Markov processes – III

- review of steady-state behavior
- probability of blocked phone calls
- calculating absorption probabilities
- calculating expected time to absorption

review of steady state behavior

- Markov chain with a single class of recurrent states, aperiodic; and some transient states; then,

$$\boxed{\lim_{n\to\infty} r_{ij}(n) = \lim_{n\to\infty} \mathbf{P}(X_n = j \mid X_0 = i) = \pi_j \qquad \forall\, i}$$

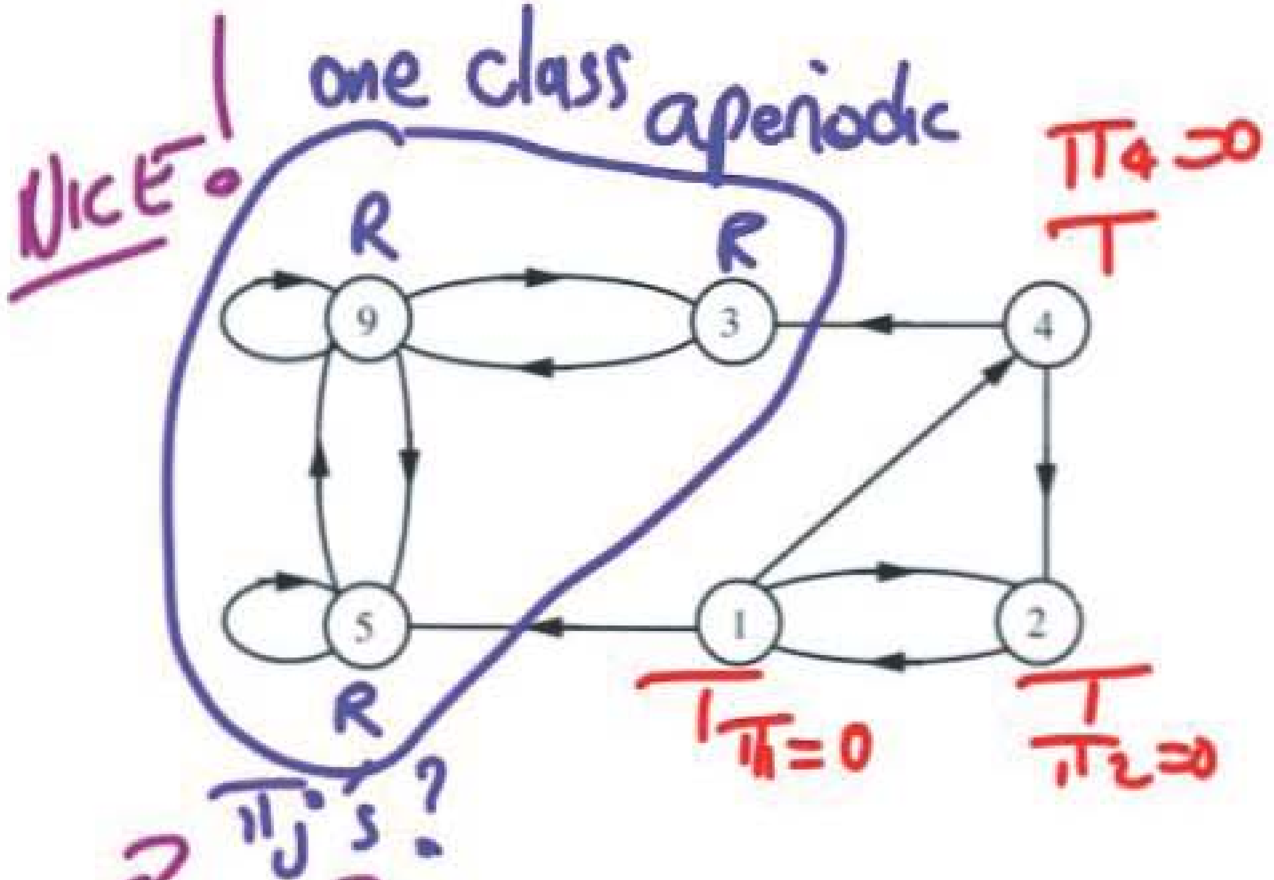
$$\mathbb{P}(X_n = j) = \sum_i r_{ij}(n) \times \mathbb{P}(X_0 = i)$$

