PROFESSOR: In the last lecture, we began the discussion of modulation. And in particular, what we focused on, was the continuous time case. And in talking about continuous time modulation, we've covered a number of topics. We talked about the properties and analysis of modulation when we had a complex exponential carrier signal. We talked about the properties and analysis in the case of a sinusoidal carrier. And in that context and related to the application associated with communications, we talked about synchronous modulation, asynchronous modulation and also the notion of single side band modulation.

In the lecture today, there are two issues that I'd like to address, broad topics. One is a parallel discussion, particularly, as associated with complex exponential and sinusoidal modulation for discrete time signals. And the second is the introduction and analysis of another kind of carriers, specifically a pulse kind of carrier in continuous time leading to the notions of pulse amplitude modulation and, eventually, a very powerful theorem and result called the sampling theorem.

Well, let me begin the lecture, though, focusing on the discrete time modulation to essentially draw your attention to the fact that the analysis in discrete time very much parallels the analysis in continuous time. Well, let's consider, in the discrete time case, just as we had in continuous time, a signal modulating a carrier signal and the resulting modulated signal is y of n. And it was in continuous time the modulation property associated with the Fourier Transform that provided the basis for the analysis. And exactly the same thing is true in the discrete time case.

In particular, what we have in discrete time, is the modulation property as it relates to the Fourier Transform, which tells us that the Fourier Transform of the modulated signal is the convolution of the Fourier Transform of the carrier and the Fourier

Transform of the modulated signal. And the only real difference at issue here is that, in the discrete time case, what we're talking about is a periodic convolution because the specter, of course is periodic. Whereas, in the continuous time case, it was an aperiodic convolution.

So let's parallel the discussion, and in particular, what we'll focus on is, first, a complex exponential carrier and second a sinusoidal carrier. And we'll see how this parallels our discussion in continuous time, and we'll make fairly brief reference as we introduce the pulse carrier for continuous time. We'll make very brief reference to the pulse carrier for discrete time indicating that, again, the analysis and discrete time and continuous time is very parallel.

So let's, first, consider complex exponential and sinusoidal carriers for the discrete time case, emphasizing the very strong parallel and similarity between discrete time and continuous time. Well we have, once, again the modulation property. And the modulation property tells us that the spectrum of the modulated signal is the periodic convolution of the two spectra. And let's consider, for example, an input, or modulating spectrum, as I've indicated here.

And since we want to consider, first of all, a complex exponential carrier, we'll consider the case of c of n equal to e to the j omega sub cn. And let me stress, by the way, as I did in the continuous time case, that I'll tend to suppress the phase angle which, of course, can be associated with the carrier also.

All right, so we have, then, the spectrum of the modulated signal. The spectra, the carrier signal, if this is the carrier, then it's spectrum is an impulse train, and that impulse train, I've indicated here. And let me stress, also, that in the discrete time case, of course, these spectra and all of the spectra involved, are periodic with a period of 2 pi.

So this then is the spectrum of the carrier signal. This is the spectrum of the input signal. The periodic convolution of these two is the spectrum the modulated signal. And the result is, then, this spectrum shifted to a center frequency, which is the carrier frequency omega sub c.

So the result of modulation with a complex exponential is a straightforward shift of the spectrum so that what occurred around zero frequency now occurs around the frequency omega sub c.

Now, in the continuous time case, we demodulated, when we had a complex exponential carrier, we demodulated by, essentially, just shifting the spectrum back. And in fact, in the discrete time case, were able to do exactly the same thing. So if we were to replace c of n which is either j omega sub cn by c of n equals e to the minus j omega sub cn, the resulting spectra would be an impulse train, as I indicate here, and the result of multiply y of n by that new carrier, in the frequency domain as a convolution of these two, and it's relatively straightforward to verify that if you convolve these with a periodic convolution, then that will get us back to the original spectrum that we started with.

So what's happened in the discrete time case, with the complex exponential, is exactly the same as in continuous time. Namely, we modulate that corresponds to shifting the spectrum. We demodulate by multiplying by the complex conjugate of the original modulated carrier and that shifts the spectrum back to where it was originally.

OK. Now let's consider the case of a sinusoidal carrier in discrete time. And again, things very much parallel what we saw in continuous time. And again, as we look at the spectra, I will choose a phase angle of zero, mainly for notational and analytical convenience.

