x_1} \right|_{x_1=0} = 0 \quad \delta_1(L) = -x_2 \left. \frac{\partial \xi}{\partial x_1} \right|_{x_1=L} = 0 \quad (b)$$

We assume solutions of the form

$$\xi(x_1, t) = \text{Re } \hat{\xi}(x_1) e^{j\omega t}, \text{ and as in problem 11.6, the solutions for } \hat{\xi}(x_1) \text{ are} \quad (c)$$

$$\hat{\xi}(x_1) = A \sin \alpha x_1 + B \cos \alpha x_1 + C \sinh \alpha x_1 + D \cosh \alpha x_1$$

$$\text{with } \alpha = \left[\omega^2 \left(\frac{3\rho}{Eb^2} \right) \right]^{1/4} \quad (d)$$

Applying the boundary conditions, we obtain

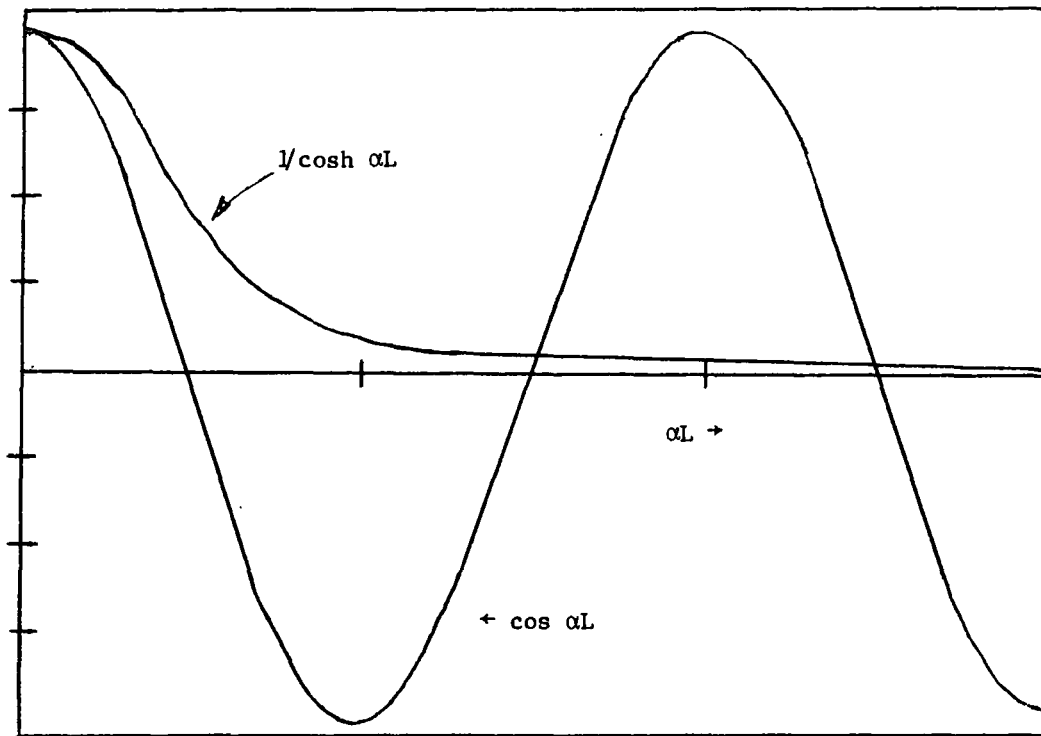
$$\begin{aligned} B + D &= 0 \\ A \sin \alpha L + B \cos \alpha L + C \sinh \alpha L + D \cosh \alpha L &= 0 \\ A + C &= 0 \\ A \cos \alpha L - B \sin \alpha L + C \cosh \alpha L + D \sinh \alpha L &= 0 \end{aligned} \quad (e)$$

The solution to this set of homogeneous equations requires that the determinant of the coefficients of A, B, C, D, equal zero. Performing this operation, we obtain

$$\cos \alpha L \cosh \alpha L = +1 \quad (f)$$

To solve for the natural frequencies, we must use a graphical procedure.

PROBLEM 11.7 (continued)



The first natural frequency is at about

$$\alpha L = \frac{3\pi}{2}$$

Thus

$$\omega^2 \left(\frac{3\rho}{Eb^2} \right) L^4 = \left(\frac{3\pi}{2} \right)^4$$

or

$$\omega = \frac{\left(\frac{3\pi}{2} \right)^2}{L^2} \left(\frac{Eb^2}{3\rho} \right)^{1/2} \quad (g)$$

Part b

We are given that $L = .5 \text{ m}$ and $b = 5 \times 10^{-4} \text{ m}$

From Table 9.1, Appendix G, the parameters for steel are:

$$E \approx 2 \times 10^{11} \text{ N/m}^2$$

$$\rho \approx 7.75 \times 10^3 \text{ kg/m}^3$$

PROBLEM 11.7 (continued)

$$\omega \approx 120 \text{ rad/sec.}$$

$$\text{Then, } f_1 = \frac{\omega}{2\pi} \approx 19 \text{ Hz.}$$

Part c

$$\text{For the next higher resonance, } \alpha L \approx \frac{5}{2} \pi$$

$$\text{Therefore, } f_2 = \left(\frac{5}{2}\right)^2 f_1 \approx 53 \text{ Hz.}$$

PROBLEM 11.8
Part a

As in Prob. 11.7, the equation of motion for the beam is

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{Eb^2}{3\rho} \frac{\partial^4 \xi}{\partial x_1^4} = 0 \quad (a)$$

At $x_1 = L$, there is a free end, so the boundary conditions are:

$$\begin{aligned} T_{11}(x_1=L) &= 0 \\ \text{and} \quad T_{21}(x_1=L) &= 0 \end{aligned} \quad (b)$$

The boundary conditions at $x_1 = 0$ are

$$M \frac{\partial^2 \xi(0,t)}{\partial t^2} = + \int_{x_1=0}^L (T_{21})_D dx_2 + \bar{f}_e + \bar{F}_o \quad (c)$$

and

$$\delta_1(x_1 = 0) = 0 \quad (d)$$

The $\bar{H}$ field in the air gap and in the plunger is

$$\bar{H} = \frac{Ni}{D} \bar{i}_1 \quad (e)$$

Using the Maxwell stress tensor

$$\bar{f}_e = - \frac{(\mu - \mu_o)}{2} \left(\frac{N^2 i^2}{D^2} \right) D^2 \bar{i}_2 = - \frac{N^2 i^2}{2} (\mu - \mu_o) \bar{i}_2 \quad (f)$$

$$\text{with } i = I_o + i_1 \cos \omega t = I_o + \text{Re } i_1 e^{j\omega t}$$

PROBLEM 11.8 (continued)

We linearize $\bar{f}^e$ to obtain

$$\bar{f}^e = -\frac{N^2}{2} (\mu - \mu_0) [I_0^2 + 2I_0 i_1 \cos \omega t] \bar{i}_2 \quad (g)$$

For equilibrium

$$\bar{F}_0 - \frac{N^2}{2} (\mu - \mu_0) I_0^2 \bar{i}_2 = 0$$

Thus

$$\bar{F}_0 = \frac{N^2}{2} (\mu - \mu_0) I_0^2 \bar{i}_2 \quad (h)$$

Part b

We write the solution to Eq. (a) in the form

$$\xi(x_1, t) = \text{Re } \hat{\xi}(x_1) e^{j\omega t}$$

where, from example 11.4.4

$$\hat{\xi}(x_1) = A_1 \sin \alpha x_1 + A_2 \cos \alpha x_1 + A_3 \sinh \alpha x_1 + A_4 \cosh \alpha x_1 \quad (i)$$

with

$$\alpha = \left[\omega^2 \left(\frac{3\rho}{Eb^2} \right) \right]^{1/4}$$

Now, from Eqs. 11.4.6 and 11.4.16

$$T_{11}(x=L) = E \frac{\partial \delta_1}{\partial x_1} = -Ex_2 \frac{\partial^2 \xi}{\partial x_1^2} = 0 \quad (j)$$

Thus

$$\frac{\partial^2 \xi}{\partial x_1^2}(x_1 = L) = 0$$

From Eq. 11.4.21

$$T_{21} = \frac{(x_2^2 - b^2)}{2} E \frac{\partial^3 \xi}{\partial x_1^3} \quad (k)$$

and from Eq. 11.4.16

$$\delta_1(x_1 = 0) = -x_2 \left(\frac{\partial \xi}{\partial x_1} \right)_{x_1=0} = 0 \quad (l)$$

Thus

$$\left(\frac{\partial \xi}{\partial x_1} \right)_{x_1=0} = 0$$

PROBLEM 11.8(continued)

Applying the boundary conditions from Eqs. (b), (c), (d) to our solution of Eq. (i), we obtain the four equations

$$\begin{aligned}
 & A_1 + A_3 = 0 \\
 & -A_1 \sin \alpha L - A_2 \cos \alpha L + A_3 \sinh \alpha L + A_4 \cosh \alpha L = 0 \\
 & -A_1 \cos \alpha L + A_2 \sin \alpha L + A_3 \cosh \alpha L + A_4 \sinh \alpha L = 0 \quad (m) \\
 & -\frac{2}{3} \alpha^3 b^3 EDA_1 + M\omega^2 A_2 + \frac{2}{3} \alpha^3 b^3 EDA_3 + M\omega^2 A_4 = +N^2 I_o i_1 (\mu - \mu_o)
 \end{aligned}$$

Now

$$v = \frac{d\lambda}{dt} = \frac{d}{dt} \left\{ \frac{N^2 I_o}{D} D [\mu_o \xi(0) + \mu(D - \xi(0))] \right\} \quad (n)$$

$$\text{or } \hat{v} = -N^2 I_o (\mu - \mu_o) j\omega (A_2 + A_4) + N^2 i_1 \mu D j\omega \quad (o)$$

We solve Eqs. (m) for A_2 and A_4 using Cramer's rule to obtain

$$A_2 = \frac{N^2 I_o i_1 (\mu - \mu_o) (-1 + \sin \alpha L \sinh \alpha L - \cos \alpha L \cosh \alpha L)}{-2M\omega^2 (1 + \cos \alpha L \cosh \alpha L) + \frac{4}{3} (\alpha b)^3 ED (\cos \alpha L \sinh \alpha L + \sin \alpha L \cosh \alpha L)} \quad (p)$$

and

$$A_4 = \frac{N^2 I_o i_1 (\mu - \mu_o) (-1 - \cos \alpha L \cosh \alpha L - \sin \alpha L \sinh \alpha L)}{-2M\omega^2 (1 + \cos \alpha L \cosh \alpha L) + \frac{4}{3} (\alpha b)^3 ED (\cos \alpha L \sinh \alpha L + \sin \alpha L \cosh \alpha L)} \quad (q)$$

Thus

$$\begin{aligned}
 Z(j\omega) = \frac{\hat{v}(j\omega)}{i_1} &= \frac{+ [N^2 I_o (\mu - \mu_o)]^2 j\omega (+2 + 2 \cos \alpha L \cosh \alpha L)}{-2M\omega^2 (1 + \cos \alpha L \cosh \alpha L) + \frac{4}{3} (\alpha b)^3 ED (\cos \alpha L \sinh \alpha L + \sin \alpha L \cosh \alpha L)} \\
 &+ N^2 \mu D j\omega \quad (r)
 \end{aligned}$$

Part c

$Z(j\omega)$ has poles when

$$+2M\omega^2 (1 + \cos \alpha L \cosh \alpha L) = \frac{4}{3} (\alpha b)^3 ED (\cos \alpha L \sinh \alpha L + \sin \alpha L \cosh \alpha L)$$

PROBLEM 11.9

Part a

The flux above and below the beam must remain constant. Therefore, the $\bar{H}$ field above is

$$\bar{H}_a = \frac{H_o (a-b)}{(a-b-\xi)} \bar{I}_1 \quad (a)$$

and the $\bar{H}$ field below is

$$\bar{H}_b = \frac{H_o (a-b)}{(a-b+\xi)} \bar{I}_1 \quad (b)$$

Using the Maxwell stress tensor, the magnetic force on the beam is

$$\begin{aligned} T_2 &= -\frac{\mu_o}{2} (H_a^2 - H_b^2) = -\frac{\mu_o}{2} H_o^2 (a-b)^2 \left(+ \frac{4\xi}{(a-b)^3} \right) \\ &= -\frac{2\mu_o H_o^2 \xi}{(a-b)} \end{aligned} \quad (c)$$

Thus, from Eq. 11.4.26, the equation of motion on the beam is

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{Eb^2}{3\rho} \frac{\partial^4 \xi}{\partial x_1^4} = -\frac{\mu_o H_o^2 \xi}{(a-b)b\rho} \quad (d)$$

Again, we let

$$\xi(x_1, t) = \text{Re } \hat{\xi}(x_1) e^{j\omega t} \quad (e)$$

with the boundary conditions

$$\begin{aligned} \xi(x_1=0) &= 0 & \xi(x_1=L) &= 0 \\ \delta_1(x_1=0) & & \delta_1(x_1=L) &= 0 \end{aligned} \quad (f)$$

Since $\delta_1 = -x_2 \partial \xi / \partial x_1$ from Eq. 11.4.16, this implies that:

$$\frac{\partial \xi}{\partial x_1} (x_1=0) = 0 \text{ and } \frac{\partial \xi}{\partial x_1} (x_1=L) = 0 \quad (g)$$

Substituting our assumed solution into the equation of motion, we have

$$-\omega^2 \hat{\xi} + \frac{Eb^2}{3\rho} \frac{\partial^4 \hat{\xi}}{\partial x_1^4} + \frac{\mu_o H_o^2}{(a-b)b\rho} \hat{\xi} = 0 \quad (h)$$

Thus we see that our solutions are again of the form

$$\hat{\xi}(x) = A \sin \alpha x + B \cos \alpha x + C \sinh \alpha x + D \cosh \alpha x \quad (i)$$

PROBLEM 11.9 (continued)

where now

$$\alpha = \left[\left(\omega^2 - \frac{\mu_o H_o^2}{(a-b)b\rho} \right) \left(\frac{3\rho}{Eb^2} \right) \right]^{1/4} \quad (j)$$

Since the boundary conditions for this problem are identical to that of problem 11.7, we can take the solutions from that problem, substituting the new value of α . From problem 11.7, the solution must satisfy

$$\cos \alpha L \cosh \alpha L = 1 \quad (k)$$

The first resonance occurs when

$$\begin{aligned} \alpha L &\approx \frac{3\pi}{2} \\ \text{or} \quad \omega^2 &= \frac{\left(\frac{3\pi}{2} \right)^4 \left(\frac{Eb^2}{3\rho} \right)}{L^4} + \frac{\mu_o H_o^2}{(a-b)b\rho} \end{aligned} \quad (l)$$

Part c

The resonant frequencies are thus shifted upward due to the stiffening effect of the constant flux constraint.

Part d

We see that, no matter what the values of the system parameters $\omega^2 > 0$, so ω will always be real, and thus stable. This is expected as the constant flux constraint imposes a force which opposes the motion.

PROBLEM 11.10
Part a

We choose a coordinate system as in Fig. 11.4.12, centered at the right end of the rod. Because $\frac{d}{D} = \frac{1}{10}$, we can neglect fringing and consider the right end of the rod as a capacitor plate. Also, since $\frac{D}{\ell} = \frac{1}{10}$, we can assume that the electrical force acts only at $x_1 = 0$. Thus, the boundary conditions at $x_1 = 0$ are

$$- \int_0^b T_{21} D dx_2 + f^e = 0 \quad (a)$$

$$\text{where } T_{21} = \frac{(x_2^2 - b^2)}{2} E \frac{\partial^3 \xi}{\partial x_1^3} \quad (\text{Eq. 11.4.21})$$

since the electrical force, f^e , must balance the shear stress T_{21} to keep the rod in equilibrium,

PROBLEM 11.10 (continued)

and

$$T_{11}(0) = -x_2 E \frac{\partial^2 \xi}{\partial x_1^2}(0) = 0 \quad (b)$$

since the end of the rod is free of normal stresses. At $x_1 = -l$, the rod is clamped so

$$\xi(-l) = 0 \quad (c)$$

and

$$\delta_1(-l) = -x_2 \frac{\partial \xi}{\partial x_1}(-l) = 0 \quad (d)$$

We use the Maxwell stress tensor to calculate the electrical force to be

$$f^e = \frac{\epsilon A}{2} \left[\frac{(v_o + v_s)^2}{[d - \xi(0)]^2} - \frac{(v_s - v_o)^2}{[d + \xi(0)]^2} \right] \quad (e)$$

$$\approx \frac{2\epsilon A v_o}{d^2} \left[v_s + \frac{v_o \xi(0)}{d} \right]$$

The equation of motion of the beam is (example 11.4.4)

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{E b^2}{3\rho} \frac{\partial^4 \xi}{\partial x_1^4} = 0 \quad (f)$$

We write the solution to Eq. (f) in the form

$$\xi(x, t) = \operatorname{Re} \hat{\xi}(x) e^{j\omega t} \quad (g)$$

where

$$\hat{\xi}(x) = A_1 \sin \alpha x + A_2 \cos \alpha x + A_3 \sinh \alpha x + A_4 \cosh \alpha x$$

with

$$\alpha = \left[\omega^2 \left(\frac{3\rho}{E b^2} \right) \right]^{1/4}$$

Applying the four boundary conditions, Eqs. (a), (b), (c) and (d), we obtain the equations

$$\begin{aligned} -A_1 \sin \alpha l + A_2 \cos \alpha l - A_3 \sinh \alpha l + A_4 \cosh \alpha l &= 0 \\ A_1 \cos \alpha l + A_2 \sin \alpha l + A_3 \cosh \alpha l - A_4 \sinh \alpha l &= 0 \\ -A_2 &+ A_4 = 0 \\ -\frac{2}{3} b^3 DE \alpha^3 A_1 + \frac{2\epsilon_o AV_o^2}{d^3} A_2 + \frac{2}{3} b^3 DE \alpha^3 A_3 + \frac{2\epsilon_o AV_o^2}{d^3} A_4 &= -\frac{2\epsilon_o AV_o \hat{v}_s}{d^2} \end{aligned} \quad (h)$$

PROBLEM 11.10 (continued)

$$\text{Now } i_s = \frac{dq_s}{dt} \quad (1)$$

$$\text{where } q_s = \frac{\epsilon_o A}{d - \xi(0)} (V_o + v_s) + \frac{\epsilon_o A (v_s - V_o)}{d + \xi(0)}$$

$$\approx \frac{2\epsilon_o A v_s}{d} + \frac{2\epsilon_o A V_o}{d^2} \xi(0) \quad (j)$$

Therefore

$$\hat{i}_s = j\omega \frac{2\epsilon_o A}{d} \left[\hat{v}_s + \frac{V_o}{d} \hat{\xi}(0) \right] \quad (k)$$

where

$$\hat{\xi}(0) = A_2 + A_4$$

We use Cramer's rule to solve Eqs. (h) for A_2 and A_4 to obtain:

$$A_2 = A_4 = \frac{-\frac{\epsilon_o A V_o \hat{v}_s}{d^2} [\cos \alpha l \sinh \alpha l - \sin \alpha l \cosh \alpha l]}{\frac{2}{3} b^3 \alpha^3 DE (1 + \cos \alpha l \cosh \alpha l) + \frac{2\epsilon_o A V_o^2}{d^3} (\cos \alpha l \sinh \alpha l - \sin \alpha l \cosh \alpha l)} \quad (l)$$

Thus, from Eq. (k) we obtain

$$Z(j\omega) = \frac{d}{j\omega 2\epsilon_o A} \left[1 + \frac{3\epsilon_o A V_o^2}{d^3 (\alpha b)^3 ED} \frac{(\cos \alpha l \sinh \alpha l - \sin \alpha l \cosh \alpha l)}{(1 + \cos \alpha l \cosh \alpha l)} \right] \quad (m)$$

Part b

We define a function $g(\alpha l)$ such that Eq. (m) has a zero when

PROBLEM 11.10 (Continued)

$$(\alpha L)^3 g(\alpha L) = \frac{(1 + \cosh \alpha l \cos \alpha l)(\alpha l)^3}{\sin \alpha l \cosh \alpha l - \cos \alpha l \sinh \alpha l} = \frac{3l^3 V_o^2 A \epsilon_o}{DEb^3 d^3} \quad (n)$$

Substituting numerical values, we obtain

$$\frac{3l^3 V_o^2 A \epsilon_o}{DEb^3 d^3} \approx \frac{3 \times 10^{-3} (10^6) 10^{-4} (8.85 \times 10^{-12})}{10^{-2} (2.2 \times 10^{11}) 10^{-9} 10^{-9}} \approx 1.2 \times 10^{-3} \quad (o)$$

In Figure 1, we plot $(\alpha l)^3 g(\alpha l)$ as a function of αl . We see that the solution to Eq. (n) first occurs when $(\alpha l)^3 g(\alpha l) \approx 0$. Thus, the solution is approximately

$$\alpha l = 1.875$$

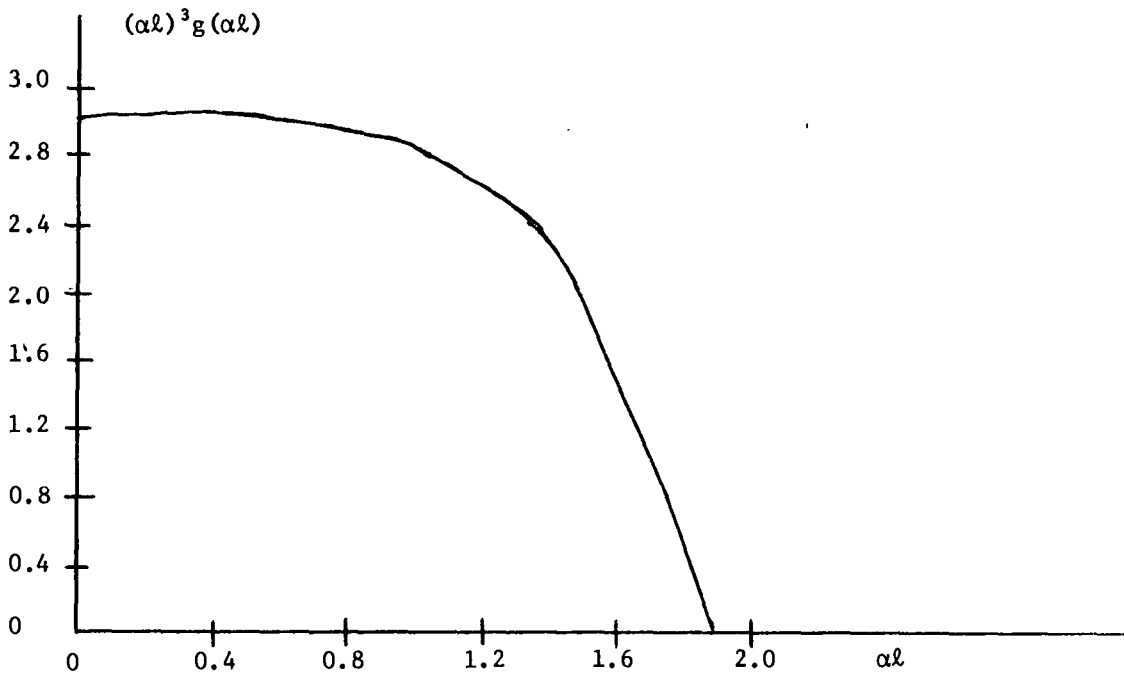


Figure 1

PROBLEM 11.10 (Continued)

From Eq. (g)

$$\alpha l = \left[\omega^2 \frac{3\rho}{Eb^2} \right]^{1/4} \quad l = 1.875$$

 Solving for ω , we obtain

$$\omega \approx 1080 \text{ rad/sec.} \quad (p)$$

Part c

The input impedance of a series LC circuit is

$$Z(j\omega) = \frac{1 - LC\omega^2}{j\omega C} \quad (q)$$

Thus the impedance has a zero when

$$\omega_o^2 = \frac{1}{LC} \quad (r)$$

 We let $\omega = \omega_o + \Delta\omega$, and expand (q) in a Taylor series around ω_o to obtain

$$Z(j\omega) \approx j \frac{2L\Delta\omega}{C\omega_o^2} = + 2j L\Delta\omega \quad (s)$$

(m) can be written in the form

$$Z(j\omega) = \frac{1}{2j\omega C_o} [1 - f(\omega)] \quad \text{where } f(\omega_o) = 1 \quad (t)$$

$$\text{and } C_o = \frac{\epsilon_o A}{d}$$

 For small deviations around ω_o

$$Z(j\omega) \approx \frac{j}{2\omega C_o} \left. \frac{\partial f}{\partial \omega} \right|_{\omega_o} \Delta\omega$$

Thus, from (q), (r) (s) and (t), we obtain the relations

$$2L = \frac{1}{2\omega C_o} \left. \frac{\partial f}{\partial \omega} \right|_{\omega_o} \quad (u)$$

$$\text{and } C = \frac{1}{\omega_o^2 L} \quad (v)$$

$$\text{now } f(\omega) = \frac{K}{(\alpha l)^3 g(\alpha l)} \quad (w)$$

$$\text{where } K = \frac{3l^3 \epsilon_o A V_o^2}{d^3 (E D b^3)} = 1.2 \times 10^{-3}$$

PROBLEM 11.10 (Continued)

$$\text{and } g(\alpha l) = \frac{1 + \cos \alpha l \cosh \alpha l}{\sin \alpha l \cosh \alpha l - \cos \alpha l \sinh \alpha l}$$

Thus, we can write

$$\left. \frac{df(\omega)}{d\omega} \right|_{\omega_0} = \left\{ \frac{d}{d(\alpha l)} \left[\frac{K}{(\alpha l)^3 g(\alpha l)} \right] \frac{d(\alpha l)}{d\omega} \right\}_{\omega_0} \quad (y)$$

Now from (g),

$$\left. \frac{d(\alpha l)}{d\omega} \right|_{\omega_0} = \left(\frac{3\rho}{Eb^2} \right)^{1/4} \frac{l}{2\omega_0^{1/2}} \quad (z)$$

and

$$\begin{aligned} \left. \frac{d}{d(\alpha l)} \left[\frac{K}{(\alpha l)^3 g(\alpha l)} \right] \right|_{\omega_0} &= \left. \frac{-K}{[(\alpha l)^3 g(\alpha l)]^2} \frac{d}{d(\alpha l)} [(\alpha l)^3 g(\alpha l)] \right|_{\omega_0} \\ &\approx \left. -\frac{1}{K} \frac{d}{d(\alpha l)} [(\alpha l)^3 g(\alpha l)] \right|_{\omega_0} \end{aligned} \quad (aa)$$

since at $\omega = \omega_0$

$$(\alpha l)^3 g(\alpha l) = K. \quad (bb)$$

Continuing the differentiating in (aa), we finally obtain

$$\begin{aligned} \left. \frac{d}{d(\alpha l)} \left[\frac{(\alpha l)^3 g(\alpha l)}{-K} \right] \right|_{\omega_0} &= \left. -\frac{1}{K} \left[g(\alpha l) 3(\alpha l)^2 + (\alpha l)^3 \frac{d}{d(\alpha l)} g(\alpha l) \right] \right|_{\omega_0} \\ &= \left. \frac{-3}{\alpha l} - \frac{(\alpha l)^3}{K} \frac{d}{d(\alpha l)} g(\alpha l) \right|_{\omega_0} \end{aligned} \quad (cc)$$

Now

$$\begin{aligned} \frac{d}{d(\alpha l)} g(\alpha l) &= \frac{-\sin \alpha l \cosh \alpha l + \cos \alpha l \sinh \alpha l}{(\sin \alpha l \cosh \alpha l - \cos \alpha l \sinh \alpha l)} \\ &= \frac{-(1 + \cos \alpha l \cosh \alpha l) + (\cos \alpha l \cosh \alpha l + \sin \alpha l \sinh \alpha l + \sin \alpha l \sinh \alpha l - \cos \alpha l \cosh \alpha l)}{(\sin \alpha l \cosh \alpha l - \cos \alpha l \sinh \alpha l)} \\ &= -1 - \frac{2g(\alpha l) (\sin \alpha l \sinh \alpha l)}{(\sin \alpha l \cosh \alpha l - \cos \alpha l \sinh \alpha l)} \end{aligned} \quad (dd)$$

PROBLEM 11.10 (Continued)

Substituting numerical values into the second term of (cc), we find it to have value much less than one at $\omega = \omega_0$.