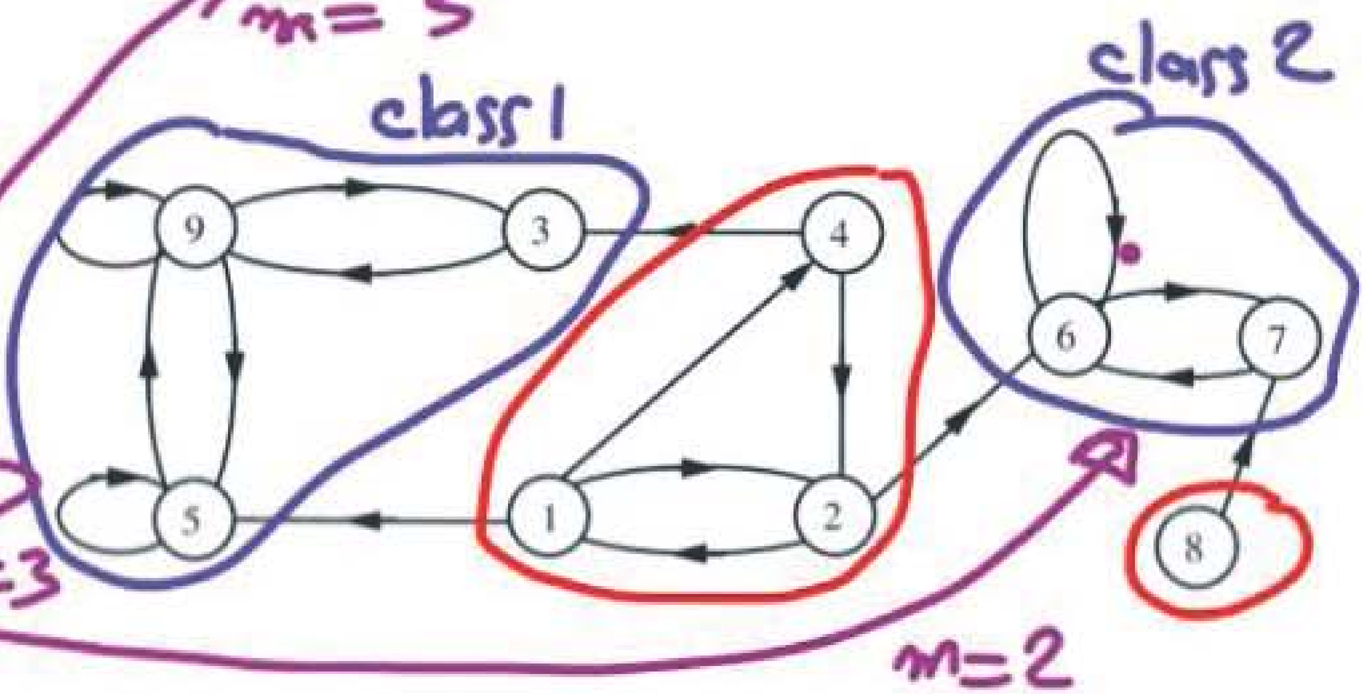
$$\pi_j \underbrace{\sum_i \mathbb{P}(X_0 = i)}_{=1}$$

- can be found as the unique solution to the balance equations

$$\pi_j = \sum_k \pi_k p_{kj}, \qquad j = 1, \ldots, m,$$

- together with $\sum_j \pi_j = 1$

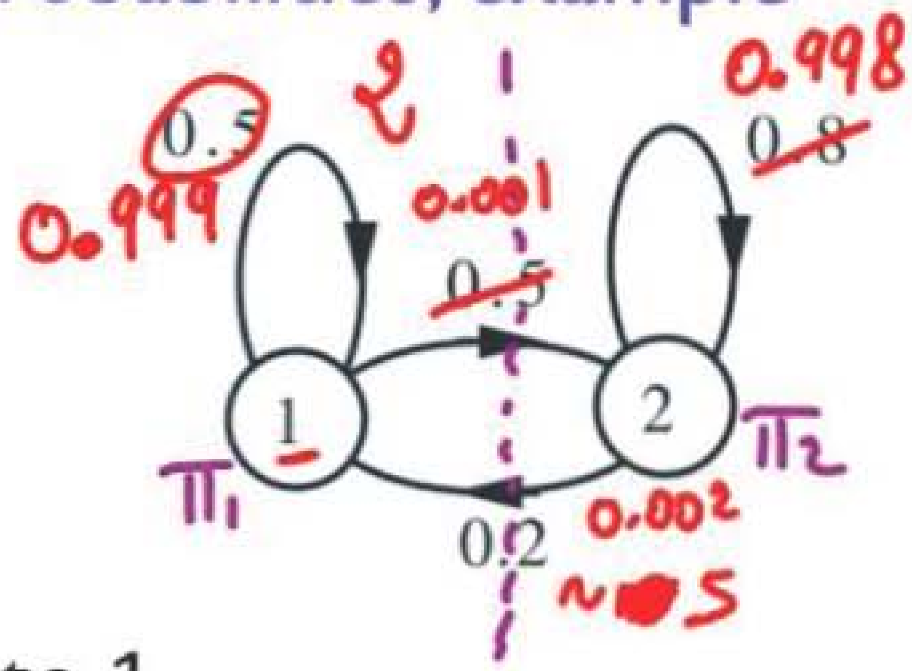




on the use of steady state probabilities, example

$$\begin{cases}\pi_1 \times 0.5 = \pi_2 \times 0.2 \\ \pi_1 + \pi_2 = 1\end{cases}$$

$$\boxed{\pi_1 = 2/7,\ \pi_2 = 5/7}$$



assume process starts in state 1

$\mathbf{P}(X_1 = 1 \text{ and } X_{100} = 1 \mid X_0 = 1) =$

$\mathbb{P}(X_1=1 \mid X_0=1) \times \mathbb{P}(X_{100}=1 \mid X_1=1, \cancel{X_0=1})$

$p_{11} \times r_{11}(99) \approx p_{11} \times \pi_1 = 0.5 \times \frac{2}{7}$

$\mathbf{P}(X_{100} = 1 \text{ and } X_{101} = 2 \mid X_0 = 1) =$

$\mathbb{P}(X_{100}=1 \mid X_0=1) \times \mathbb{P}(X_{101}=2 \mid X_{100}=1, \cancel{X_0=1})$

