

16

Sampling

The sampling theorem, which is a relatively straightforward consequence of the modulation theorem, is elegant in its simplicity. It basically states that a bandlimited time function can be exactly reconstructed from equally spaced samples provided that the sampling rate is sufficiently high—specifically, that it is greater than twice the highest frequency present in the signal. A similar result holds for both continuous time and discrete time. One of the important consequences of the sampling theorem is that it provides a mechanism for exactly representing a bandlimited continuous-time signal by a sequence of samples, that is, by a discrete-time signal. The reconstruction procedure consists of processing the impulse train of samples by an ideal lowpass filter.

Central to the sampling theorem is the assumption that the sampling frequency is greater than twice the highest frequency in the signal. The reconstructing lowpass filter will always generate a reconstruction consistent with this constraint, even if the constraint was purposely or inadvertently violated in the sampling process. Said another way, the reconstruction process will always generate a signal that is bandlimited to less than half the sampling frequency and that matches the given set of samples. If the original signal met these constraints, the reconstructed signal will be identical to the original signal. On the other hand, if the conditions of the sampling theorem are violated, then frequencies in the original signal above half the sampling frequency become reflected down to frequencies less than half the sampling frequency. This distortion is commonly referred to as *aliasing*, a name suggestive of the fact that higher frequencies (above half the sampling frequency) take on the alias of lower frequencies.

The concept of aliasing is perhaps best understood in the context of simple sinusoidal signals. Given samples of a sinusoidal signal, many continuous-time sinusoids can be threaded through the samples. For example, if the samples were all of equal height, they could correspond to samples of a sinusoid of zero frequency or in fact a sinusoid at any frequency that is an integer multiple of the sampling frequency. From the samples alone there is clearly no way of determining which of the continuous sinusoids was sampled. The reconstruction filter, however, makes the assumption that the samples also correspond to a frequency less than half the sampling frequency; so for this particular example, the reconstructed output will be a constant. If, in fact, the

original signal was a sinusoid at the sampling frequency, then through the sampling and reconstruction process we would say that a sinusoid at a frequency equal to the sampling frequency is aliased down to zero frequency (DC).

Thus, as we demonstrate in this lecture, if we sample the output of a sinusoidal oscillator and then reconstruct with a lowpass filter, as the oscillator frequency increases from zero, the output of the lowpass filter will correspondingly increase. The output frequency will match the input frequency until the oscillator frequency reaches half the sampling frequency. As the oscillator frequency continues to increase, the output of the lowpass filter will begin to decrease in frequency.

It is important to understand that in sampling and reconstruction with an ideal lowpass filter, the reconstructed output will not be equal to the original input in the presence of aliasing, but samples of the reconstructed output will always match the samples of the original signal. This relationship is emphasized in this lecture through a computer movie. It is also important to recognize that aliasing is not necessarily undesirable. As we illustrate with a hopefully enjoyable and entertaining visit with Dr. Harold Edgerton at MIT's Strobe Laboratory, stroboscopy heavily exploits the concept of aliasing. Specifically, by using pulses of light, motion too fast for the eye to follow can be aliased down to much lower frequencies. In this case, the strobe light represents the sampler, and the lowpass filtering is accomplished visually.

Suggested Reading

Section 8.0, Introduction, pages 513–514

Section 8.1, Representation of a Continuous-Time Signal by Its Samples: The Sampling Theorem, pages 514 to mid-519

Section 8.3, The Effect of Undersampling: Aliasing, pages 527–531

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16.1

Sampling Theorem

- Equally Spaced Samples of $x(t)$

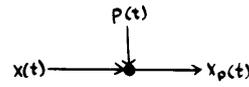
$$x(nT) \quad n=0, \pm 1, \pm 2, \dots$$

- $x(t)$ Bandlimited

$$X(\omega) = 0 \quad |\omega| > \omega_M$$

$$\text{If } \frac{2\pi}{T} \triangleq \omega_s > 2\omega_M$$

Then $x(t)$ uniquely recoverable



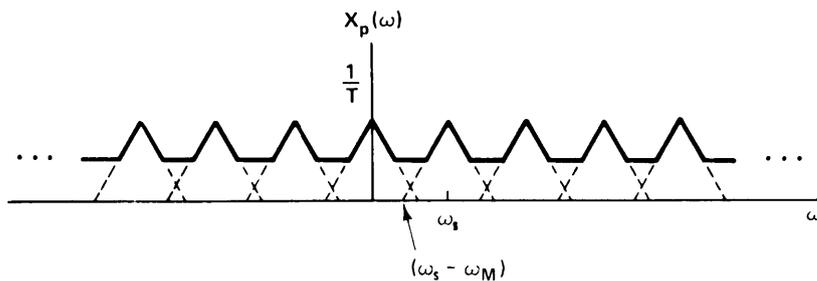
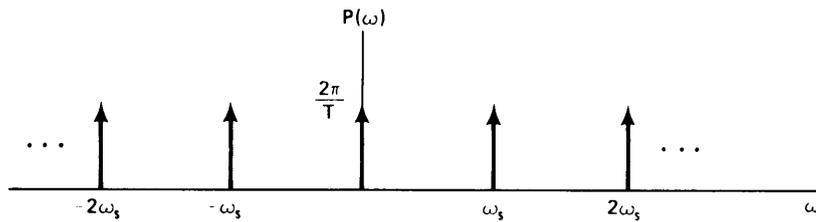
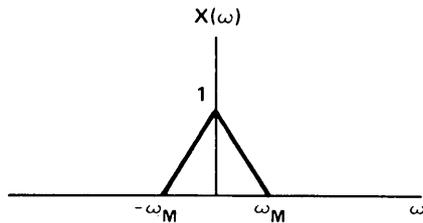
$$x_p(t) = x(t) \sum_{n=-\infty}^{+\infty} \delta(t-nT)$$

$$= \sum_{n=-\infty}^{+\infty} x(nT) \delta(t-nT)$$

$$X_p(\omega) = \frac{1}{2\pi} [X(\omega) * P(\omega)]$$

$$P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\omega - k \frac{2\pi}{T})$$