Thus,

$$\frac{d}{d(\alpha l)} g(\alpha l) \approx -1 \quad (ee)$$

Thus, using (y), (z), (aa), (bb) and (dd), we have

$$\left. \frac{df}{d\omega} \right|_{\omega_0} \approx \left(\frac{3\rho}{Eb^2} \right)^{1/4} \frac{l}{2\omega_0^{1/2}} \left[- \left. \frac{3}{\alpha l} \right|_{\omega_0} + \left. \frac{(\alpha l)^3}{K} \right|_{\omega_0} \right] \approx 4.8 \quad (ff)$$

Thus, from (v) and (w)

$$L \approx \frac{4.8 \times 10^{-3}}{4(1080)(8.85 \times 10^{-12})(10^{-4})} = 1.25 \times 10^9 \text{ henries}$$

and

$$C \approx \frac{1}{1.25 \times 10^9 (1080)^2} = 6.8 \times 10^{-16} \text{ farads.}$$

PROBLEM 11.11

From Eq. (11.4.29), the equation of motion is

$$\rho \frac{\partial^2 \delta_3}{\partial t^2} = G \left(\frac{\partial^2 \delta_3}{\partial x_1^2} + \frac{\partial^2 \delta_3}{\partial x_2^2} \right) \quad (a)$$

We let

$$\delta_3 = \text{Re } \hat{\delta}(x_2) e^{j(\omega t - kx_1)} \quad (b)$$

Substituting this assumed solution into the equation of motion, we obtain

$$-\rho \omega^2 \hat{\delta} = G \left(-k^2 \hat{\delta} + \frac{\partial^2 \hat{\delta}}{\partial x_2^2} \right) \quad (c)$$

or

$$\frac{\partial^2 \hat{\delta}}{\partial x_2^2} + \left(\frac{\rho \omega^2}{G} - k^2 \right) \hat{\delta} = 0 \quad (d)$$

$$\text{If we let } \beta^2 = \frac{\rho \omega^2}{G} - k^2 \quad (e)$$

the solutions for $\hat{\delta}$ are:

$$\hat{\delta}(x_2) = A \sin \beta x_2 + B \cos \beta x_2 \quad (f)$$

The boundary conditions are

$$\hat{\delta}(0) = 0 \quad \text{and} \quad \hat{\delta}(d) = 0 \quad (g)$$

This implies that $B = 0$

and that $\beta d = n\pi$.

Thus, the dispersion relation is

$$\omega^2 \frac{\rho}{G} - k^2 = \left(\frac{n\pi}{d} \right)^2 \quad (h)$$

Part b

The sketch of the dispersion relation is identical to that of Fig. 11.4.19. However, now the $n=0$ solution is trivial, as it implies that

$$\hat{\delta}(x_2) = 0$$

Thus, there is no principal mode of propagation.

PROBLEM 11.12

From Eq. (11.4.1), the equation of motion is

$$\rho \frac{\partial^2 \delta}{\partial t^2} = (2G + \lambda) \nabla(\nabla \cdot \delta) - G \nabla \times (\nabla \times \delta) \quad (a)$$

We consider motions

$$\delta = \delta_\theta(r, z, t) \bar{i}_\theta \quad (b)$$

Thus, the equation of motion reduces to

$$\rho \frac{\partial^2 \delta_\theta}{\partial t^2} - G \left[\frac{\partial^2 \delta_\theta}{\partial z^2} + \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} r \delta_\theta \right) \right] = 0 \quad (c)$$

We assume solutions of the form

$$\delta_\theta(r, z, t) = \text{Re } \hat{\delta}(r) e^{j(\omega t - kz)} \quad (d)$$

which, when substituted into the equation of motion, yields

$$\frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} r \hat{\delta}(r) \right] + \left(\frac{\rho \omega^2}{G} - k^2 \right) \hat{\delta}(r) = 0 \quad (e)$$

From page 207 of Ramo, Whinnery and Van Duzer, we recognize solutions to this equation as

$$\hat{\delta}(r) = A J_1 \left[\left(\frac{\rho \omega^2}{G} - k^2 \right)^{1/2} r \right] + B N_1 \left[\left(\frac{\rho \omega^2}{G} - k^2 \right)^{1/2} r \right] \quad (f)$$

On page 209 of this reference there are plots of the Bessel functions J_1 and N_1 . We must have $B = 0$ as at $r = 0$, N_1 goes to $-\infty$. Now, at $r = R$

$$\hat{\delta}(R) = 0 \quad (g)$$

This implies that

$$J_1 \left[\left(\frac{\rho \omega^2}{G} - k^2 \right)^{1/2} R \right] = 0 \quad (h)$$

If we denote α_1 as the zeroes of J_1 , i.e.

$$J_1(\alpha_1) = 0$$

we have the dispersion relation as

$$\frac{\rho}{G} \omega^2 - k^2 = \frac{\alpha_1^2}{R^2} \quad (i)$$

PROBLEM 12.1Part a

Since we are in the steady state ($\partial/\partial t = 0$), the total forces on the piston must sum to zero. Thus

$$pLD + (f^e)_x = 0 \quad (a)$$

where $(f^e)_x$ is the upwards vertical component of the electric force

$$(f^e)_x = - \frac{\epsilon_0 V_o^2}{2x^2} LD \quad (b)$$

Solving for the pressure p , we obtain

$$p = \frac{\epsilon_0 V_o^2}{2x} \quad (c)$$

Part b

Because $\frac{d}{L} \ll 1$, we approximate the velocity of the piston to be negligibly small. Then, applying Bernoulli's equation, Eq. (12.2.11) right below the piston and at the exit nozzle where the pressure is zero, we obtain

$$\frac{1}{2} \rho V_p^2 = \frac{\epsilon_0 V_o^2}{2x^2} \quad (d)$$

Solving for V_p , we have

$$V_p = \frac{V_o}{x} \sqrt{\frac{\epsilon_0}{\rho}} \quad (e)$$

Part c

The thrust T on the rocket is then

$$\begin{aligned} T &= V_p \frac{dM}{dt} = V_p^2 \rho dD \\ &= \frac{\epsilon_0 V_o^2}{x^2} dD \end{aligned} \quad (f)$$

PROBLEM 12.2Part a

The forces on the movable piston must sum to zero. Thus

$$pwD - f^e = 0 \quad (a)$$

where f^e is the component of electrical force normal to the piston in the direction of V , and p is the pressure just to the right of the piston.

$$f^e = \frac{\mu_0}{2} \frac{I^2 D}{w} \quad (b)$$

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PROBLEM 12.2 (Continued)

Therefore

$$p = \frac{\mu_0 I^2}{2w^2} \quad (c)$$

Assuming that the velocity of the piston is negligible, we use Bernoulli's law, Eq. (12.2.11), just to the right of the piston and at the exit orifice where the pressure is zero, to write

$$\frac{1}{2} \rho V^2 = p \quad (d)$$

or

$$V = \frac{I}{w} \sqrt{\frac{\mu_0}{\rho}} \quad (e)$$

Part b

The thrust T is

$$T = V \frac{dM}{dt} = V^2 \rho dW = \frac{\mu_0 I^2 d}{w} \quad (f)$$

Part c

For $I = 10^3 \text{ A}$

$d = .1 \text{ m}$

$w = 1 \text{ m}$

$\rho = 10^3 \text{ kg/m}^3$

the exit velocity is

$$V = 3.5 \times 10^{-2} \text{ m/sec.}$$

and the thrust is

$$T = .126 \text{ newtons.}$$

Within the assumption that the fluid is incompressible, we would prefer a dense material, for although the thrust is independent of the fluid's density, the exhaust velocity would decrease with increasing density, and thus the rocket will work longer. Under these conditions, we would prefer water in our rocket, since it is much more dense than air.

PROBLEM 12.3

Part a

From the results of problem 12.2, we have that the pressure p , acting just to the left of the piston, is

$$p = \frac{\mu_0 I^2}{2w^2} \quad (a)$$

The exit velocity at each orifice is obtained by using Bernoulli's law just to the left of the piston and at either orifice, from which we obtain

PROBLEM 12.3 (Continued)

$$V = \left(\frac{\mu_0}{\rho} \right)^{1/2} \frac{I}{w} \quad (b)$$

at each orifice.

Part b

The thrust is

$$T = 2V \frac{dM}{dt} = 2V^2 \rho dw \quad (c)$$

$$T = \frac{2\mu_0 I^2 d}{w} \quad (d)$$

PROBLEM 12.4Part a

In the steady state, we choose to integrate the momentum theorem, Eq. (12.1.29), around a rectangular surface, enclosing the system from $-L \leq x_1 \leq +L$.

$$-\rho V_o^2 a + \rho [V(L)]^2 b = P_o a - P(L)b + F \quad (a)$$

where F is the x_1 component force per unit length which the walls exert on the fluid. We see that there is no x_1 component of force from the upper wall, therefore F is the force purely from the lower wall.

In the steady state, conservation of mass, (Eq. 12.1.8), yields

$$V(L) = V_o \frac{a}{b} \quad (b)$$

Bernoulli's equation gives us

$$\frac{1}{2} \rho V_o^2 + P_o = \frac{1}{2} \rho V_o^2 \frac{a^2}{b^2} + P(L) \quad (c)$$

Solving (c) for $P(L)$, and then substituting this result and that of (b) into (a), we finally obtain

$$F = P_o(b-a) + \rho V_o^2 \left(-a + \frac{b}{2} + \frac{a^2}{2b} \right) \quad (d)$$

The problem asked for the force on the lower wall, which is just the negative of F .

Thus

$$F_{\text{wall}} = -P_o(b-a) - \rho V_o^2 \left(-a + \frac{a^2}{2b} + \frac{b}{2} \right) \quad (e)$$

PROBLEM 12.5Part a

We recognize this problem to be analogous to a dielectric or high-permeability cylinder placed in a uniform electric or magnetic field. The solutions are then dipole fields. We expect similar results here. As in Eqs. (12.2.1 - 12.2.3), we

PROBLEM 12.5 (continued)

define

$$\bar{V} = -\nabla\phi$$

and since

$$\nabla \cdot \bar{v} = 0$$

then $\nabla^2\phi = 0$.

Using our experience from the electromagnetic field problems, we guess a solution of the form

$$\phi = \frac{A}{r} \cos \theta + Br \cos \theta$$

Then

$$\bar{V} = \left(\frac{A}{r^2} \cos \theta - B \cos \theta\right)\bar{i}_r + \left(\frac{A}{r^2} \sin \theta + B \sin \theta\right)\bar{i}_\theta$$

Now, as $r \rightarrow \infty$

$$V = V_0 \bar{i}_1 = V_0 (\cos \theta \bar{i}_r - \bar{i}_\theta \sin \theta)$$

Therefore

$$B = -V_0$$

The other boundary condition at $r = a$ is that

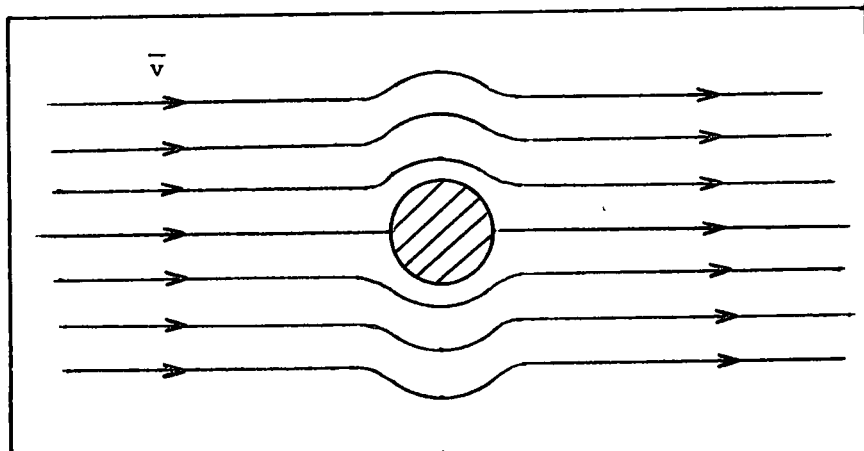
$$V_r(r=a) = 0$$

Thus

$$A = B a^2 = -V_0 a^2$$

Therefore

$$\bar{V} = V_0 \cos \theta \left(1 - \frac{a^2}{r^2}\right)\bar{i}_r - V_0 \sin \theta \left(1 + \frac{a^2}{r^2}\right)\bar{i}_\theta$$

Part b

PROBLEM 12.5 (continued)Part c

Using Bernoulli's law, we have

$$\frac{1}{2} \rho v_o^2 + p_o = \frac{1}{2} \rho v_o^2 \left(1 + \frac{a^4}{r^4} - \frac{2a^2}{r^2} \cos 2\theta \right) + p$$

Therefore the pressure is

$$p = p_o - \frac{1}{2} \rho v_o^2 \left(\frac{a^4}{r^4} - \frac{2a^2}{r^2} \cos 2\theta \right)$$

Part d

We choose a large rectangular surface which encloses the cylinder, but the sides of which are far away from the cylinder. We write the momentum theorem as

$$\int_S \rho \bar{v} (\bar{v} \cdot \bar{n}) da = - \int_S p d\bar{a} + \bar{F}$$

where $\bar{F}$ is the force which the cylinder exerts on the fluid. However, with our surface far away from the cylinder

$$\bar{v} = v_o \bar{i}_1$$

and the pressure is constant

$$p = p_o.$$

Thus, integrating over the closed surface

$$\bar{F} = 0$$

The force which is exerted by the fluid on the cylinder is $-\bar{F}$, which, however, is still zero.

PROBLEM 12.6Part a

This problem is analogous to 12.5, only we are now working in spherical coordinates. As in Prob. 12.5,

$$\bar{V} = -\nabla\phi$$

In spherical coordinates, we try the solution to Laplace's equation

$$\phi = Ar \cos \theta + \frac{B}{r} \cos \theta \quad (a)$$

Theta is measured clockwise from the x_1 axis.

Thus

$$\bar{V} = \left(-A \cos \theta + \frac{2B}{r^3} \cos \theta \right) \bar{i}_r + \bar{i}_\theta \left(A + \frac{B}{r^3} \right) \sin \theta \quad (b)$$

As $r \rightarrow \infty$

$$\bar{V} \rightarrow V_o (\bar{i}_r \cos \theta - \bar{i}_\theta \sin \theta) \quad (c)$$

$$\text{Therefore } A = -V_o \quad (d)$$

At $r = a$

$$V_r(a) = 0 \quad (e)$$

$$\text{Thus } \frac{2B}{a^3} = A = -V_o$$

or

$$B = -\frac{V_o a^3}{2} \quad (f)$$

Therefore

$$\bar{V} = V_o \left(1 - \frac{a^3}{r^3} \right) \cos \theta \bar{i}_r - V_o \left(1 + \frac{a^3}{2r^3} \right) \sin \theta \bar{i}_\theta \quad (g)$$

with

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

Part b

$$\text{At } r = a, \theta = \pi, \text{ and } \phi = -\frac{\pi}{2}$$

we are given that $p = 0$

At this point

$$\bar{V} = 0$$

Therefore, from Bernoulli's law

$$p = -\frac{1}{2} \rho V_o^2 \left[\left(1 - \frac{a^3}{r^3} \right)^2 \cos^2 \theta + \sin^2 \theta \left(1 + \frac{a^3}{2r^3} \right)^2 \right] \quad (h)$$

Part c

We realize that the pressure force acts normal to the sphere in the $-\bar{i}_r$ direction.

PROBLEM 12.6 (continued)at $r = a$

$$p = -\frac{9}{8} \rho V_o^2 \sin^2 \theta$$

We see that the magnitude of p remains unchanged if, for any value of θ , we look at the pressure at $\theta + \pi$. Thus, by the symmetry, the force in the x_1 direction is zero,

$$\bar{f}_1 = 0.$$

PROBLEM 12.7Part a

We are given the potential of the velocity field as

$$\phi = \frac{V_o}{a} x_1 x_2, \quad \bar{v} = -\nabla\phi = -\frac{V_o}{a} (x_2 \bar{i}_1 + x_1 \bar{i}_2)$$

If we sketch the equipotential lines in the $x_1 x_2$ plane, we know that the velocity distribution will cross these lines at right angles, in the direction of decreasing potential.

Part b

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla) \bar{v}$$

$$= \left(\frac{V_o^2}{a} \right) (x_1 \bar{i}_1 + x_2 \bar{i}_2) \quad (a)$$

$$\bar{a} = \left(\frac{V_o^2}{a} \right)^2 r \bar{i}_r \quad (b)$$

where

$$r = \sqrt{x_1^2 + x_2^2} \text{ and } \bar{i}_r \text{ is a unit vector in the radial direction.}$$

Part c

This flow could represent a fluid impinging normally on a flat plate, located along the line

$$x_1 + x_2 = 0. \quad \text{See sketches on next page.}$$

PROBLEM 12.8Part a

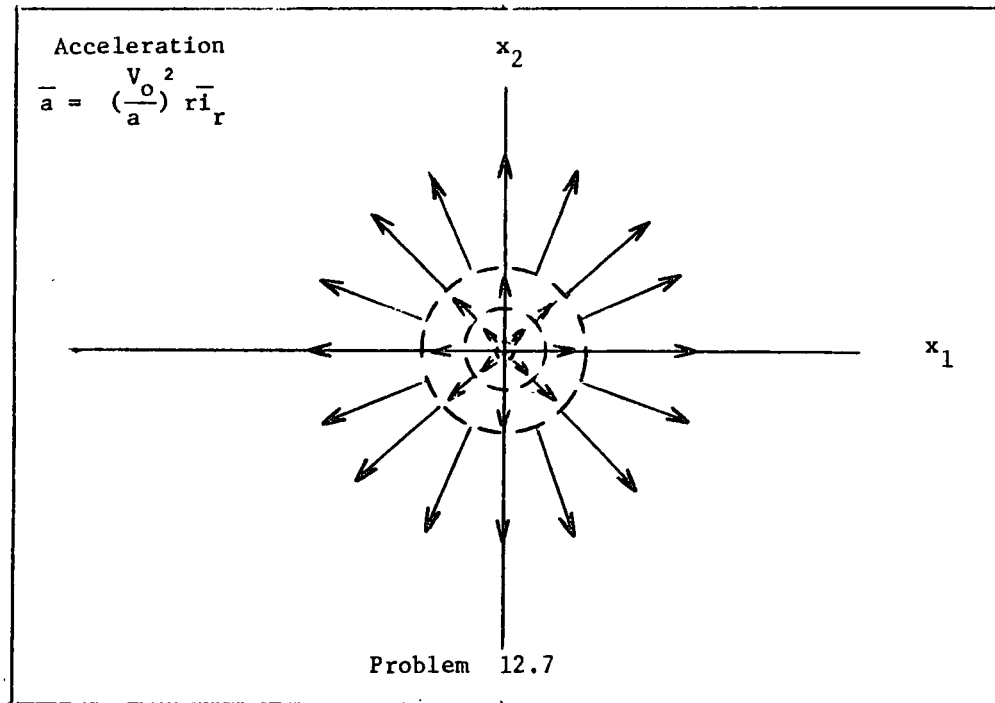
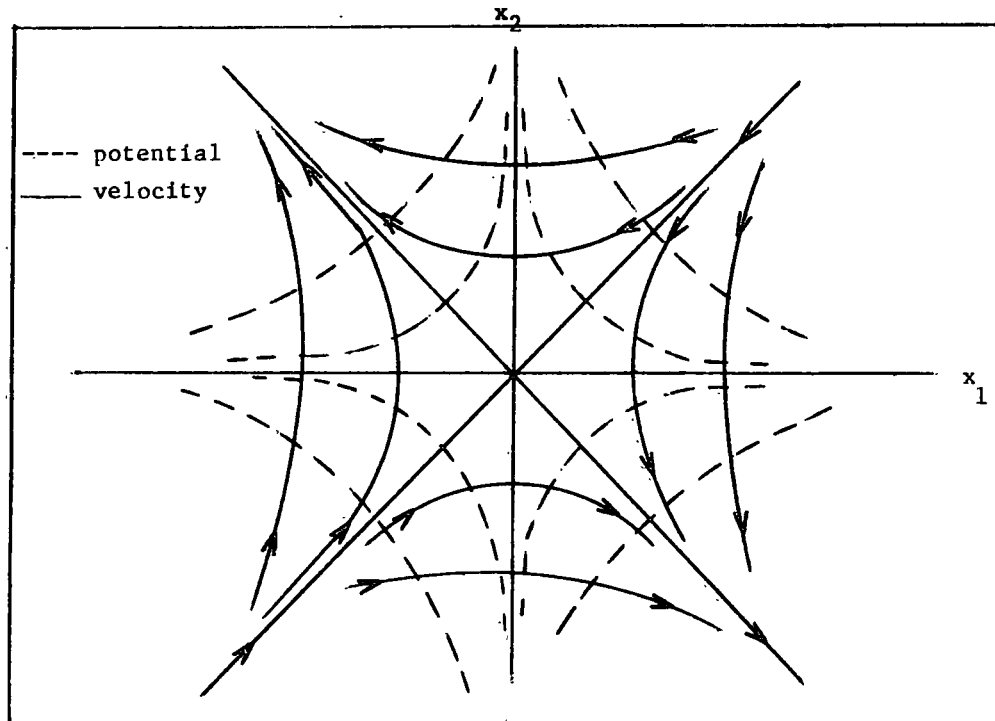
Given that

$$\bar{v} = \bar{i}_1 v_o \frac{x_2}{a} + \bar{i}_2 v_o \frac{x_1}{a} \quad (a)$$

we have that

$$\begin{aligned} \bar{a} &= \frac{d\bar{v}}{dt} = \frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla) \bar{v} \\ &= \left(v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} \right) \bar{v} \end{aligned} \quad (b)$$

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PROBLEM 12.8 (Continued)

Thus

$$\bar{a} = v_o^2 \frac{x_1}{a^2} \bar{i}_1 + \left(\frac{v_o}{a^2}\right)^2 x_2 \bar{i}_2 \quad (c)$$

Part b

Using Bernoulli's law, we have

$$p_o = \frac{1}{2} \rho \left(\frac{v_o}{a}\right)^2 (x_2^2 + x_1^2) + p \quad (d)$$

$$\begin{aligned} p &= p_o - \frac{1}{2} \rho \left(\frac{v_o}{a}\right)^2 (x_2^2 + x_1^2) \\ &= p_o - \frac{1}{2} \rho v_o^2 \frac{r^2}{a^2} \end{aligned} \quad (e)$$

where

$$r = \sqrt{x_1^2 + x_2^2}$$

PROBLEM 12.9

Part a

The addition of a gravitational force will not change the velocity from that of Problem 12.8. Only the pressure will change. Therefore,

$$\bar{v} = \bar{i}_1 \frac{v_o}{a} x_2 + \bar{i}_2 \frac{v_o}{a} x_1 \quad (a)$$

Part b

The boundary conditions at the walls are that the normal component of the velocity must be zero at the walls. Consider first the wall

$$x_2 - x_1 = 0 \quad (b)$$

We take the gradient of this expression to find a normal vector to the curve. (Note that this normal vector does not have unit magnitude.)

$$\bar{n} = \bar{i}_2 - \bar{i}_1 \quad (c)$$

Then

$$\bar{v} \cdot \bar{n} = \frac{v_o}{a} (x_1 - x_2) = 0 \quad (d)$$

Thus, the boundary condition is satisfied along this wall.

