

**Note:** All references to Figures and Equations whose numbers are *not* preceded by an “S” refer to the textbook.

Because the operational amplifier is ideal, having infinite open-loop gain and contributing no dynamics, its connection forms an ideal integrator. Further, due to the infinite open-loop gain, the inverting input terminal is forced to ground potential, and thus the 10 k $\Omega$  input resistor has no effect on the system outer-loop transmission. Thus, the integrator connection contributes a term of  $\frac{-1}{RCs}$  (where  $R = 10 \text{ k}\Omega$ ) to the outer-loop transmission. Then, the negative of the outer-loop transmission is given by

$$-\text{L.T.} = \frac{1}{RCs} \times \frac{1}{(10^{-6}s + 1)(10^{-7}s + 1)} \quad (\text{S9.1})$$

To set the phase margin to 45°, this term must have a phase angle of  $-135^\circ$  at the loop-transmission crossover frequency. The integrator contributes  $-90^\circ$  at all frequencies. Thus, the crossover frequency,  $\omega_c$ , must be such that

$$\angle \left[ \frac{1}{(10^{-6}s + 1)(10^{-7}s + 1)} \right] \Big|_{s=j\omega_c} = -45^\circ \quad (\text{S9.2})$$

A rough estimate of  $\omega_c$  is  $10^6 \text{ rad/sec}$ , because at this frequency the pole at  $10^6 \text{ rad/sec}$  contributes  $-45^\circ$ , while the pole at  $10^7 \text{ rad/sec}$  contributes little phase shift. This estimate may be refined by recognizing that the pole at  $10^7 \text{ rad/sec}$  will contribute  $-5^\circ$  of phase when  $\omega$  is about  $10^6 \text{ rad/sec}$  (i.e., a decade below the upper pole). Thus, we look for the frequency where the pole at  $10^6 \text{ rad/sec}$  contributes  $-40^\circ$  of phase. That is,

$$\angle \left( \frac{1}{10^{-6}s + 1} \right) \Big|_{s=j\omega_c} = -40^\circ \quad (\text{S9.3})$$

or

$$\tan^{-1} 10^{-6}\omega_c = 40^\circ \quad (\text{S9.4})$$

### Solution 9.1 (P5.3)

which is solved by  $\omega_c = 8.4 \times 10^5$  rad/sec. For a more exact answer, we can program the expression for the phase angle of Equation S9.2, and search numerically to find that the angle is  $-45^\circ$  at  $\omega_c = 8.45 \times 10^5$  rad/sec, verifying the accuracy of the approximation of Equation S9.4. The ability to make useful approximations is important in this subject, both for efficiency of computation and for understanding the impact of individual terms on system performance. For instance, here the pole at  $10^7$  rad/sec has only a minor effect on system behavior. It is the pole at  $10^6$  rad/sec that is most significant.

Now, we pick the value of  $C$  to set crossover at  $8.4 \times 10^5$  rad/sec. At this frequency, the poles at  $10^6$  and  $10^7$  rad/sec contribute a magnitude of 0.76. Thus, to set the crossover point, we must have

$$\frac{1}{RC\omega_c} \times 0.76 = 1 \quad (\text{S9.5})$$

or

$$\begin{aligned} C &= \frac{1}{R\omega_c} \times 0.76 = \frac{0.76}{10^4 \times 8.4 \times 10^5} \\ &= 91 \text{ pF} \end{aligned} \quad (\text{S9.6})$$

### Solution 9.2 (P5.4)

- (a) We begin by drawing the block diagram for the motor. Let the motor torque be  $T_m$ . Then, from the model of Figure 5.25b, we have

$$T_m = 0.1 I_a = 0.1 \frac{(V_a - 0.1\Omega_s)}{1} \quad (\text{S9.7})$$

where  $V_a$  is the voltage at the output of the operational amplifier. Recall from physics that angular velocity is related to torque by integration, through the rotational equivalent of Newton's  $F = ma$ . That is,

