S O L U T I O N S

Root locus 5

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 5.1 (P4.5)

(a) Rule 2 is all that is required to find the branches, and Rule 1 tells us that the branches terminate on the two zeros. Thus the root locus is as sketched in Figure S5.1*a*.

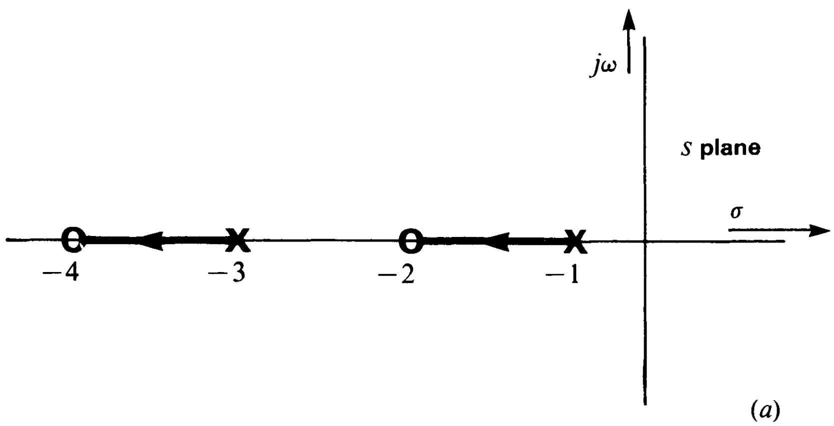


Figure S5.1 Root loci for Problem 5.1 (P4.5). (*a*) Root locus for Figure 4.27*a*.

(b) Rule 4 tells us that the average distance of the poles from the origin remains constant at $\bar{s} = -\dfrac{2+1+1}{3} = -4/3$. By Rule 5, the two complex poles approach asymptotes of $\pm 60°$. By Rule 6, the angle of the branch in the vicinity of the upper complex pole is

$$\begin{aligned}\theta_p &= 180° + \Sigma \sphericalangle z - \Sigma \sphericalangle p \\ &= 180° + 0 - 90° - 45° \qquad \textbf{(S5.1)} \\ &= 45°\end{aligned}$$

At the lower complex pole the angle will be $-45°$.

The point at which the complex pair crosses the imaginary axis can be estimated by using the asymptotes that intersect the real axis at $s = -4/3$. They intersect the imaginary axis at $s = \pm 4/3 \tan 60° = \pm j2.31$. The complex pole pair will cross the imaginary axis near this point.

The exact intersection of the root loci with the imaginary axis can be solved for using the Routh criterion. By inspection of the pole-zero diagram the characteristic equation (after clearing fractions) is

$$\begin{aligned} P(s) &= (s+2)(s+1+j)(s+1-j) + a_o f_o \\ &= (s+2)(s^2+2s+2) + a_o f_o \\ &= s^3 + 4s^2 + 6s + 4 + a_o f_o \end{aligned} \tag{S5.2}$$

The Routh array is

$$\begin{array}{cc} 1 & 6 \\ 4 & 4 + a_o f_o \\ 5 - \dfrac{a_o f_o}{4} & 0 \\ 4 + a_o f_o & 0 \end{array} \tag{S5.3}$$

The value $a_o f_o = 20$ will make the third row all zero. With this value of $a_o f_o$, we use the second row to write the auxiliary equation

$$4s^2 + 24 = 0 \tag{S5.4}$$

This has solutions at

$$s = \pm j\sqrt{6} = \pm j2.45 \tag{S5.5}$$

This exact value is close to our earlier estimate of $s = \pm j2.31$, as expected. In many cases, such an estimate will be sufficiently accurate, and the Routh computation may be avoided.

Using all of the above information we sketch the root locus in Figure S5.1*b*.

(c) Following the development in the textbook of Rule 7, for moderate values of $a_o f_o$, we may ignore the pole at $s = -1000$, and sketch the locus of the three other poles. Using Rule 3, the breakaway point between the poles at $s = -1$ and $s = -2$ will occur at the solution of

$$\frac{d((s+1)(s+2)(s+3))}{ds} = 0 \tag{S5.6}$$

Multiplying gives

$$\frac{d(s^3 + 6s^2 + 11s + 6)}{ds} = 0 \tag{S5.7}$$

Figure S5.1 (*b*) Root locus for Figure 4.27*b*.