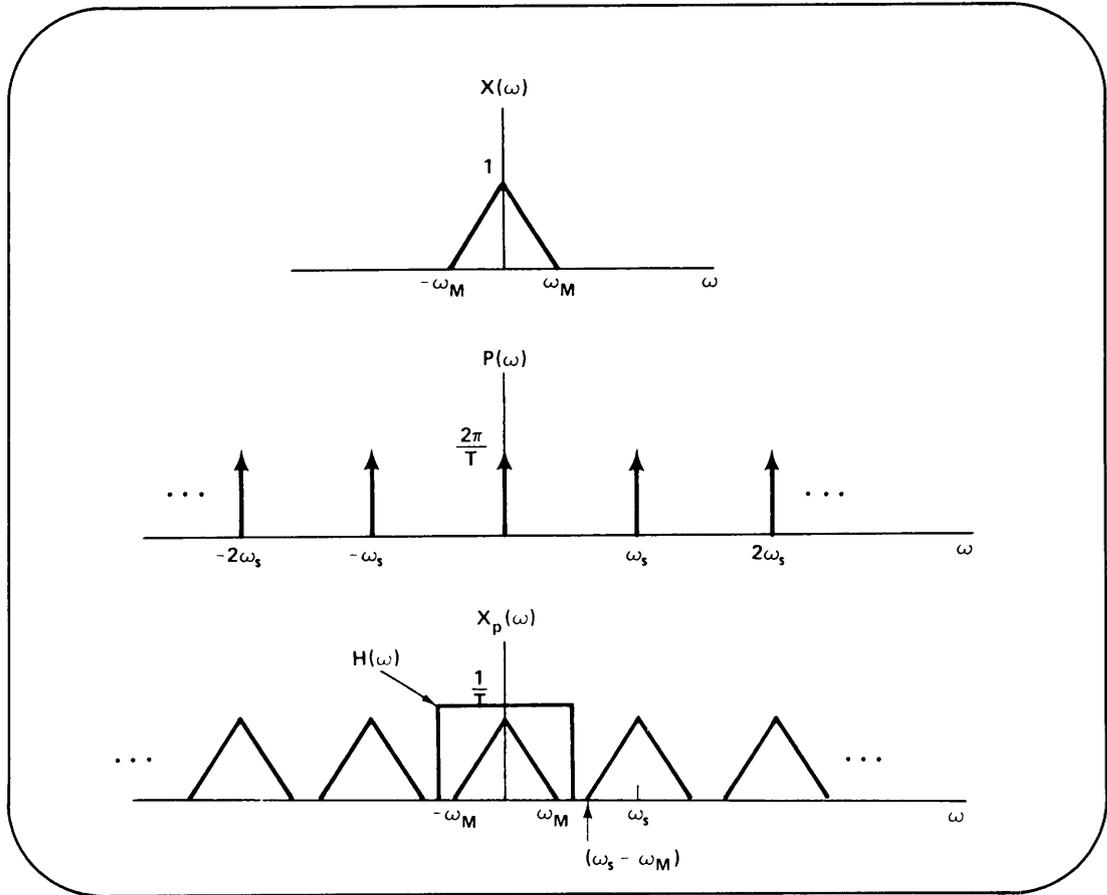
$$X_p(\omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X(\omega - k \frac{2\pi}{T})$$