$r_{11}(100) \times p_{12} \approx \pi_1 \times p_{12} = \frac{2}{7} \times 0.5$

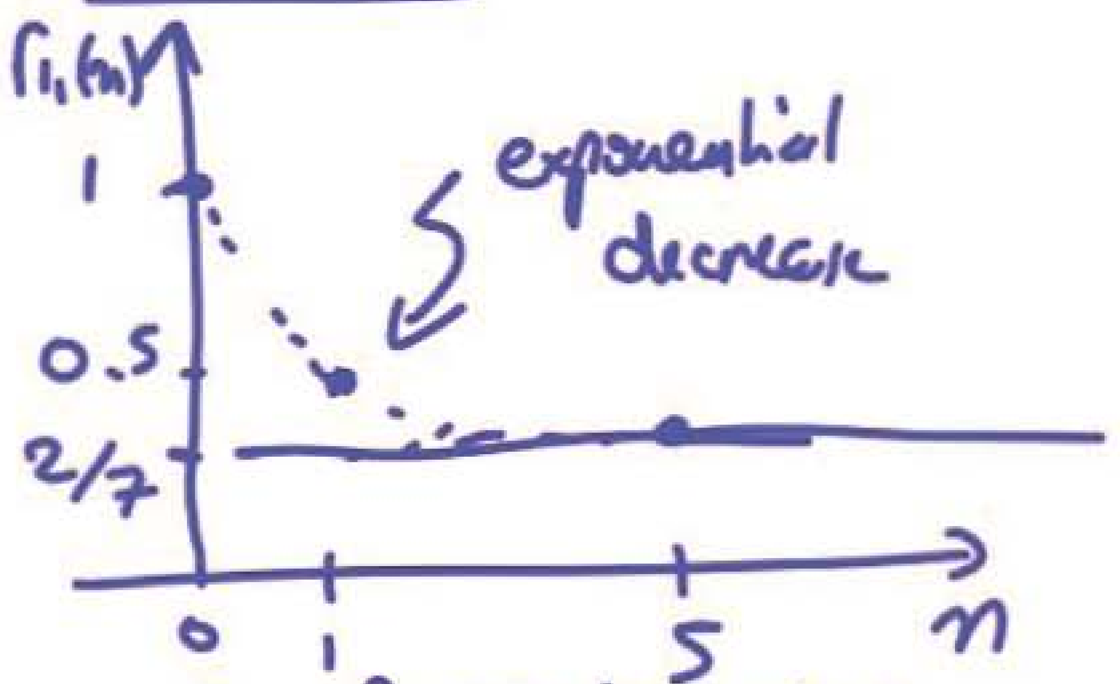
$\mathbf{P}(X_{100} = 1 \text{ and } X_{200} = 1 \mid X_0 = 1) =$

$\mathbb{P}(X_{100}=1 \mid X_0=1) \times \mathbb{P}(X_{200}=1 \mid X_{100}=1, \cancel{X_0=1})$

$= r_{11}(100) \times r_{11}(100) \approx \pi_1 \times \pi_1 = \pi_1^2 = (2/7)^2$

is $n = 99, 100$ large enough?