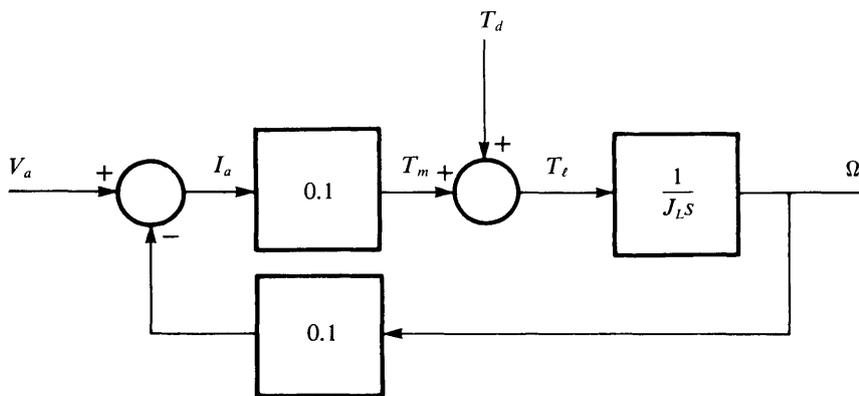
$$\Omega_s = \frac{1}{J_L} \int T_e dt \quad (\text{S9.8})$$

where  $T_\ell$  is the torque applied to the load inertia. In the Laplace domain, this becomes

$$\Omega_s(s) = \frac{T_\ell}{J_L s} \quad (\text{S9.9})$$

Now, because  $T_\ell$  is the net torque applied to the load, it is the sum of the motor torque and the disturbance torque. That is,

$$T_\ell = T_m + T_d \quad (\text{S9.10})$$



**Figure S9.1** Block diagram for motor and load.

Combining Equations S9.7, S9.9, and S9.10, we draw the block diagram for the motor and load as shown in Figure S9.1.

Now, the remaining portion of the loop including the operational amplifier is shown in Figure S9.2.

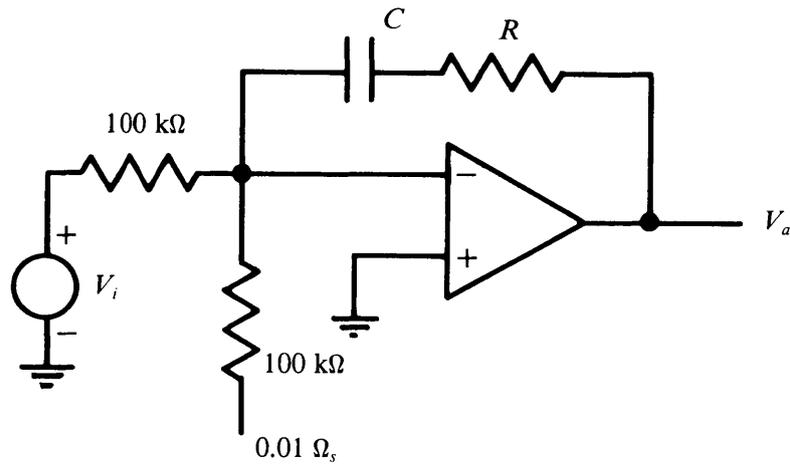
Because the op amp is ideal, the inverting terminal is held by feedback at ground potential. The current through the series  $RC$  is then  $\frac{(V_i + 0.01\Omega_s)}{10^5}$ , and the output  $V_a$  is given by

$$V_a(s) = -\frac{(V_i + 0.01\Omega_s)}{10^5} \times \left( R + \frac{1}{Cs} \right) \quad (\text{S9.11})$$

or

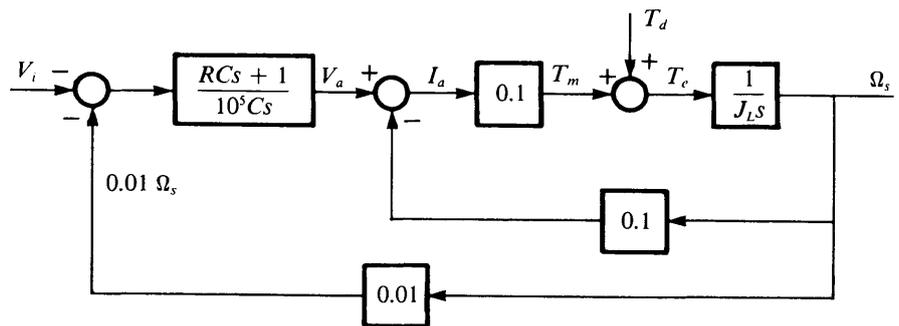
$$V_a(s) = -(V_i + 0.01\Omega_s) \left( \frac{RCs + 1}{10^5 Cs} \right) \quad (\text{S9.12})$$

