# LECTURE 9: Conditioning on an event; Multiple continuous r.v.'s

- Conditioning a r.v. on an event
  - Conditional PDF
  - Conditional expectation and the expected value rule
  - Exponential PDF: memorylessness
  - Total probability and expectation theorems
  - Mixed distributions
- Jointly continuous r.v.'s and joint PDFs
  - From the joints to the marginals
  - Uniform joint PDF example
  - The expected value rule and linearity of expectations
  - The joint CDF

## Conditional PDF, given an event

$$p_X(x) = \mathbf{P}(X = x)$$

$$f_X(x) \cdot \delta \approx \mathbf{P}(x \le X \le x + \delta)$$

$$p_{X|A}(x) = \mathbf{P}(X = x \mid A)$$

$$f_{X|A}(x) \cdot \delta \approx \mathbf{P}(x \le X \le x + \delta \mid A)$$

$$\mathbf{P}(X \in B) = \sum_{x \in B} p_X(x)$$

$$\mathbf{P}(X \in B) = \int_B f_X(x)\, dx$$

$$\mathbf{P}(X \in B \mid A) = \sum_{x \in B} p_{X|A}(x)$$

$$\mathbf{P}(X \in B \mid A) = \int_B f_{X|A}(x)\, dx$$

$$\sum_x p_{X|A}(x) = 1$$

$$\int f_{X|A}(x)\, dx = 1$$

## Conditional PDF of $X$, given that $X \in A$

$$\mathbf{P}(x \le X \le x + \delta \mid X \in A) \approx f_{X|X\in A}(x) \cdot \delta$$

$$f_{X|X\in A}(x) = \begin{cases} 0, & \text{if } x \notin A \\ \dfrac{f_X(x)}{\mathbf{P}(A)}, & \text{if } x \in A \end{cases}$$

## Conditional expectation of $X$, given an event

$$\mathbf{E}[X] = \sum_x x p_X(x)$$

$$\mathbf{E}[X] = \int x f_X(x)\, dx$$

$$\mathbf{E}[X \mid A] = \sum_x x p_{X|A}(x)$$

$$\mathbf{E}[X \mid A] = \int x f_{X|A}(x)\, dx$$

## Expected value rule:

$$\mathbf{E}[g(X)] = \sum_x g(x) p_X(x)$$

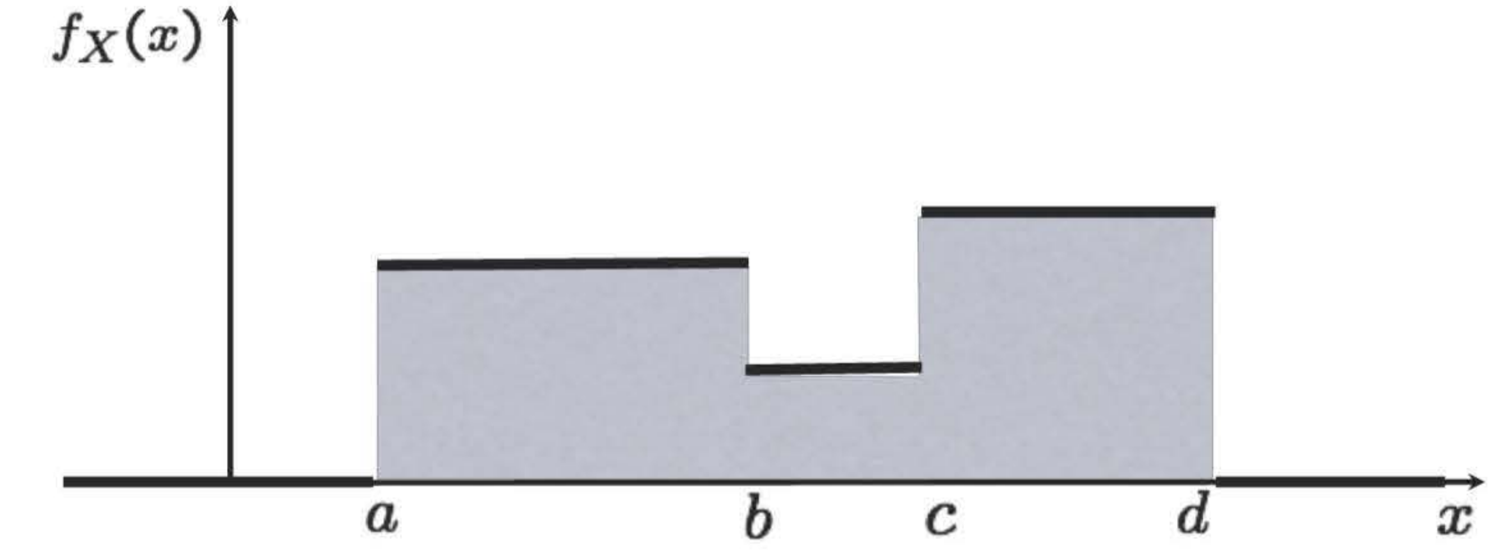
$$\mathbf{E}[g(X)] = \int g(x) f_X(x)\, dx$$