1) simulation



$n=5$ 2 correct decimal

$n=10$ correct up to 5 decimal

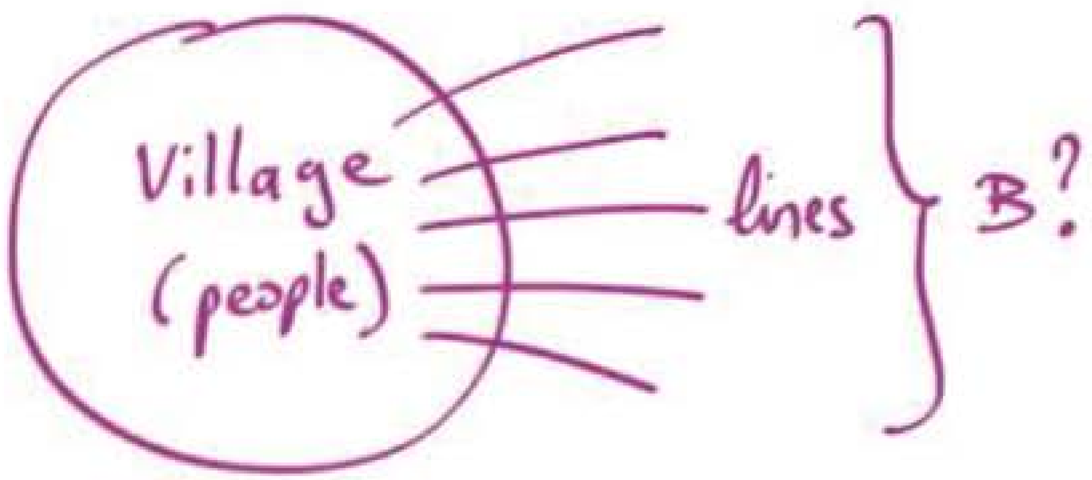
2) order of magnitude

3) by theory

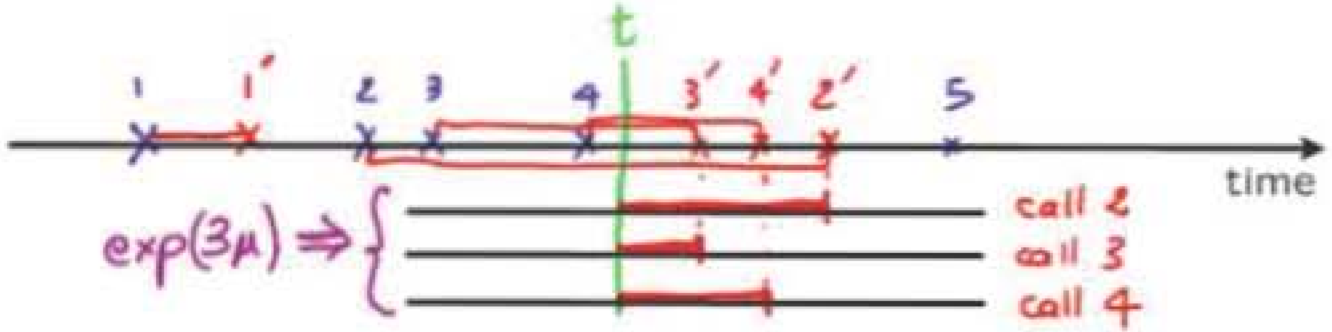
$(c)^n \quad 0<c<1$

- $c = 0.3$
- $c = 0.997$

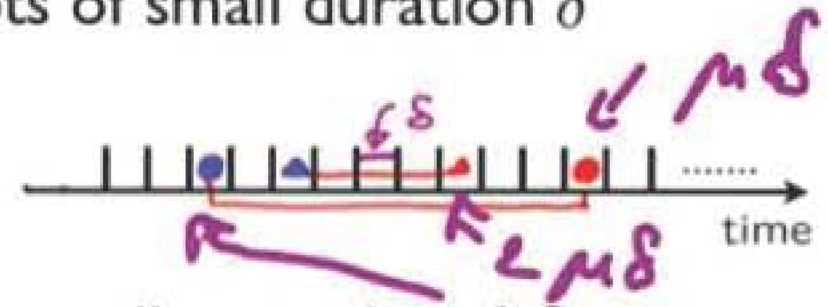
design of a phone system
(Erlang)



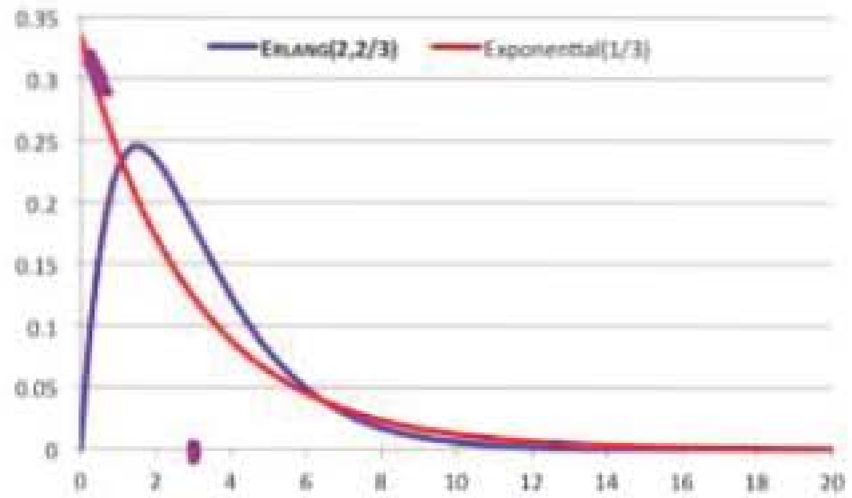
- calls originate as a Poisson process, rate λ
- each call duration is exponential (parameter μ)
- need to decide on how many lines, B?



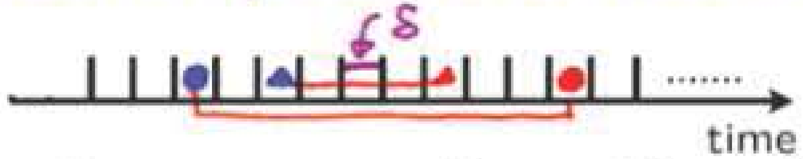
- for time slots of small duration δ

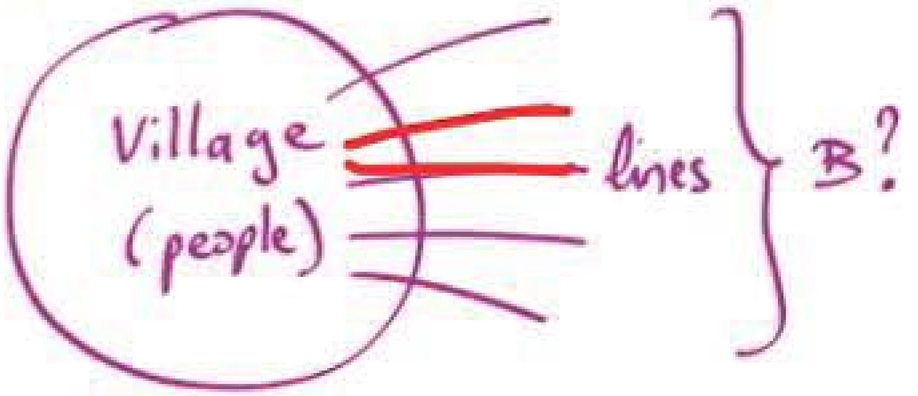


  - P(a new call arrives) $\approx \lambda\delta$
  - if you have i active calls, then P(a departure) $\approx i\mu\delta$



design of a phone system, a discrete time approximation





- approximation: discrete time slots of (small) duration δ

$\mathbb{P}(1 \text{ new call}) \approx \lambda\delta \; ; \; \mathbb{P}(1 \text{ call ends} \mid i \text{ busy}) \approx i\mu\delta$