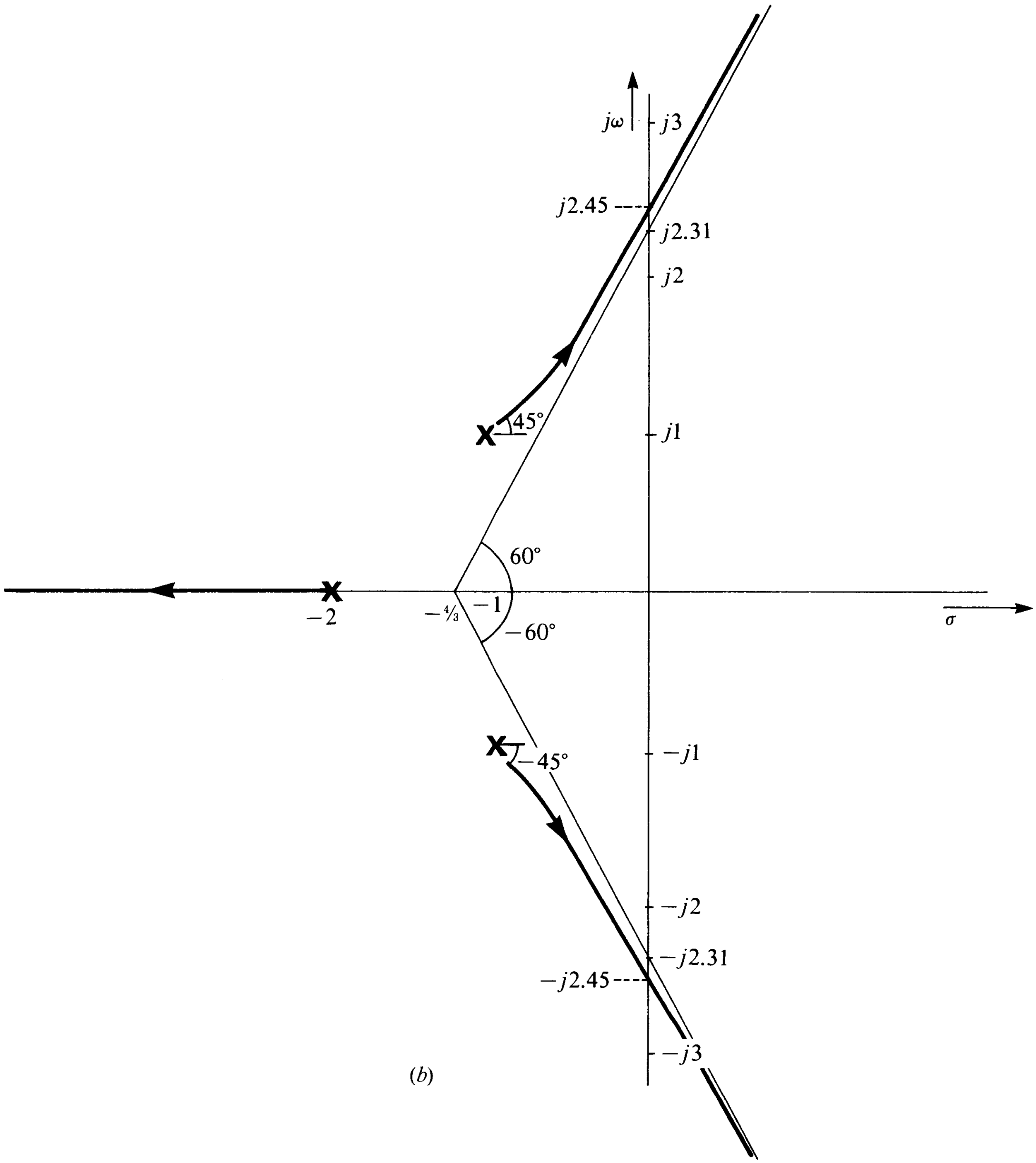