Similarly, along the wall

$$x_2 + x_1 = 0 \quad (e)$$

$$\bar{n} = \bar{i}_2 + \bar{i}_1 \quad (f)$$

and

$$\bar{v} \cdot \bar{n} = \frac{v_o}{a} (x_1 + x_2) = 0 \quad (g)$$

Thus, the boundary condition is satisfied here. Along the parabolic wall

$$x_2^2 - x_1^2 = a^2 \quad (h)$$

$$\bar{n} = x_2 \bar{i}_2 - x_1 \bar{i}_1 \quad (i)$$

PROBLEM 12.9 (Continued)

$$\vec{v} \cdot \vec{n} = \frac{v_0}{a} (x_1 x_2 - x_1 x_2) = 0 \quad (j)$$

Thus, we have shown that along all the walls, the fluid flows purely tangential to these walls.

PROBLEM 12.10Part a

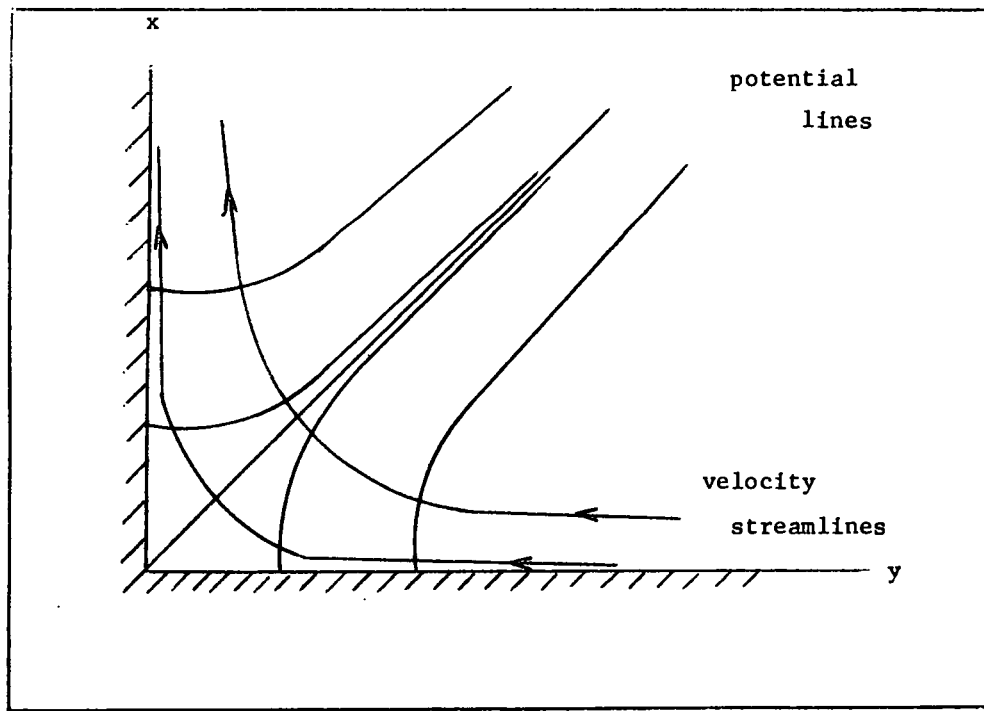
Along the lines $x = 0$ and $y = 0$, the normal component of the velocity must be zero. In terms of the potential, we must then have

$$\left. \frac{\partial \phi}{\partial x} \right|_{x=0} = 0 \quad (a)$$

and

$$\left. \frac{\partial \phi}{\partial y} \right|_{y=0} = 0 \quad (b)$$

To aid in the sketch of $\phi(x,y)$, we realize that since at the boundary the velocity must be purely tangential, the potential lines must come in normal to the walls.

Part b

For the fluid to be irrotational and incompressible, the potential must obey

PROBLEM 12.10 (Continued)

Laplace's equation

$$\nabla^2 \phi = 0 \quad (c)$$

From our sketch of part (a), and from the boundary conditions, we guess a solution of the form

$$\phi = -\frac{v_o}{a} (x^2 - y^2) \quad (d)$$

where $\frac{v_o}{a}$ is a scaling constant. By direct substitution, we see that this solution satisfies all the conditions.

Part c

For the potential of part (b), the velocity is

$$\vec{v} = -\nabla\phi = 2 \frac{v_o}{a} (x\vec{i}_x - y\vec{i}_y) \quad (e)$$

Using Bernoulli's equation, we obtain

$$p_o = p + 2 \left(\frac{v_o}{a} \right)^2 (x^2 + y^2) \quad (f)$$

The net force on the wall between $x=c$ and $x=d$ is

$$\vec{f} = \int_{z=0}^{z=w} \int_{x=c}^{x=d} (p_o - p) dx dz \vec{i}_y \quad (g)$$

where w is the depth of the wall.

Thus

$$\begin{aligned} \vec{f} &= + \frac{\left(\frac{v_o}{a} \right)^2}{6} w \int_c^d x^2 dx \vec{i}_y \\ &= + \frac{\left(\frac{v_o}{a} \right)^2}{6} w (d^3 - c^3) \vec{i}_y \end{aligned} \quad (h)$$

Part d

The acceleration is

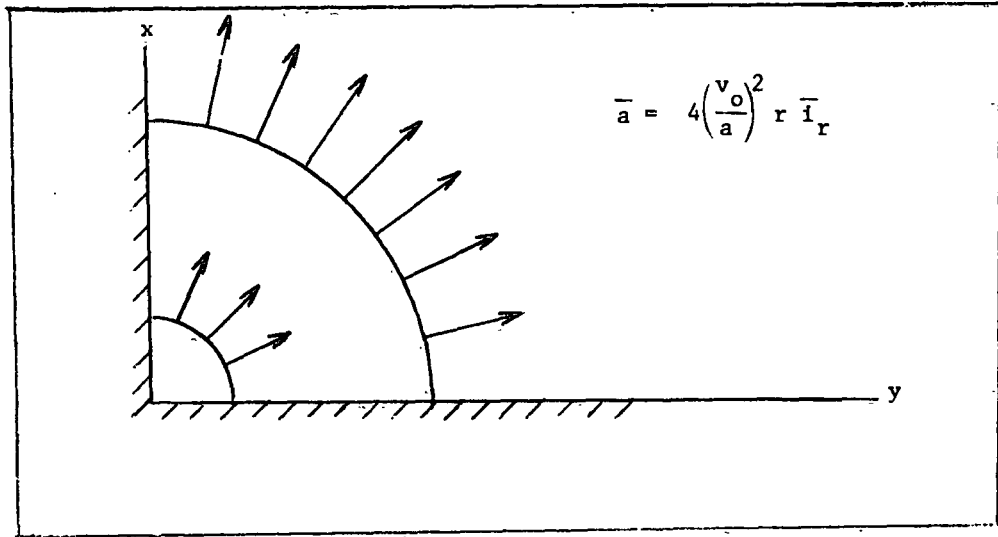
$$\vec{a} = (\vec{v} \cdot \nabla) \vec{v} = 2 \frac{v_o}{a} x \left(2 \frac{v_o}{a} \vec{i}_x \right) - 2 \frac{v_o}{a} y \left(-2 \frac{v_o}{a} y \vec{i}_y \right).$$

or

$$\vec{a} = 4 \left(\frac{v_o}{a} \right)^2 (x\vec{i}_x + y\vec{i}_y) \quad (i)$$

or in cylindrical coordinates

$$\vec{a} = 4 \left(\frac{v_o}{a} \right)^2 r \vec{i}_r \quad (j)$$

PROBLEM 12.10 (Continued)PROBLEM 12.11Part a

Since the $\nabla \cdot \bar{v} = 0$, we must have

$$V_o h = v_x(x)(h - \xi) \quad (a)$$

or

$$v_x(x) = \frac{V_o h}{h - \xi} \approx V_o \left(1 + \frac{\xi}{h} \right) \quad (b)$$

Part b

Using Bernoulli's law, we have

$$\frac{1}{2} \rho V_o^2 + p_o = \frac{1}{2} \rho [v_x(x)]^2 + P \quad (c)$$

$$P = P_o + \frac{1}{2} \rho V_o^2 - \frac{1}{2} \rho V_o^2 \left(1 + \frac{\xi}{h} \right)^2 \quad (d)$$

Part c

We linearize P around $\xi = 0$ to obtain

$$P \approx P_o - \rho V_o^2 \frac{\xi}{h} \quad (e)$$

Thus

$$T_z = -P + P_o = \rho V_o^2 \frac{\xi}{h} \quad (f)$$

PROBLEM 12.11 (continued)

$$\begin{aligned} \text{Thus } T_z &= C\xi \\ \text{with } C &= \frac{\rho V_o^2}{h} \end{aligned} \quad (g)$$

Part d

We can write the equations of motion of the membrane as

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \frac{\partial^2 \xi}{\partial x^2} + T_z \quad (h)$$

$$= S \frac{\partial^2 \xi}{\partial x^2} + C\xi \quad (i)$$

We assume

$$\xi(x,t) = \text{Re } \hat{\xi} e^{j(\omega t - kx)} \quad (j)$$

Solving for the dispersion relation, we obtain

$$-\sigma_m \omega^2 = -Sk^2 + C \quad (k)$$

or

$$\omega = \left[\frac{S}{\sigma_m} k^2 - \frac{C}{\sigma_m} \right]^{1/2} \quad (l)$$

Now, since the membrane is fixed at $x = 0$ and $x = L$, we know that

$$k = \frac{n\pi}{L} \quad n = 1, 2, 3, \dots \quad (m)$$

Now if

$$S \left(\frac{\pi}{L} \right)^2 - C < 0 \quad (n)$$

we realize that the membrane will become unstable.

So for

$$\frac{\rho V_o^2}{h} < S \left(\frac{\pi}{L} \right)^2 \quad (o)$$

we have stability.

Part e

As ξ increases, the velocity of the flow above the membrane increases, since the fluid is incompressible. Through Bernoulli's law, the pressure on the membrane must decrease, thereby increasing the net upwards force on the membrane, which tends to make ξ increase even further, thus making the membrane become unstable.

PROBLEM 12.12Part a

We wish to write the equation of motion for the membrane.

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \frac{\partial^2 \xi}{\partial x^2} + p_1(\xi) - p_o + T^e - \sigma_m g \quad (a)$$

where

$$T^e = \frac{\epsilon_o}{2} \left(\frac{V_o}{d-\xi} \right)^2 \approx \frac{\epsilon_o}{2} \frac{V_o^2}{d^2} \left(1 + \frac{2\xi}{d} \right)$$

is the electric force per unit area on the membrane.

In the equilibrium $\xi(x,t) = 0$, we must have

$$p_1(0) = p_o - \frac{\epsilon_o}{2} \left(\frac{V_o}{d} \right)^2 + \sigma_m g \quad (b)$$

As in example 12.1.3

$$p_1 = -\rho g y + C$$

and, using the boundary condition of (b), we obtain

$$p_1 = -\rho g y + \sigma_m g + p_o - \frac{\epsilon_o}{2} \left(\frac{V_o}{d} \right)^2 \quad (c)$$

Part b

We are interested in calculating the perturbations in p_1 for small deflections of the membrane. From Bernoulli's law, a constant of motion of the fluid is D , where D equals

$$D = \frac{1}{2} \rho U^2 + \sigma_m g + p_o - \frac{\epsilon_o}{2} \left(\frac{V_o}{d} \right)^2 \quad (d)$$

For small perturbations $\xi(x,t)$, the velocity in the region $0 \leq x \leq L$ is

$$v = \frac{Ud}{d+\xi}$$

We use Bernoulli's law to write

$$\frac{1}{2} \rho v^2 + p_1(\xi) + \rho g \xi = D \quad (e)$$

Since we have already taken care of the equilibrium terms, we are interested only in small changes of p_1 , so we omit constant terms in our linearization of p_1 .

Thus

$$p_1(\xi) = -\rho g \xi + \frac{\rho U^2 \xi}{d} \quad (f)$$

Thus, our linearized force equation is

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \frac{\partial^2 \xi}{\partial x^2} + \left(\frac{\rho U^2}{d} - \rho g + \frac{\epsilon_o V_o^2}{d^3} \right) \xi \quad (g)$$

We define

$$C = -\rho g + \frac{\rho U^2}{d} + \frac{\epsilon_o V_o^2}{d^3}$$

and assume solutions of the form

$$\xi(x,t) = \text{Re } \hat{\xi} e^{j(\omega t - kx)}$$

PROBLEM 12.12 (Continued)

from which we obtain the dispersion relation

$$\omega = \left(\frac{S}{\sigma_m} k^2 - \frac{C}{\sigma_m} \right)^{1/2} \quad (h)$$

Since the membrane is fixed at $x=0$ and at $x=L$

$$k = \frac{n\pi}{L} \quad n = 1, 2, 3, \dots \quad (i)$$

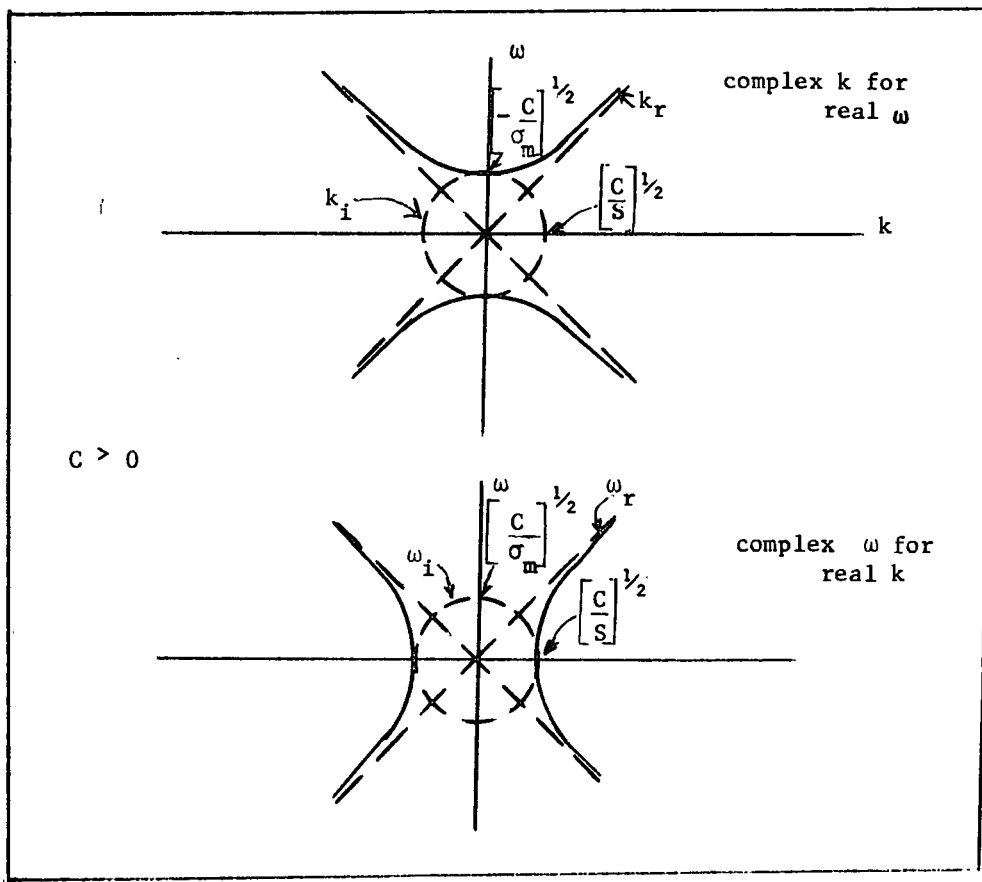
If $C < 0$, then ω is always real, and we can have oscillation about the equilibrium.
For $C > S(\frac{\pi}{L})^2$, then ω will be imaginary, and the system is unstable.

Part c

The dispersion relation is thus

$$\omega = \left(\frac{S}{\sigma_m} k^2 - \frac{C}{\sigma_m} \right)^{1/2}$$

Consider first $C < 0$



PROBLEM 12.12 (Continued)Part d

Since the membrane is not moving, one wave propagates upstream and the other propagates downstream. Thus, to find the solution we need two boundary conditions, one upstream and one downstream. If, however, both waves had propagated downstream, then causality does not allow us to apply a downstream boundary condition. This is not the case here.

PROBLEM 12.13Part a

Since $\nabla \cdot \vec{v} = 0$, in the region $0 \leq x \leq L$,

$$v_x = \frac{V_o d}{d + \xi_1 - \xi_2} \approx V_o \left[1 - \frac{(\xi_1 - \xi_2)}{d} \right] \quad (a)$$

where d is the spacing between membranes. Using Bernoulli's law, we can find the pressure p_1 right below membrane 1, and pressure p_2 right above membrane 2.

Thus

$$\frac{1}{2} \rho V_o^2 + p_o = \frac{1}{2} \rho v_x^2 + p_1 \quad (b)$$

and

$$\frac{1}{2} \rho V_o^2 + p_o = \frac{1}{2} \rho v_x^2 + p_2 \quad (c)$$

Thus

$$p_1 = p_2 \approx p_o + \frac{\rho V_o^2 (\xi_1 - \xi_2)}{d} \quad (d)$$

We may now write the equations of motion of the membranes as

$$\sigma_m \frac{\partial^2 \xi_1}{\partial t^2} = S \frac{\partial^2 \xi_1}{\partial x^2} + (p_1 - p_o) = S \frac{\partial^2 \xi_1}{\partial x^2} + \frac{\rho V_o^2 (\xi_1 - \xi_2)}{d} \quad (e)$$

$$\sigma_m \frac{\partial^2 \xi_2}{\partial t^2} = S \frac{\partial^2 \xi_2}{\partial x^2} + p_o - p_2 = S \frac{\partial^2 \xi_2}{\partial x^2} - \frac{\rho V_o^2 (\xi_1 - \xi_2)}{d} \quad (f)$$

Assume solutions of the form

$$\begin{aligned} \xi_1 &= \text{Re } \hat{\xi}_1 e^{j(\omega t - kx)} \\ \xi_2 &= \text{Re } \hat{\xi}_2 e^{j(\omega t - kx)} \end{aligned} \quad (g)$$

Substitution of these assumed solutions into our equations of motion will yield the dispersion relation

$$\begin{aligned} -\sigma_m \omega^2 \hat{\xi}_1 &= -S k^2 \hat{\xi}_1 + \frac{\rho V_o^2}{d} (\hat{\xi}_1 - \hat{\xi}_2) \\ -\sigma_m \omega^2 \hat{\xi}_2 &= -S k^2 \hat{\xi}_2 + \frac{\rho V_o^2}{d} (\hat{\xi}_2 - \hat{\xi}_1) \end{aligned} \quad (h)$$

These equations may be rewritten as

PROBLEM 12.13 (Continued)

$$\begin{aligned} \hat{\xi}_1 \left[-\sigma_m \omega^2 + Sk^2 - \frac{\rho V_o^2}{d} \right] + \hat{\xi}_2 \left[+ \frac{\rho V_o^2}{d} \right] &= 0 \\ \hat{\xi}_1 \left[\frac{\rho V_o^2}{d} \right] + \hat{\xi}_2 \left[-\sigma_m \omega^2 + Sk^2 - \frac{\rho V_o^2}{d} \right] &= 0 \end{aligned} \quad (i)$$

For non-trivial solution, the determinant of coefficients of ξ_1 and ξ_2 must be zero.

$$\text{Thus} \quad \left[-\sigma_m \omega^2 + Sk^2 - \frac{\rho V_o^2}{d} \right]^2 = \left[\frac{\rho V_o^2}{d} \right]^2 \quad (j)$$

$$\text{or} \quad -\sigma_m \omega^2 + Sk^2 - \frac{\rho V_o^2}{d} = \pm \frac{\rho V_o^2}{d} \quad (k)$$

If we take the upper sign (+) on the right-hand side of the above equation, we obtain

$$\omega = \left[\frac{S}{\sigma_m} k^2 - \frac{2\rho V_o^2}{\sigma_m d} \right]^{1/2} \quad (l)$$

We see that if V_o is large enough, ω can be imaginary. This can happen when

$$V_o^2 > \frac{Sk^2 d}{2\rho} \quad (m)$$

Since the membranes are fixed at $x=0$ and $x=L$

$$k = \frac{n\pi}{L} \quad n = 1, 2, 3, \dots \quad (n)$$

So the membranes first become unstable when

$$V_o^2 > \frac{S \left(\frac{\pi}{L} \right)^2 d}{2\rho} \quad (o)$$

For this choice of sign (+), $\xi_1 = -\xi_2$, so we excite the odd mode. If we had taken the negative sign, then the even mode would be excited

$$\xi_1 = \xi_2.$$

However, the dispersion relation is then

$$\omega = \pm \frac{S}{\sigma_m} k$$

and then we would have no instability.

Part b

The odd mode is unstable.



or



PROBLEM 12.14Part a

The force equation in the y direction is

$$\frac{\partial p}{\partial y} = -\rho g \quad (a)$$

Thus

$$p = -\rho g(y-\xi) \quad (b)$$

where we have used the fact that at $y = \xi$, the pressure is zero.

Part b

$\nabla \cdot \vec{v} = 0$ implies

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad (c)$$

Integrating with respect to y, we obtain

$$v_y = -\frac{\partial v_x}{\partial x} y + C \quad (d)$$

where C is a constant of integration to be evaluated by the boundary condition at $y = -a$, that

$$v_y(y = -a) = 0$$

since we have a rigid bottom at $y = -a$.

Thus

$$v_y = -\frac{\partial v_x}{\partial x} (y+a) \quad (e)$$

Part c

The x-component of the force equation is

$$\rho \frac{\partial v_x}{\partial t} = -\frac{\partial p}{\partial x} = -\rho g \frac{\partial \xi}{\partial x} \quad (f)$$

or

$$\frac{\partial v_x}{\partial t} = -g \frac{\partial \xi}{\partial x} \quad (g)$$

Part d

At $y = \xi$,

$$v_y = \frac{\partial \xi}{\partial t} \quad (h)$$

Thus, from part (b), at $y = \xi$

$$\frac{\partial \xi}{\partial t} = -\frac{\partial v_x}{\partial x} (\xi+a) \quad (i)$$

However, since $\xi \ll a$, and v_x and v_y are small perturbation quantities, we can approximately write

$$\frac{\partial \xi}{\partial t} = -a \frac{\partial v_x}{\partial x} \quad (j)$$

Part e

Our equations of motion are now

PROBLEM 12.14 (Continued)

$$\frac{\partial \xi}{\partial t} = -a \frac{\partial v_x}{\partial x} \quad (k)$$

and

$$\frac{\partial v_x}{\partial t} = -g \frac{\partial \xi}{\partial x} \quad (l)$$

 If we take $\partial/\partial x$ of (k) and $\partial/\partial t$ of (l) and then simplify, we obtain

$$\frac{\partial^2 v_x}{\partial t^2} = ag \frac{\partial^2 v_x}{\partial x^2} \quad (m)$$

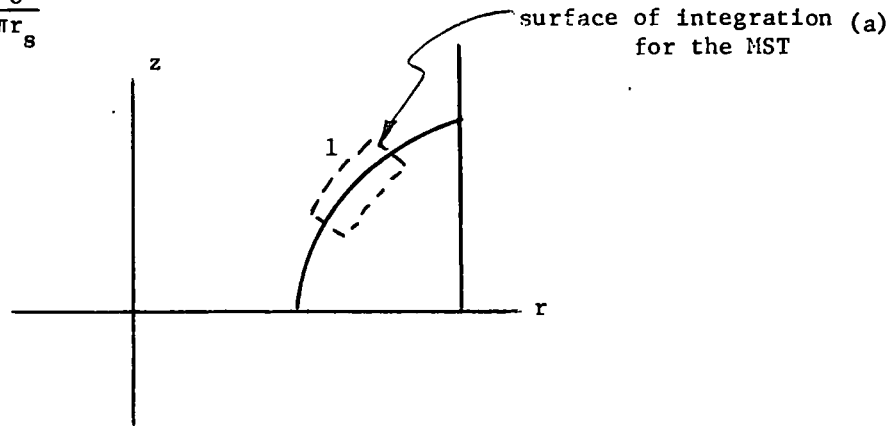
We recognize this as the wave equation for gravity waves, with phase velocity

$$v_p = \sqrt{ag} \quad (n)$$

PROBLEM 12.15
Part a

 As shown in Fig. 12P.15b, the H field is in the $-\hat{i}$ direction with magnitude:

$$|H_s| = \frac{I_o}{2\pi r_s}$$



If we integrate the MST along the surface defined in the above figure, the only contribution will be along surface (1), so we obtain for the normal traction

$$\tau_n = -\frac{1}{2} \mu_o |H_s|^2 = -\frac{1}{8} \frac{\mu_o I_o^2}{\pi^2 r_s^2} \quad (b)$$

Part b

Since the net force on the interface must be zero, we must have

$$\tau_n + p_{int} - p_o = 0 \quad (c)$$

 where p_{int} is the hydrostatic pressure on the fluid side of the interface.