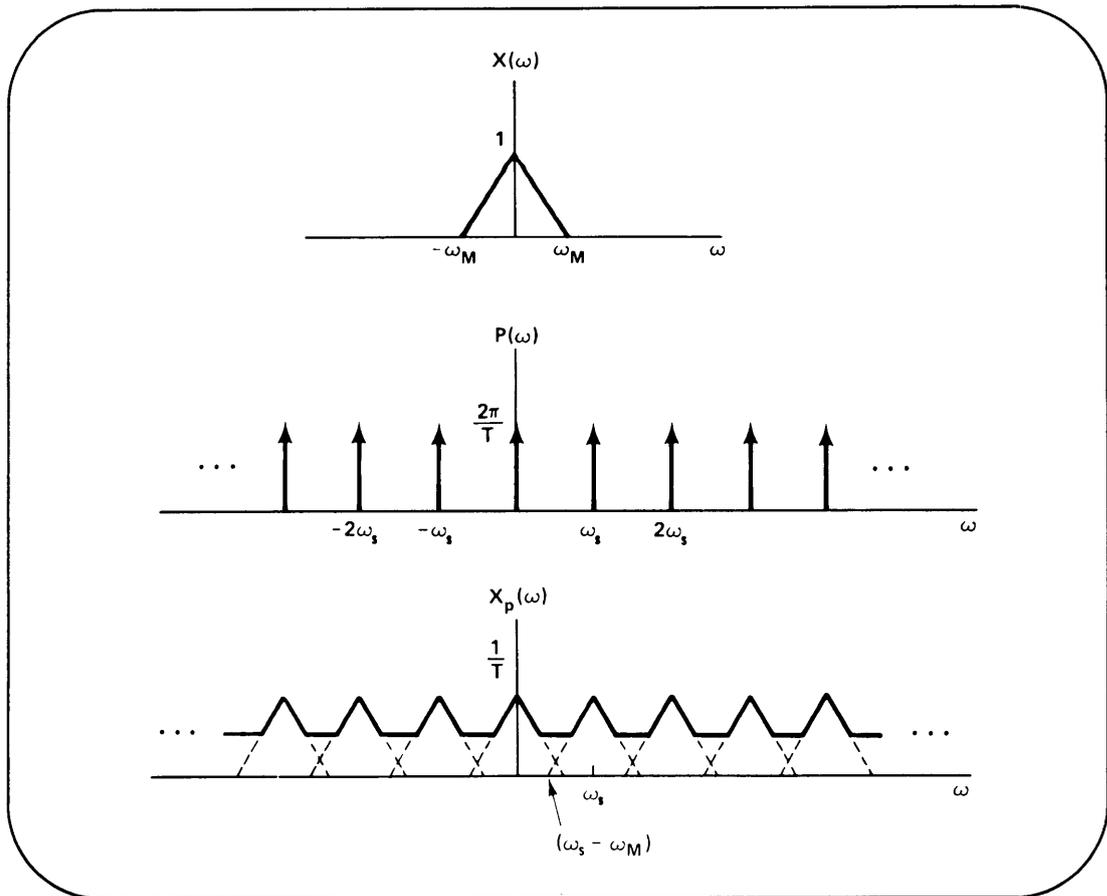
TRANSPARENCY
16.1

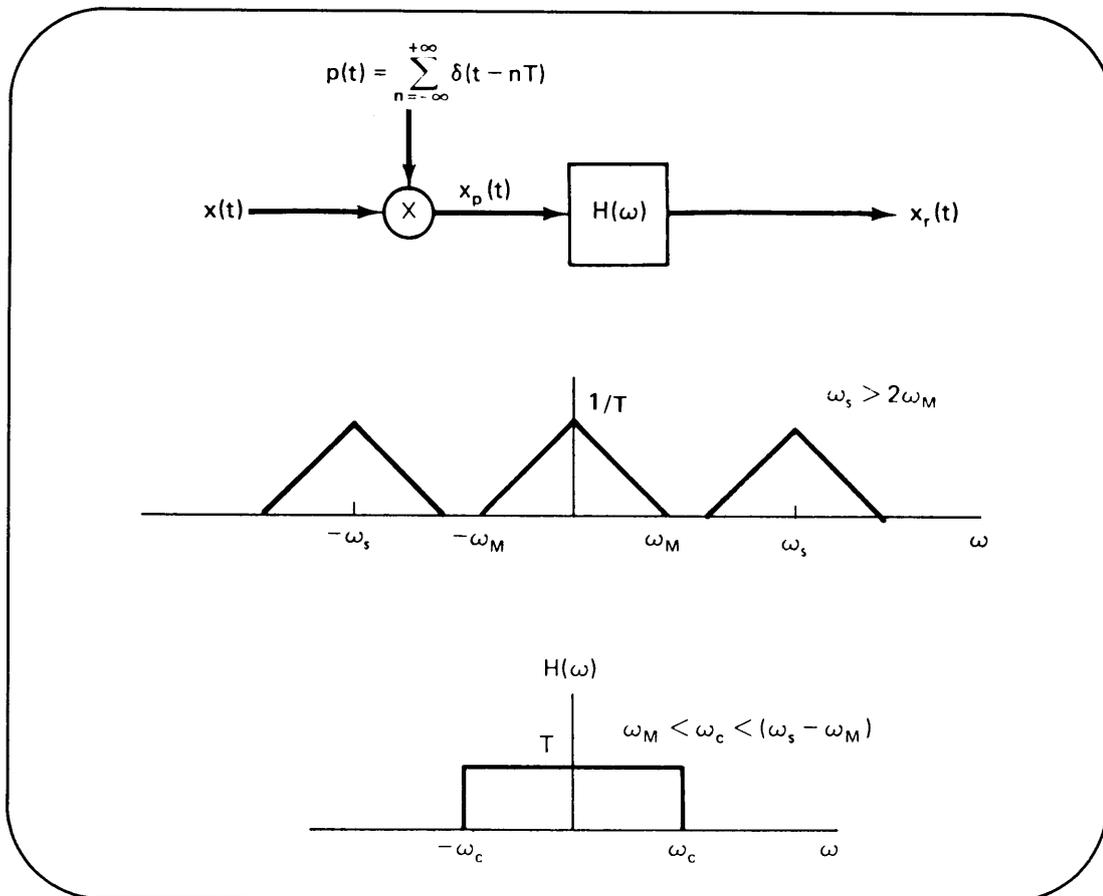
Spectra associated with sampling a signal based on amplitude modulation with an impulse train carrier.

TRANSPARENCY 16.2
 Lowpass filter used for recovery of the input signal.