$$\mathbf{E}[g(X) \mid A] = \sum_x g(x) p_{X|A}(x)$$

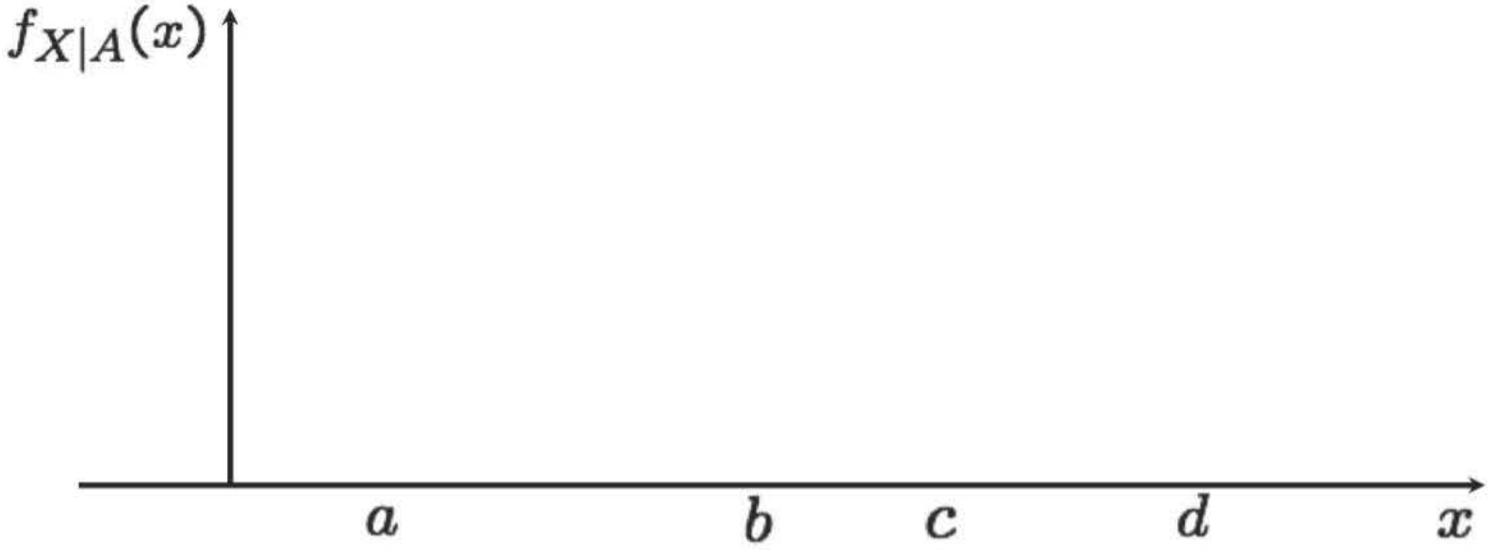
$$\mathbf{E}[g(X) \mid A] = \int g(x) f_{X|A}(x)\, dx$$

## Example

$$A: \quad \frac{a+b}{2} \leq X \leq b$$




$$\mathbf{E}[X \mid A] =$$




$$\mathbf{E}[X^2 \mid A] =$$

## Memorylessness of the exponential PDF

- Do you prefer a used or a new "exponential" light bulb? **Probabilistically identical!**

- Bulb lifetime $T$: exponential($\lambda$)