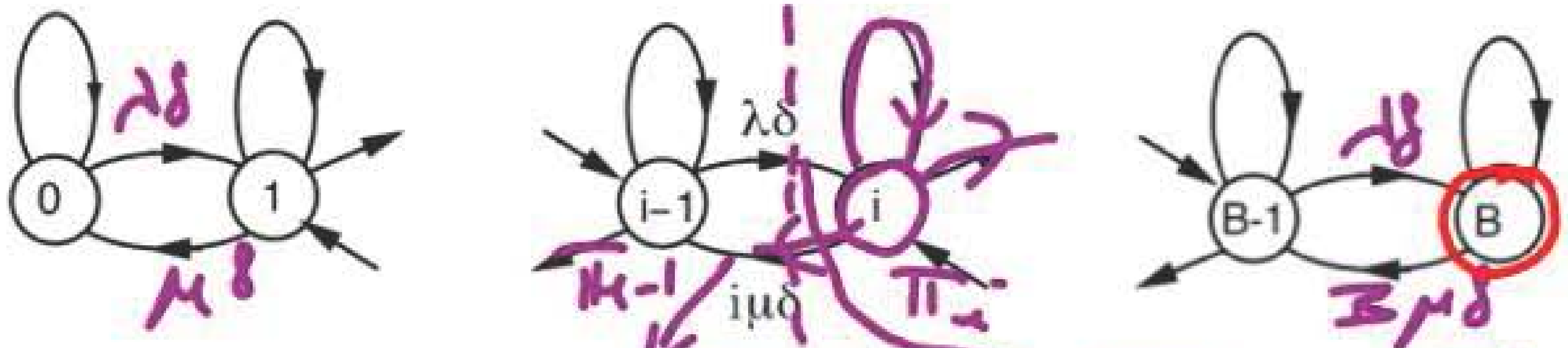
$\lambda = 30$ calls/minute

$\mu = 1/3$

3 minutes

90 calls

$B = 90$?



$\pi_i (i\mu\cancel{\delta}) = \pi_{i-1} \times \lambda\cancel{\delta}$

- balance equations

$\boxed{\lambda\pi_{i-1} = i\mu\pi_i}$

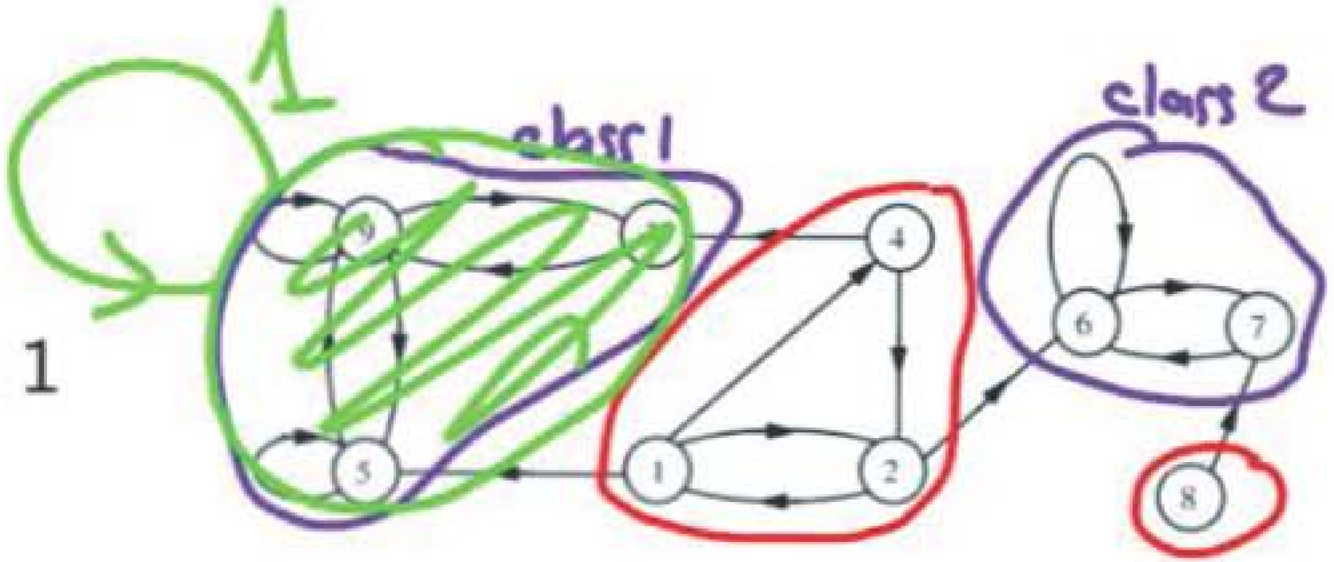
$\pi_0 \to \pi_1 \to \pi_2 \cdots\cdots$

$\boxed{\pi_i = \pi_0 \frac{\lambda^i}{\mu^i i!}}$ $\qquad \sum_i \pi_i = 1 \Rightarrow \boxed{\pi_0 = 1 / \sum_{i=0}^{B} \frac{\lambda^i}{\mu^i i!}} \Rightarrow \pi_i = f(B, \lambda, \mu)$

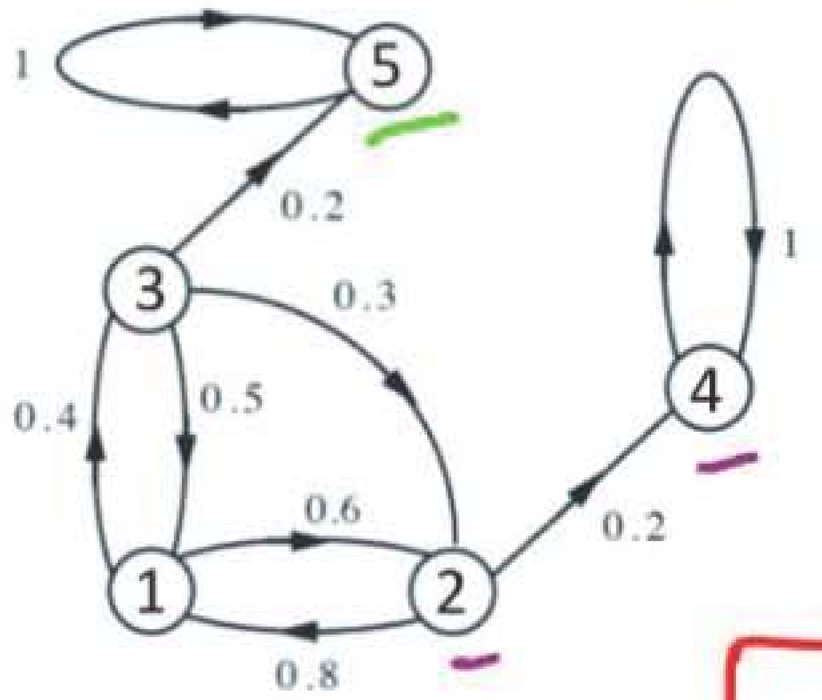
- P(arriving customer finds busy system) is $\boxed{\pi_B}$