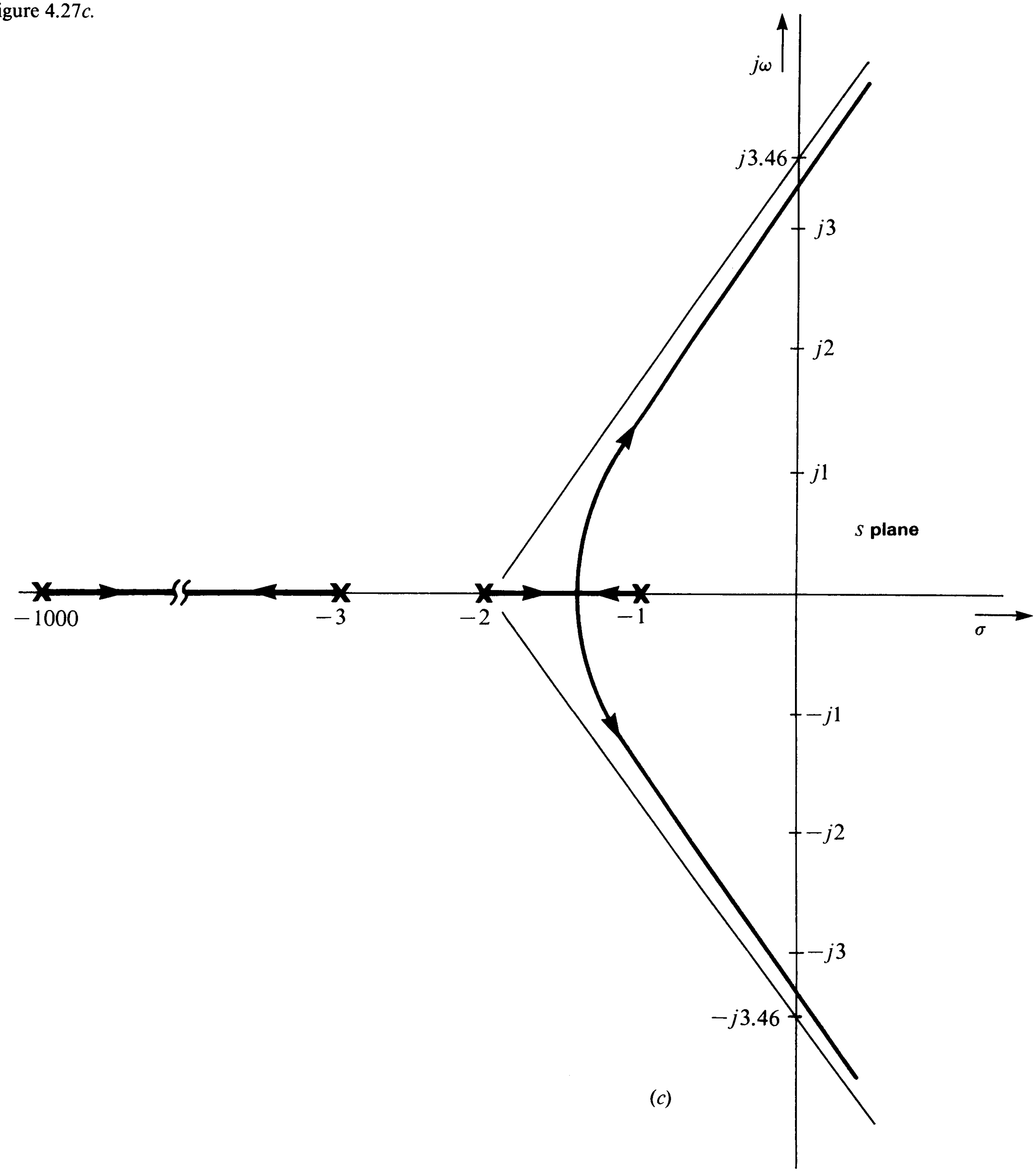


