In this last video, we illustrate how to use the techniques we have recently learned in order to answer some questions about the following classical problem-- consider a gambler putting a bet of $1 in a game that has a pay off of one dollar if she wins.

We assume that this is a fair game, so the probability of winning $1 on each play of the game is one-half.

And so the probability of losing the bet is also one-half.

Suppose that she starts with i dollars and continues to play the game until either she reaches a goal of n dollars, or she has no money left, whatever comes first.

Let us consider a first question, which is the following-- what is the probability that she ends up with having her goal of n dollars?

Now, how to go about solving this problem?

Can we think of a Markov chain representation for this problem?

But in that case, what would be good choices for the definition of the states?

Let us think.

At any point in time, the only relevant information is the amount of money the gambler has available.

How she got to that amount in the past is irrelevant.

And if this amount is neither zero nor n, then she will play again.

And the next state will be a number which will be increased or decreased by one unit, depending on winning or losing the next bet.

So we can represent the possible evolution of this game with the following probability transition graph.

So we have n plus 1 states.

This is the state where she loses all her money.

This is the state where she has i amount of money before the next betting.

And here, this is the state n where she reaches her goal and she can leave.

In terms of the transition probability, assuming that you are at a given time in that state, that means that you have i money in your pocket, and you play the next bet.

With a probability one-half, you will gain or win.

And so your amount of money is i plus 1.

Or you lose and your money is i minus 1.

And you keep repeating until you reach either n or zero.

And then you stop.

So this is a Markov chain, and that state 0 and that state n are absorbing states.

Once you reach them, you stay there forever.

So what this question is asking is the probability a of i of-- starting from i, what is the probability that you will end up in that absorbing state?

And we have done this calculation previously.

So let us repeat the technique very briefly.

Clearly here, if you start with 0 dollars, you will never reach that.

So it's going to be 0.

On the other hand, if you start with your desired goal, you don't play anymore.

So your probability is 1.

Now of course, what is of interest is if i is strictly between 0 and n.

And now the question is how to calculate that probability.

And we have seen that the way to do that is to look at the situation, and say let's assume that you are in that state i.

And what happens next?

Well with a probability 0.5, you will move to that state.

And now you are in that level with i plus 1 amount of money.

And what is the probability that, given that you're here, you're going to end up in n?

It's going to be a i plus one.

So it's a i plus one.

Plus the other alternative is that you are going to lose money and end up in that state.

And there, the remaining probability to reach the time n is a i minus 1.

So you have this kind of equation.

This is valid for all i between 0 and n.

And this is a system of equations that you can solve.

It's not very difficult to solve.

Actually, you can look in the textbook.

There will be some trick to do that.

There are many, many ways to do that.

We're not going to spend our time going into details, but essentially if you solve that system, you will see that the answer will be that a of i is i over n.

So if you start with i amount of money, the probability that you're going to reach your goal here is i over n.

So here clearly, if you're extremely greedy, and you have a very, very, very high goal, that means n is very, very large-- so large that compared to your initial amount i, n can be considered to be infinity.

Then the probability that you're going to reach your high goal will go to 0, where n goes to infinity.

So again, if you are extremely greedy, no matter how much your fixed amount of initial money is, the probability that you will stop the game reaching your goal is going to be increasingly small.

And since the other state is 1 minus this one, the priority that you're going to get ruined is going to be closer to 1.

All right, so we have that answer here.

What about the next question?

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Next question is the following-- what would be the expected wealth at the end?

Again, this is a Markov chain where there are two absorbing states.

All the others are transient.

You're guaranteed with probability 1 that you will reach either 0 or n.

So it's a valid question to know once you reach either 0 or n, what is the expected wealth at the end?

Well, if you arrive here, it's going to be 0.

And if you arrive here, it's going to be n.

So the expected value of that wealth will be 0 times the probability of ending in that, plus n times the probability of getting there.

So it's going to be that expression-- 0 times 1 minus a of i, plus n times a of i.

And here what we then get is n times i over n, which is i.

Which is quite interesting.

This is exactly how you started.

So in some sense, in expectation there is no free lunch here.

The next question is-- how long does the gambler expect to stay in the game?

We know that eventually, he will either reach 0 or n with probability 1.

The question is-- how long is it going to take?

Again, we have seen in a previous video that this is essentially calculating the expectation to absorption.

And we know how to do that.

So let's recap what we had said.

If you define mu of i to be the expected number of plays starting from i, what do you have?

Well, for i equal to 0 or i equals n, either way-- if you start from here, or you start from here-- the expected number of plays is 0, right?

Because you're done.

And otherwise, you use the same kind of derivation that we had.

If you start at i between 1 and n, then you will see that mu of i, after one transition, plus 1, you will either be in state i plus 1-- in that case, this expectation will be mu i plus 1 -- or you will be in state i minus 1.

In that case, the expectation is mu i minus 1.

So this is an equation that you have, which is almost the same as this one, except that you have a plus 1 in it.

And as we had discussed before, in general this is the kind of formula that you have.

Now you can solve the system.

I will let you do that.

There are many ways to do this.

But the solution that you're going to have is that mu i will be equals to i times n minus i.

This is the result.

Finally what would be the case if you didn't have a fair game-- for example, unfair to the gambler or unfair to the casino?

What we mean here is that the probability p is different from 0.5.

So here, instead of 0.5, you have p everywhere.

And here, of course, you have 1 minus p everywhere on this side.

So you have a probability p of winning, and probability 1 minus p of losing each bet.

So you might ask the same question-- well, for the probability a of i, you still have 0 here.

You still have 1 here.

The formula that you would write here, instead of writing it this way, it would be-- you start from here with a probability p.

You end up here.

And with a probability of 1 minus p, you end up there.

And the expression that you get for a of i-- if you define r to be 1 minus p over p-- you will see that a of i is going to be 1 minus r to the power of i over 1 minus r to the power n.

And what would be the expected amount of time she will play?

Instead of that equation, if you solve it, you would have mu of i equals r plus 1 divided by r minus 1 times i minus n times 1 minus r to the i divided by 1 minus r to the power n.

Because you would have here again p, and 1 minus p here.

And you can see that when p is strictly less than one-half-- in other words, it's even worse for this gambler-- then a of i-- which is the probability of getting to this favorable state-- will also go to 0 when n goes to infinity.

And in the case where p is strictly greater than 0.5-- that means that she has some favored odd on her favor-- then in that case, this number r to the power n will go to zero.

And 1 minus r of i will represent the probability that she would become infinitely rich.

In other words, being very greedy and n going to infinity.

This will go to 0 and 1 minus r to the power of i is the probability that she would get infinitely rich.