  $\mathbf{P}(T > x) = e^{-\lambda x}$, for $x \geq 0$

  - we are told that $T > t$

  - r.v. $X$: remaining lifetime

  $\mathbf{P}(X > x \,|\, T > t) = e^{-\lambda x}$, for $x \geq 0$

## Memorylessness of the exponential PDF

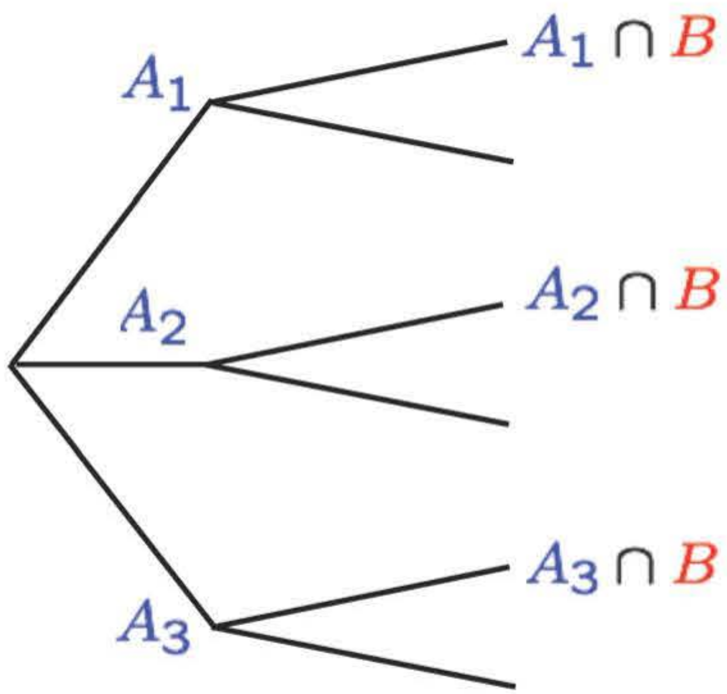
$f_T(x) = \lambda e^{-\lambda x}, \quad$ for $x \geq 0$

$\mathbf{P}(0 \leq T \leq \delta)$

$\mathbf{P}(t \leq T \leq t + \delta \mid T > t)$

similar to an independent coin flip,
every $\delta$ time steps,
with $\mathbf{P}(\text{success}) \approx \lambda\delta$

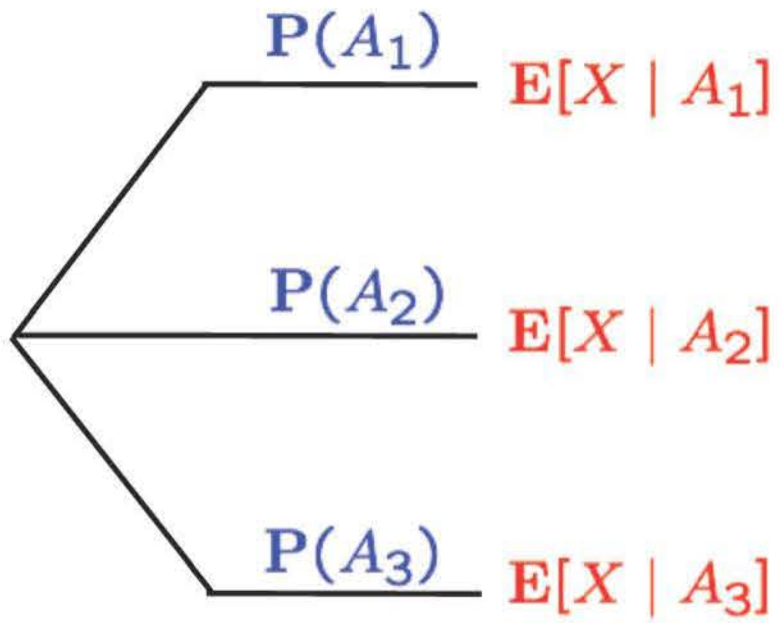
## Total probability and expectation theorems




$$\mathbf{P}(B) = \mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \cdots + \mathbf{P}(A_n)\mathbf{P}(B \mid A_n)$$

$$p_X(x) = \mathbf{P}(A_1)\, p_{X|A_1}(x) + \cdots + \mathbf{P}(A_n)\, p_{X|A_n}(x)$$

