

All right.

So, as we've seen in the previous clip, another way that initial conditions may matter is if a chain has a periodic structure.

There are many ways of defining periodicity.

Some of them are more mathematical than others.

Let us consider one of the most intuitive ways of doing that.

So here is the definition.

The states in a recurrent class are periodic if they can be lumped together, or grouped, into several subgroups so that all transitions from one group lead to the next group.

So what does that really mean?

Let us try to parse out this by looking at the given example.

So here we would have a situation, a structure of a diagram, in which  $d$  is equal to 2.

Whenever you are at a given time in a state in that group, in the next transition you will go to that group.

And if you are in that group, at the next transition, you will go to that group.

So in some sense, there is periodicity, and there is somewhat of a deterministic behavior, according to this transition.

But there is also some randomness left.

We can be in any of these states within that sub-group.

But one thing that is for sure is that whenever, at a given time, you are into one of these states, here, in that group, if the Markov chain has this specific structure, in the next transition you will transition to one of the states, here.

And the next transition after, you will go back here.

So essentially, if your Markov chain started in one of these states here, at time  $n$  equals zero, then at time  $n$  equals 1, it would be here.

And then at time  $n$  equals 2, it would be here.

Times them into 3, and so on and so forth.

So every time you would have an even number, then you are guaranteed that the Markov chain will be in one of these states, here.

Clearly, with this kind of structure, it is impossible to have convergence of the steady state probabilities.

This is another example where you would have a structure of a diagram in which you have a period of 3, here.

So in that case here,  $d$  equals 3.

So here again, if you are in one of these states at a given time, then at the next time the following transition will guarantee to bring you in that group.

And during the next transition you will go to that group.

And then again, here, and again you have this kind of behavior in a very systematic way.

So if you started here at time  $n$  equals 0, you would be in that group, here, at times  $n$  equals 1.

And you would be here at  $n$  equals 2.

And again,  $n$  equals 3 here,  $n$  equals 4,  $n$  equals 5, 6, 7, 8.

And you see the pattern.

If you look at any time in the future, if the time is of the form 3 times  $k$  for any  $k$  greater than or equal to 0, you're guaranteed that your chain will be in one of these states.

Otherwise, it will be here, or here.

So here again, you do not have convergence of the steady states, because if you are told that you started in one of the states here, then you know that whenever you have a time that is the form  $3k$  plus 1 you will be here.

And so the probability of being here would be 0.

All right, so we have been able to explain a little bit what a periodic state is, using this definition.

Now given a Markov chain, how can we tell whether a Markov chain is periodic or not?

There are, in fact, systematic ways of doing it mathematically.

But usually within the types of examples we see in this class-- most of them, these will be easy to see-- we just eyeball the chain and we tell whether it is periodic or not.

So let us see if it is that easy to do, and consider that chain, here.

OK, so let's see.

Is this chain periodic or not?

I will give you a couple of seconds to think about it.

Now I'm going to help a little bit.

I'll just decide that this one will be red, this one will be red, this one will be red, and this one will be red.

Now I'm asking the same question.

What do you think?

Well, from the structure it is clear that this Markov chain is periodic and has a period of  $d$  equals 2.

In some sense, the red state, here, could be one of these groups.

These are the red ones.

And the white, here, would be those, here.

And it is clear, if you look at this diagram, that whenever you are in a red state, at the next transition you are guaranteed to be in a white state.

From this one, you will be in white.

From this one, you will be in the white.

From this one, as well.

And from this one, as well.

And whenever you are in a white state, this one, this one, this one, or this one, at the next period you are guaranteed to be in a red state.

So this tells you that sometimes it is not that easy to eyeball and decide if the Markov chain is periodic or not.

If you have, for example, lots of, lots of states, you might have some trouble doing this exercise.

On the other hand, something very useful to know.

Sometimes, it's extremely easy to tell that the chain is not periodic, even if you have a lot of states.

What is that case?

Well, look at this case, here, and suppose for a moment that you have a self transition here.

Well in that case, you would have a transition from a white state to a white state.

And you're not guaranteed anymore that from a white state you would go to a red state.

In that sense, as soon as you have a self transition, the Markov chain is aperiodic.

It cannot be periodic.

So whenever you have a self transition, this implies that the chain is not periodic.

And usually that's the simplest and easiest way that we can tell, most of the time, that the chain is not periodic.