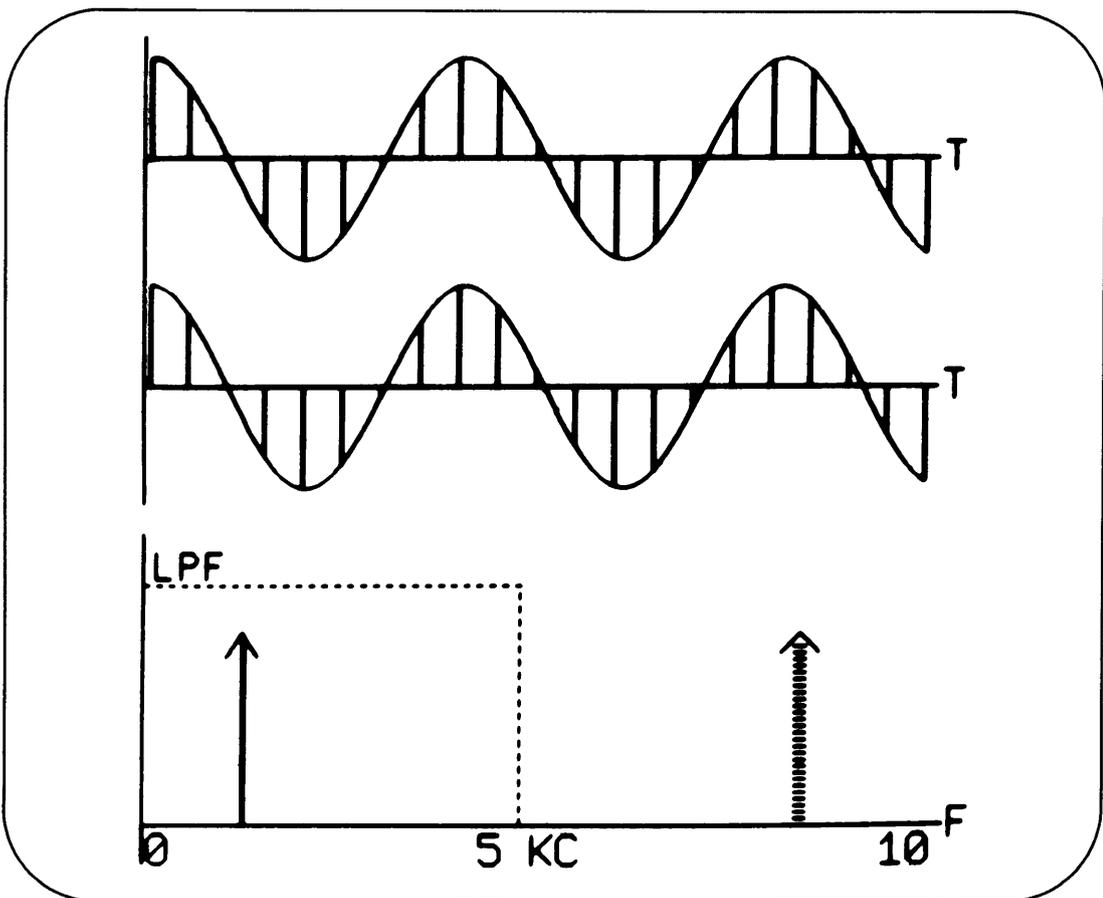


TRANSPARENCY 16.3
 Illustration of aliasing when the sampling frequency is too low.



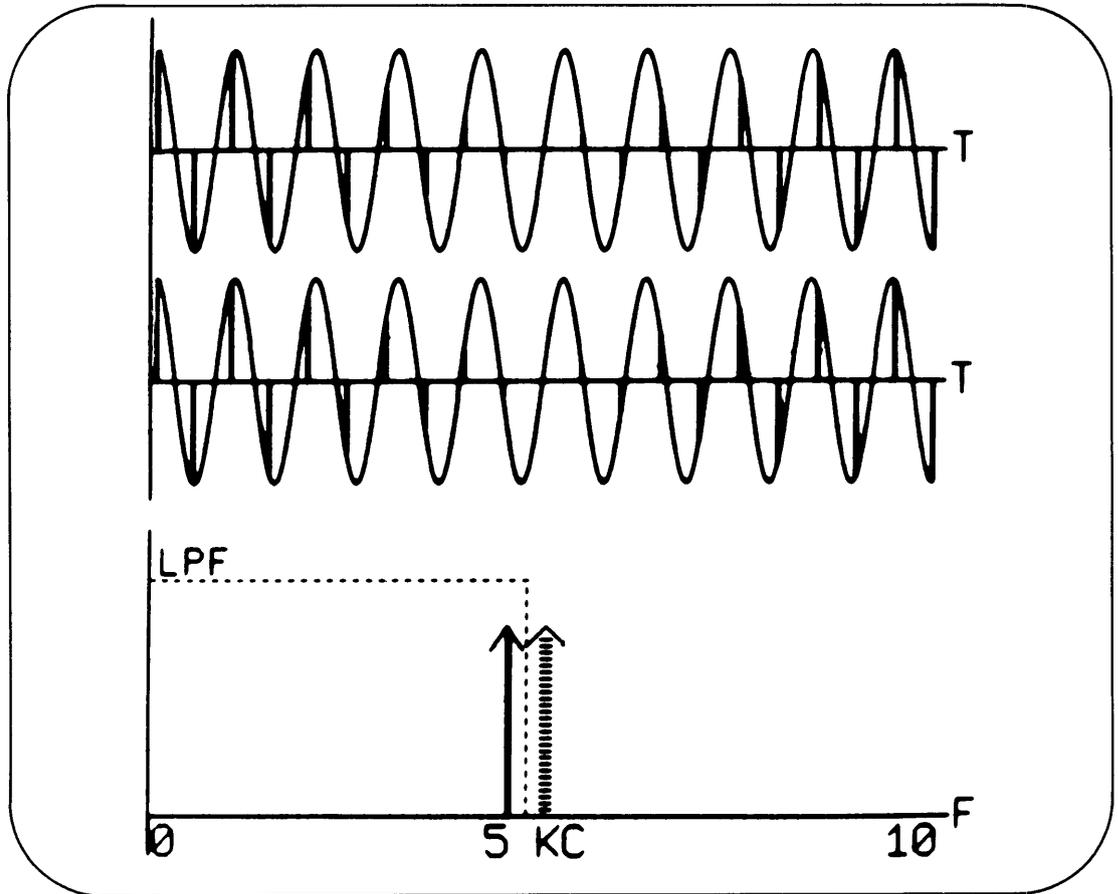


TRANSPARENCY 16.4
 Block diagram of sampling and reconstruction using an ideal lowpass filter.