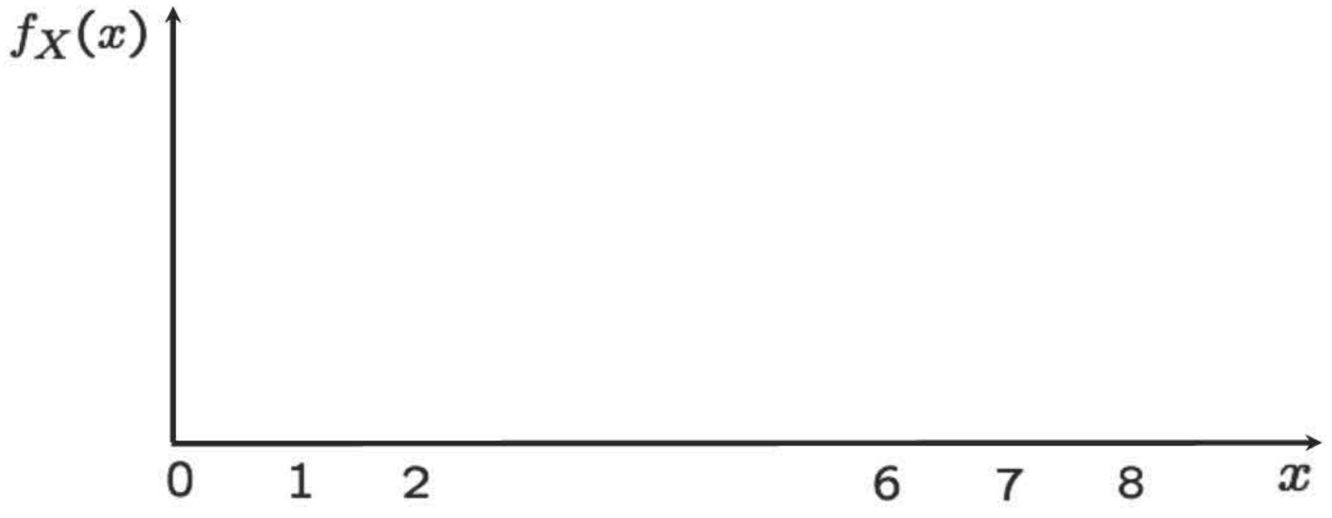
$$f_X(x) = \mathbf{P}(A_1) f_{X|A_1}(x) + \cdots + \mathbf{P}(A_n) f_{X|A_n}(x)$$




$$\mathbf{E}[X] = \mathbf{P}(A_1)\mathbf{E}[X \mid A_1] + \cdots + \mathbf{P}(A_n)\mathbf{E}[X \mid A_n]$$

## Example

- Bill goes to the supermarket shortly, with probability 1/3,
  at a time uniformly distributed between 0 and 2 hours from now;
  or with probability 2/3, later in the day
  at a time uniformly distributed between 6 and 8 hours from now




$$f_X(x) = \mathbf{P}(A_1) f_{X|A_1}(x) + \cdots + \mathbf{P}(A_n) f_{X|A_n}(x)$$

$$\mathbf{E}[X] = \mathbf{P}(A_1)\mathbf{E}[X \mid A_1] + \cdots + \mathbf{P}(A_n)\mathbf{E}[X \mid A_n]$$

## Mixed distributions

$$X = \begin{cases} \text{uniform on } [0,2], & \text{with probability } 1/2 \\ 1, & \text{with probability } 1/2 \end{cases}$$

Is $X$ discrete?

Is $X$ continuous?

$Y$ discrete

$Z$ continuous

$$X = \begin{cases} Y, & \text{with probability } p \\ Z, & \text{with probability } 1-p \end{cases}$$