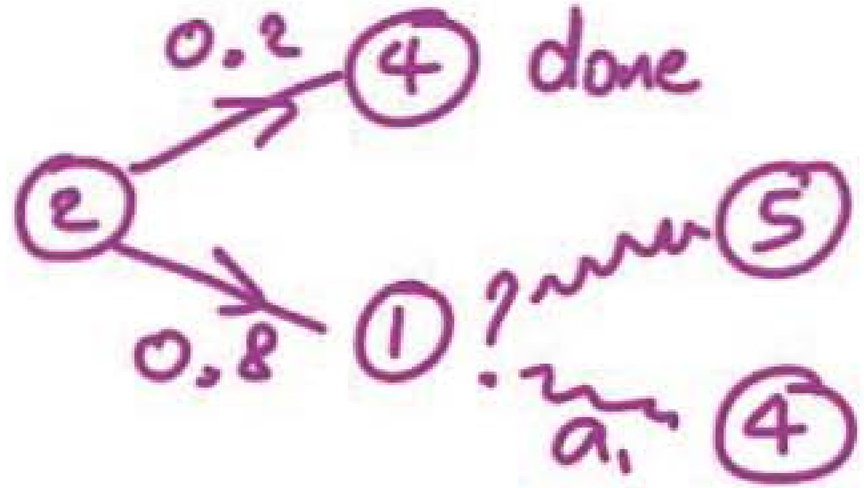
$\pi_B \leq 1\% \Rightarrow B \geq 106$

calculating absorption probabilities



- absorbing state: recurrent state k with $p_{kk} = 1$
- what is the probability a_i (b_i) that the chain eventually settles in ~~4~~ 5 given it started in i?





$i = 4,\ a_i = 1$

$i = 5,\ a_i = 0$

otherwise, $a_i = ?$

$a_1 = 18/28$
$a_2 = 20/28$
$a_3 = 15/28$

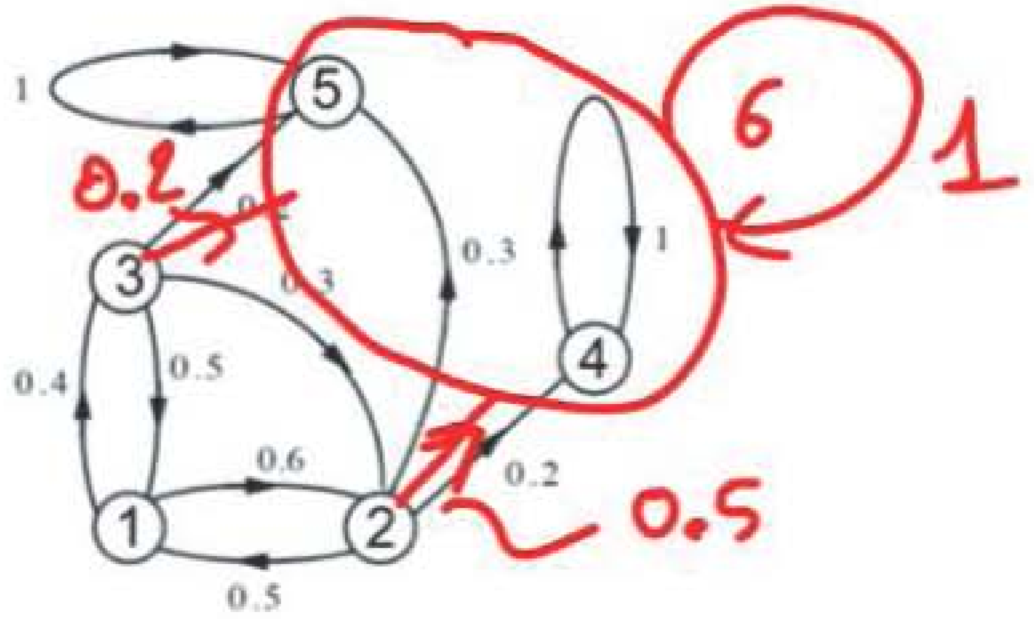
$$\Leftarrow \begin{cases} a_2 = 0.2\,\underbrace{a_4}_{1} + 0.8\,a_1 \\ a_1 = 0.6\,a_2 + 0.4\,a_3 \\ a_3 = 0.3\,a_2 + 0.5\,a_1 + 0.\cancel{2}\,\underbrace{a_5}_{0} \end{cases}$$