So in this case, now, rather than a carrier signal, which is a single complex exponential, it's now a sinusoidal carrier and the sinusoidal carrier is the sum of two complex exponential. And so if we consider a modulated spectrum, that is the spectrum of x of n, something of the type that I indicate here, and the spectrum of the carrier, now, since the carrier is sinusoidal rather than a complex exponential consists of two impulses, one at plus omega sub c and one at minus omega sub c, convolving this spectrum with this spectrum gives us a replication of x of omega around plus and minus omega sub c. And incidentally, with an amplitude change of a half.

So again, things have worked as they did in continuous time. In continuous time or in discrete time, modulating with a sinusoidal carrier would correspond to a replication of the spectrum around, plus the carrier frequency and a replication of the spectrum around minus the carrier frequency, in both cases, as long as the carrier frequency is large enough compared with the bandwidth of the signal so that those two replication don't overlap, then it's reasonable to suppose that we should be able to recover the original signal.

Well, in fact, to demodulate in the discrete time case, we would again follow very much the strategy that we did in continuous time. In particular, let's consider demodulating by taking the modulated signal and, again, putting that through a modulator, again, with the carrier which is cosine omega sub cn. If we do that, we have a demodulator or what will turn out to be part of a demodulator, as I indicate here, the spectrum of the input signal is, as I had just developed, a replication of the original spectrum around plus and minus omega sub c with an amplitude of a half.

When this is, again, convolved with the spectrum of the carrier, then we get a replication of the original spectrum, first around zero frequency, as I indicate here, and then around twice the carrier frequency and minus twice the carrier frequency. And as long as the carrier frequency is large enough compared with the width of the original signal, then, as you can see, by extracting this part of the spectrum with a low pass filter, we can, in principle, recover the spectrum associated with the original signal. And again, just as in continuous time, because this amplitude is a half, we would want to choose, for scaling purposes, a low pass filter amplitude which is 2 to compensate for this factor of a half.

So once again things work out basically the same way as they had in continuous time. We have sinusoidal modulation which consists of using a sinusoidal carrier. And we have the demodulator which consists of taking a modulated signal, multiplying by the carrier, and then processing that with a low pass filter to extract the portion of the spectrum, which is around zero frequency, as I indicate in the spectrum below and the result, then, that this low pass filter having a gain of 2 is that we've recovered the original spectrum, x of omega, which is the spectrum that we started with.

Now several other things to stress. This is a fairly quick tour through sinusoidal modulation for discrete time. There are very similar issues that arise in the discrete time case in terms of having phase synchronization and frequency synchronization between the modulator and demodulator. And we had discussed that in a fair amount of detail for the continuous time case.

In some sense, in practical terms, that becomes much more of an issue in continuous time than it does in discrete time, in part, because synchronization between a modulator and demodulator is often much harder in a continuous time system, which is essentially an analog system as compared with a digital system. Another very important reason and it's important to stress this at the outset is that, whereas the theory involving the use of complex exponential and sinusoidal modulation parallels very strongly in the continuous time and the discrete time case. In practical terms, it has much more significance in continuous time than it does in discrete time. That is, the notion called sinusidal modulation, in the context of communication systems, is extremely important for continuous time systems, and less so in discrete time systems.

Now as a preview of a point to be raised later on, I should modify that slightly with the statement that one very important place in which sinusoidal modulation in a discrete time context arises, is in a class of systems called transmultiplexers or transmodulation systems. And this surface is basically because so many communication systems are now becoming digital and, specifically, discrete time, although the actual transmission is continuous time, the signal processing manipulation and switching is discrete time.

And so, in fact, it turns out to be very important and useful to take a discrete time representation of the analog signals or continuous time signals and, in a discrete time, or digital representation, to convert them from one modulation scheme or one multiplexing scheme to another. And although I said a lot there that really requires much more detail to develop in any sense at all, you should get the notion that discrete time modulation systems become very important, in part, because of implementational issues.

OK, now, there is another application that we have discussed for both continuous time and actually, previously, for discrete time, amplitude modulation with sinusoidal complex exponential carriers. And let me just remind you of that, because, in fact, it becomes a very important one in the case of discrete time systems. And that is the notion of using modulation together with fixed filtering to implement a filter, which either has a variable cut off or converts, let's say, a low pass filter to a high pass filter.