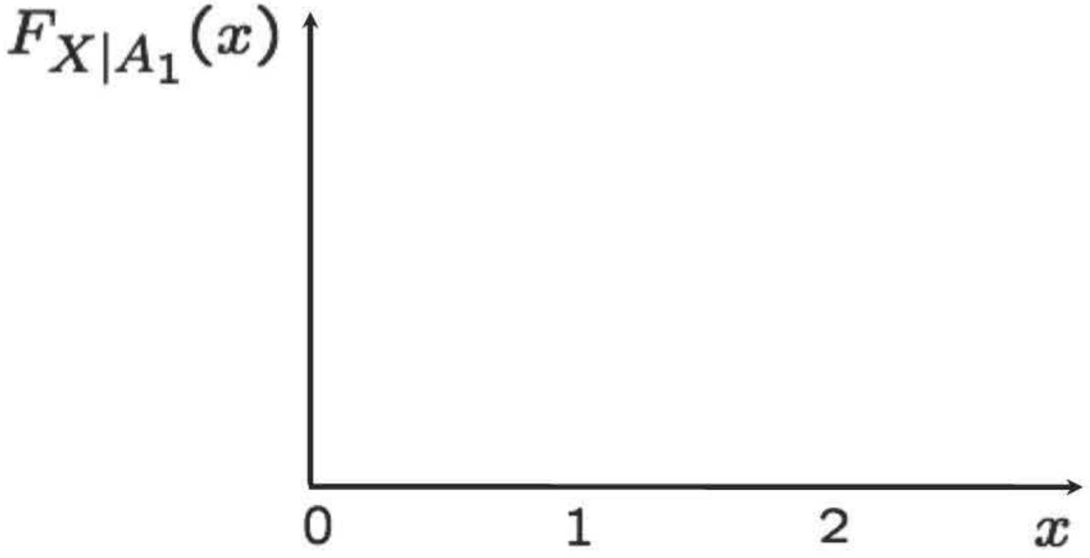
$X$ is mixed

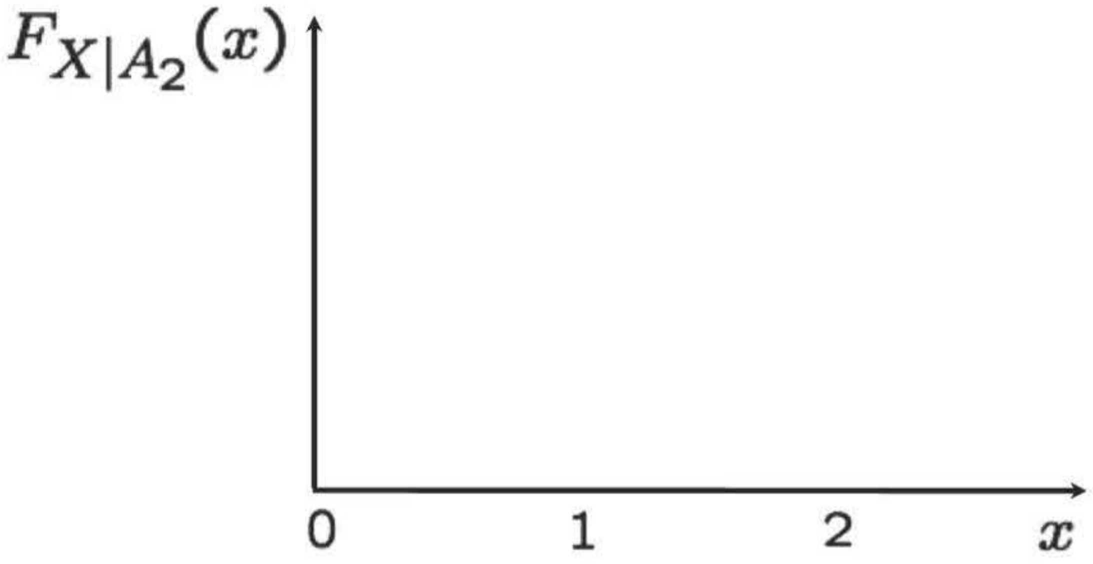
$F_X(x) =$

$\mathbf{E}[X] =$

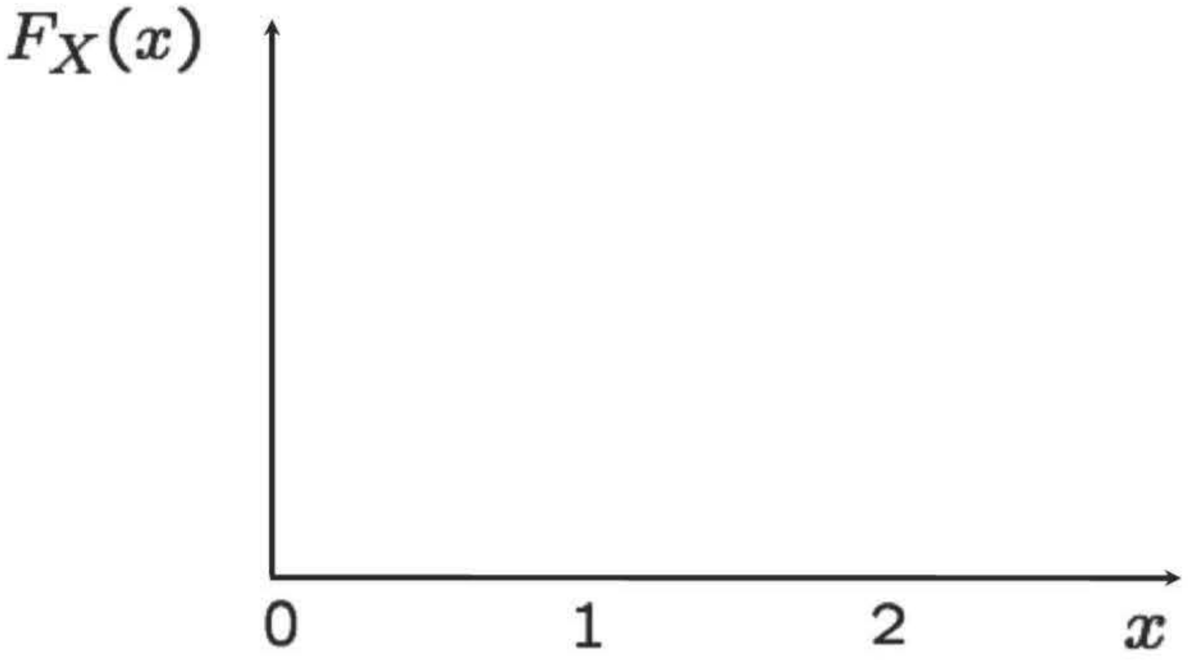
## Mixed distributions

$$X = \begin{cases} \text{uniform on } [0,2], & \text{with probability } 1/2 \\ 1, & \text{with probability } 1/2 \end{cases}$$




$$F_X(x) = \mathbf{P}(A_1)F_{X|A_1}(x) + \mathbf{P}(A_2)F_{X|A_2}(x)$$