**Figure S9.2** Amplifier connection from Figure 5.25.



The block-diagram representation of Equation S9.12 is added to the block diagram of Figure S9.1 to give the complete system block diagram as shown in Figure S9.3.

**Figure S9.3** System block diagram for Problem 9.2 (P5.4).



(b) By inspection of Figure S9.3, for  $T_d = 0$ ,

$$\frac{\Omega_s(s)}{V_a(s)} = \frac{0.1}{J_L s} = \frac{10}{100J_L s + 1} \quad (\text{S9.13})$$

Then, the system loop transmission is

$$\text{L.T.} = -0.01 \left( \frac{RCs + 1}{10^5 Cs} \right) \left( \frac{10}{100J_L s + 1} \right) \quad (\text{S9.14})$$

The problem requires that the loop transmission be  $-100/s$ . To achieve this, the zero due to the series  $RC$  is used to cancel the motor pole, and capacitor  $C$  is used to set the loop-transmission magnitude. That is, we require

$$RC = 100J_L \quad (\text{S9.15})$$

and

$$10^6 C = 0.01 \quad (\text{S9.16})$$

Thus,  $C = 0.01 \mu\text{F}$  and  $R = 10^{10} J_L$ .

- (c) Here, we must examine the relation between  $T_d$  and  $\Omega_s$ . It will be shown that this transfer function has a zero at the origin, thus the shaft velocity  $\Omega_s$  is unaffected by any constant  $T_d$ .

For a system in standard form we recall that the transfer function is  $\frac{a(s)}{1 + a(s)f(s)}$ . Applying this to the block diagram of Figure S9.3, we identify

$$a(s) = \frac{1}{J_L s} \quad (\text{S9.17})$$

and

$$f(s) = 0.1 \left[ 0.1 + 0.01 \left( \frac{RCs + 1}{10^5 Cs} \right) \right] \quad (\text{S9.18})$$

But, from part b,  $RC = 100J_L$  and  $C = 0.01 \mu\text{F}$ , so

$$f(s) = 0.01 + \frac{100J_L s + 1}{s} \quad (\text{S9.19})$$

Applying Equations S9.17 and S9.19 to the standard form, we have

$$\begin{aligned} \frac{\Omega_s(s)}{T_d(s)} &= \frac{1/J_L s}{1 + \frac{1}{J_L s} \left( 0.01 + \frac{100J_L s + 1}{s} \right)} \\ &= \frac{s}{J_L s^2 + 0.01s + 100J_L s + 1} \quad (\text{S9.20}) \\ &= \frac{s}{(100J_L s + 1)(0.01s + 1)} \end{aligned}$$

The d-c response is given by

$$\lim_{s \rightarrow 0} \frac{\Omega_s(s)}{T_d(s)} = 0 \quad (\text{S9.21})$$

Thus the response to a constant  $T_d$  is zero, as stated earlier.

### Solution 9.3 (P5.5)

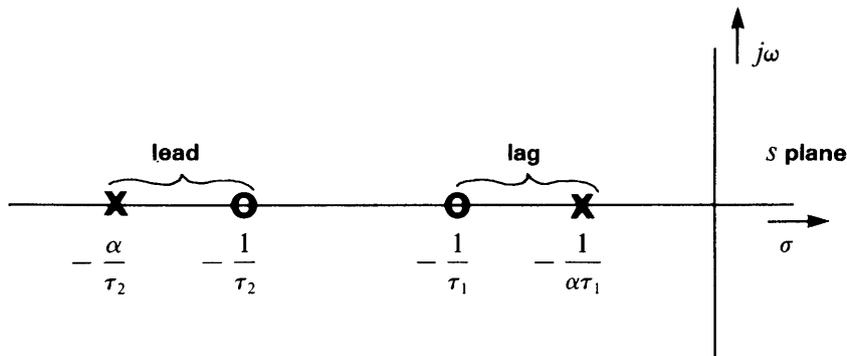
The transfer function for the network of Figure 5.26 is