PROBLEM 12.15 (continued)

Thus
$$p_{int} = p_o + \frac{1}{8} \frac{\mu_o I_o^2}{\pi^2 r^2} \quad (d)$$

Within the fluid, the pressure p must obey the relation

$$\frac{\partial p}{\partial z} = -\rho g \quad (e)$$

or
$$p = -\rho g z + C \quad (f)$$

Let us look at the point $z = z_o$, $r = R_o$. There

$$p = -\rho g z_o + C = p_o + \frac{1}{8} \frac{\mu_o I_o^2}{\pi^2 R_o^2} \quad (g)$$

Therefore

$$C = \rho g z_o + p_o + \frac{1}{8} \frac{\mu_o I_o^2}{\pi^2 R_o^2} \quad (h)$$

Now let's look at any point on the interface with coordinates z_s , r_s

Then, by Bernoulli's law,

$$p_o + \frac{1}{8} \frac{\mu_o I_o^2}{\pi^2 R_o^2} + \rho g z_o = \frac{1}{8} \frac{\mu_o I_o^2}{\pi^2 r_s^2} + p_o + \rho g z_s \quad (i)$$

Thus, the equation of the surface is

$$\rho g z_s + \frac{1}{8} \frac{\mu_o I_o^2}{\pi^2 r_s^2} = \rho g z_o + \frac{1}{8} \frac{\mu_o I_o^2}{\pi^2 R_o^2} \quad (j)$$

Part c

The total volume of the fluid is

$$V = \pi [R_o^2 - (\frac{b}{2})^2] a. \quad (k)$$

We can find the value of z_o by finding the volume of the deformed fluid in terms of z_o , and then equating this volume to V .

Thus
$$V = \pi [R_o^2 - (\frac{b}{2})^2] a = 2\pi \int_{r=r_o}^{R_o} \int_{z=0}^{z_o + \frac{1}{8} \frac{\mu_o I_o^2}{\rho g \pi^2} \left(\frac{1}{R_o^2} - \frac{1}{r^2} \right)} r dr dz \quad (l)$$

where

r_o is that value of r when $z = 0$, or

$$r_o = \left[\frac{\frac{1}{8} \frac{\mu_o I_o^2}{\pi^2}}{\rho g z_o + \frac{1}{8} \frac{\mu_o I_o^2}{\pi^2 R_o^2}} \right]^{1/2} \quad (m)$$

Evaluating this integral, and equating to V , will determine z_o .

PROBLEM 12.16

We do an analysis similar to that of Sec. 12.2.1a, to obtain

$$E = - \bar{i}_y \frac{V}{w} \quad (a)$$

and
$$\bar{J} = \bar{i}_y \sigma \left(-\frac{V}{w} + vB \right) = \frac{I}{\ell d} \bar{i}_y \quad (b)$$

Here
$$V = IR + V_o \quad (c)$$

Thus
$$I = \frac{vBw - V_o}{R + \frac{w}{\ell d \sigma}}$$

The electric power out is

$$P_e = VI = (IR + V_o)I = \left[V_o + \frac{R(vBw - V_o)}{R + \frac{w}{\ell d \sigma}} \right] \left[\frac{vBw - V_o}{R + \frac{w}{\ell d \sigma}} \right] \quad (e)$$

From equations (12.2.23 - 12.2.25)

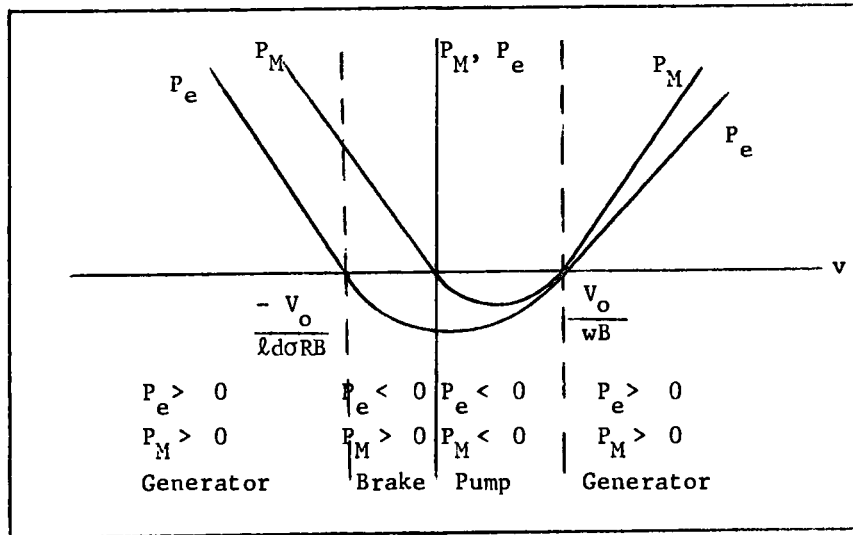
we have

$$\Delta p = p(0) - p(\ell) = \frac{IB}{d} \quad (f)$$

Thus, the mechanical power in is

$$P_M = (\Delta p w d) v = \frac{Bw(vBw - V_o)v}{R + \frac{w}{\ell d \sigma}} \quad (g)$$

Plots of P_E and P_M versus v specify the operating regions of the MHD machine.



PROBLEM 12.17Part a

The mechanical power input is

$$P_M = - \int_{z=0}^L \int_{y=0}^w \int_{x=0}^d \nabla p v_o dx dy dz \quad (a)$$

The force equation in the steady state is

$$-\nabla p + f^e = 0 \quad (b)$$

where

$$f^e = -J_y B_o \quad (c)$$

Thus

$$P_M = \int_{z=0}^L \int_{y=0}^w \int_{x=0}^d J_y B_o v_o dx dy dz \quad (d)$$

Now

$$J_y = \sigma(E_y + v_o B_o) = \sigma(-\frac{\partial \phi}{\partial y} + v_o B_o) \quad (e)$$

Integrating, we obtain

$$\begin{aligned} P_M &= \sigma v_o^2 B_o L w d - \sigma B_o v_o V L d \\ &= \frac{v_{oc}^2}{R_i} - \frac{V v_{oc}}{R_i} = \frac{1}{R_i} (v_{oc} - V) v_{oc} \end{aligned} \quad (f)$$

Part b

$$\text{Defining } \eta = \frac{P_{out}}{P_M}$$

we have

$$\eta = \frac{(v_{oc} - V)V - aV^2}{(v_{oc} - V)v_{oc}} \quad (g)$$

First, we wish to find what terminal voltage maximizes P_{out} . We take

$$\frac{\partial P_{out}}{\partial V} = 0 \text{ and find that}$$

$$V = \frac{v_{oc}}{2(1+a)} \text{ maximizes } P_{out}.$$

For this value of V , η equals

$$\eta = \frac{1}{2} \frac{1}{(1+2a)} \quad (h)$$

Plotting η vs. $\frac{1}{a}$ gives



PROBLEM 12.17(Continued)

Now, we wish to find what voltage will give maximum efficiency, so we take

$$\frac{\partial \eta}{\partial V} = 0$$

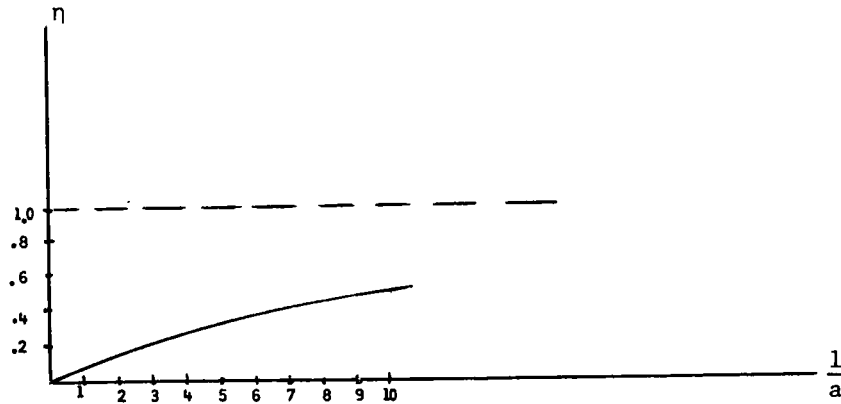
Solving for the maximum, we obtain

$$V = V_{oc} \left[1 \pm \sqrt{\frac{a}{1+a}} \right] \quad (i)$$

We choose the negative sign, since $V \leq V_{oc}$ for generator operation. We thus obtain

$$\eta = 1 + 2a - 2\sqrt{a(1+a)} \quad (j)$$

Plotting η vs. $\frac{1}{a}$, we obtain

PROBLEM 12.18

From Fig. 12P.18, we have

$$\bar{E} = \frac{V}{w} \bar{i}_y$$

and

$$\bar{J} = \bar{i}_y \sigma \left[\frac{V}{w} + vB \right] = \frac{I}{LD} \bar{i}_y \quad (b)$$

The z component of the force equation is

$$-\frac{\partial p}{\partial z} - \frac{I}{LD} B = 0 \quad (c)$$

$$\text{or} \quad \Delta p = p_1 - p_0 = \frac{IB}{D} = \Delta p_0 \left(1 - \frac{v}{v_0} \right) \quad (d)$$

Solving for v , we obtain

$$v = \left(1 - \frac{IB}{D \Delta p_0} \right) v_0 \quad (e)$$

PROBLEM 12.18 (Continued)

Thus, we have

$$\frac{I}{LD\sigma} = \frac{V}{w} + B\left(1 - \frac{IB}{D\Delta p_o}\right)v_o \quad (f)$$

or

$$V = I\left(\frac{w}{LD\sigma} + \frac{B^2 v_o w}{D\Delta p_o}\right) - v_o Bw \quad (g)$$

Thus, for our equivalent circuit

$$R'_1 = \frac{w}{LD\sigma} + \frac{v_o w B^2}{D\Delta p_o} \quad (h)$$

and

$$V_{oc} = -v_o w B \quad (i)$$

We notice that the current I in Fig. 12P.18b is not consistent with that of Fig. 12P.18a. It should be defined flowing in the other direction.

PROBLEM 12.19

Using Ampere's law

$$H_o = \frac{N_o I_o + N_L I_L}{d} \quad (a)$$

Within the fluid

$$\bar{J} = \frac{I_L}{\ell d} \bar{i}_z = \sigma\left(-\frac{V_L}{w} + v_o \mu_o H_o\right) \bar{i}_z \quad (b)$$

Simplifying, we obtain

$$I_L \left[\frac{1}{\ell d} - \frac{\sigma v_o \mu_o N_L}{d} \right] = \frac{\sigma v_o \mu_o N_o I_o}{d} - \frac{\sigma V_L}{w} \quad (c)$$

 For V_L to be independent of I_L , we must have

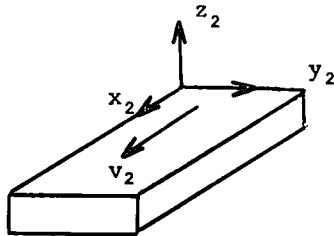
$$\frac{\sigma v_o \mu_o N_L}{d} = \frac{1}{\ell d} \quad (d)$$

or

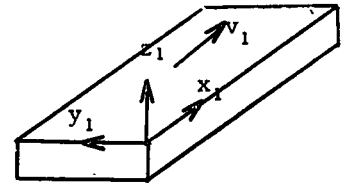
$$N_L = \frac{1}{\ell \sigma v_o \mu_o} \quad (e)$$

PROBLEM 12.20

We define coordinate systems as shown below.



MHD #2.



MHD # 1

PROBLEM 12.20 (Continued)

Now, since $\nabla \cdot \bar{v} = 0$, we have

$$v_1 w_1 d_1 = v_2 w_2 d_2$$

In system (2),

$$\bar{J}_2 = \bar{i}_{y2} \frac{I_2}{\ell_2 d_2} = -\sigma \left(\frac{V_2}{w_2} + v_2 B \right) \bar{i}_{y2} \quad (a)$$

and

$$\Delta p_2 = p(0_+) - p(\ell_{2-}) = -\frac{I_2 B}{d_2} \quad (b)$$

In system (1),

$$\bar{J}_1 = \bar{i}_{y1} \frac{I_1}{\ell_1 d_1} = \sigma \left(\frac{V_1}{w_1} - v_1 B \right) \quad (c)$$

and

$$\Delta p_1 = p(0_+) - p(\ell_{1-}) = -\frac{I_1 B}{d_1} \quad (d)$$

By applying Bernoulli's law at the points $x_1 = 0_-$ (right before MHD system 1) and at $x_1 = \ell_{1+}$ (right after MHD system 1), we obtain

$$\frac{1}{2} \rho v_1^2 + p_1(0_-) = \frac{1}{2} \rho v_1^2 + p_1(\ell_{1+}) \quad (e)$$

or

$$p_1(0_-) = p_1(\ell_{1+}) \quad (f)$$

Similarly on MHD system (2):

$$p_2(0_-) = p_2(\ell_{2+}) \quad (g)$$

Now,

$$\oint_C \nabla p \cdot d\ell = 0$$

Applying this relation to a closed contour which follows the shape of the channel, we obtain

$$\begin{aligned} \oint_C \nabla p \cdot d\ell &= \int_{x_1=0_+}^{\ell_{1-}} \nabla p \cdot d\ell + \int_{x_1=\ell_{1+}}^{x_2=0_-} \nabla p \cdot d\ell + \int_{x_2=0_+}^{x_2=\ell_{2-}} \nabla p \cdot d\ell + \int_{x_2=\ell_{2+}}^{x_1=0_-} \nabla p \cdot d\ell \\ &= p_1(\ell_{1-}) - p_1(0_+) + p_2(0_-) - p_1(\ell_{1+}) + p_2(\ell_{2-}) \\ &\quad - p_2(0_+) + p_1(0_-) - p_2(\ell_{2+}) \end{aligned} \quad (h)$$

From (f) and (g) we reduce this to

$$\Delta p_1 + \Delta p_2 = 0 \quad (i)$$

or

$$\frac{I_1}{d_1} = -\frac{I_2}{d_2} \quad (j)$$

PROBLEM 12.20 (Continued)

Thus, we may express v_1 as

$$v_1 = \left(+ \frac{I_2}{\ell_1 d_2 \sigma} + \frac{V_1}{w_1} \right) \frac{1}{B} \quad (k)$$

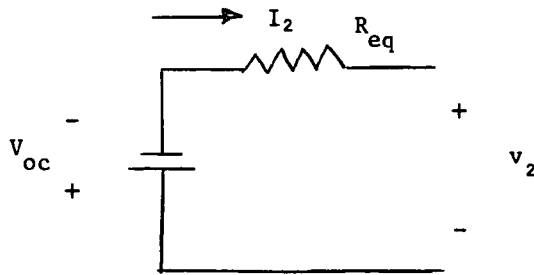
We substitute this into our original equation for J_2 (a), to obtain

$$\frac{I_2}{\ell_2 d_2} = -\sigma \frac{V_2}{w_2} - \sigma \left(\frac{w_1 d_1}{w_2 d_2 \ell_1} \right) \left(\frac{I_2}{\ell_1 d_2 \sigma} + \frac{V_1}{w_1} \right) \quad (l)$$

This may be rewritten as

$$V_2 = -I_2 \frac{w_2}{\sigma} \left[\frac{1}{\ell_2 d_2} + \frac{w_1 d_1}{w_2 \ell_1 d_2} \right] - \frac{d_1}{d_2} V_1 \quad (m)$$

The Thevenin equivalent circuit is:



where

$$V_{oc} = \frac{d_1}{d_2} V_1$$

and

$$R_{eq} = \frac{w_2}{\sigma d_2} \left[\frac{1}{\ell_2} + \frac{w_1 d_1}{w_2 d_2 \ell_1} \right]$$

PROBLEM 12.21

For the MHD system

$$|\vec{J}| = \frac{I}{LW} = \sigma \left(\frac{V_o}{D} - v \mu_o H_o \right) \quad (a)$$

and

$$\Delta p = p_1 - p_2 = + \frac{I \mu_o H_o}{w} \quad (b)$$

Now, since

$$\oint_C \vec{V}_p \cdot d\vec{\ell} = 0 \quad (c)$$

we must have

$$\Delta p = kv = \mu_o H_o L \sigma \left(\frac{V_o}{D} - v \mu_o H_o \right) \quad (d)$$

Solving for v , we obtain

$$v = \frac{\mu_o H_o L \sigma V_o}{D [k + (\mu_o H_o)^2 L \sigma]} \quad (e)$$

PROBLEM 12.22Part a

We assume that the fluid flows in the $+x$ direction with velocity v .

Thus

$$\vec{J} = \vec{I}_3 \quad \frac{I}{Lw} = \sigma \left(\frac{V}{d} + v\mu_o H_o \right) i_3 \quad (a)$$

where I is defined as flowing out of the positive terminal of the voltage source V_o .

We write the x_1 component of the force equation as

$$-\frac{\partial p}{\partial x_1} - \frac{I\mu_o H_o}{Lw} - \rho g = 0 \quad (b)$$

Thus

$$p = - \left(\frac{I\mu_o H_o}{Lw} + \rho g \right) x_1 \quad (c)$$

For $\Delta p = p(0) - p(L) = 0$

Then

$$\frac{I\mu_o H_o}{Lw} = -\rho g \quad (d)$$

For the external circuit shown,

$$V = -IR + V_o \quad (e)$$

Solving for I we get

$$I = \frac{\frac{V_o}{d} + v\mu_o H_o}{\frac{1}{\sigma Lw} + \frac{R}{d}} = \frac{-\rho g Lw}{\mu_o H_o} \quad (f)$$

Solving for the velocity, v , we get

$$v = \frac{-\frac{\rho g Lw}{\mu_o H_o} \left(\frac{1}{\sigma Lw} + \frac{R}{d} \right) - \frac{V_o}{d}}{\mu_o H_o} \quad (g)$$

For $v > 0$, then

$$V_o < \frac{-\rho g}{\mu_o H_o} \left(\frac{d}{\sigma} + RLw \right) \quad (h)$$

Part b

If the product $V_o I > 0$, then we are supplying electrical power to the fluid. From part (a), (f) and (h), V_o is always negative, but so is I . So the product $V_o I$ is positive.

PROBLEM 12.23

Since the electrodes are short-circuited,

$$\bar{J} = \bar{i}_z \frac{I}{\ell d} = \sigma v B_o \bar{i}_z \quad (a)$$

In the upper reservoir

$$p_1 = p_o + \rho g(h_1 - y) \quad (b)$$

while in the lower reservoir

$$p_2 = p_o + \rho g(h_2 - y) \quad (c)$$

The pressure drop within the MHD system is

$$\Delta p = p(0) - p(\ell) = \frac{IB}{d} \quad (d)$$

Integrating along the closed contour from $y=h_1$ through the duct to $y=h_2$, and then back to $y=h_1$ we obtain

$$\oint_C \nabla p \cdot d\ell = 0 = -\rho g(h_1 - h_2) + \frac{IB}{d} \ell \quad (e)$$

Thus

$$I = \frac{\rho g(h_1 - h_2)d}{B\ell} \quad (f)$$

and so

$$v = \frac{I}{\sigma \ell d B_o} = \frac{\rho g(h_1 - h_2)}{\sigma \ell^2 B_o} \quad (g)$$

PROBLEM 12.24Part a

We define the velocity v_h as the velocity of the fluid at the top interface, where

$$v_h = -\frac{dh}{dt} \quad (a)$$

Since $\nabla \cdot v = 0$, we have

$$v_h A = v_e w D \quad (b)$$

where v_e is the velocity of flow through the MHD generator (assumed constant). We assume that accelerations of the fluid are negligible. When we obtain the solution, we must check that these approximations are reasonable. With these approximations, the pressure in the storage tank is

$$p = -\rho g(y-h) + p_o \quad (c)$$

where p_o is the atmospheric pressure and y the vertical coordinate. The pressure drop in the MHD generator is

$$\Delta p = \frac{I \mu_o H_o}{D} \quad (d)$$

where I is defined positive flowing from right to left within the generator in the end view of Fig. 12P.24.

PROBLEM 12.24 (continued)

We have also assumed that within the generator, v_e does not vary with position. The current within the generator is

$$\frac{I}{L_o D} = \sigma \left(-\frac{IR}{w} + v_e \mu_o H_o \right) \quad (e)$$

Solving for I , we obtain

$$I = \frac{v_e \mu_o H_o}{\frac{1}{\sigma L_o D} + \frac{R}{w}} \quad (f)$$

Now, since $\oint \nabla p \cdot d\mathbf{l} = 0$, we have

$$\Delta p - \rho gh = 0 \quad (g)$$

Thus, using (d), (f) and (g), we obtain

$$-\rho gh + \frac{(\mu_o H_o)^2}{D} \left[\frac{1}{\frac{R}{w} + \frac{1}{\sigma L_o D}} \right] v_e = 0 \quad (h)$$

Using (b), we finally obtain

$$\frac{dh}{dt} + sh = 0 \quad (i)$$

where

$$s = \frac{\rho g w}{(\mu_o H_o)^2} \frac{D}{A} \left[\frac{RD}{w} + \frac{1}{\sigma L_o} \right]$$

Thus $h = 10 e^{-st}$, until time τ , when the valve closes
at $h = 5$. (j)

Numerically

$$s = 7.1 \times 10^{-3}, \text{ thus } \tau \approx 100 \text{ seconds.}$$

For our approximations to be valid, we must have

$$\left| \rho \frac{\partial v_h}{\partial t} \right| \ll \rho g \quad (k)$$

or

$$s^2 h \ll g.$$

Also, we must have

$$\left| \frac{1}{2} \rho v_h^2 \right| \ll \left| \rho gh \right|$$

or

$$\frac{1}{2} s^2 h \ll g \quad (l)$$

Our other approximation was

$$\left| \rho L_o \frac{\partial v_e}{\partial t} \right| \ll \left| \frac{I \mu_o H_o}{D} \right| \quad (m)$$

which implies from (f) that

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PROBLEM 12.24 (continued)

$$\rho s L_o \ll \frac{(\mu_o H_o)^2}{D \left[\frac{R}{w} + \frac{1}{\sigma L_o D} \right]} \quad (n)$$

Substituting numerical values, we see that our approximations are all reasonable.

Part b

From (b) and (f)

$$I = \frac{\mu_o H_o A}{w d \left[\frac{1}{\sigma L_o D} + \frac{R}{w} \right]} \frac{\partial h}{\partial t}$$

$$= - 650 \times 10^3 e^{-st} \text{ amperes.}$$

until $t = 100$ seconds, where $I = -325 \times 10^3$ amperes. Once the valve is closed, $I = 0$.