We had originally talked about this when we introduce the modulation property in the context of converting a low pass filter to a high pass filter. And the notion was that, if we modulate the signal with a carrier which is minus 1 to the n, and that's just simply a complex exponential or sinusoidal carrier with a carrier frequency of pi, then that, in effect, interchanges the low frequencies and the high frequencies. And if, after modulation, that is processed with a low pass filter, and then the result is demodulated, using exactly the same carrier, namely a carrier which is minus 1 to the n, then the effect of that is equivalent to high pass filtering on the original signal.

And a generalization of that would involve, instead of this specific choice of minus 1 to the n, would involve a choice, in general, of e to the j omega sub cn, that is a more general carrier frequency, and a demodulator which is e to the minus j omega sub cn. And as I've represented it here, and as we had talked about it when we talked about the modulation property for discrete time signals, we had specifically chosen the conversion of a low pass to a high pass filter.

Well, let me continue the review of that just by reminding you of the details of what happens with the spectra, and, specifically, the notion, if we take this particular case of omega sub c is equal to pi, or equivalently, a carrier signal which is minus 1 to the n, then if we have the original spectra and the spectrum of the carrier signal, the spectrum of the carrier signal convolved with this spectrum will then, in effect, shift this by pi. And so, after modulating, the result that we have is a shift of that spectrum so that what happened in low frequencies now happens at high frequencies, namely around pi, and what happened at high frequencies now happens at low frequencies.

Well, if that's processed now, with a low pass filter, and this dashed line indicates the low pass filter, then the result that we get is shown here, where we've extracted the low frequency portion of the modulated signal. And now when we modulate or demodulate back, then this spectrum is shifted back to where it belongs. Namely, it's shifted back to be centered around minus pi and around plus pi.

So if we just compare this spectrum with the original spectrum at the top, what we can see is that, in effect, what we've done is to extract a portion of the spectrum equivalent to processing with a high pass filter. And, again, this is very similar to what we did in continuous time and all of the analytical processes and convolution involved are very much the same.

Really, the biggest difference between continuous time discrete time has to do, not so much with the details of the analysis, but perhaps has more to do with issues of practical applications.

OK, well, so what we've done, so far, for continuous time and discrete time, is to talk about modulation, amplitude modulation with complex exponential and sinusoidal carriers. We saw that the analysis is very similar, although applications are slightly different.

And now what I'd like to turn to is a different choice of carrier signal. And the carrier signal, in this particular case, is a pulse train rather than a sinusoidal signal. Now the idea is the following. In general, of course, the modulator consists of all of multiplying x of t by whatever the carrier signal is. And previously, we've talked about a carrier signal which is sinusoidal signal. The carrier signal that we want to consider now is a carrier signal which, in fact, is a pulse train. And so, in fact, what we want to do is multiply the input signal by a pulse train and, in effect, then, the modulated signal consists of the original signal, simply with time slices pulled out of it, as I've indicated in the bottom curve.

So what we have now is a modulated signal that is a chopped or sliced version of the original waveform and that is what's referred to as pulse amplitude modulation. Now it seems like it's kind of a crazy idea. The idea is to chop out slices of the wave form and hope that you could put things back together again. And the amazing thing about it, as we'll see, is that, in fact, under fairly broad general and applicable conditions, you really can put the waveform back together again if you just have these time slices.

Not only that, but that basic notion, as we'll see, is independent, in fact, of what the width of those time slices are. In fact the width can go to zero. And, in fact, we're going to make it go to zero, and really only dependent on what the frequency of the pulse train is.

So let's explore that in some detail. And what we want to look at is the analysis, but let me, first, just comment, very briefly, that all of the analysis we go through, as has been true in the case of sinusoidal modulation, all of the analysis then we go through holds just as well with, essentially minor analytical modifications, to discrete time pulse amplitude modulation as it does to continuous time pulse amplitude modulation. And so we'll really only go through this in terms of tracking the wave forms and spectra for the continuous time case. But bear in mind that the results are basically similar for discrete time.

OK, well, let's see how so we get the basic result that we want to get. What we have is modulated signal which is a pulse train, basically a square wave, and as we've seen in previous lectures, the spectra or Fourier transform associated with that is an impulse train. And the envelope of that impulse train is on the form of a sine x over x function. The Fourier transform is impulses. And the spacing of the impulses is associated with the fundamental frequency of the pulse train and that's omega sub p.