$$\frac{V_o(s)}{V_i(s)} = \frac{R_2 + \frac{1}{C_2 s}}{R_2 + \frac{1}{C_2 s} + \frac{\frac{R_1}{C_1 s}}{R_1 + \frac{1}{C_1 s}}} \quad (\text{S9.22})$$

After some algebraic manipulation, this becomes

$$\frac{V_o(s)}{V_i(s)} = \frac{(R_1 C_1 s + 1)(R_2 C_2 s + 1)}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2) s + 1} \quad (\text{S9.23})$$

At this point, attempting to find the pole locations by factoring the denominator of Equation S9.23 simply leads to an algebraic mess. We can avoid this difficulty by examining the general characteristics of the lag-lead network. Such a network will have a pole-zero pair and a zero-pole pair at higher frequencies, forming the lag and lead components respectively. Furthermore, the network of Figure 5.26 imposes the constraint that the  $\alpha$  of the lag and lead pairs must be identical. (See Section 5.2.3 in the textbook for the definition of  $\alpha$ .) This is required because the magnitude of  $\frac{V_o(s)}{V_i(s)}$  is unity both in the limit as  $s \rightarrow 0$  and in the limit as  $s \rightarrow \infty$ . (Consider the limiting cases where the capacitors are open-circuited [ $s \rightarrow 0$ ], or short-circuited [ $s \rightarrow \infty$ ].) If the lag and lead  $\alpha$  parameters were not equal, this would not be possible. Following this argument, the singularities will be placed as shown in Figure S9.4, where  $\tau_1$  and  $\tau_2$  are the time constants associated with the two zeros.



**Figure S9.4** Singularities for circuit of Figure 5.26.

The transfer function that realizes the singularity pattern of Figure S9.4 is

$$H(s) = \frac{(\tau_1 s + 1)(\tau_2 s + 1)}{(\alpha \tau_1 s + 1) \left( \frac{\tau_2}{\alpha} s + 1 \right)} \quad (\text{S9.24})$$

This reduces to

$$H(s) = \frac{(\tau_1 s + 1)(\tau_2 s + 1)}{\tau_1 \tau_2 s^2 + \left( \alpha \tau_1 + \frac{\tau_2}{\alpha} \right) s + 1} \quad (\text{S9.25})$$

By means of the above argument, we have restricted the form of the transfer function we are trying to realize, thereby simplifying the design process. At this point, we match coefficients in the denominators of Equations S9.23 and S9.25 to find that

$$R_1 C_1 R_2 C_2 = \tau_1 \tau_2 \quad (\text{S9.26})$$

and

$$R_1 C_1 + R_2 C_2 + R_1 C_2 = \alpha \tau_1 + \frac{\tau_2}{\alpha} \quad (\text{S9.27})$$

Further, matching numerators requires that either  $R_1 C_1 = \tau_1$  and  $R_2 C_2 = \tau_2$ , or  $R_1 C_1 = \tau_2$  and  $R_2 C_2 = \tau_1$ . The choice between these two solutions is arbitrary. That is, for a given transfer function of the form of Equation S9.25, there are two sets of network element values that will realize the desired transfer function. We arbitrarily choose the solution where

$$R_1 C_1 = \tau_1 \quad (\text{S9.28a})$$

and

$$R_2 C_2 = \tau_2 \quad (\text{S9.28b})$$

Equation S9.26 is identically satisfied, and substitution into Equation S9.27 yields:

$$\tau_1 + \tau_2 + R_1 C_2 = \alpha \tau_1 + \frac{\tau_2}{\alpha} \quad (\text{S9.29})$$

which reduces to

$$R_1 C_2 = (\alpha - 1)\tau_1 + \left(\frac{1}{\alpha} - 1\right)\tau_2 \quad (\text{S9.30})$$

Note that the equivalence of Equations S9.23 and S9.25 has imposed only three independent constraints, while the network of Figure 5.26 has four elements. This means that the choice of one network element value is arbitrary. Once this element value is picked, the other three values are determined by Equations S9.28 and S9.30. Thus, it is shown that the network of Figure 5.26 may be used to realize an arbitrary lag-lead network.