Figure S5.1 (*c*) Root locus for Figure 4.27*c*.



which gives

$$3s^2 + 12s + 11 = 0 \qquad \textbf{(S5.8)}$$

This is solved by

$$s = -1.42$$

and

$$s = -2.58 \qquad \textbf{(S5.9)}$$

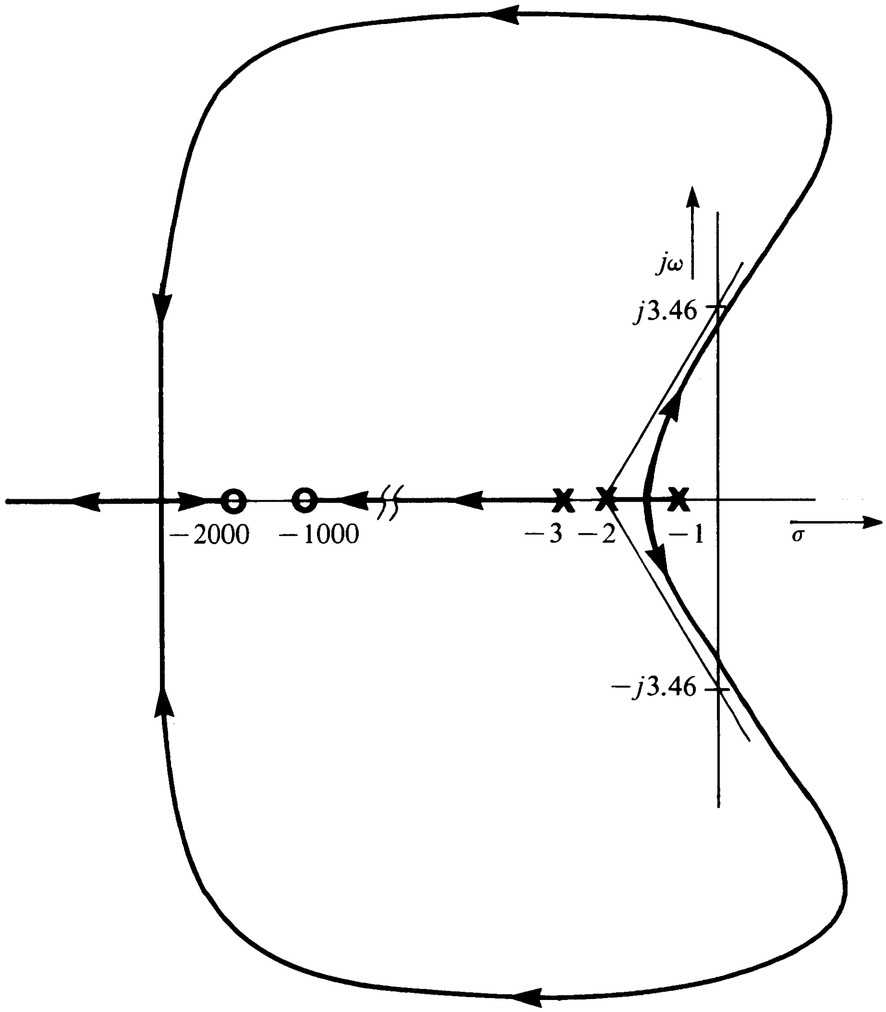
Only the solution at $s = -1.42$ is meaningful here (see p. 127 in the textbook), so the breakaway point is at $s = -1.42$.

By Rule 5, the loci approach asymptotes of $\pm 60°$, and these asymptotes intersect the real axis at $s = -2$. The asymptotes cross the imaginary axis at $s = \pm j2 \tan 60° = \pm j3.46$. Using this information, we sketch the root locus as shown in Figure S5.1*c*.

(d) By Rule 7, for low values of $a_o f_o$, the root locus will be the same as in part c. However, by Rule 2, we know that there is a branch of the locus to the left of the zero at $s = -2000$. Thus, while the two complex poles initially enter the right half of the s plane, they must at some point turn around, reenter the left half of the s plane, and rejoin the real axis at some point to the left of the zero at $s = -2000$. After rejoining the real axis, one pole will move to the right to the zero at $s = -2000$, and the other pole will move off to the left to infinity. Thus, the root locus appears as sketched in Figure S5.1*d*.