So omega sub p is pi divided by the period of the pulse train. And the amplitude and shape of this envelope is dictated by the parameter delta, which has to do with how wide the pulses are. OK, so we have a time function. It's multiplied by this pulse train. Now we're talking continuous time.

So, in the frequency domain, we have the Fourier transform of the time function convolved with this Fourier transform for the pulse train. And let's see what that looks like. If we were to consider, let's say, a Fourier transform, which I have chosen as more or less a general one, then in fact, when we convert all of this with this impulse train, what we end up with is a replication of this spectrum at the places in the frequency domain where the individual impulses occurred.

So we can see that this spectrum is replicated at each of these locations. And as long as the frequency of the pulse train is large enough, compared with the maximum frequency in the original signal, x of t, so that there's no overlap between these triangles, then what you can see, in fact, somewhat amazingly is that, simply by low pass filtering the result, we can get back, except for amplitude factor, we can get back to the original signal.

Now it's amazing. It really is amazing that all that this depends on is the original signal being band limited and the frequency of the pulse train being high enough so that when you replicate the spectrum the frequency domain, there's no overlap between these individual replications. And we'll have address that a little more in a few minutes.

But let me, first of all, point out that this has a whole variety of very important implications. One is, in the context of communications, it leads to another very important multiplexing scheme for communications. We had talked last time about frequency division multiplexing, where individual signals were put into individual frequencies slots by choosing different carrier frequencies for a sinusoidal modulating signal.

What this suggests is that what we can put different signals into, non-overlapping time slots and, in fact, be able to recover the original signals back again. So in particular, suppose that I had a signal which I modulated with a pulse train and I chose another signal, modulated with another pulse train, where the time slot was different, and I continued this process. And after I'd done this with some number of channels, simply added all those together as I indicate here. Then as long as I knew what time slots to associate with what signal, I could get the original modulated signals back again.

And then as long as the frequency of the impulse train was such that I was able to do this reconstruction by simply low pass filtering, then I would be able to demodulate. So it's a very different very important modulation scheme called time division multiplexing in contrast to frequency division multiplexing as we had talked about last time.

I had made reference earlier to the concept of trans-multiplexing. And in fact, what happens in many communication systems is that the signals are represented, in fact, in discrete time. The analog and continuous time signals are represented in discrete time. And very often the conversion from frequency division multiplexing to time division multiplexing and back is done, in fact, in the discrete time domain.

OK, so what we have then, is the notion that we can multiply a time function by a pulse train, as I indicate here. And from the output I can, if the frequency of this pulse train is high enough in relation to this bandwidth, from the output, which consists of time slices, from those time slices I can recover the original signal. Stressing again the reason it relates to the spectra, and the reason is that the original spectra is simply replicated at multiples of the fundamental frequency of the pulse train.

Now there's a very important thing to observe here, which is that the ability to do the reconstruction is associated with the notion of whether we can extract that central triangle. I happened to choose a triangular shape but obviously I could be talking about any shape, as long as it's band limited, the ability to extract that. And notice that, in this modulated output spectrum, the ability to recover this is totally independent called what the value of delta is.

In other words, if we look back at the modulator, then, in fact, we can make delta, the width of these pulses, arbitrarily small. And, in theory, that doesn't affect our ability to do the reconstruction. Now in practical terms it might.

Looking back once more at the spectrum of the output, notice that this amplitude is proportional to delta. And what that suggests is that, as we make delta smaller and smaller, which we might, in fact, want to do, if you want to time division multiplex lots of channels, in principle, in theory, you could make it an infinite number of channels just by making that infinitesimally small. The smaller it is, in some sense, the less energy there is. And again, in practical terms, this one of those things if you push down here pops up there, namely, you eventually run into issues such as noise problems.

So, more typically what's done is to, in fact, eliminate this scale factor of delta. And the way that that's done is very simply. It's done by choosing the width of the pulses, and the height of the pulses, in such a way that the area is constant, even as we make delta get arbitrarily small. So we can just modify our argument so that what we're referring to is a modulated pulse train, which is a pulse train with pulses of width delta and height, 1 over delta.

In that case, as delta gets arbitrarily small, then, in fact, what these rectangles become are impulses, in which case, what we're talking about is a carrier signal which, in fact, is an impulse train. And the resulting modulated signal is an impulse train for which the amplitudes of the impulses are proportional to the original input waveform at the times at which these impulses occur.