Now, we turn to the question of realizing the given transfer function

$$\frac{V_o(s)}{V_i(s)} = \frac{(0.1s + 1)(10^{-2}s + 1)}{(s + 1)(10^{-3}s + 1)} \quad (\text{S9.31})$$

Comparing this with Equation S9.24 gives

$$\tau_1 = 0.1 \quad (\text{S9.31a})$$

$$\tau_2 = 10^{-2} \quad (\text{S9.31b})$$

and

$$\alpha = 10 \quad (\text{S9.31c})$$

Combining these with Equations S9.28 and S9.30 gives

$$R_1 C_1 = \tau_1 = 0.1 \quad (\text{S9.32a})$$

and

$$R_2 C_2 = \tau_2 = 10^{-2} \quad (\text{S9.32b})$$

and

$$\begin{aligned} R_1 C_2 &= (\alpha - 1)\tau_1 + \left(\frac{1}{\alpha} - 1\right)\tau_2 \\ &= 9 \times 0.1 - 0.9 \times 10^{-2} = 0.891 \end{aligned} \quad (\text{S9.32c})$$

Use the one arbitrary component value choice to let  $R_1 = 10 \text{ k}\Omega$ .

Then, by S9.32a,

$$C_1 = 10 \mu\text{F} \quad (\text{S9.33a})$$

by S9.32c,

$$C_2 = 89.1 \mu\text{F} \quad (\text{S9.33b})$$

and by S9.32b,

$$R_2 = 112 \Omega \quad (\text{S9.33c})$$

This is one set of possible component values. Due to the several arbitrary choices involved in this design process, there are an infinite number of other sets of component values that will give the desired transfer function.

Because  $L(s) = -\frac{10^6}{s^2}$ , it contributes a constant phase shift of  $-180^\circ$  to the  $af$  product. Thus, to maximize phase margin, we must place the loop-transmission crossover frequency at the point where the positive phase shift of the lead network is maximum. Recall from Section 5.2.3, that the point of maximum positive phase shift occurs at the geometric mean of the pole and zero locations. For the lead network, let the pole location be  $\omega_p$ . Then, given that  $\alpha = 10$ , the zero is at  $\frac{\omega_p}{10}$ , and the maximum positive phase shift occurs at  $\omega_{\max} = \frac{\omega_p}{\sqrt{10}}$ , the geometric mean of these locations.

At this frequency, the lead-network gain is  $\frac{1}{\sqrt{\alpha}} = \frac{1}{\sqrt{10}}$ . As stated earlier, we must set crossover frequency  $\omega_c$  at  $\omega_{\max}$ , to achieve maximum phase margin. Because the lead network contributes a gain of  $\frac{1}{\sqrt{10}}$  at  $\omega_{\max}$ , we require

$$|L(j\omega_{\max})| = \frac{10^6}{\omega_{\max}^2} = \sqrt{10} \quad (\text{S9.34})$$

#### Solution 9.4 (P5.6)

to set the loop-transmission magnitude to unity at  $\omega_{\max}$ . This is solved by  $\omega_{\max} = 10^{2.75} = 562$  rad/sec, which will be the system crossover frequency. From Equation 5.6 in the textbook, at this frequency, with  $\alpha = 10$ , the lead network contributes a positive phase shift of

$$\phi_{\max} = \sin^{-1} \left( \frac{9}{11} \right) = 54.9^\circ \quad (\text{S9.35})$$

Because, as stated earlier, the rest of the loop contributes a phase of  $-180^\circ$ , the loop phase margin is  $54.9^\circ$ .

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