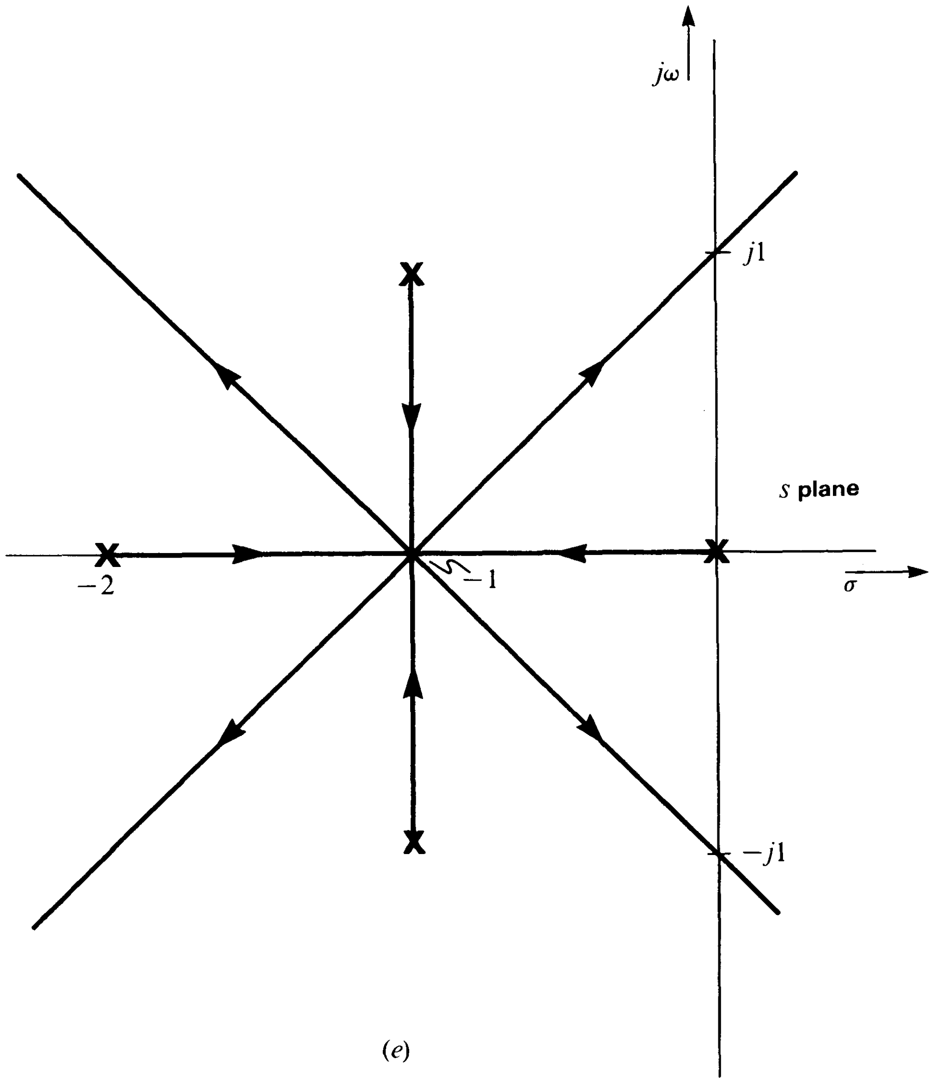
This system has the interesting property of being stable for low and high values of $a_o f_o$, but unstable for intermediate values of $a_o f_o$.

Figure S5.1 (*d*) Root locus for Figure 4.27*d*.



(*d*)

(e) By Rule 2, branches exist on the real axis between $s = 0$ and $s = -2$. By Rule 5, because there are four poles and no zeros, all four poles eventually (for large $a_o f_o$) approach infinity along asymptotes of $\pm 45°$ and $\pm 135°$. The asymptotes intersect at $s = -2$. By Rule 6, the loci leave the pole at $s = -1 + j$ at an angle of $-90°$, and leave the pole at $s = -1 - j$ at an angle of $+90°$. Combining this information we sketch the root locus as in Figure S5.1*e*.



(e)

Figure S5.1 (*e*) Root locus for Figure 4.27*e*.

Solution 5.2 (P4.7)

The noninverting configuration is as shown in Figure S5.2.