OK well, let's look at the analysis of that. And so now, what we're talking about, is a spectrum that consists of the result of the spectrum we talked about before with the sine x over x envelope, except that, now, as delta goes to zero that becomes flat. In other words, the modulated signal is an impulse train. And so as we look at the spectrum of the modulated signal, that is, then, an impulse train in the frequency domain.

The height is proportional to the frequency of the impulse train and omega sub s now denotes the frequency of the impulse train. And the resulting output of the modulator has a spectrum which is this original spectrum, again, replicated around each of these impulses, in other words, replicated in multiples of the sampling frequency

Now this is very much identical to the more general case. We have this replication of the spectra. And as long as the frequency of the impulse train is large enough, compared with the bandwidth of the signal so that these triangles don't overlap, I can extract this portion of the spectrum by low pass filtering, in fact, would then give us back the original signal.

Now if, instead, this frequency omega sub m is greater than omega sub s minus omega sub m, we would have a spectrum that looked something more like this. And what's happened, in this case, is that, because we have an overlap here, we've destroyed the ability to recover the original signal from the impulse train. And that would be true, also in a more general case, of pulse amplitude modulation with pulses of non-zero width.

This effect by the way, is one that we'll be exploring in considerably more detail in the next lecture. And it's a phenomenon or distortion refer to as aliasing which, in fact, is an important and interesting topic.

But going back to the case in which we've chosen the frequency of the impulse train high enough, then we would recover the original signal by processing it through a low pass filter. And in that case, what this says is, that if we have a signal, and we modulate it with an impulse train, if we then process that impulse train through an idea low pass filter, given the right conditions on the frequency of impulse train and the bandwidth of the signal, we can recover the original signal back again.

Now let me stress, just going back to the picture in which we had done this modulation, that this process, where the modulation, where the carrier signal involves an impulse train, is often referred to as sampling. And what that means, specifically, is that, if we notice, this resulting impulse train is, in fact, a sequence of samples of the original continuous time signal.

In other words, what we've done, in effect, is taken instantaneous sample of this wave form. And the implication is that, if we do that at a rapid enough rate in relation to the bandwidth of the signal, then we can, in fact, recover the original signal back again.

And, finally to remind you of the argument once more, we have an original signal and we have its spectrum. When we've sampled it, and this is now the sampled signal, it's an impulse train whose instantaneous values are samples of the original waveform, the spectrum of that is the original one replicated. And when that is processed, through a low pass filter, to extract this part of the spectrum, then, after the low pass filter, we can recover the original signal back again.

OK well, in fact, although if you follow through the spectra and the wave forms, this all seems fairly straightforward and, perhaps or perhaps not, obvious, it's really worth reflecting on how amazing the result really is. We began this discussion by talking about modulation. And in fact modulation and sinusoidal of modulation is important in its own right.

We ended the discussion by talking about first pulse amplitude modulation, and then showing how, under the right set of conditions, you can, in fact, take a wave form and sample it with a set of instantaneous samples. And that set of instantaneous samples, in fact, are sufficient to totally represent and reconstruct the signal. What in fact, the formal statement that is, is refer to as the sampling theorem, a very powerful theorem that says, if we're given equally spaced samples of a time function, and if that time function is band limited, and if the bandwidth and if the sampling frequency is chosen in the right way, in relation to the bandwidth, then, in fact, the original time function is uniquely recoverable with a low pass filter.

Now the sampling theorem is, I would say, a watershed or cornerstone of a lot of the discussion that we've been having for a whole variety of reasons. It, first of all, drops out almost as a straightforward obvious statement. But more importantly what it says is, if I have a continuous time signal which satisfies the right set of conditions, I could represent it by what it does at sampling instance or, equivalently, at discrete instance of time.

Now what that leads to is a whole host of things. One of which is this statement that says, if we have a continuous time signal, I could in fact, represent it as a discrete time signal. And I could even think of processing a continuous time signal using discrete time concepts. And when I'm all done converting back, through the power of the sampling theorem, converting back to a continuous time signal.

So the sampling theorem provides us with a very major important bridge between continuous time and discrete time implementations and ideas. In the next several lectures, we will be exploring some of this in considerable detail. First, to focus in more, next time, on some of the specific issues and distortions associated with sampling. And following that, a discussion of what is referred to discrete time processing of continuous time signals. Thank you.