- unique solution from $\boxed{a_i = \sum_{j=1}^{m} p_{ij} a_j}$ $\forall i$

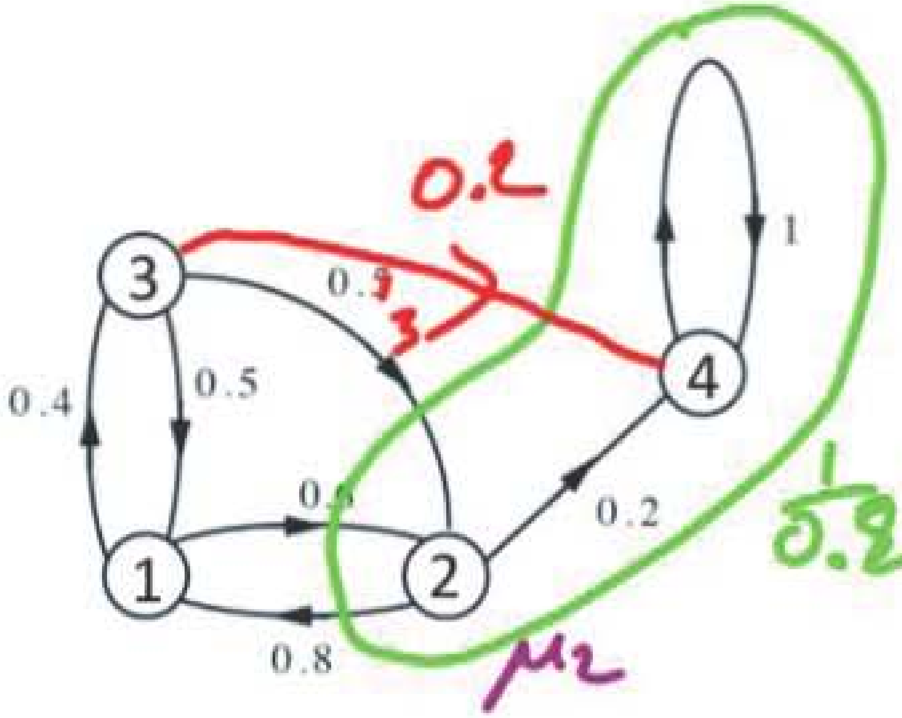
state s: $a_s = 1$, and $a_{s'} = 0$ for the other absorbing state

$a_i + b_i = 1 \quad \forall i$

expected time to absorption



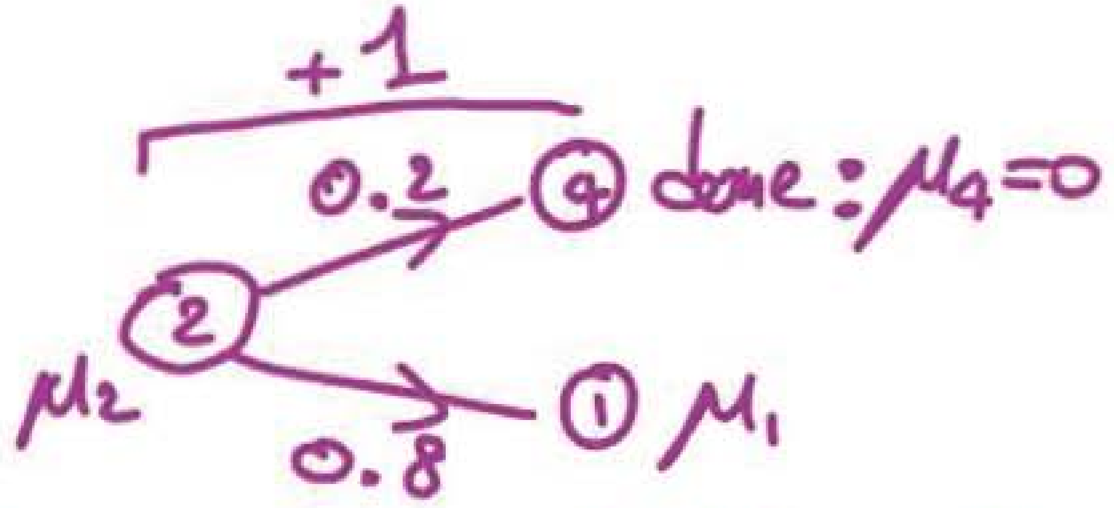
- find expected number of transitions μ_i until reaching 4, given that the initial state is i



$\mu_i = 0$ for $i = 4$

for all others, $\mu_i = ?$

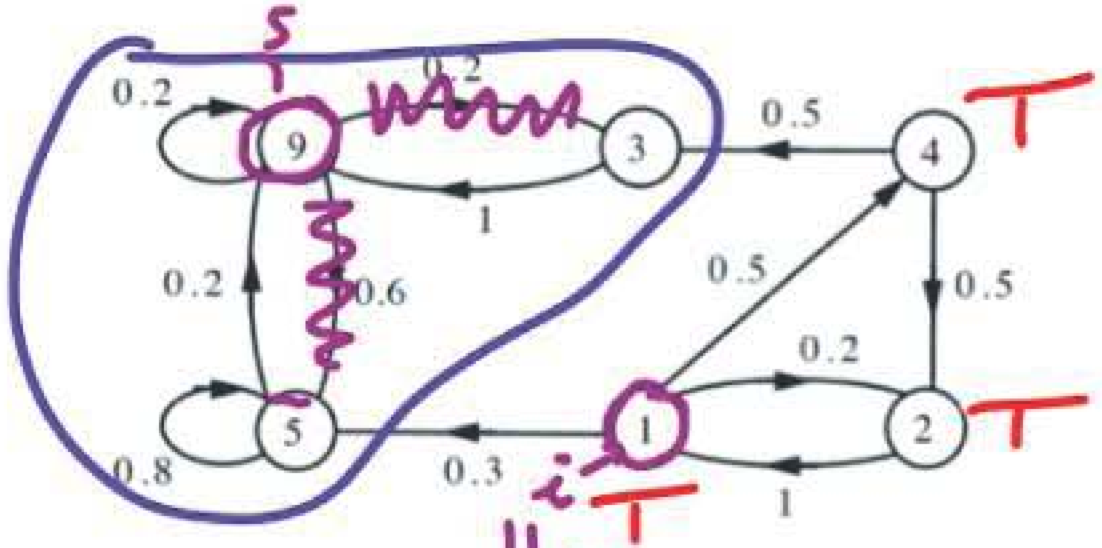
$$\frac{1}{0.2} = 5 \quad \begin{cases} \mu_1 = 110/8 \\ \mu_2 = 96/8 = 12 \\ \mu_3 = 111/8 \end{cases}$$