The loop transmission for this topology is $-a(s)f_o$, where $f_o = \dfrac{R_2}{R_1 + R_2}$. As long as the $a_o f_o$ product is large, that is, as long as $f_o \gg 10^{-6}$, the closed-loop gain is well approximated by $\dfrac{1}{f_o} = \dfrac{R_1 + R_2}{R_2}$.

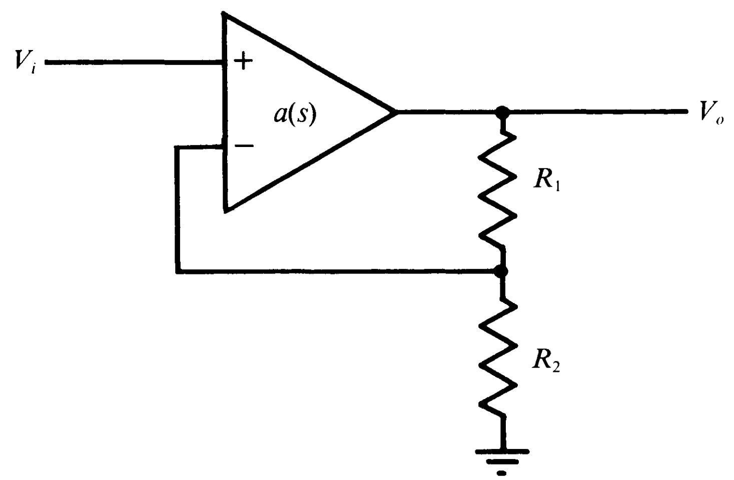


Figure S5.2 Noninverting amplifier.

At this point, a sketch of the root locus is helpful for understanding the system behavior as f_o is varied. The closed-loop poles start at the poles of $a(s)$ for small values of f_o, and move as shown in Figure S5.3 as f_o is increased. (Note that the axes are not drawn to scale.)

The root locus has been drawn by recognizing that two of the poles will approach asymptotes of $\pm 60°$, and eventually enter the right half of the s plane, as f_o grows. The third pole follows the real axis off to the left. Other details, such as breakaway points, are not important for this analysis, and we do not solve for them.

The dashed lines indicate points where the damping ratio for a complex pair is equal to 0.5. From the root-locus sketch it is apparent that there is some value of f_o for which the poles will lie on this line. Following the development on p. 128 of the textbook, when the system damping ratio is 0.5, its characteristic equation is given by Equation 4.62 as

$$P'(s) = s^3 + (\gamma + 2\beta)s^2 + 2\beta(\gamma + 2\beta)s + 4\gamma\beta^2 \quad \textbf{(S5.10)}$$

The closed-loop characteristic equation for the amplifier is 1 minus the loop transmission, which is (when cleared of fractions)