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PROBLEM 12.25

Part a

Within the MHD system

$$\bar{J} = \frac{-i}{L_1 D} \bar{i}_3 = -\sigma \left(\frac{V}{w} - v \mu_o H_o \right) \bar{i}_3 \quad \text{where } V = -iR + V_o \quad (a)$$

$$\text{and } \Delta p = p(0) - p(-L_1) = \frac{i \mu_o H_o}{D} \quad (b)$$

We are considering static conditions ($v=0$) so the pressure in tank 1 is

$$p_1 = -\rho g(x_2 - h_1) + p_o \quad (c)$$

and in tank 2 is

$$p_2 = -\rho g(x_2 - h_2) + p_o \quad (d)$$

where p_o is the atmospheric pressure,

thus

$$i = \frac{V_o}{w \left[\frac{1}{\sigma L_1 D} + \frac{R}{w} \right]} \quad (e)$$

Now since $\oint_C \nabla p \cdot d\mathbf{l} = 0$, we must have

$$+ \rho g h_1 + \frac{i \mu_o H_o}{D} - \rho g h_2 = 0 \quad (f)$$

Solving in terms of V_o we obtain

$$V_o = \frac{\rho g (h_2 - h_1) w D}{(\mu_o H_o)} \left(\frac{1}{\sigma L_1 D} + \frac{R}{w} \right) \quad (g)$$

For $h_2 = .5$ and $h_1 = .4$ and substituting for the given values of the parameters,

we obtain

$$V_o = 6.3 \text{ millivolts}$$

Under these static conditions, the current delivered is

$$i = \frac{\rho g (h_2 - h_1) D}{\mu_o H_o} = 210 \text{ amperes}$$

and the power delivered is

$$P_e = V_o i = \left[\frac{\rho g (h_2 - h_1) D}{\mu_o H_o} \right]^2 w \left[\frac{1}{\sigma L_1 D} + \frac{R}{w} \right] = 1.33 \text{ watts}$$

Part b

We expand h_1 and h_2 around their equilibrium values h_{10} and h_{20} to obtain

$$\begin{aligned} h_1 &= h_{10} + \Delta h_1 \\ h_2 &= h_{20} + \Delta h_2 \end{aligned}$$

PROBLEM 12.25 (Continued)

Since the total volume of the fluid remains constant

$$\Delta h_2 = -\Delta h_1$$

Since we are neglecting the acceleration in the storage tanks, we may still write

$$p_1 = -\rho g(x_2 - h_1) + p_0 \quad (h)$$

$$p_2 = -\rho g(x_2 - h_2) + p_0$$

Within the MHD section, the force equation is

$$\rho \frac{\partial v}{\partial t} = -\nabla p_{\text{MHD}} + \frac{i\mu_o H_o}{L_1 D} \quad (i)$$

Integrating with respect to x_1 , we obtain

$$\Delta p_{\text{MHD}} = p(0) - p(-L) = \frac{i\mu_o H_o}{L_1 D} - \rho L_1 \frac{\partial v}{\partial t} \quad (j)$$

The pressure drop over the rest of the pipe is

$$\Delta p_{\text{pipe}} = -L_2 \rho \frac{dv}{dt}$$

Again, since $\oint_C \nabla p \cdot d\mathbf{l} = 0$, we have

$$\rho g(h_1 - h_2) + \Delta p_{\text{MHD}} + \Delta p_{\text{pipe}} = 0 \quad (k)$$

For $t > 0$ we have

$$i = \frac{\frac{2V_o}{w} - v\mu_o H_o}{\frac{1}{\sigma L_1 D} + \frac{R}{w}} \quad (l)$$

and substituting into the above equation, we obtain

$$\rho g(h_1 - h_2) - \rho(L_1 + L_2) \frac{\partial v}{\partial t} + \left(\frac{\frac{2V_o}{w} - v\mu_o H_o}{\left[\frac{1}{\sigma L_1 D} + \frac{R}{w} \right]} \right) \frac{\mu_o H_o}{D} = 0 \quad (m)$$

We desire an equation just in Δh_2 . From the $\nabla \cdot \mathbf{v} = 0$, we obtain

$$vwD = \frac{d\Delta h_2}{dt} A \quad (n)$$

Making these substitutions, the resultant equation of motion is

$$\begin{aligned} \frac{d^2 \Delta h_2}{dt^2} + \frac{(\mu_o H_o)^2}{\rho(L_1 + L_2)D \left[\frac{1}{\sigma L_1 D} + \frac{R}{w} \right]} \frac{d\Delta h_2}{dt} + \frac{2gwd\Delta h_2}{(L_1 + L_2)A} \\ = \frac{V_o \mu_o H_o}{\rho(L_1 + L_2)A \left[\frac{1}{\sigma L_1 D} + \frac{R}{w} \right]} \end{aligned} \quad (o)$$

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PROBLEM 12.25 (continued)

Solving, we obtain

$$\Delta h_2 = \frac{V_o \mu_o H_o}{2\rho g w d \left(\frac{1}{\sigma L_1 D} + \frac{R}{w} \right)} + B_1 e^{s_1 t} + B_2 e^{s_2 t} \quad (p)$$

where B_1 and B_2 are arbitrary constants to be determined by initial conditions and

$$s_{1,2} = - \frac{[(\mu_o H_o)^2]}{2\rho(L_1 + L_2)D \left(\frac{1}{\sigma L_1 D} + \frac{R}{w} \right)} \pm \sqrt{\left(\frac{[(\mu_o H_o)^2]}{2\rho(L_1 + L_2)D \left(\frac{1}{\sigma L_1 D} + \frac{R}{w} \right)} \right)^2 - \frac{2gwd}{(L_1 + L_2)A}} \quad (q)$$

Substituting values, we obtain approximately

$$\begin{aligned} s_1 &= - .025 \text{ sec.}^{-1} \\ s_2 &= - .94 \text{ sec.}^{-1} \end{aligned}$$

The initial conditions are

$$\begin{aligned} \Delta h_2(t=0) &= 0 \\ \text{and } \frac{d\Delta h_2}{dt}(t=0) &= 0 \end{aligned}$$

Thus, solving for B_1 and B_2 we have

$$B_1 = \frac{- V_o \mu_o H_o}{2\rho g w d \left[\frac{1}{\sigma L_1 D} + \frac{R}{w} \right] \left(1 - \frac{s_1}{s_2} \right)} = - .051 \quad (r)$$

$$B_2 = \frac{- V_o \mu_o H_o}{2\rho g w d \left[\frac{1}{\sigma L_1 D} + \frac{R}{w} \right] \left(1 - \frac{s_2}{s_1} \right)} = + 1.36 \times 10^{-3}$$

Thus

$$h_2(t) = h_{20} + \Delta h_2(t) = .55 + 1.36 \times 10^{-3} e^{-.94t} - .051 e^{-.025t} \quad (s)$$

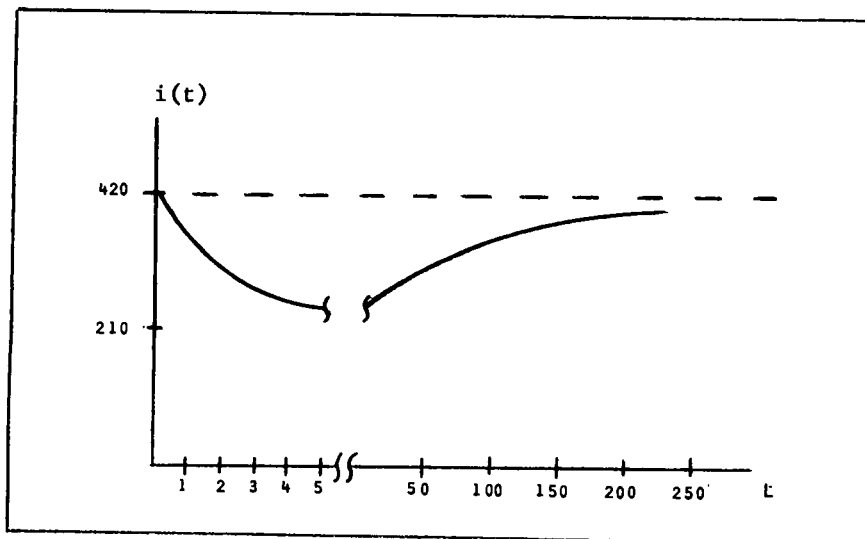
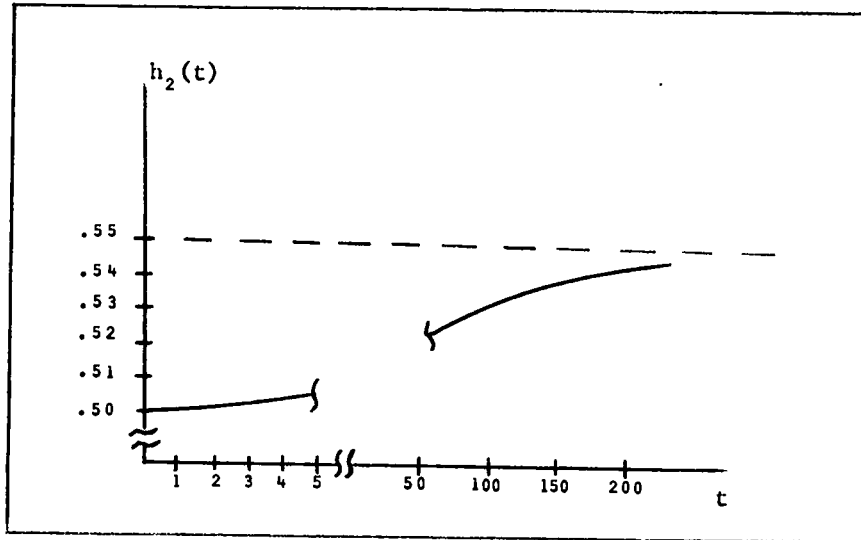
From (l) we have

$$i = \frac{\frac{2V_o}{w} - v\mu_o H_o}{\frac{R}{w} + \frac{1}{\sigma L_1 D}} \quad (t)$$

Substituting numerical values, we obtain

$$\begin{aligned} i &= 420 - 2.08 \times 10^5 (B_1 s_1 e^{s_1 t} + B_2 s_2 e^{s_2 t}) \\ &= 420 - 268 (e^{-.025t} - e^{-.94t}) \end{aligned} \quad (u)$$

PROBLEM 12.25 (continued)



PROBLEM 12.25 (continued)

Our approximations were made in (h) and (k). For them to be valid, the following relations must hold:

$$\frac{\frac{\partial^2 \Delta h_2}{\partial t^2}}{gh_2} \ll 1$$

and

$$\int \frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla) \bar{v} ds \approx \frac{\partial \bar{v}}{\partial t} \sqrt{A} \ll L_2 \frac{\partial \bar{v}}{\partial t}$$

transition
region

Substituting values, we find the first ratio to be about .001, so there our approximation is good to about .1%. In the second approximation

$$\frac{\sqrt{A}}{L_2} \approx \frac{.3}{2} \approx .15$$

Here, our approximation is good only to about 15%, which provides us with an idea of the error inherent in the approximation.

PROBLEM 12.26Part a

We use the same coordinate system as defined in Fig. 12P.25. The magnetic field through the pump is

$$\bar{B} = \frac{N\mu_o}{d} \bar{i}_2 \quad (a)$$

We integrate Newton's law across the length ℓ to obtain

$$\begin{aligned} \rho \ell \frac{\partial v}{\partial t} &= p(0) - p(\ell) + JB\ell = -\frac{\Delta p_o}{v_o} v + \frac{1}{d} B \\ &= -\frac{\Delta p_o}{v_o} v + \frac{N\mu_o}{d^2} i^2 \end{aligned} \quad (b)$$

Thus

$$\frac{\partial v}{\partial t} + \frac{\Delta p_o}{\rho \ell v_o} v = \frac{N\mu_o}{d^2 \rho \ell} I^2 \sin^2 \omega t = \frac{N\mu_o}{2d^2 \rho \ell} I^2 (1 - \cos 2\omega t) \quad (c)$$

Solving, we obtain

$$v = \frac{N\mu_o I^2}{2d^2 \rho \ell} \left[\frac{v_o \rho \ell}{\Delta p_o} - \frac{\left(\frac{\Delta p_o}{\rho \ell v_o} \cos 2\omega t + 2\omega \sin 2\omega t \right)}{\left(\frac{\Delta p_o}{\rho \ell v_o} \right)^2 + 4\omega^2} \right] \quad (d)$$

Part b

The ratio R of ac to dc velocity components is:

$$R = \frac{\Delta p_o / v_o \rho \ell}{\left[\left(\frac{\Delta p_o}{v_o \rho \ell} \right)^2 + 4\omega^2 \right]^{1/2}} \quad (e)$$

PROBLEM 12.27Part a

The magnetic field in generator (1) is upward, with magnitude

$$B_1 = \frac{N i_1 \mu_0}{a} - \frac{N i_2 \mu_0}{a} \quad (a)$$

and in generator (2) upward with magnitude

$$B_2 = \frac{N i_1 \mu_0}{a} + \frac{N i_2 \mu_0}{a} \quad (b)$$

We define the voltages V_1 and V_2 across the terminals of the generators.

Applying Kirchoff's voltage law around the loops of wire with currents i_1 and i_2 we have

$$V_1 + N \frac{d\lambda_1}{dt} + N_m \frac{d\lambda_2}{dt} + i_1 R_L = 0 \quad (c)$$

and

$$V_2 + N \frac{d\lambda_2}{dt} - N_m \frac{d\lambda_1}{dt} + i_2 R_L = 0 \quad (d)$$

where

$$\lambda_1 = B_1 w b \quad (e)$$

$$\lambda_2 = B_2 w b$$

From conservation of current we have

$$\frac{i_1}{ab\sigma} = \frac{V_1}{w} + VB_1 \quad (f)$$

and

$$\frac{i_2}{ab\sigma} = \frac{V_2}{w} + VB_2 \quad (g)$$

Combining these relations, we obtain

$$(N^2 + N_m^2) \frac{wb\mu_0}{a} \frac{di_1}{dt} + i_1 \left[\frac{w}{ab\sigma} + R_L - \frac{w\mu_0 NV}{a} \right] + \frac{\mu_0 w}{a} VN_m i_2 = 0 \quad (h)$$

and

$$(N^2 + N_m^2) \frac{wb\mu_0}{a} \frac{di_2}{dt} + i_2 \left[\frac{w}{ab\sigma} + R_L - \frac{VN\mu_0 w}{a} \right] - \frac{N\mu_0}{a} wV i_1 = 0 \quad (i)$$

Part b

We combine these two first-order differential equations to obtain one second-order equation.

$$a_1 \frac{d^2 i_2}{dt^2} + a_2 \frac{di_2}{dt} + a_3 i_2 = 0 \quad (j)$$

where

$$a_1 = \frac{\left[(N^2 + N_m^2) \frac{wb\mu_0}{a} \right]^2}{\frac{wN_m V\mu_0}{a}} \quad (k)$$

PROBLEM 12.27 (continued)

$$a_2 = 2 \left[\frac{w}{ab\sigma} + R_L - \frac{w\mu_o NV}{a} \right] \left[\frac{(N^2 + N_m^2)b}{N_m V} \right]$$

$$a_3 = \frac{VN_m \mu_o w}{a}$$

If we assume solutions of the form

$$i_2 = Ae^{st} \quad (\ell)$$

Then we must have

$$a_1 s^2 + a_2 s + a_3 = 0 \quad (m)$$

or

$$s = \frac{-a_2 \pm \sqrt{a_2^2 - 4a_1 a_3}}{2a_1}$$

For the generators to be stable, the real part of s must be negative.

Thus

$$a_2 > 0 \quad \text{for stability}$$

which implies the condition for stability is

$$\text{Part c} \quad \frac{w}{ab\sigma} + R_L > \frac{w\mu_o NV}{a} \quad (n)$$

When $a_2 = 0$

$$\frac{w}{ab\sigma} + R_L = \frac{w\mu_o NV}{a} \quad (o)$$

then s is purely imaginary, so the system will operate in the sinusoidal steady state.

Then

$$\begin{aligned} s &= \pm j \sqrt{\frac{a_3}{a_1}} \\ &= \pm j \frac{N_m V}{b(N^2 + N_m^2)} \end{aligned} \quad (p)$$

The length b necessary for sinusoidal operation is

$$b = \frac{w}{a\sigma \left[\frac{w\mu_o NV}{a} - R_L \right]} \quad (q)$$

Substituting values, we obtain

$$b = 4 \text{ meters.}$$

Part d

Thus, the frequency of operation is

$$\omega = \frac{4000}{8} = 500 \text{ rad/sec.}$$

$$\text{or } f = \frac{\omega}{2\pi} \approx 80 \text{ Hz.}$$

PROBLEM 12.28Part a

The magnetic field within the generator is

$$\vec{B} = \frac{\mu_0 N i}{w} \vec{i}_2 \quad (a)$$

The current through the generator is

$$\vec{J} = \vec{i}_1, \frac{i}{\ell w} = \sigma \left(\frac{v}{D} + vB \right) \vec{i}_1 \quad (b)$$

Solving for v , the voltage across the channel, we obtain

$$v = \left(\frac{D}{\sigma \ell w} - \frac{v \mu_0 N}{w} D \right) i \quad (c)$$

We apply Faraday's law around the electrical circuit to obtain

$$v + \frac{1}{C} \int i dt + i R_L = - \frac{\mu_0 N^2}{w} \ell d \frac{di}{dt} \quad (d)$$

Differentiating and simplifying this equation we finally obtain

$$\frac{d^2 i}{dt^2} + \left(\frac{R_L w}{\mu_0 N^2 \ell d} + \frac{D}{\sigma L w} - \frac{\mu_0 N D v}{w} \right) \frac{di}{dt} + \frac{w}{\mu_0 N^2 \ell d C} i = 0 \quad (e)$$

We assume that $i = \text{Re } \hat{I} e^{st}$.

Substituting this assumed solution back into the differential equation, we obtain

$$s^2 + \left(\frac{R_L w}{\mu_0 N^2 \ell d} + \frac{D}{\sigma L w} - \frac{\mu_0 N D v}{w} \right) s + \frac{w}{\mu_0 N^2 \ell d C} = 0 \quad (f)$$

Solving, we have

$$s = - \frac{\left(\frac{R_L w}{\mu_0 N^2 \ell d} + \frac{D}{\sigma L w} - \frac{\mu_0 N D v}{w} \right)}{2} \pm \sqrt{\left(\frac{R_L w}{\mu_0 N^2 \ell d} + \frac{D}{\sigma L w} - \frac{\mu_0 N D v}{w} \right)^2 - \frac{4w}{\mu_0 N^2 \ell d C}} \quad (g)$$

For the device to be a pure ac generator, we must have that s is purely imaginary, or

$$R_L = \left(\frac{\mu_0 N D v}{w} - \frac{D}{\sigma L w} \right) \frac{\mu_0 N^2 \ell d}{w} \quad (h)$$

Part b

The frequency of operation is then

$$\omega = \frac{w}{\mu_0 N^2 \ell d C} \quad (i)$$

PROBLEM 12.29Part a

The current within the MHD generator is

$$\vec{J} = \frac{i}{\ell d} \vec{i}_y = \sigma \left(\frac{v}{w} + v B_o \right) \vec{i}_y \quad (a)$$

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PROBLEM 12.29 (continued)

where V is the voltage across the channel. The pressure drop along the channel is

$$\Delta p = p_1 - p_0 = \frac{i B_0}{d} + \rho \frac{\partial v}{\partial t} \ell \quad (b)$$

where we assume that v does not vary with distance along the channel. With the switch open, we apply Faraday's law around the circuit, for which we obtain

$$V + 2iR = 0 \quad (c)$$

Since the pressure drop is maintained constant, we solve for v to obtain

$$\left(\frac{2\sigma R}{w} + \frac{1}{\ell d} \right) \frac{\rho d \ell}{B_0} \frac{\partial v}{\partial t} + \sigma v B_0 = \left(\frac{1}{\ell d} + \frac{2\sigma R}{w} \right) \frac{d}{B_0} \Delta p \quad (d)$$

In the steady state

$$v = \left(\frac{1}{\sigma \ell d} + \frac{2R}{w} \right) \frac{d}{B_0^2} \Delta p \quad (e)$$

and

$$i = \frac{d}{B_0} \Delta p \quad (f)$$

Part b

For $t > 0$, the differential equation for v is

$$\left(\frac{\sigma R}{w} + \frac{1}{\ell d} \right) \frac{\rho \ell d}{B_0} \frac{\partial v}{\partial t} + \sigma v B_0 = \left(\frac{1}{\ell d} + \frac{\sigma R}{w} \right) \frac{d}{B_0} \Delta p \quad (g)$$

The general solution for v is

$$v = \left(\frac{1}{\sigma \ell d} + \frac{R}{w} \right) \frac{d}{B_0^2} \Delta p + A e^{-t/\tau} \quad (h)$$

where $\tau = \left(\frac{\sigma R}{w} + \frac{1}{\ell d} \right) \frac{\ell d}{\sigma B_0^2}$

We evaluate A by realizing that at $t = 0$, the velocity must be continuous.

Therefore

$$v = \left(\frac{1}{\sigma \ell d} + \frac{R}{w} \right) \frac{d}{B_0^2} \Delta p + \frac{R}{w} \frac{d}{B_0^2} \Delta p e^{-t/\tau} \quad (i)$$

and

$$\begin{aligned} i &= \Delta p \left(1 + \frac{\rho \ell}{\tau} \frac{R}{w} \frac{d}{B_0^2} e^{-t/\tau} \right) \frac{d}{B_0} \\ &= \Delta p \left(1 + \frac{R \sigma e^{-t/\tau}}{w \left[\frac{\sigma R}{w} + \frac{1}{\ell d} \right]} \right) \frac{d}{B_0} \end{aligned} \quad (j)$$

PROBLEM 12.30

Part a

The magnetic field in the generator is

$$B = \frac{\mu_0 N i}{d} \quad (a)$$

The current within the generator is

$$\frac{i}{\ell d} = \sigma \left(\frac{V}{w} + v B \right) \quad (b)$$

PROBLEM 12.30 (continued)

where V is the voltage across the channel. The pressure drop in the channel is

$$\Delta p = p_1 - p_o = \Delta p_o \left(1 - \frac{v}{v_o}\right) = \frac{iB}{d} \quad (c)$$

Applying Faraday's law around the external circuit, we obtain

$$V + i(R_L + R_C) = - \frac{d(NB\ell w)}{dt} = - \frac{\ell w}{d} \mu_o N^2 \frac{di}{dt} \quad (d)$$

Using (a), (b), (c) and (d), the differential equation for i is then

$$\frac{\ell \mu_o N^2}{d} \frac{di}{dt} + i \left[\frac{R_L + R_C}{w} + \frac{1}{\sigma \ell d} - \frac{\mu_o N}{d} v_o \right] + \frac{\left(\frac{\mu_o N}{d}\right)^2 v_o i^3}{d \Delta p_o} = 0 \quad (e)$$

In the steady state, we have

$$i^2 = - \frac{\left[\frac{R_L + R_C}{w} + \frac{1}{\sigma \ell d} - \frac{\mu_o N v_o}{d} \right] d \Delta p_o}{\left[\frac{\mu_o N}{d} \right]^2 v_o} \quad (f)$$

The power dissipated in R_L is

$$P = i^2 R_L$$

For $P = 1.5 \times 10^6$, then

$$i^2 = .6 \times 10^8 \text{ (amperes)}^2$$

Substituting in values for the parameters in (f), we obtain

$$i^2 = .6 \times 10^8 = - \frac{(.125 + 2.5 \times 10^{-6} N^2 - 6.3 \times 10^{-4} N) 40 \times 10^3}{N^2 (4 \times 10^{-8})} \quad (g)$$

Rearranging (g), we obtain

$$N^2 - 102N + 2.04 \times 10^3 = 0$$

or $N = 75, 27$

The most efficient solution is that one which dissipates the least power in the coil's resistance. Thus, we choose

$$N = 27$$

Part b

Substituting numerical values into (e), using $N = 27$, we obtain

$$(1.27 \times 10^7) \frac{di}{dt} - (6 \times 10^7) i + i^3 = 0 \quad (h)$$

or, rewriting, we have

$$\frac{dt}{1.27 \times 10^7} = \frac{di}{i(6 \times 10^7 - i^2)} \quad (i)$$

Integrating, we obtain

$$9.4t + C = \log \left(\frac{i^2}{6 \times 10^7 - i^2} \right) \quad (j)$$

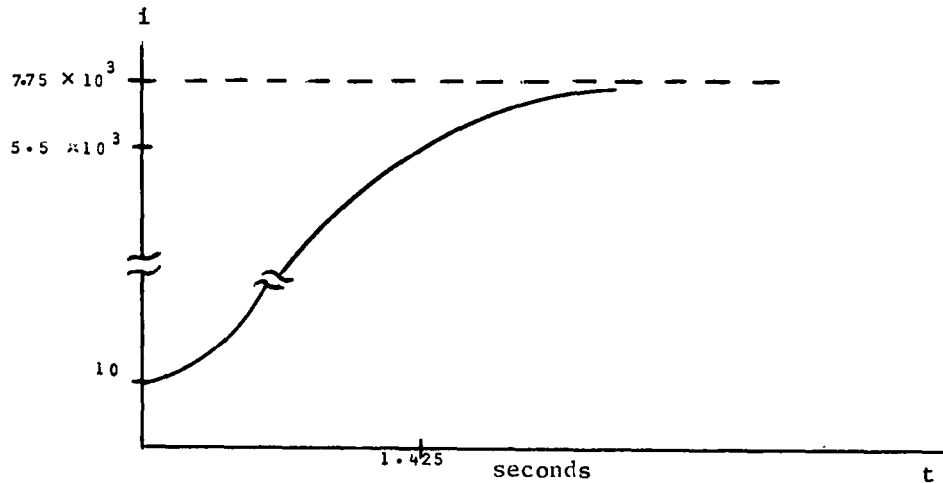
We evaluate the arbitrary constant C by realizing that at $t=0$, $i = 10$ amps

PROBLEM 12.30 (continued)

Thus $C = -13.3$

We take the anti-log of both sides of (j), and solve for i^2 to obtain

$$i^2 = \frac{6 \times 10^7}{1 + e^{(13.3 - 9.4t)}} \quad (k)$$

Part c

For $N = 27$, in the steady state, we use (f) to write

$$P = i^2 R_L = \frac{-\left[\frac{R_L + R_C}{w} + \frac{1}{\sigma l d} - \frac{\mu_o N v_o}{d}\right] d \Delta p_o R_L}{\left(\frac{\mu_o N}{d}\right)^2 v_o}$$

or

$$P = a_1 R_L - a_2 R_L^2$$

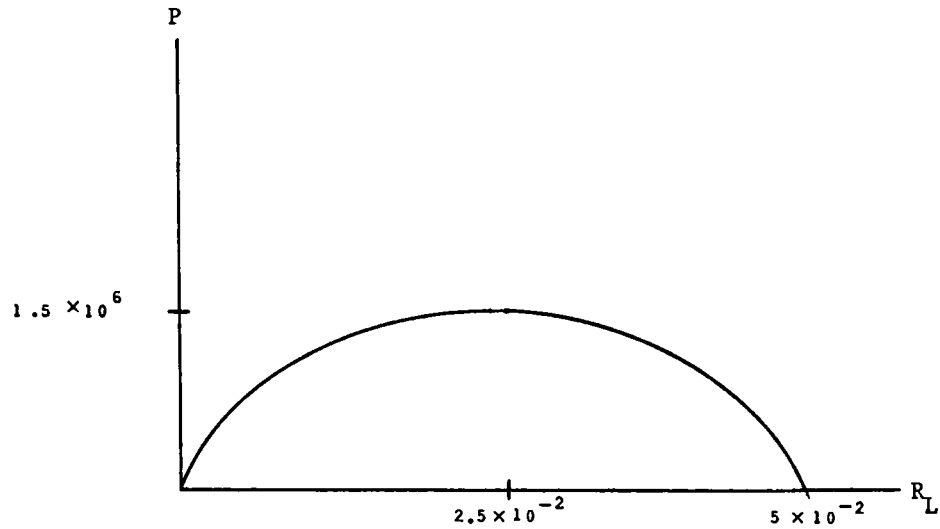
where

$$a_1 = - \frac{d \Delta p_o \left(\frac{R_C}{w} + \frac{1}{\sigma l d} - \frac{\mu_o N v_o}{d}\right)}{\left(\frac{\mu_o N}{d}\right)^2 v_o} \approx 1.47 \times 10^8$$

and

$$a_2 = \frac{d \Delta p_o}{\left(\frac{\mu_o N}{d}\right)^2 v_o} \approx \frac{1}{2.85 \times 10^{-10}}$$