## Jointly continuous r.v.'s and joint PDFs

| | |
|---|---|
| $p_X(x)$ | $f_X(x)$ |
| $p_{X,Y}(x,y)$ | $f_{X,Y}(x,y)$ |

$$p_{X,Y}(x,y) = \mathbf{P}(X = x \text{ and } Y = y) \geq 0$$

$$f_{X,Y}(x,y) \geq 0$$

$$\mathbf{P}\big((X,Y) \in B\big) = \sum_{(x,y)\in B}\sum p_{X,Y}(x,y)$$

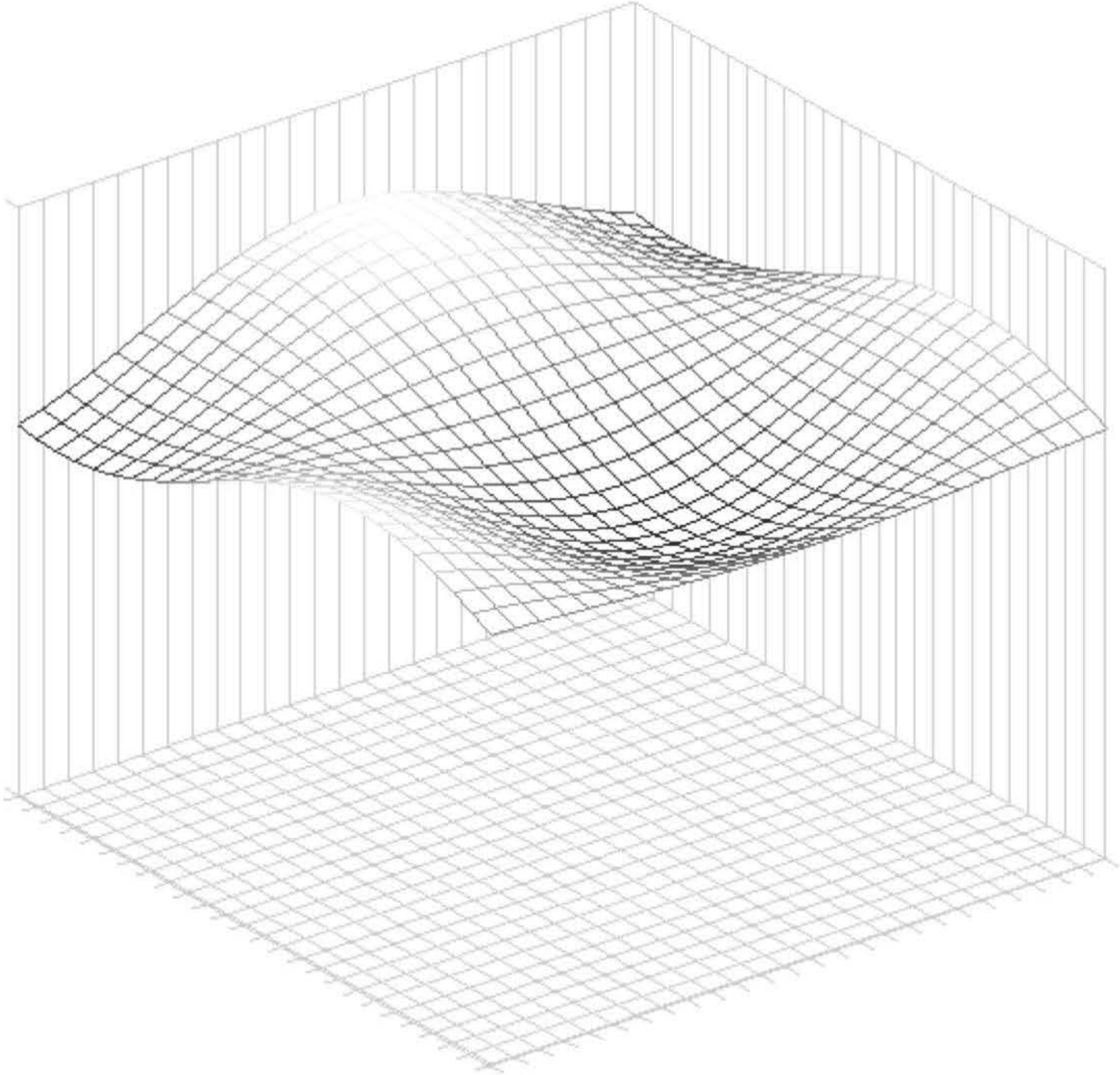
$$\mathbf{P}\big((X,Y) \in B\big) = \int\int_{(x,y)\in B} f_{X,Y}(x,y)\,dx\,dy$$