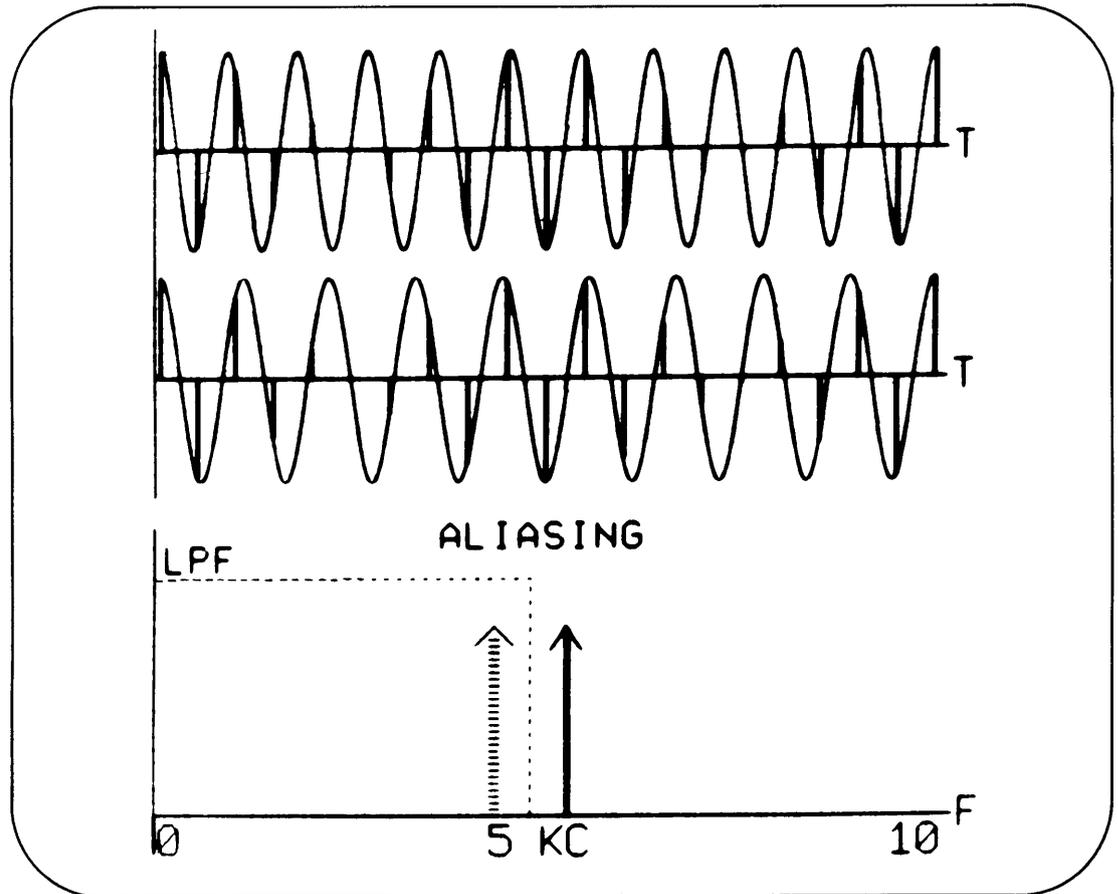


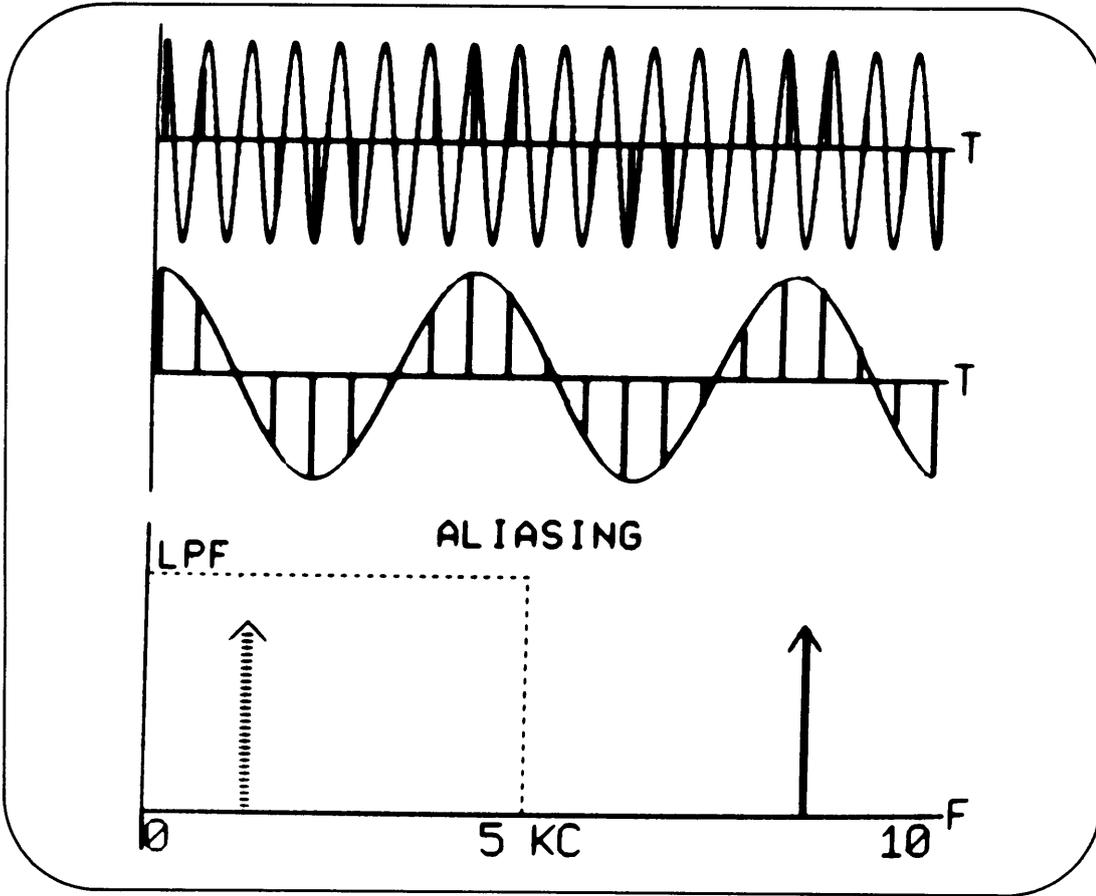
TRANSPARENCY 16.5
 Time waveforms and spectra for a sampled sinusoidal signal, with no aliasing.