PROBLEM 12.30 (continued)



PROBLEM 12.31

Part a

With the switch open, the current through the generator is

$$\bar{J} = 0 = \frac{i}{\ell d} \bar{i}_y = \sigma \left(-\frac{V}{w} + v B_0 \right) \bar{i}_y \quad (a)$$

where V is the voltage across the channel. In the steady state, the pressure drop in the channel is

$$\Delta p = p_i - p_o = \frac{i B}{d} = 0 = \Delta p_o \left(1 - \frac{v}{v_o} \right) \quad (b)$$

Thus, $v = v_o$ and the voltage across the channel is

$$V = v_o B_o w. \quad (c)$$

Part b

With the switch closed, applying Faraday's law around the circuit we obtain

$$V = i R_L \quad (d)$$

Thus

$$\frac{i}{\ell d} = -\frac{\sigma R_L}{w} i + \sigma v B_o \quad (e)$$

and

$$\Delta p = \frac{i B}{d} + \rho \frac{\partial v}{\partial t} \ell = \Delta p_o \left(1 - \frac{v}{v_o} \right) \quad (f)$$

Obtaining an equation in v , we have

$$\rho \ell \frac{\partial v}{\partial t} + v \left[\frac{\Delta p_o}{v_o} + \frac{\sigma B_o}{\frac{1}{\ell d} + \frac{\sigma R_L}{w}} \right] = \Delta p_o \quad (g)$$

PROBLEM 12.31 (continued)

Solving for v we obtain

$$v = Ae^{-t/\tau} + \frac{\Delta p_o}{\left(\frac{\Delta p_o}{v_o} + \frac{B_o w}{R_L + R_1}\right)} \quad \text{where } R_1 = \frac{w}{\sigma \ell d} \quad (h)$$

and where

$$\tau = \frac{\rho \ell}{\left[\frac{\Delta p_o}{v_o} + \frac{w B_o}{R_L + R_1}\right]} \quad (i)$$

at $t = 0$, the velocity must be continuous. Therefore,

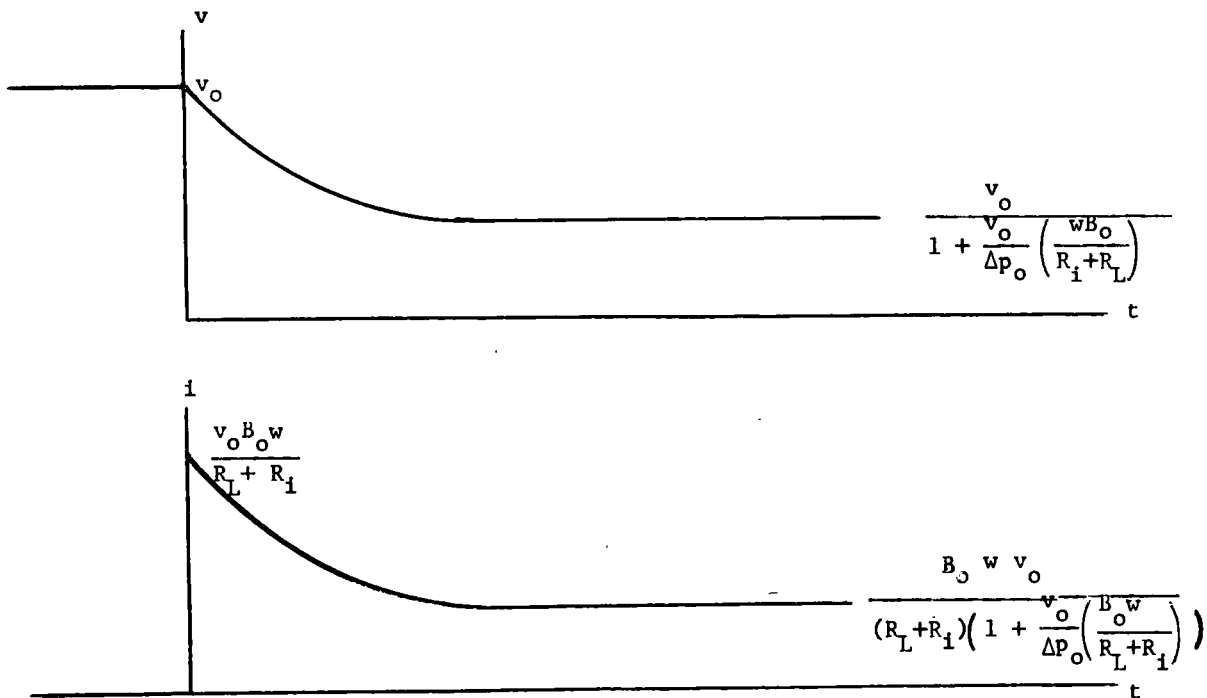
$$A = v_o - \frac{\Delta p_o}{\left(\frac{\Delta p_o}{v_o} + \frac{w B_o}{R_L + R_1}\right)}$$

Now, the current is

$$i = \frac{w B_o v}{R_L + R_1} \quad (k)$$

Thus

$$i = \left(\frac{w B_o}{R_L + R_1}\right) \left[\frac{\Delta p_o}{\left(\frac{\Delta p_o}{v_o} + \frac{w B_o}{R_L + R_1}\right)} (1 - e^{-t/\tau}) + v_o e^{-t/\tau} \right] \quad (l)$$



PROBLEM 12.32

The current in the generator is

$$\frac{i}{\ell_1 d} = \sigma \left(\frac{V}{w} - vB \right) \quad (a)$$

where we assume that the $\vec{B}$ field is up and that the fluid flows counter-clockwise.

We integrate Newton's law around the channel to obtain

$$\rho \ell \frac{\partial v}{\partial t} = JB \ell_1 = \frac{i}{d} B \quad (b)$$

or, using (a),

$$\frac{\partial V}{\partial t} = \frac{w}{d \ell_1 \sigma} \frac{\partial i}{\partial t} + \frac{B^2 w}{d \rho \ell} i \quad (c)$$

Integrating, we have

$$V = \frac{w}{d \ell_1 \sigma} i + \frac{B^2 w}{d \rho \ell} \int_0^\infty i dt \quad (d)$$

Defining $R_i = \frac{w}{\sigma \ell_1 d}$

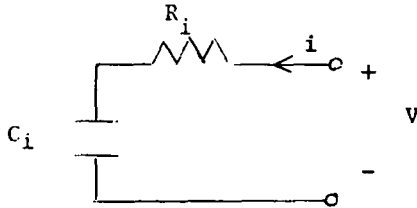
and

$$C_i = \frac{\rho \ell d}{w B^2}$$

we rewrite (d) as

$$V = i R_i + \frac{1}{C_i} \int_0^\infty i dt \quad (e)$$

The equivalent circuit implied by (e) is


PROBLEM 12.33
Part a

We assume that the capacitor is initially uncharged when the switch is closed at $t = 0$. The current through the capacitor is

$$i = C \frac{dV_C}{dt} = \sigma \ell d \left(-\frac{V_C}{w} + v_o B_o \right) \quad (a)$$

or

$$\frac{dV_C}{dt} + \frac{\sigma \ell d}{w C} V_C = \frac{\sigma \ell d v_o}{C} B_o \quad (b)$$

PROBLEM 12.33 (Continued)

The solution for V_C is

$$V_C = v_o B_o w (1 - e^{-t/\tau}) \quad (c)$$

with $\tau = \frac{wC}{\sigma l d}$, where we have used the initial condition that at $t = 0$, the voltage cannot change instantaneously across the capacitor. The energy stored as $t \rightarrow \infty$, is

$$W_e = \frac{1}{2} C V_C^2 = \frac{1}{2} C (v_o B_o w)^2 \quad (d)$$

Part b

The pressure drop along the fluid is

$$\Delta p = \frac{1 B_o}{d} = B_o^2 v_o \sigma l e^{-t/\tau} \quad (e)$$

The total energy supplied by the fluid source is

$$\begin{aligned} W_f &= \int_0^\infty \Delta p v_o dw dt \\ &= \int_0^\infty (v_o B_o)^2 \sigma l w e^{-t/\tau} dt \\ &= -\sigma l (v_o B_o)^2 \tau w e^{-t/\tau} \Big|_0^\infty \end{aligned} \quad (f)$$

$$W_f = C (w v_o B_o)^2 \quad (g)$$

Part c

We see that the energy supplied by the fluid source is twice that stored in the capacitor. The rest of the energy has been dissipated by the conducting fluid. This dissipated energy is

$$\begin{aligned} W_d &= \int_0^\infty V_C i dt \\ &= \int_0^\infty + (v_o B_o)^2 w (1 - e^{-t/\tau}) \sigma l d e^{-t/\tau} dt \\ &= \sigma l d w (v_o B_o)^2 \left[-\tau e^{-t/\tau} + \frac{\tau}{2} e^{-2t/\tau} \right] \Big|_0^\infty \\ &= \sigma l d w (v_o B_o)^2 \frac{\tau}{2} \end{aligned} \quad (h)$$

Therefore

$$W_d = \frac{1}{2} C (v_o B_o w)^2 \quad (i)$$

Thus

$$W_{\text{fluid}} = W_{\text{elec}} + W_{\text{dissipated}} \quad (j)$$

As we would expect from conservation of energy.

PROBLEM 12.34

The current through the generator is

$$\frac{1}{\ell_1 d} = \sigma \left(\frac{V}{w} - v B_o \right) \quad (a)$$

Since the fluid is incompressible, and the channel has constant cross-sectional area, the velocity of the fluid does not change with position. Thus, we write Newton's law as in Eq. (12.2.41) as

$$\rho \frac{\partial \bar{v}}{\partial t} = -\nabla(p+U) + \bar{J} \times \bar{B} \quad (b)$$

where U is the potential energy due to gravity. We integrate this expression along the length of the tube to obtain

$$\rho \frac{\partial v}{\partial t} \ell = \frac{i B_o}{d} - \rho g(x_a + x_b) \quad (c)$$

Realizing that $x_a = x_b$

$$\text{and} \quad v = \frac{dx_a}{dt} \quad (d)$$

We finally obtain

$$\frac{d^2 x_a}{dt^2} + \frac{\sigma B_o^2 \ell_1}{\rho \ell} \frac{dx_a}{dt} + \frac{2g}{\ell} x_a = \frac{\sigma B_o V}{w \rho} \frac{\ell_1}{\ell} \quad (e)$$

We assume the transient solution to be of the form

$$x_a = \hat{x} e^{st} \quad (f)$$

Substituting into the differential equation, we obtain

$$s^2 + \frac{\sigma B_o^2 \ell_1}{\rho \ell} s + \frac{2g}{\ell} = 0 \quad (g)$$

Solving for s , we obtain

$$s = -\frac{\sigma B_o^2 \ell_1}{2\rho \ell} \pm \sqrt{\left(\frac{\sigma B_o^2 \ell_1}{2\rho \ell}\right)^2 - \frac{2g}{\ell}} \quad (h)$$

Substituting the given numerical values, we obtain

$$\begin{aligned} s_1 &= -29.4 \\ s_2 &= -.665 \end{aligned} \quad (i)$$

In the steady state

$$x_a = \frac{\sigma B_o V \ell_1}{w \rho 2g} \approx .075 \text{ meters} \quad (j)$$

Thus the general solution is of the form

$$x_a = .075 + A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad (k)$$

where the initial conditions to solve for A_1 and A_2 are

PROBLEM 12.34 (continued)

$$x_a(t=0) = .0$$

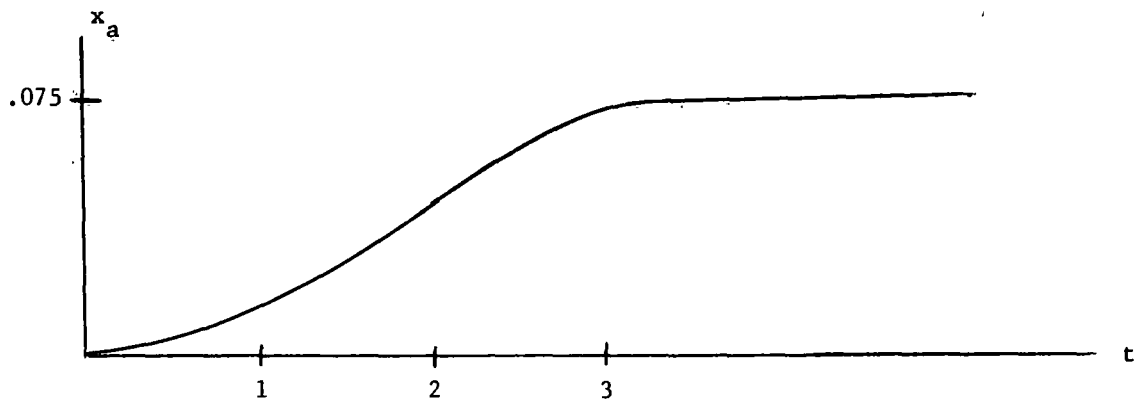
$$\frac{dx_a}{dt}(t=0) = 0$$

(l)

$$\text{Thus, } A_2 = \frac{.075 s_1}{s_2 - s_1} = -.0765 \text{ and } A_1 = -\frac{.075 s_2}{s_2 - s_1} = .00174$$

Thus, we have:

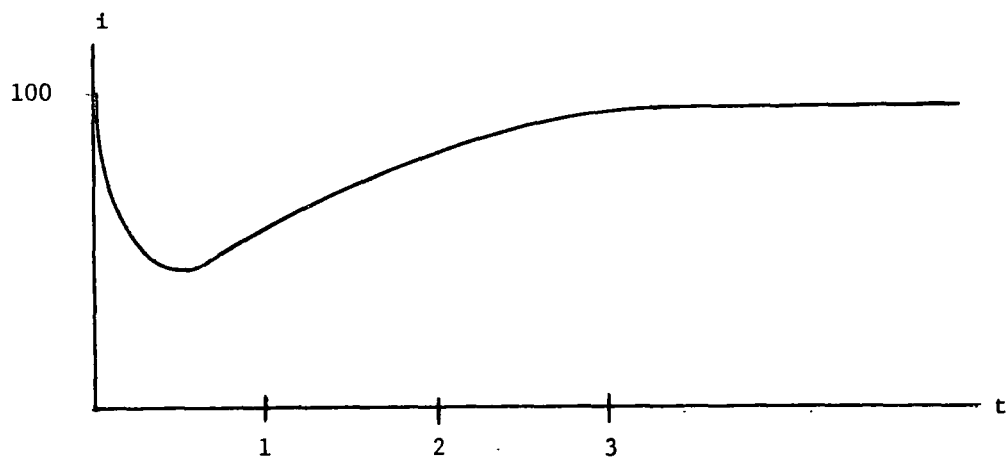
$$x_a = .075 + .00174e^{-29.4t} - .0765e^{-.665t}$$



Now the current is

$$\begin{aligned} i &= \ell_1 d\sigma \left(\frac{V}{w} - B_0 \frac{dx}{dt} \right) \\ &= \ell_1 d\sigma \left[\frac{V}{w} - B_0 (s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t}) \right] \\ &= 100 - 2 \times 10^3 (s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t}) \text{ amperes} \\ &= 100(1 + e^{-29.4t} - e^{-.665t}) \end{aligned} \quad (m)$$

Sketching, we have



PROBLEM 12.35

The currents I_1 and I_2 are determined by the resistance of the fluid between the electrodes. Thus

$$I_1 = \frac{V_o \sigma D x}{w} \quad (a)$$

and

$$I_2 = \frac{V_o \sigma D y}{w} \quad (b)$$

The magnetic field produced by the circuit is

$$\bar{B} = \frac{\mu_o N}{w} (I_2 - I_1) \bar{i}_2 \quad (c)$$

or

$$\bar{B} = \frac{\mu_o N}{w^2} V_o \sigma D (y - x) \bar{i}_2 \quad (d)$$

From conservation of mass,

$$y = (L - x) \quad (e)$$

Thus

$$\bar{B} = \frac{\mu_o N V_o \sigma D}{w^2} (L - 2x) \bar{i}_2 \quad (f)$$

The momentum equation is

$$\rho \frac{\partial \bar{v}}{\partial t} = -\nabla(p + U) + \bar{J} \times \bar{B} \quad (g)$$

Integrating the equation along the conduit's length, we obtain

$$\rho \frac{\partial v}{\partial t} (2L + 2a) = -\rho g(y - x) - J_o B L \quad (h)$$

Now

$$v = -\frac{\partial x}{\partial t} \quad (i)$$

so we write:

$$2\rho(L + a) \frac{\partial^2 x}{\partial t^2} + \left(\rho g + J_o \frac{\mu_o N V_o \sigma D L}{w^2} \right) (2x - L) = 0 \quad (j)$$

We assume solutions of the form

$$x = \operatorname{Re} \hat{x} e^{st} + \frac{L}{2} \quad (k)$$

Thus

$$s^2 + \frac{g}{(L + a)} + \frac{\mu_o N V_o \sigma D}{\rho w^2 (L + a)} J_o L = 0 \quad (l)$$

Defining

$$\omega_o^2 = \frac{g}{(L + a)} + \frac{\mu_o N V_o \sigma D J_o L}{\rho w^2 (L + a)} \quad (m)$$

we have our solution in the form

$$x = A \sin \omega_o t + B \cos \omega_o t + \frac{L}{2} \quad (n)$$

Applying the initial conditions

$$x(0) = L \quad \text{and} \quad \frac{dx(0)}{dt} = 0 \quad (o)$$

we obtain

$$x = \frac{L}{2} (1 + \cos \omega_o t) \quad (p)$$

PROBLEM 12.36

As from Eqs. (12.2.88 - 12.2.91), we assume that

$$\begin{aligned}\bar{v} &= \bar{i}_\theta v_\theta \\ \bar{B} &= B_0 \bar{i}_z + \bar{i}_\theta B_\theta \\ \bar{J} &= \bar{i}_r J_r + \bar{i}_z J_z \\ \bar{E} &= \bar{i}_r E_r + \bar{i}_z E_z\end{aligned}\tag{a}$$

As derived in Sec. 12.2.3, Eq. (12.2.102), we know that the equation governing Alfvén waves is

$$\frac{\partial^2 v_\theta}{\partial t^2} = \frac{B_0^2}{\mu_0 \rho} \frac{\partial^2 v_\theta}{\partial z^2}\tag{b}$$

For our problem, the boundary conditions are:

$$\begin{aligned}\text{at } z = 0 & \quad E_r = 0 \\ \text{at } z = \ell & \quad v_\theta = \operatorname{Re}[\Omega r e^{j\omega t}]\end{aligned}\tag{c}$$

As in section 12.2.3, we assume

$$v_\theta = \operatorname{Re}[A(r) \hat{v}_\theta(z) e^{j\omega t}]\tag{d}$$

Thus, the pertinent differential equation reduces to

$$\frac{d^2 \hat{v}_\theta}{dz^2} + k^2 \hat{v}_\theta = 0\tag{e}$$

where $k = \omega \sqrt{\frac{\mu_0 \rho}{B_0^2}}$

The solution is

$$\hat{v}_\theta = C_1 \cos kz + C_2 \sin kz\tag{f}$$

Imposing the boundary condition at $z = \ell$, we obtain

$$A(r)[C_1 \cos k\ell + C_2 \sin k\ell] = \Omega r\tag{g}$$

We let

$$A(r) = \frac{r}{R}\tag{h}$$

and thus

$$\Omega R = C_1 \cos k\ell + C_2 \sin k\ell\tag{i}$$

Now

$$E_r = -v_\theta B_0\tag{j}$$

Thus, applying the second boundary condition, we obtain

$$v_\theta(z=0) = 0$$

or $C_1 = 0\tag{k}$

Thus $C_2 = \frac{\Omega R}{\sin k\ell}\tag{l}$

Now, using the relations

$$E_r = -v_\theta B_0\tag{m}$$

PROBLEM 12.36 (continued)

$$E_z = 0 \quad (n)$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = - \frac{\partial B_\theta}{\partial t} \quad (o)$$

$$- \frac{1}{\mu_o} \frac{\partial B_\theta}{\partial z} = J_r \quad (p)$$

$$\frac{1}{\mu_o r} \frac{\partial(rB_\theta)}{\partial r} = J_z \quad (q)$$

we obtain

$$v_\theta = \text{Re} \left[\frac{\Omega r}{\sin k\ell} \sin kz e^{j\omega t} \right] \quad (r)$$

$$B_\theta = \text{Re} \left[\frac{\Omega r B_o k}{j \omega \sin k\ell} \cos kz e^{j\omega t} \right] \quad (s)$$

$$J_r = \text{Re} \left[\frac{\Omega r B_o k^2}{\mu_o j \omega \sin k\ell} \sin kz e^{j\omega t} \right] \quad (t)$$

$$J_z = \text{Re} \left[\frac{2 \Omega B_o k}{\mu_o j \omega \sin k\ell} \cos kz e^{j\omega t} \right] \quad (u)$$

PROBLEM 12.37Part a

We perform a similar analysis as in section 12.2.3, Eqs. (12.2.84 - 12.2.88).

From Maxwell's equation

$$\nabla \times \bar{E} = - \frac{\partial \bar{B}}{\partial t} \quad (a)$$

which yields

$$\frac{\partial E_y}{\partial z} = \frac{\partial}{\partial t} B_x \quad (b)$$

Now, since the fluid is perfectly conducting,

$$\bar{E}' = \bar{E} + \bar{v} \times \bar{B} = 0 \quad (c)$$

$$\text{or } E_y = v_x B_o \quad (d)$$

Substituting, we obtain

$$B_o \frac{\partial v_x}{\partial z} = \frac{\partial B_x}{\partial t} \quad (e)$$

The x component of the force equation is

$$\rho \frac{\partial v_x}{\partial t} = \frac{\partial T_{xz}}{\partial z} \quad (f)$$

where

$$T_{xz} = \frac{B_o}{\mu_o} B_x \quad (g)$$

PROBLEM 12.37 (continued)

Thus

$$\rho \frac{\partial v_x}{\partial t} = \frac{B_o}{\mu_o} \frac{\partial B_x}{\partial z} \quad (h)$$

Eliminating B_x and solving for v_x , we obtain

$$\frac{\partial^2 v_x}{\partial t^2} = \frac{B_o^2}{\mu_o \rho} \frac{\partial^2 v_x}{\partial z^2} \quad (i)$$

or eliminating and solving for H_x , we have

$$\frac{\partial^2 H_x}{\partial t^2} = \frac{B_o^2}{\mu_o \rho} \frac{\partial^2 H_x}{\partial z^2} \quad (j)$$

where

$$B_x = \mu_o H_x \quad (k)$$

Part b

The boundary conditions are

$$v_x(-l, t) = \text{Re } V e^{j\omega t} \quad (l)$$

$$E_y(0, t) = 0 \rightarrow v_x(0, t) = 0 \quad (m)$$

We write the solution in the form

$$v_x = A e^{j(\omega t - kz)} + B e^{j(\omega t + kz)} \quad (n)$$

where

$$k = \omega \sqrt{\frac{\mu_o \rho}{B_o^2}}$$

Applying the boundary conditions, we obtain

$$v_x(l, t) = \text{Re} \left[-\frac{V \sin kz}{\sin kl} \right] e^{j\omega t} \quad (o)$$

Now

$$B_o \frac{\partial v_x}{\partial z} = \frac{\partial B_x}{\partial t} \quad (p)$$

or

$$\frac{-B_o V k \cos kz}{\sin kl} = j\omega \mu_o \hat{H}_x \quad (q)$$

Thus

$$H_x = \text{Re} \left[\frac{-B_o V k \cos kz}{j\omega \mu_o \sin kl} e^{j\omega t} \right] \quad (r)$$

Part c

From Maxwell's equations

$$\nabla \times \vec{H} = \vec{i}_y \frac{\partial H_x}{\partial z} = \vec{J} \quad (s)$$

Thus

$$\vec{J} = \vec{i}_y \text{Re} \left[\frac{B_o V k^2 \sin kz}{j\omega \mu_o \sin kl} e^{j\omega t} \right] \quad (t)$$

PROBLEM 12.37 (continued)

Since $\nabla \cdot \vec{J} = 0$, the current must have a return path, so the walls in the x-z plane must be perfectly conducting.

Even though the fluid has no viscosity, since it is perfectly conducting, it interacts with the magnetic field such that for any motion of the fluid, currents are induced such that the magnetic force tends to restore the fluid to its original position. This shearing motion sets the neighboring fluid elements into motion, whereupon this process continues throughout the fluid.