$$\sum_x \sum_y p_{X,Y}(x,y) = 1$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

**Definition:** Two random variables are **jointly continuous** if they can be described by a joint PDF

# Visualizing a joint PDF



$$\mathbf{P}\Big((X,Y)\in B\Big) = \int\int_{(x,y)\in B} f_{X,Y}(x,y)\,dx\,dy$$

# On joint PDFs

$$\mathbf{P}\big((X,Y)\in B\big) = \iint_{(x,y)\in B} f_{X,Y}(x,y)\,dx\,dy$$

$$\mathbf{P}(a \le X \le b,\, c \le Y \le d) = \int_c^d \int_a^b f_{X,Y}(x,y)\,dx\,dy$$

$$\mathbf{P}(a \le X \le a+\delta,\, c \le Y \le c+\delta) \approx f_{X,Y}(a,c)\cdot\delta^2$$

$f_{X,Y}(x,y)$: probability per unit area

$\text{area}(B) = 0 \quad \Rightarrow \quad \mathbf{P}\big((X,Y)\in B\big) = 0$

## From the joint to the marginals

$$p_X(x) = \sum_y p_{X,Y}(x, y) \qquad f_X(x) = \int f_{X,Y}(x, y)\, dy$$

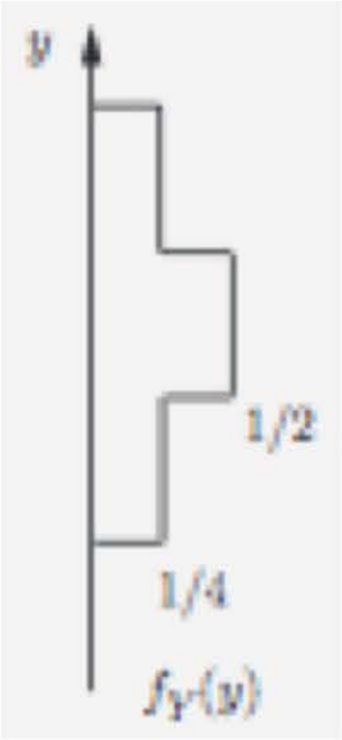
$$p_Y(y) = \sum_x p_{X,Y}(x, y) \qquad f_Y(y) = \int f_{X,Y}(x, y)\, dx$$