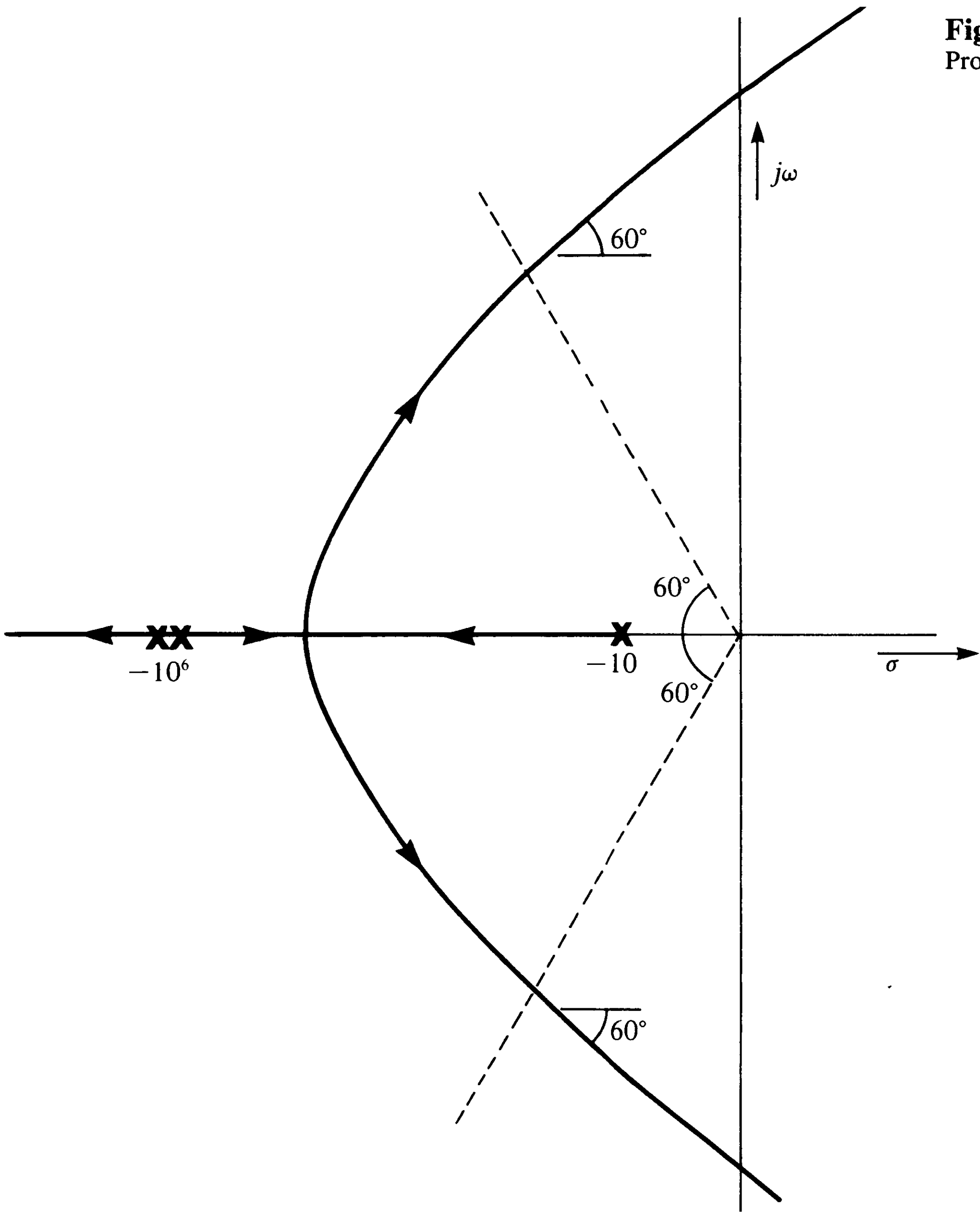


Figure S5.3 Root locus for Problem 5.2 (P4.7).

$$
\begin{aligned}
P(s) &= (0.1s + 1)(10^{-6}s + 1)^2 + 10^6 f_o \\
&\simeq 10^{-13}s^3 + 2 \times 10^{-7}s^2 + 0.1s + 10^6 f_o
\end{aligned}
\tag{S5.11}
$$

where three insignificant terms have been dropped. Now, we multiply $P(s)$ through by 10^{13} and equate with $P'(s)$

$$
\begin{aligned}
s^3 + 2 \times 10^6 s^2 + 10^{12}s + 10^{19} f_o &= s^3 \\
&+ (\gamma + 2\beta)s^2 + 2\beta(\gamma + 2\beta)s + 4\gamma\beta^2
\end{aligned}
\tag{S5.12}
$$

Equating the coefficients of s^2 yields

$$\gamma + 2\beta = 2 \times 10^6 \tag{S5.13}$$

Equating the coefficients of s, and substituting in from Equation S5.13 gives

$$2\beta(2 \times 10^6) = 10^{12} \tag{S5.14}$$

Thus,

$$\beta = 2.5 \times 10^5$$

and

$$\gamma = 1.5 \times 10^6 \tag{S5.15}$$

Then the remaining term is used to find that

$$f_o = 3.75 \times 10^{-2} \tag{S5.16}$$

With this value of f_o, the low-frequency loop transmission is large, and the low-frequency closed-loop gain is given by $\frac{1}{f_o}$ which is equal to 26.7. Resistor values to realize this gain can be chosen by setting

$$\frac{R_1 + R_2}{R_2} = 26.7 \tag{S5.17}$$

or

$$R_1 = 25.7R_2$$