TRANSPARENCY 16.6
The same as Transparency 16.5, with the frequency increasing.



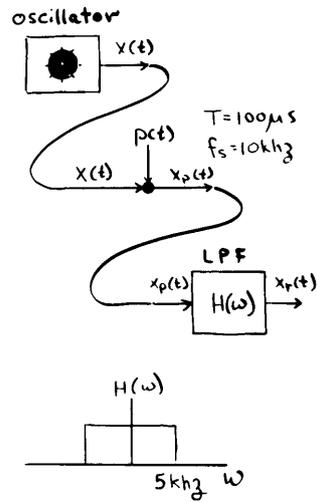
TRANSPARENCY 16.7
The input frequency increased to the point where aliasing occurs.





TRANSPARENCY
16.8
An input frequency
considerably above
that at which aliasing
occurs.

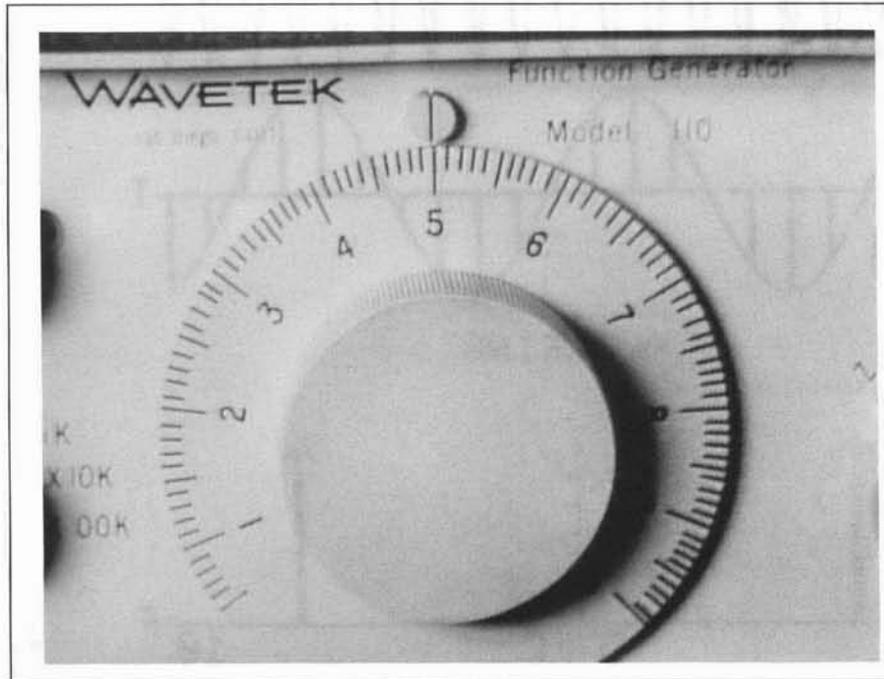
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16.2



DEMONSTRATION

16.1

Audio demonstration of aliasing with a sinusoidal oscillator.

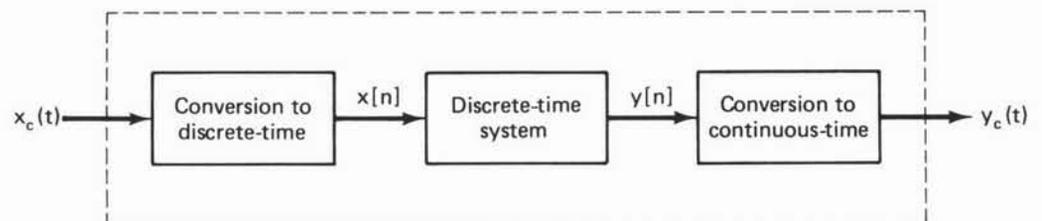


TRANSPARENCY

16.9

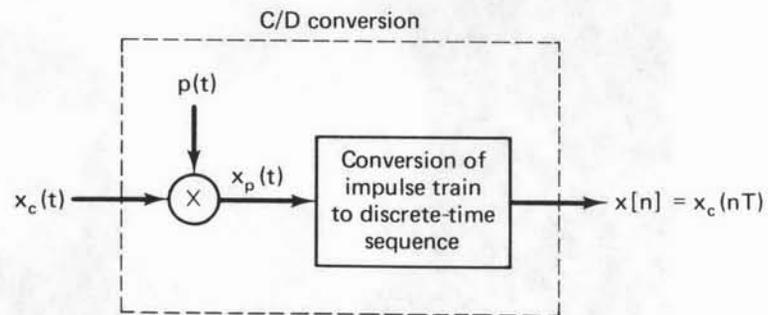
Discrete-time processing of continuous-time signals.