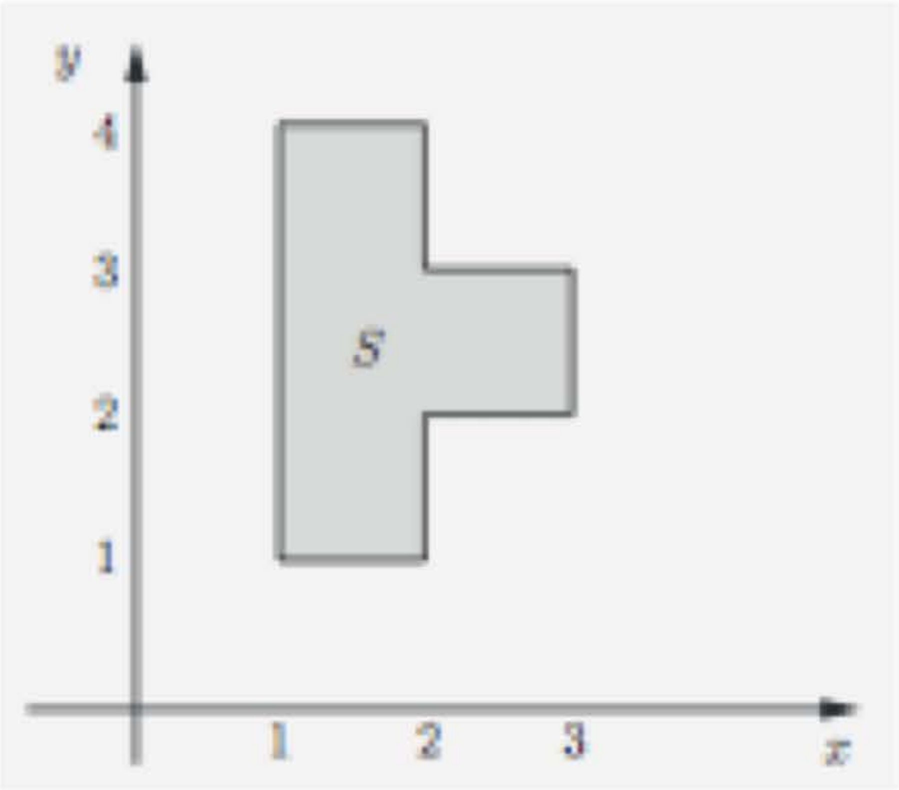
## Uniform joint PDF on a set $S$

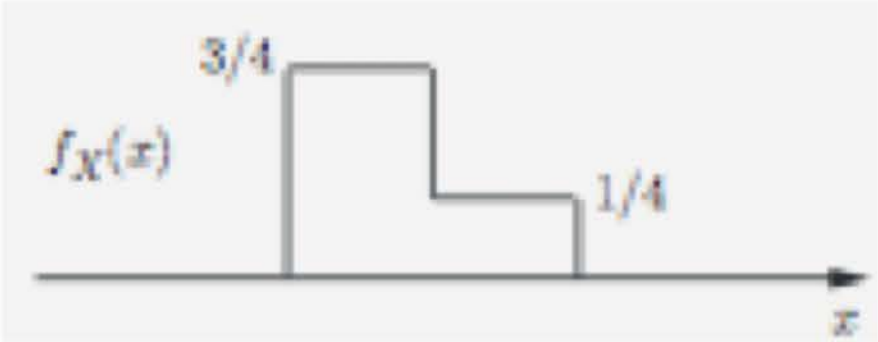
$$f_X(x) = \int f_{X,Y}(x,y)\,dy$$

$$f_Y(y) = \int f_{X,Y}(x,y)\,dx$$

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\text{area of } S}, & \text{if } (x,y) \in S, \\ 0, & \text{otherwise.} \end{cases}$$



## More than two random variables

$p_{X,Y,Z}(x,y,z)$

$f_{X,Y,Z}(x,y,z)$

$$\sum_x \sum_y \sum_z p_{X,Y,Z}(x,y,z) = 1$$

$$p_X(x) = \sum_y \sum_z p_{X,Y,Z}(x,y,z)$$

$$p_{X,Y}(x,y) = \sum_z p_{X,Y,Z}(x,y,z)$$

## Functions of multiple random variables

$Z = g(X, Y)$

### Expected value rule:

$$\mathbf{E}[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y)$$

$$\mathbf{E}[g(X,Y)] = \int \int g(x,y) f_{X,Y}(x,y)\, dx\, dy$$

### Linearity of expectations

$$\mathbf{E}[aX + b] = a\mathbf{E}[X] + b$$

$$\mathbf{E}[X + Y] = \mathbf{E}[X] + \mathbf{E}[Y]$$

$$\mathbf{E}[X_1 + \cdots + X_n] = \mathbf{E}[X_1] + \cdots + \mathbf{E}[X_n]$$

## The joint CDF

$$F_X(x) = \mathbf{P}(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt$$

$$f_X(x) = \frac{dF_X}{dx}(x)$$

$$F_{X,Y}(x,y) = \mathbf{P}(X \leq x, Y \leq y)$$

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}}{\partial x\,\partial y}(x,y)$$