PROBLEM 13.1

In static equilibrium, we have

$$-\nabla p - \rho g \bar{i}_1 = 0 \quad (a)$$

Since $p = \rho RT$, (a) may be rewritten as

$$RT \frac{d\rho}{dx_1} + \rho g = 0 \quad (b)$$

Solving, we obtain

$$\rho = \rho_0 e^{-\frac{g}{RT} x_1} \quad (c)$$

PROBLEM 13.2

Since the pressure is a constant, Eq. (13.2.25) reduces to

$$\rho v \frac{dv}{dz} = -J_y B \quad (a)$$

where we use the coordinate system defined in Fig. 13P.4. Now, from Eq. (13.2.21) we obtain

$$J_y = \sigma(E_y + vB) \quad (b)$$

If the loading factor K , defined by Eq. (13.2.32) is constant, then

$$-KvB = +E \quad (c)$$

$$\text{Thus, } J_y = \sigma v B(1-K) \quad (d)$$

$$\text{Then } \rho v \frac{dv}{dz} = -\sigma v B^2(1-K) \quad (e)$$

$$\text{or } \rho \frac{dv}{dz} = -\sigma B^2(1-K) = -\sigma(1-K) \frac{B_1^2 A_1}{A(z)} \quad (f)$$

From conservation of mass, Eq. (13.2.24), we have

$$\rho_1 v_1 A_1 = \rho A(z) v \quad (g)$$

Thus

$$\frac{\rho_1 v_1 A_1}{v} \frac{dv}{dz} = -\sigma(1-K) B_1^2 A_1 \quad (h)$$

Integrating, we obtain

$$\ln v = \frac{-\sigma(1-K) B_1^2}{\rho_1 v_1} z + C \quad (i)$$

or

$$v = v_i e^{-\frac{z}{\ell_d}} \quad (j)$$

where $\ell_d = \frac{\rho_1 v_1}{\sigma(1-K) B_1^2}$ and we evaluate the arbitrary constant by realizing that

$$v = v_i \text{ at } z = 0.$$

PROBLEM 13.3
Part a

We assume T , B_0 , w , σ , c_p and c_v are constant. Since the electrodes are short-circuited, $E = 0$, and so

$$J_y = v B_0 \quad (a)$$

We use the coordinate system defined in Fig. 13P.4. Applying conservation of energy, Eq. (13.2.26), we have

$$\rho v \frac{d}{dz} \left(\frac{1}{2} v^2 \right) = 0, \text{ where we have set } h = \text{constant}. \quad (b)$$

Thus, v is a constant, $v = v_1$. Conservation of momentum, Eq. (13.2.25), implies

$$\frac{dp}{dz} = -v_1 B_0^2 \quad (c)$$

$$\text{Thus, } p = -v_1 B_0^2 z + p_1 \quad (d)$$

The mechanical equation of state, Eq. (13.1.10) then implies

$$\rho = \frac{n}{RT} = -\frac{v_1 B_0^2 z + p_1}{RT} = \rho_1 - \frac{v_1 B_0^2 z}{RT} \quad (e)$$

From conservation of mass, we then obtain

$$\rho_1 v_1 w d_1 = \left(-\frac{v_1 B_0^2 z}{RT} + \rho_1 \right) v_1 w d(z) \quad (f)$$

Thus

$$d(z) = \frac{\rho_1 d_1}{\left(\rho_1 - \frac{v_1 B_0^2 z}{RT} \right)} \quad (g)$$

Part b

Then

$$\rho(z) = \rho_1 - \frac{v_1 B_0^2 z}{RT} \quad (h)$$

PROBLEM 13.4

Note:

There are errors in Eqs. (13.2.16) and (13.2.31). They should read:

$$\frac{1}{M^2} \frac{d(M^2)}{dx_1} = \frac{\{(\gamma-1)(1+\gamma M^2)E_3 + \gamma[2 + (\gamma-1)M^2]v_1 B_2\} J_3}{(1-M^2)\gamma p v_1} \quad (13.2.16)$$

and

$$\frac{1}{M^2} \frac{d(M^2)}{dx_1} = \frac{1}{(1-M^2)} \left\{ [(\gamma-1)(1+\gamma M^2)E_3 + \gamma[2 + (\gamma-1)M^2]v_1 B_2] \frac{J_3}{\gamma p v_1} - \frac{[2 + (\gamma-1)M^2]dA}{A} \frac{dA}{dx_1} \right\} \quad (13.2.31)$$

Part a

We assume that σ , γ , B_0 , K and M are constant along the channel. Then, from the corrected form of Eq. (13.2.31), we must have

PROBLEM 13.4 (continued)

$$0 = \frac{1}{1-M^2} \left\{ [(\gamma-1)(1+\gamma M^2)(-K) + \gamma(2+(\gamma-1)M^2)] \frac{v B_o^2 \sigma (1-K)}{\gamma p} - \frac{[2 + (\gamma-1)M^2]}{A} \frac{dA}{dz} \right\} \quad (a)$$

Now, using the relations

$$v^2 = M^2 \gamma R T$$

and $p = \rho R T$

we write

$$\frac{v}{\gamma p} = \frac{M^2}{\rho v} \quad (b)$$

Thus, we obtain

$$\frac{1}{A^2} \frac{dA}{dz} = \frac{[(\gamma-1)(1+\gamma M^2)(-K) + \gamma(2+(\gamma-1)M^2)] \frac{B_o^2 \sigma (1-K) M^2}{\rho v A}}{2 + (\gamma-1)M^2} \quad (c)$$

From conservation of mass,

$$\rho v A = \rho_1 v_1 A_1 \quad (d)$$

Using (d), we integrate (c) and solve for $\frac{A(z)}{A_1}$

to obtain

$$\frac{A(z)}{A_1} = \frac{1}{1 - \beta_1 z} \quad (e)$$

where

$$\beta_1 = \frac{[(\gamma-1)(1+\gamma M^2)(-K) + \gamma(2+(\gamma-1)M^2)] \sigma B_o^2 M^2 (1-K)}{\rho_1 v_1 [2 + (\gamma-1)M^2]}$$

We now substitute into Eq. (13.2.27) to obtain

$$\frac{1}{v} \frac{dv}{dz} = \frac{1}{(1-M^2)} [(\gamma-1)(-K) + \gamma] \frac{v B_o^2 (1-K) \sigma}{\gamma p} - \frac{1}{A} \frac{dA}{dz} \quad (f)$$

Thus may be rewritten as

$$\frac{1}{v} \frac{dv}{dz} = \frac{1}{(1-M^2)} \left[[(\gamma-1)(-K) + \gamma] \frac{\sigma B_o^2 (1-K) M^2}{\rho_1 v_1 A_1} - \frac{\beta_1}{A_1} \right] A \quad (g)$$

Solving, we obtain

$$\ln v = -\frac{\beta_2}{\beta_1} \ln(1 - \beta_1 z) + \ln v_1 \quad (h)$$

or

$$\frac{v(z)}{v_1} = (1 - \beta_1 z)^{-\beta_2/\beta_1} \quad (i)$$

where $\beta_2 = \frac{1}{(1-M^2)} \frac{[(\gamma-1)(-K) + \gamma] \sigma B_o^2 (1-K) M^2 - \beta_1}{\rho_1 v_1}$

Now the temperature is related through Eq. (13.2.12), as

PROBLEM 13.4 (continued)

$$M^2 \gamma R T = v^2 \quad (j)$$

Thus

$$\frac{T(z)}{T_i} = \left(\frac{v}{v_i} \right)^2 \quad (k)$$

From (d), we have

$$\frac{\rho(z)}{\rho_i} = \frac{v_i}{v} \frac{A_i}{A} \quad (l)$$

Thus, from Eq. (13.1.10)

$$\frac{p(z)}{p_i} = \frac{v_i}{v} \frac{A_i}{A} \frac{T}{T_i} \quad (m)$$

Since the voltage across the electrodes is constant,

$$E = - \frac{V}{w(z)} = - K v(z) B_o \quad (n)$$

or

$$w(z) = \frac{K v_i B_o w_i}{K v(z) B_o} = \frac{v_i}{v(z)} w_i \quad (o)$$

Thus,

$$\frac{w(z)}{w_i} = \frac{v_i}{v(z)} \quad (p)$$

Then

$$\frac{d(z)}{d_i} = \frac{A(z)}{A_i} \frac{w_i}{w(z)} \quad (q)$$

Part b

We now assume that σ , γ , B_o , K and v are constant along the channel. Then, from Eq. (13.2.27) we have

$$0 = \frac{1}{(1-M^2)} \left\{ [(\gamma-1)(-K) + \gamma] v_i B_o^2 \frac{(1-K)\sigma}{\gamma p} - \frac{1}{A} \frac{dA}{dz} \right\} \quad (r)$$

But, from Eq. (13.2.25) we know that

$$\frac{p}{p_i} = 1 - \frac{(1-K)\sigma v_i B_o^2 z}{p_i} = 1 - \beta_3 z \quad (s)$$

where $\beta_3 = (1-K) \frac{\sigma v_i B_o^2}{p_i}$

Substituting the results of (b), into (a) and solving for $\frac{A(z)}{A_i}$, we obtain

$$\frac{A(z)}{A_i} = \left(\frac{p}{p_i} \right)^{-\beta_4 / \beta_3} \quad (t)$$

where $\beta_4 = [(\gamma-1)(-K) + \gamma] \frac{v_i B_o^2}{\gamma p_i} (1-K)\sigma$

From conservation of mass,

$$\frac{\rho(z)}{\rho_i} = \frac{A_i}{A(z)} \quad (u)$$

PROBLEM 13.4 (continued)

and so, from Eq. (13.1.10)

$$\frac{T(z)}{T_i} = \frac{p(z)}{p_i} \frac{\rho_i}{\rho(z)} \quad (v)$$

As in (p)

$$\frac{w(z)}{w_i} = \frac{v_i}{v(z)} = 1 \quad (w)$$

Thus

$$\frac{d(z)}{d_i} = \frac{A(z)}{A_i} \quad (x)$$

Part c

We wish to find the length ℓ such that

$$\frac{C_p T(\ell) + \frac{1}{2} [v(\ell)]^2}{C_p T(o) + \frac{1}{2} [v(o)]^2} = .9 \quad (y)$$

For the constant M generator of part (a), we obtain from (i) and (k)

$$\frac{C_p \left[\frac{v(\ell)}{v_i} \right]^2 T_i + \frac{1}{2} [v(\ell)]^2}{C_p \left[\frac{v(o)}{v_i} \right]^2 T_i + \frac{1}{2} [v(o)]^2} = \frac{C_p (1 - \beta_1 \ell)^{-2\beta_2/\beta_1} T_i + \frac{1}{2} [v_i (1 - \beta_1 \ell)]^2}{C_p T_i + \frac{1}{2} v_i^2} = .9 \quad (z)$$

Reducing, we obtain

$$(1 - \beta_1 \ell)^{-2\beta_2/\beta_1} = .9 \quad (aa)$$

Substituting the given numerical values, we have

$$\beta_1 = .396 \quad \text{and} \quad \beta_2/\beta_1 = -7.3 \times 10^{-2}$$

We then solve (aa) for ℓ , to obtain

$$\ell \approx 1.3 \text{ meters}$$

For the constant v generator of part (b), we obtain from (s), (t), (u) and (v)

$$\frac{C_p T_i \left[\frac{p(\ell)}{p_i} \frac{\rho_i}{\rho(\ell)} \right] + \frac{1}{2} v_i^2}{C_p T_i + \frac{1}{2} v_i^2} = .9 \quad (bb)$$

or

$$\frac{C_p T_i (1 - \beta_3 \ell)^{(1 - \beta_4/\beta_3)} + \frac{1}{2} v_i^2}{C_p T_i + \frac{1}{2} v_i^2} = .9 \quad (cc)$$

Substituting the given numerical values, we have

PROBLEM 13.4 (continued)

$$\beta_3 = .45 \quad \text{and} \quad \beta_4/\beta_3 = .857$$

Solving for l , we obtain

$$l \sim 1.3 \text{ meters.}$$

PROBLEM 13.5

We are given the following relations:

$$\frac{B(z)}{B_i} = \frac{E(z)}{E_i} = \frac{w_i}{w(z)} = \frac{d_i}{d(z)} = \left(\frac{A_i}{A(z)} \right)^{1/2}$$

and that v , σ , γ , and K are constant.

Part a

From Eq. (13.2.33),

$$J = (1-K)\sigma v B \tag{a}$$

For constant velocity, conservation of momentum yields

$$\frac{dp}{dz} = - (1-K)\sigma v B^2 \tag{b}$$

Conservation of energy yields

$$\rho v c_p \frac{dT}{dz} = - K(1-K)\sigma (vB)^2 \tag{c}$$

Using the equation of state,

$$p = \rho R T \tag{d}$$

we obtain

$$T \frac{dp}{dz} + p \frac{dT}{dz} = - \frac{(1-K)}{R} \sigma v B^2 \tag{e}$$

or

$$T \frac{dp}{dz} + \frac{(-K)(1-K)\sigma v B^2}{c_p} = - \frac{(1-K)\sigma v B^2}{R} \tag{f}$$

Thus,

$$T \frac{dp}{dz} = \sigma v B^2 (1-K) \left(-\frac{1}{R} + \frac{K}{c_p} \right) \tag{g}$$

Also

$$B^2 = \frac{B_i^2 (A_i)}{A(z)}$$

and

$$\rho_i A_i = \rho(z) A(z)$$

$$\text{Therefore} \quad T \frac{dp}{dz} = \frac{\sigma v B_i^2 (1-K) \left(-\frac{1}{R} + \frac{K}{c_p} \right)}{\rho_i} \rho(z) \tag{h}$$

and

$$\rho c_p \frac{dT}{dz} = -K(1-K)\sigma v \frac{B_i^2}{\rho_i} \tag{i}$$

PROBLEM 13.5 (continued)

and so

$$\frac{dT}{dz} = - \frac{K(1-K)\sigma v B_1^2}{\rho_1 c_p} \quad (j)$$

Therefore

$$T = - K(1-K) \frac{\sigma v B_1^2}{\rho_1 c_p} z + T_1 \quad (k)$$

Let

$$\alpha = \frac{-K(1-K)\sigma v B_1^2}{\rho_1 c_p} \quad (l)$$

Then

$$T = T_1 \left(\frac{\alpha z}{T_1} + 1 \right) \quad (m)$$

$$\frac{d\rho}{\rho} = \frac{+ \sigma v B_1^2 (1-K) \left(\frac{K}{c_p} - \frac{1}{R} \right)}{\rho_1 (\alpha z + T_1)} dz \quad (n)$$

We let

$$\begin{aligned} \beta &= \frac{+ \sigma v B_1^2 (1-K) \left(\frac{K}{c_p} - \frac{1}{R} \right)}{\rho_1 \alpha} \\ &= \frac{c_p}{KR} - 1 \end{aligned}$$

Integrating (n), we then obtain

$$\ln \rho = \beta \ln(\alpha z + T_1) + \text{constant}$$

or

$$\rho = \rho_1 \left(\frac{\alpha z}{T_1} + 1 \right)^\beta \quad (o)$$

Therefore

$$A(z) = \frac{A_1}{\left(\frac{\alpha z}{T_1} + 1 \right)^\beta} \quad (p)$$

Part b

From (m),

$$\frac{T(l)}{T_1} = \frac{\alpha l + T_1}{T_1} = .8$$

or

$$\frac{\alpha l}{T_1} = -.2$$

Now

$$\frac{\alpha}{T_1} = - \frac{K(1-K)\sigma v B_1^2}{\rho_1 c_p T_1}$$

But

$$c_p T_1 = \frac{R T_1}{\left(1 - \frac{1}{\gamma}\right)} = \frac{p_1}{\rho_1 \left(1 - \frac{1}{\gamma}\right)} = 2.5 \times 10^6$$

PROBLEM 13.5 (Continued)

Thus

$$\frac{\alpha}{T_1} = \frac{-.5(.5)50(700)16}{.7(2.5 \times 10^6)} = -8.0 \times 10^{-2}$$

Solving for ℓ , we obtain

$$\ell = \frac{.2}{8} \times 10^2 = 1.25 \text{ meters}$$

Part c

$$\rho = \rho_1 \left(\frac{\alpha z}{T_1} + 1 \right)^\beta$$

Numerically

$$\beta = \frac{c_p}{KR} - 1 = \frac{1}{(1 - \frac{1}{\gamma})K} - 1 \approx 6.$$

Thus

$$\rho(z) = .7(1 - .08z)^6$$

Then it follows:

$$p(z) = \rho RT = p_1(1 - .08z)^7 = 5 \times 10^5(1 - .08z)^7$$

$$T(z) = T_1(1 - .08z)$$

From the given information, we cannot solve for T_1 , only for

$$RT_1 = \frac{p_1}{\rho_1} = \frac{v_1^2}{\gamma M_1^2} \approx 7 \times 10^5$$

Now

$$\begin{aligned} M^2(z) &= \frac{v_1^2}{\gamma RT(z)} = \frac{v_1^2}{\gamma p(z)} \rho(z) = \frac{v_1^2}{\gamma} \frac{\rho_1 \left(\frac{\alpha z}{T_1} + 1 \right)^\beta}{p_1 \left(\frac{\alpha z}{T_1} + 1 \right)^{(\beta+1)}} \\ &= \frac{.5}{1 - .08z} \end{aligned}$$

Part d

The total electric power drawn from this generator is

$$\begin{aligned} p^e &= VI = -E(z)w(z)J(z)\ell d(z) \\ &= -E(z)(1-K)\sigma v B(z)\ell d(z)w(z) \\ &= -E_1 w_1 (1-K)\sigma v B_1 d_1 \ell \end{aligned}$$

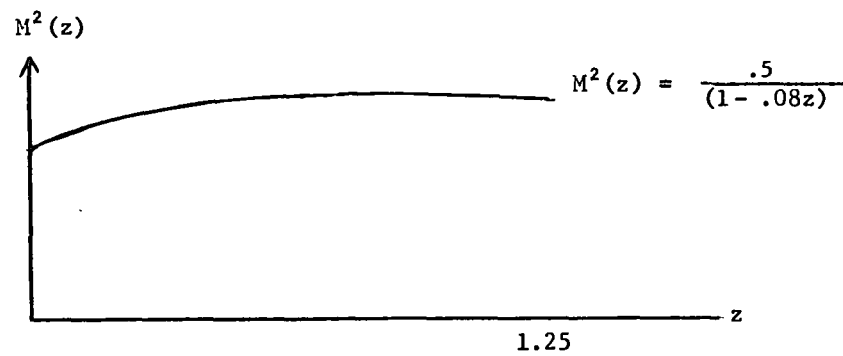
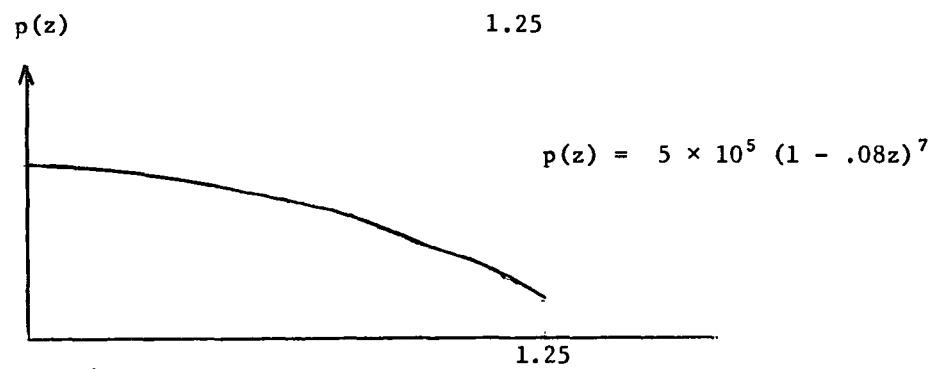
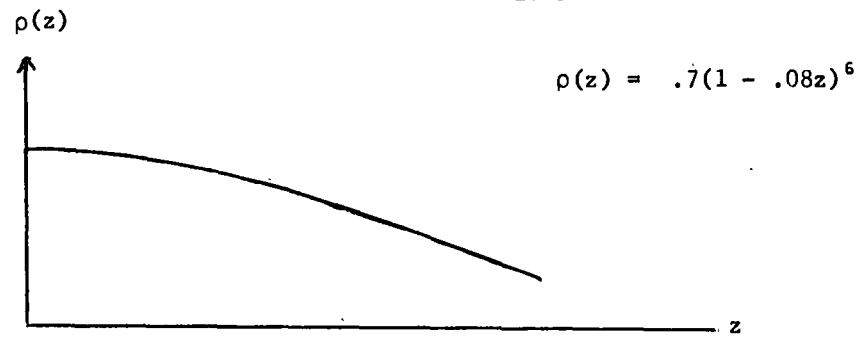
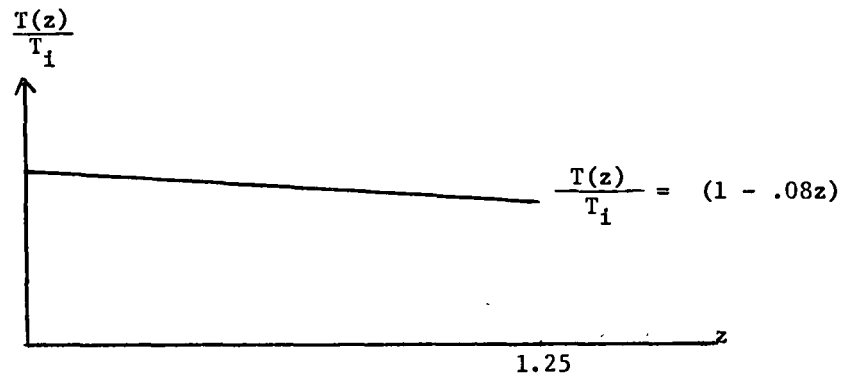
But

$$E_1 = -K v B_1$$

Thus

$$\begin{aligned} p^e &= K(v B_1)^2 w_1 d_1 \sigma (1-K)\ell \\ &= .5(700)^2 16(.5)50(.5)1.25 \\ &= 61.3 \times 10^6 \text{ watts} = 61.3 \text{ megawatts} \end{aligned}$$

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PROBLEM 13.6Part a

We are given that

$$\bar{E} = \bar{i}_x \frac{4}{3} \frac{V_o}{L^{1/3}} x^{1/3} \quad (a)$$

and

$$\rho_e = \frac{4}{9} \frac{\epsilon_o V_o}{L^{1/3} x^{2/3}} \quad (b)$$

The force equation in the steady state is

$$\rho_m v_x \frac{dv_x}{dx} \bar{i}_x = \rho_e \bar{E} \quad (c)$$

Since $\rho_e/\rho_m = q/m = \text{constant}$, we can write

$$\frac{d}{dx} \left(\frac{1}{2} v_x^2 \right) = \frac{q}{m} \frac{4}{3} \frac{V_o}{L^{1/3}} x^{1/3} \quad (d)$$

Solving for v_x we obtain

$$v_x = \sqrt{\frac{2q}{m} V_o \left(\frac{x}{L} \right)^{2/3}} \quad (e)$$

Part b

The total force per unit volume acting on the accelerator system is

$$\bar{F} = \rho_e \bar{E} \quad (f)$$

Thus, the total force which the fixed support must exert is

$$\begin{aligned} \bar{f}_{\text{total}} &= - \int_0^L F dV \bar{i}_x \\ &= - \int_0^L \frac{16}{27} \frac{\epsilon_o V_o^2}{L^{1/3}} x^{-1/3} A dx \bar{i}_x \\ \bar{f}_{\text{total}} &= - \frac{8}{9} \frac{\epsilon_o V_o^2}{L^2} A \bar{i}_x \end{aligned}$$

PROBLEM 13.7Part a

We refer to the analysis performed in section 13.2.3a. The equation of motion for the velocity is, Eq. (13.2.76),

$$\frac{\partial^2 v}{\partial t^2} = a^2 \frac{\partial^2 v}{\partial x_1^2} \quad (a)$$

The boundary conditions are

$$\begin{aligned} v(-L) &= V_o \cos \omega t \\ v(0) &= 0 \end{aligned}$$

We write the solution in the form

PROBLEM 13.7 (continued)

$$v(x_1, t) = \operatorname{Re}[A e^{j(\omega t - kx_1)} + B e^{j(\omega t + kx_1)}] \quad (b)$$

where $k = \frac{\omega}{a}$

Using the boundary condition at $x_1 = 0$, we can alternately write the solution as

$$v = \operatorname{Re}[A \sin kx_1 e^{j\omega t}]$$

Applying the other boundary condition at $x_1 = -L$, we finally obtain

$$v(x_1, t) = -\frac{V_0}{\sin kL} \sin kx_1 \cos \omega t. \quad (d)$$

The perturbation pressure is related to the velocity through Eq. (13.2.74)

$$\rho_0 \frac{\partial v'}{\partial t} = -\frac{\partial p'}{\partial x_1} \quad (e)$$

Solving, we obtain

$$\frac{\rho_0 V_0 \omega}{\sin kL} \sin kx_1 \sin \omega t = -\frac{\partial p'}{\partial x_1} \quad (f)$$

or

$$p' = \frac{\rho_0 V_0 \omega}{k \sin kL} \cos kx_1 \sin \omega t \quad (g)$$

where ρ_0 is the equilibrium density, related to the speed of sound a , through Eq. (13.2.83).

Thus, the total pressure is

$$p = p_0 + p' = p_0 + \frac{\rho_0 V_0 \omega}{k \sin kL} \cos kx_1 \sin \omega t \quad (h)$$

and the perturbation pressure at $x_1 = -L$ is

$$p'(-L, t) = \frac{\rho_0 V_0 a}{\sin kL} \cos kL \sin \omega t \quad (i)$$

Part b

There will be resonances in the pressure if

$$\sin kL = 0 \quad (j)$$

$$\text{or} \quad kL = n\pi \quad n = 1, 2, 3, \dots \quad (k)$$

Thus

$$\omega = \frac{n\pi}{L} a \quad (l)$$

PROBLEM 13.8Part a

We carry through an analysis similar to that performed in section 13.2.3b.