DISCRETE-TIME PROCESSING OF CONTINUOUS-TIME SIGNALS

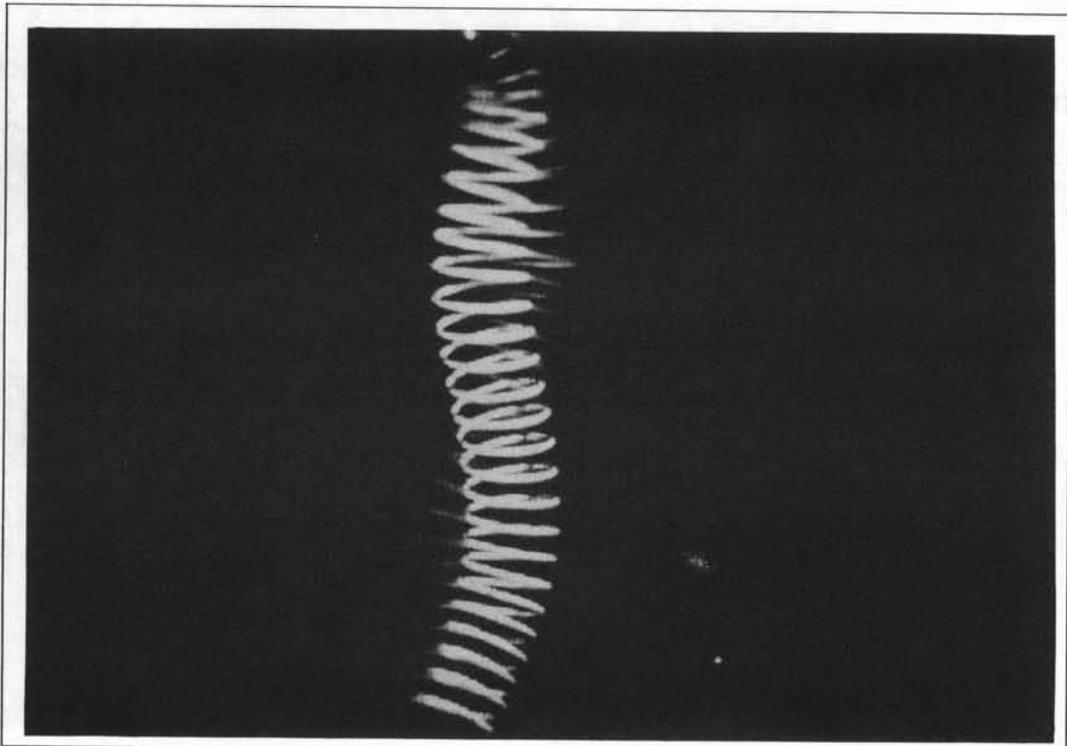


TRANSPARENCY**16.10**

Conversion of a continuous-time signal to a discrete-time signal interpreted in two steps. The continuous-time signal is first sampled with a periodic impulse train, and the impulse train values are then converted to a discrete-time sequence.

**DEMONSTRATION****16.2**

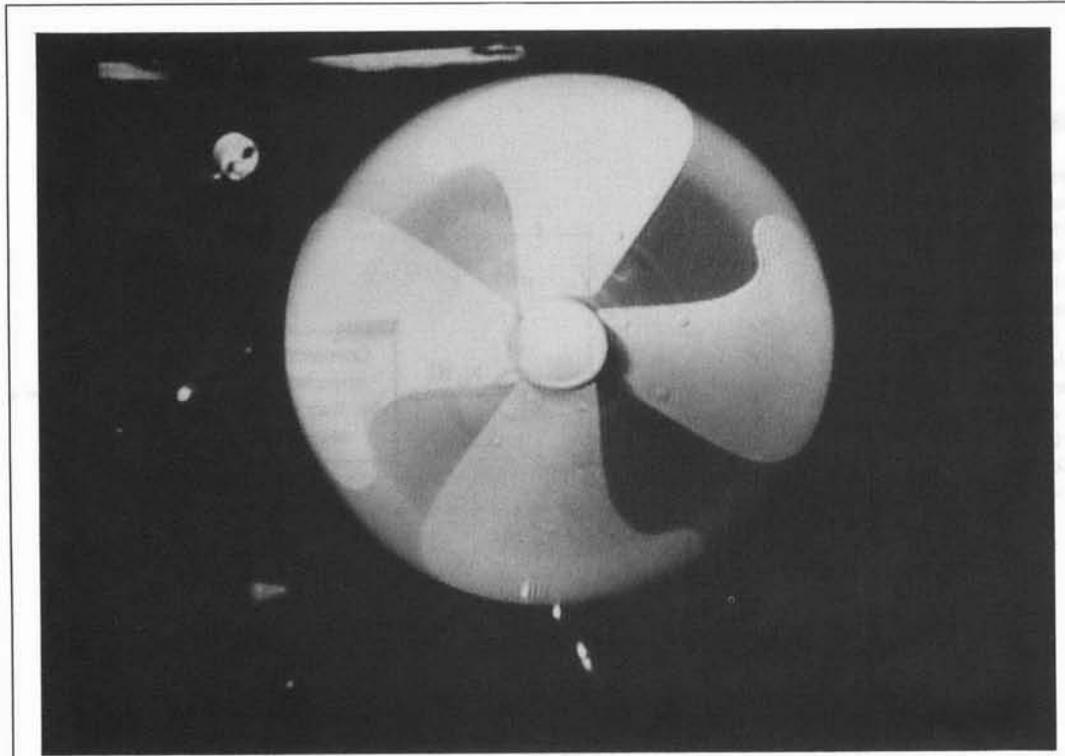
Use of a strobe to view the motion of an oscillating spring.