We assume that

$$\bar{E} = \bar{i}_2 E_2(x_1, t)$$

$$\bar{J} = \bar{i}_2 J_2(x_1, t)$$

PROBLEM 13.8

$$\bar{B} = \bar{i}_3 [\mu_0 H_0 + \mu_0 H'_3(x_1, t)]$$

Conservation of momentum yields

$$\rho \frac{Dv_1}{Dt} = - \frac{\partial p}{\partial x_1} + J_2 \mu_0 (H_0 + H'_3) \quad (a)$$

Conservation of energy gives us

$$\rho \frac{D}{Dt} \left(u + \frac{1}{2} v_1^2 \right) = - \frac{\partial}{\partial x_1} (p v_1) + J_2 E_2 \quad (b)$$

We use Ampere's and Faraday's laws to obtain

$$\frac{\partial H'_3}{\partial x_1} = - J_2 \quad (c)$$

and

$$\frac{\partial E_2}{\partial x_1} = - \frac{\mu_0 \partial H'_3}{\partial t} \quad (d)$$

while

Ohm's law yields

$$J_2 = \sigma [E_2 - v_1 B_3] \quad (e)$$

Since $\sigma \rightarrow \infty$

$$E_2 = v_1 B_3 \quad (f)$$

We linearize, as in Eq. (13.2.91), so $E_2 \approx v_1 \mu_0 H_0$

Substituting into Faraday's law

$$\mu_0 H_0 \frac{\partial v_1}{\partial x_1} = - \mu_0 \frac{\partial H'_3}{\partial t} \quad (g)$$

Linearization of the conservation of mass yields

$$\frac{\partial \rho'}{\partial t} = - \rho_0 \frac{\partial v_1}{\partial x_1} \quad (h)$$

Thus, from (g)

$$\frac{\mu_0 H_0}{\rho_0} \frac{\partial \rho'}{\partial t} = \mu_0 \frac{\partial H'_3}{\partial t} \quad (i)$$

Then

$$\frac{H_0}{H'_3} = \frac{\rho_0}{\rho'}$$

Linearizing Eq. (13.2.71), we obtain

$$\frac{Dp'}{Dt} = \frac{\gamma p_0}{\rho_0} \frac{D\rho'}{Dt} \quad (k)$$

PROBLEM 13.8 (continued)

Defining the acoustic speed

$$a_s = \left(\frac{\gamma p_o}{\rho_o} \right)^{1/2} \text{ where } p_o \text{ is the equilibrium pressure,}$$

$$p_o = p_1 - \frac{\mu_o H_o^2}{2}$$

we have

$$p' = a_s^2 \rho' \quad (l)$$

Linearization of conservation of momentum (a) yields

$$\rho_o \frac{\partial v_1}{\partial t} = - \frac{\partial p'}{\partial x_1} - \frac{\partial H'}{\partial x_1} \mu_o H_o \quad (m)$$

or, from (j) and (l),

$$\rho_o \frac{\partial v_1}{\partial t} = \frac{\partial \rho'}{\partial x_1} \left(-a_s^2 - \frac{\mu_o H_o^2}{\rho_o} \right) \quad (n)$$

Differentiating (n) with respect to time, and using conservation of mass (h), we finally obtain

$$\frac{\partial^2 v_1}{\partial t^2} = \left(a_s^2 + \frac{\mu_o H_o^2}{\rho_o} \right) \frac{\partial^2 v_1}{\partial x_1^2} \quad (o)$$

Defining

$$a^2 = a_s^2 + \frac{\mu_o H_o^2}{\rho_o} \quad (p)$$

we have

$$\frac{\partial^2 v_1}{\partial t^2} = a^2 \frac{\partial^2 v_1}{\partial x_1^2} \quad (q)$$

Part b

We assume solutions of the form

$$v_1 = \operatorname{Re} [A_1 e^{j(\omega t - kx_1)} + A_2 e^{j(\omega t + kx_1)}] \quad (r)$$

where $k = \frac{\omega}{a}$

The boundary condition at $x_1 = -L$ is

$$v(-L, t) = v_s \cos \omega t = v_s \operatorname{Re} e^{j\omega t} \quad (s)$$

and at $x_1 = 0$

$$M \frac{dv_1}{dt}(0, t) = p' A \Big|_{x_1=0} + \mu_o H_o H'_3 A \Big|_{x_1=0} \quad (t)$$

From (h), (j) and (l),

$$\frac{1}{a_s^2} \frac{\partial p'}{\partial t} = -\rho_o \frac{\partial v_1}{\partial x_1} \quad (u)$$

PROBLEM 13.8 (continued)

$$\frac{H'_3}{H_0} = \frac{p'}{a_s^2 \rho_0} \quad (v)$$

Thus

$$M \frac{dv_1(0,t)}{dt} = A \left(\frac{\mu_0 H_0^2}{a_s^2 \rho_0} + 1 \right) p' = A \frac{a^2}{a_s^2} p' \quad (w)$$

From (u), we solve for p' to obtain:

$$p' \Big|_{x_1=0}^{x_1=0} = - \frac{\rho_0 a_s^2 k}{w} (A_2 - A_1) e^{j\omega t} \quad (x)$$

Substituting into (s) and (t), we have

$$Mjw(A_1 + A_2) = A \left(\frac{a}{a_s} \right)^2 \left(\frac{\rho_0 a_s^2 k}{w} \right) (A_1 - A_2)$$

and

$$A_1 e^{+jkl} + A_2 e^{-jkl} = v_s \quad (y)$$

Solving for A_1 and A_2 , we obtain

$$A_1 = \frac{(Mjw + Aa\rho_0)v_s}{2(-Mw \sin kl + Aa\rho_0 \cos kl)}$$

$$A_2 = \frac{(Aa\rho_0 - Mjw)v_s}{2(-Mw \sin kl + Aa\rho_0 \cos kl)} \quad (z)$$

Thus, the velocity of the piston is

$$v_1(0,t) = \text{Re} [A_1 + A_2] e^{j\omega t}$$

$$v_1(0,t) = \frac{Aa\rho_0 v_s}{-Mw \sin kl + Aa\rho_0 \cos kl} \cos \omega t \quad (aa)$$

PROBLEM 13.9Part a

The differential equation for the velocity as derived in problem 13.8 is

$$\frac{\partial^2 v_1}{\partial t^2} = a^2 \frac{\partial^2 v_1}{\partial x_1^2} \quad (a)$$

where

$$a^2 = a_s^2 + \frac{\mu_0 H_0^2}{\rho_0}$$

with $a_s^2 = \left(\frac{\gamma p_0}{\rho_0} \right)^{1/2}$ where $p_0 = p_1 - \frac{\mu_0 H_0^2}{2}$

Part b

We assume a solution of the form

PROBLEM 13.9 (continued)

$$V(x_1, t) = \operatorname{Re} [D e^{j(\omega t - k x_1)}] \quad \text{where } k = \frac{\omega}{a}$$

We do not consider the negatively traveling wave, as we want to use this system as a delay line without distortion. The boundary condition at $x_1 = -L$ is

$$V(-L, t) = \operatorname{Re} V_s e^{j\omega t}$$

and at $x_1 = 0$ is

$$M \frac{dV(0, t)}{dt} = p'(0, t)A - B V_1(0, t) + \mu_o H_o H'_3 A \quad (b)$$

From problem 13.8, (h), (j) and (l)

$$p' = a_s^2 \rho' \quad , \quad \frac{\partial \rho'}{\partial t} = -\rho_o \frac{\partial v_1}{\partial x_1} \quad \text{and} \quad \frac{H'_3}{H_o} = \frac{p'}{a_s^2 \rho_o}$$

Thus, (b) becomes

$$-B D e^{j\omega t} + \left(\frac{a}{a_s}\right)^2 p' A = 0 \quad (c)$$

where

$$p' \Big|_{x_1=0} = -\frac{\rho_o D(-jk)}{j \omega} a_s^2 e^{j\omega t} \quad (d)$$

Thus, for no reflections

$$-B + \left(\frac{a}{a_s}\right)^2 \frac{A \rho_o a_s^2}{a} = 0 \quad (e)$$

or

$$B = A a \rho_o \quad (f)$$

PROBLEM 13.10

The equilibrium boundary conditions are

$$T[-(L_1 + L_2 + \Delta), t] = T_o$$

$$T[-(L_1 + \Delta), t] A_s = -p_o A_c$$

Boundary conditions for incremental motions are

$$1) \quad T[-(L_1 + L_2 + \Delta), t] = T_s(t)$$

$$2) \quad T[-(L_1 + \Delta), t] A_s - p(-L_1, t) A_c = M \frac{d}{dt} v_\ell(-L_1, t)$$

$$3) \quad v_\ell(-L_1, t) = v_e[-(L_1 + \Delta), t] \quad \text{since the mass is rigid}$$

$$\text{and } 4) \quad v_\ell(0, t) = 0 \quad \text{since the wall at } x=0 \text{ is fixed.}$$

PROBLEM 13.11

Part a

We can immediately write down the equation for perturbation velocity, using equations (13.2.76) and (13.2.77) and the results of chapters 6 and 10.

PROBLEM 13.11 (continued)

We replace $\partial/\partial t$ by $\partial/\partial t + \mathbf{v} \cdot \nabla$ to obtain

$$\left(\frac{\partial}{\partial t} + V_o \frac{\partial}{\partial x} \right)^2 v' = a_s^2 \frac{\partial^2 v'}{\partial x^2}$$

Letting $v' = \text{Re } \hat{v} e^{j(\omega t - kx)}$

we have

$$(\omega - kV_o)^2 = a_s^2 k^2$$

Solving for ω , we obtain

$$\omega = k(V_o \pm a_s)$$

Part b

Solving for k , we have

$$k = \frac{\omega}{V_o \pm a_s}$$

For $V_o > a_s$, both waves propagate in the positive x - direction.

PROBLEM 13.12Part a

We assume that

$$\begin{aligned} \vec{E} &= \hat{i}_z E_z(x,t) \\ \vec{J} &= \hat{i}_z J_z(x,t) \\ \vec{B} &= \hat{i}_y \mu_o [H_o + H'_y(x,t)] \end{aligned}$$

We also assume that all quantities can be written in the form of Eq. (13.2.91) .

$$\rho_o \frac{\partial v_x}{\partial t} = - \frac{\partial p'}{\partial x} - J_z \mu_o H_o \quad \text{(conservation of momentum linearized)} \quad (a)$$

The relevant electromagnetic equations are

$$\frac{\partial H'_y}{\partial x} = J_z \quad (b)$$

and

$$\frac{\partial E_z}{\partial x} = \mu_o \frac{\partial H'_y}{\partial t} \quad (c)$$

and the constitutive law is

$$J_z = \sigma(E_z + v_x \mu_o H_o) \quad (d)$$

We recognize that Eqs. (13.2.94), (13.2.96) and (13.2.97) are still valid, so

$$\frac{1}{\rho_o} \frac{\partial \rho'}{\partial t} = - \frac{\partial v_x}{\partial x} \quad (e)$$

PROBLEM 13.12 (continued)

and

$$p' = a_s^2 \rho' \quad (f)$$

Part b

We assume all perturbation quantities are of the form

$$v_x = \text{Re}[\hat{v} e^{j(\omega t - kx)}]$$

Using (b), (a) may be rewritten as

$$\rho_o j\omega \hat{v} = + jk \hat{p} + \mu_o H_o jk \hat{H} \quad (g)$$

and (c) may now be written as

$$- jk \hat{E} = \mu_o j\omega \hat{H} \quad (h)$$

Then, from (b) and (d)

$$- jk \hat{H} = \sigma(\hat{E} + \hat{v} \mu_o H_o) \quad (i)$$

Solving (g) and (h) for $\hat{H}$ in terms of $\hat{v}$, we have

$$\hat{H} = \frac{\hat{v} \mu_o H_o}{\left(- jk + \sigma \mu_o \frac{\omega}{k} \right)} \quad (j)$$

From (e) and (f), we solve for $\hat{p}$ in terms of $\hat{v}$ to be

$$\hat{p} = \frac{k}{\omega} \rho_o a_s^2 \hat{v} \quad (k)$$

Substituting (j) and (k) back into (g), we find

$$\hat{v} \left[\rho_o j\omega - \frac{j k^2}{\omega} \rho_o a_s^2 - \frac{j k (\mu_o H_o)^2 \sigma}{\left[- jk + \frac{\sigma \mu_o \omega}{k} \right]} \right] = 0 \quad (l)$$

Thus, the dispersion relation is

$$(\omega^2 - k^2 a_s^2) - \frac{j (\mu_o H_o)^2 \omega k^2}{\left(+ \frac{k^2}{\sigma} + j \mu_o \omega \right) \rho_o} = 0 \quad (m)$$

We see that in the limit as $\sigma \rightarrow \infty$, this dispersion relation reduces to the lossless dispersion relation

$$\omega^2 - k^2 \left(a_s^2 + \frac{\mu_o H_o^2}{\rho_o} \right) = 0 \quad (n)$$

Part cIf σ is very small, we can approximate (m) as

$$\omega^2 - k^2 a_s^2 - j (\mu_o H_o)^2 \frac{\omega \sigma}{\rho_o} \left(1 - \frac{j \omega \mu_o \sigma}{k^2} \right) = 0 \quad (o)$$

for which we can rewrite (o) as

PROBLEM 13.12 (continued)

$$k^4 a_s^2 - k^2 \left[\omega^2 - j\omega\sigma \frac{(\mu_o H_o)^2}{\rho_o} \right] + \left(\frac{\mu_o H_o}{\rho_o} \right)^2 \omega^2 \sigma^2 \mu_o = 0 \quad (p)$$

Solving for k^2 , we obtain

$$k^2 = \frac{\omega^2 - j\omega\sigma \frac{(\mu_o H_o)^2}{\rho_o}}{2 a_s^2} \pm \sqrt{\left[\frac{\omega^2 - j\omega\sigma \frac{(\mu_o H_o)^2}{\rho_o}}{2 a_s^2} \right]^2 - \frac{\left(\frac{\mu_o \omega^2 \sigma^2}{\rho_o} \right) (\mu_o H_o)^2}{a_s^2}} \quad (q)$$

Since σ is very small, we expand the radical in (q) to obtain

$$k^2 = \frac{\left[\omega^2 - j\omega\sigma \frac{(\mu_o H_o)^2}{\rho_o} \right]}{2 a_s^2} \pm \left[\frac{\omega^2 - j\omega\sigma \frac{(\mu_o H_o)^2}{\rho_o}}{2 a_s^2} - \frac{\left(\frac{\mu_o \omega^2 \sigma^2}{\rho_o} \right) (\mu_o H_o)^2}{\left[\omega^2 - j\omega\sigma \frac{(\mu_o H_o)^2}{\rho_o} \right]} \right] \quad (r)$$

Thus, our approximate solutions for k^2 are

$$k^2 \approx \frac{\left[\omega^2 - j\omega\sigma \frac{(\mu_o H_o)^2}{\rho_o} \right]}{a_s^2} \quad (s)$$

and

$$k^2 \approx \frac{\left(\frac{\mu_o \omega^2 \sigma^2}{\rho_o} \right) (\mu_o H_o)^2}{\left[\omega^2 - j\omega\sigma \frac{(\mu_o H_o)^2}{\rho_o} \right]} \approx \left(\frac{\mu_o \sigma^2}{\rho_o} \right) (\mu_o H_o)^2 \quad (t)$$

The wavenumbers for the first pair of waves are approximately:

$$k \approx \pm \left(\frac{\omega - j \frac{\sigma}{2\rho_o} (\mu_o H_o)^2}{a_s} \right) \quad (u)$$

while for the second pair, we obtain

$$k \approx \pm \sigma (\mu_o H_o) \sqrt{\frac{\mu_o}{\rho_o}} \quad (v)$$

The wavenumbers from (u) represent a forward and backward traveling wave, both with amplitudes exponentially decreasing. Such waves are called 'diffusion waves'. The wavenumbers from (v) represent pure propagating waves in the forward and reverse directions.

PROBLEM 13.12 (continued)Part d

If σ is very large, then (m) reduces to

$$\omega^2 - k^2 a^2 - j \frac{H_o^2}{\rho_o} \frac{k^4}{\sigma \omega} = 0 ; \quad a^2 = a_s^2 + \frac{\mu_o H_o^2}{\rho_o} \quad (w)$$

This can be put in the form

$$k^2 = \frac{\omega^2}{a^2} - j \frac{f(\omega, k)}{\sigma} \quad (x)$$

where

$$f(\omega, k) = \frac{H_o^2 k^4}{\rho_o \omega a^2}$$

As σ becomes very large, the second term in (x) becomes negligible, and so

$$k^2 \sim \frac{\omega^2}{a^2} \quad (y)$$

However, it is this second term which represents the damping in space; that is,

$$k \sim \pm \left[\frac{\omega}{a} - j \frac{f(\omega, k)}{2\sigma \omega} a \right] \quad (z)$$

Thus, the approximate decay rate, k_i , is

$$k_i \sim \frac{f(\omega, k) a}{2\sigma \omega} = \frac{H_o^2 k^4}{2\rho_o \omega a^2 \sigma \omega} \quad (aa)$$

or

$$k_i \sim \frac{H_o^2}{2\rho_o a \sigma} \frac{k^4}{\omega^2} = \frac{H_o^2}{2\rho_o a^3 \sigma} \omega^2$$

PROBLEM 14.1Part a

We can specify the relevant variables as

$$\begin{aligned}\bar{v} &= \bar{i}_1 v_1(x_2) \\ \bar{E} &= \bar{i}_2 E_2(x_2) + \bar{i}_3 E_3(x_2) \\ \bar{J} &= \bar{i}_2 J_o \\ \bar{B} &= \bar{i}_2 B_o + \bar{i}_1 B_1(x_2)\end{aligned}\tag{a}$$

The x_1 component of the momentum equation is

$$0 = \mu \frac{\partial^2 v_1}{\partial x_2^2}\tag{b}$$

with solution

$$v_1 = C_1 x_2 + C_2$$

Applying the boundary conditions

$$\begin{aligned}v_1 &= 0 & @ & x_2 = 0 \\ v_1 &= v_o & @ & x_2 = d\end{aligned}\tag{c}$$

We obtain

$$v_1 = \frac{v_o x_2}{d}\tag{d}$$

We note that there is no magnetic force density since the imposed current and magnetic field are colinear. We apply Ohm's law for a moving fluid

$$\bar{J} = \sigma(\bar{E} + \bar{v} \times \bar{B})\tag{e}$$

in the x_2 and x_3 directions to obtain

$$J_o = \sigma E_2\tag{f}$$

and

$$0 = \sigma(E_3 + v_1 B_o)\tag{g}$$

since no current can flow in the x_3 direction.

Thus

$$E_2 = \frac{J_o}{\sigma}\tag{h}$$

and

$$E_3 = -\frac{v_o x_2 B_o}{d}\tag{i}$$

As from Eq. (14.2.5),

$$V = \int_0^d E_2 dx_2 = \frac{J_o}{\sigma} d\tag{j}$$

Thus, the electrical input p_e per unit area in an $x_1 - x_3$ plane is

$$p_e = J_o V = \frac{J_o^2 d}{\sigma}\tag{k}$$

PROBLEM 14.1 (continued)

We see that this power is dissipated as Ohmic loss. The moving fluid looks just like a resistor from the electrical terminals. The traction that must be applied to the upper plate to maintain the steady motion is

$$\tau = \mu \left. \frac{\partial v_1}{\partial x_2} \right|_{x_2=d} = \frac{\mu v_0}{d} \quad (l)$$

Again we note no contribution from the magnetic forces.

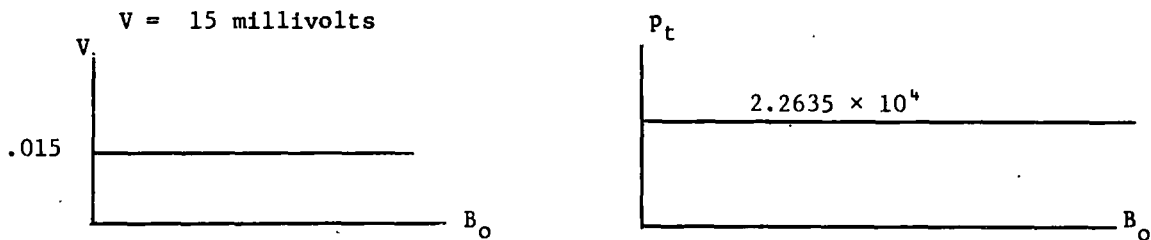
The mechanical input power per unit area is then

$$p_m = \tau_1 v_0 = \frac{\mu v_0^2}{d} \quad (m)$$

The total input power per unit area is thus

$$p_t = p_e + p_m = \frac{\mu v_0^2}{d} + \frac{J_0^2 d}{\sigma} \quad (n)$$

The first term is due to viscous loss that results from simple shear flow, while the second term is simply the Joule loss associated with Ohmic heating. There is no electromechanical coupling. Using the parameters from Table 14.2.1, we obtain



and

$$p_t = 2.2635 \times 10^{-4} \text{ watts/m}^2, \text{ independent of } B_0.$$

These results correspond to the plots of Fig. 14.2.3 in the limit as

$$B_0 \rightarrow 0.$$

We see that the brush losses and brush voltage are much less for this configuration than for that analyzed in Sec. 14.2.1. This is because the electrical and mechanical equations were uncoupled when the applied flux density was in the x_2 direction. This configuration is better, because low voltages at the brush eliminate arcing, and because the net power input per unit area is less no matter the field strength B_0 .

The only effect of applying a flux density in the x_2 direction was to cause an electric field in the x_3 direction. However, since there was no current flow in the x_3 direction, there was no additional dissipated power. However, if E_3 became too large, the fluid might experience electrical breakdown, resulting in corona arcs.

PROBLEM 14.2

The momentum equation for the fluid is

$$\rho \frac{\partial \bar{\mathbf{v}}}{\partial t} + \rho(\bar{\mathbf{v}} \cdot \nabla) \bar{\mathbf{v}} = -\nabla p + \mu \nabla^2 \bar{\mathbf{v}} \quad (\text{a})$$

We consider solutions of the form

$$\bar{\mathbf{v}} = \bar{\mathbf{i}}_z v_z(r)$$

and $p = p(z).$

Then in the steady state, we write the z component of (a) in cylindrical coordinates

as
$$\frac{\partial p}{\partial z} = \mu \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial v_z}{\partial r} \quad (\text{b})$$

Now, the left side of (b) is only a function of z, while the right side is only a function of r. Thus, from the given information

$$\frac{\partial p}{\partial z} = \frac{p_2 - p_1}{L} \quad (\text{c})$$

Using the results of (c) in (b), we solve for $v_z(r)$ in the form

$$v_z(r) = \frac{p_2 - p_1}{4L\mu} r^2 + A \ln r + B \quad (\text{d})$$

where A and B are arbitrary constants to be evaluated by the boundary conditions

$$v_z(r = 0) \text{ is finite}$$

and $v_z(r = R) = 0$

Thus the solution is

$$v_z(r) = \frac{(p_2 - p_1)}{4\mu L} (r^2 - R^2) \quad (\text{e})$$

We can also find relations between the flow rate and the pressure difference, since

$$\int_0^R v_z 2\pi r dr = Q$$

PROBLEM 14.3Part a

We are given the pressure drop Δp , the magnetic field B_0 , the conductivity σ , and the dimensions of the system.

Now

$$i = \int_{-d}^{+d} J dx_2 = \sigma \int_{-d}^{+d} (E_3 + v_1 B_0) dx_2 = \frac{V}{R} \quad (\text{a})$$

where

$$V = -\frac{E}{w} \text{ is defined as the voltage across the resistor.}$$

From Eq. (14.2.29), we have the solution for the velocity v_1 . We then perform the integrations of (a) and solve for the voltage V to obtain

PROBLEM 14.3(continued)

$$V = \frac{\frac{(\Delta p) 2d}{B_o} \left(1 - \frac{\tanh M}{M}\right)}{\frac{1}{R} + \frac{2\sigma l d}{w} \frac{\tanh M}{M}} \quad (b)$$

where

$$M = B_o d \left(\frac{\sigma}{\mu}\right)^{1/2}$$

Then, the power p^e dissipated in the resistor is

$$p^e = \frac{V^2}{R} = \frac{\left(\frac{\Delta p 2d}{B_o}\right)^2 \left(1 - \frac{\tanh M}{M}\right)^2}{\left(\frac{1}{R} + \frac{1}{R_i} \frac{\tanh M}{M}\right)^2 R} \quad (c)$$

where we have defined the internal resistance R_i as

$$R_i = \frac{w}{2\sigma l d}$$

Part b

To maximize p^e , we differentiate (c) with respect to R , solve for that value of R which makes this quantity zero, and then check that this value does indeed maximize p^e . Performing these operations, we obtain

$$R_{\max} = \frac{M R_i}{\tanh M} \quad (d)$$

Part c

We must convert the given numerical values to MKS units, using the conversions

$$10,000 \text{ gauss} = 1 \text{ Weber/meter}^2$$

$$\text{and } 100 \text{ cm} = 1 \text{ meter}$$

For mercury

$$\sigma = 10^6 \text{ mhos/m}$$

$$\text{and } \mu = 1.5 \times 10^{-3} \text{ kg/m-sec.}$$

Thus

$$M = B_o d \left(\frac{\sigma}{\mu}\right)^{1/2} = 2 \times 10^{-2} \left(\frac{1}{1.5} \times 10^9\right)^{1/2}$$

$$M = 520$$

Then $\tanh M \approx 1$

and so

$$R_{\max} = 520 \left(\frac{10^{-1}}{2 \times 10^6 \times 10^{-2}} \right) \approx 2.60 \times 10^{-3} \text{ ohms.}$$

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