DEMONSTRATION

16.3

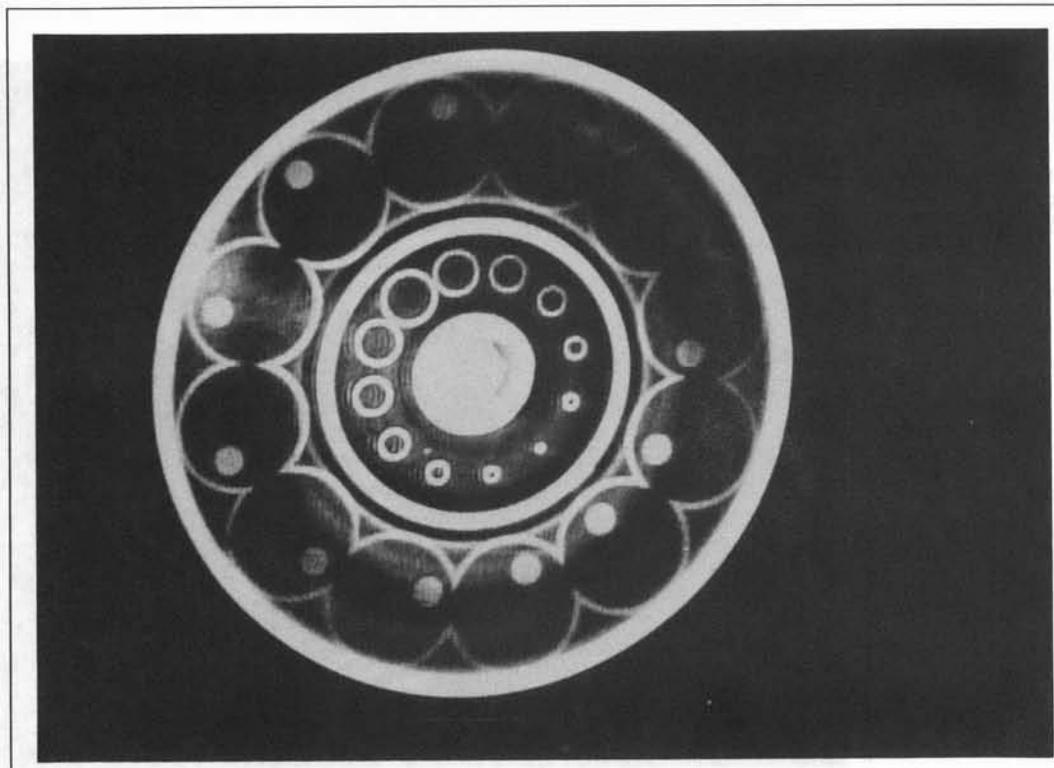
Use of a strobe to view the motion of a rotating fan.

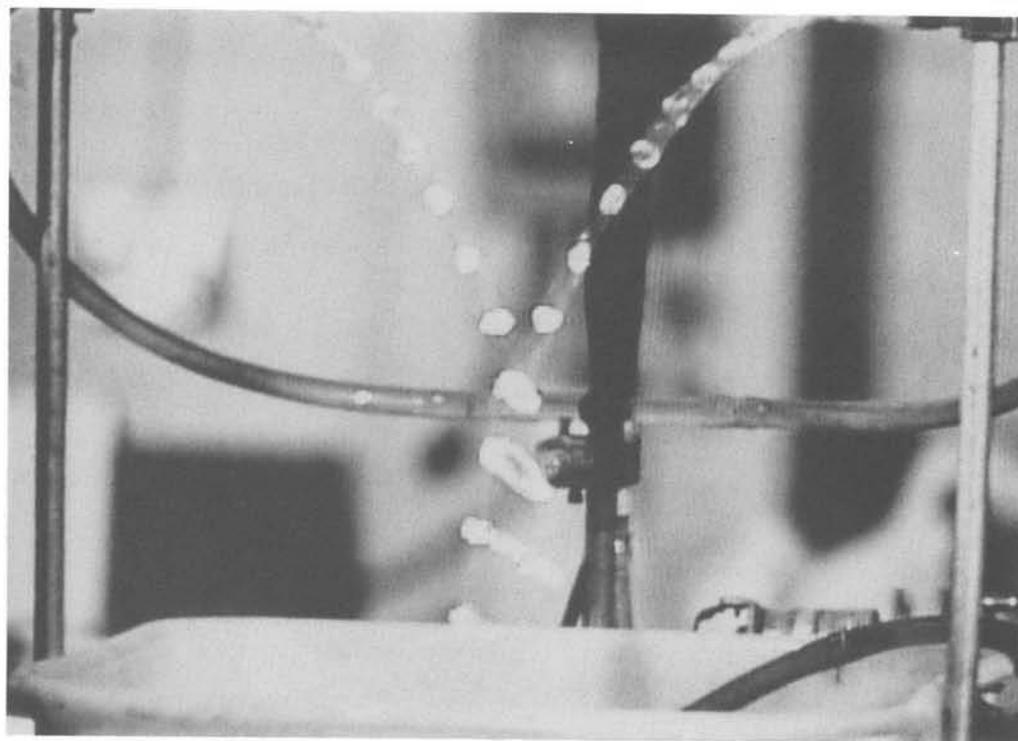


DEMONSTRATION

16.4

A rotating disk observed with a strobe.





DEMONSTRATION
16.5
Water drops as seen
with a strobe.

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Resource: Signals and Systems
Professor Alan V. Oppenheim

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