$$\begin{cases} \mu_2 = 1 + 0.2\cancel{\mu_4} + 0.8\mu_1 \\ \mu_2 = 1 + 0.8\mu_1 \\ \mu_1 = 1 + 0.6\mu_2 + 0.4\mu_3 \\ \mu_3 = 1 + 0.5\mu_1 + 0.5\mu_2 \end{cases}$$

- unique solution from $\boxed{\mu_i = 1 + \sum_j p_{ij}\mu_j}$

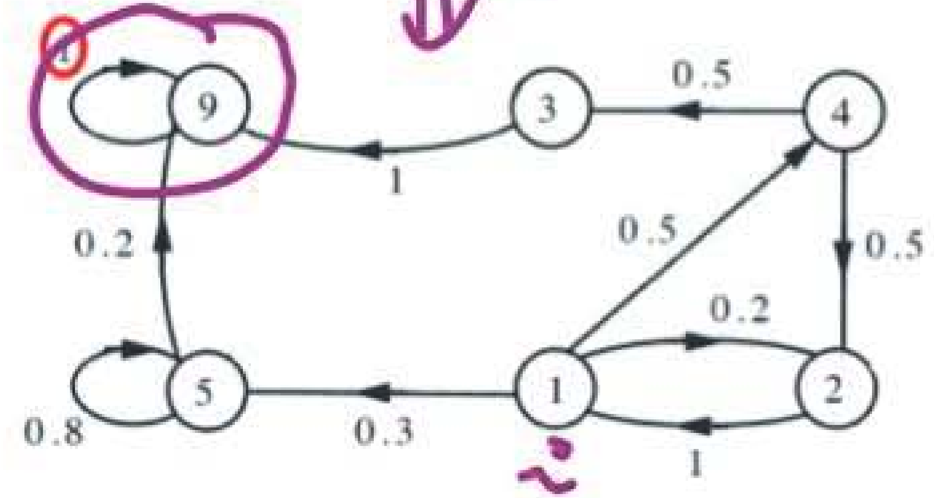
mean first passage and recurrence times



- chain with one recurrent class; fix a recurrent state s

- mean first passage time from i to s :

$$t_i = \mathbf{E}[\min\{n \geq 0 \text{ such that } X_n = s\} \mid \underline{X_0 = i}]$$



– unique solution to:

$$\begin{aligned} t_s &= 0, \\ t_i &= 1 + \sum_j p_{ij}\, t_j, \quad \text{for all } i \neq s \end{aligned}$$

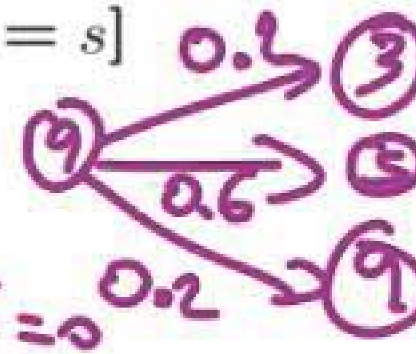
- mean recurrence time of s

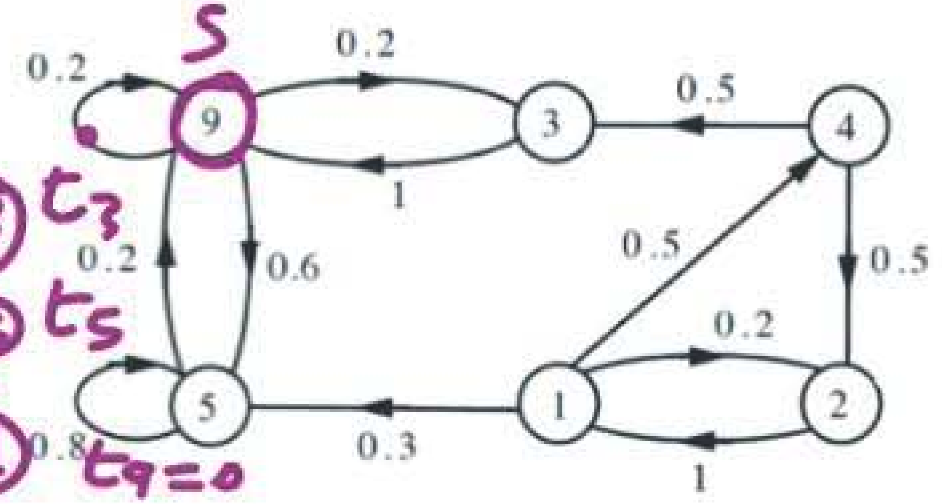
$$t_s^* = \mathbf{E}[\min\{n \geq 1 \text{ such that } X_n = s\} \mid X_0 = s]$$

– solution to:

$$t_s^* = 1 + \textstyle\sum_j p_{sj}\, t_j$$

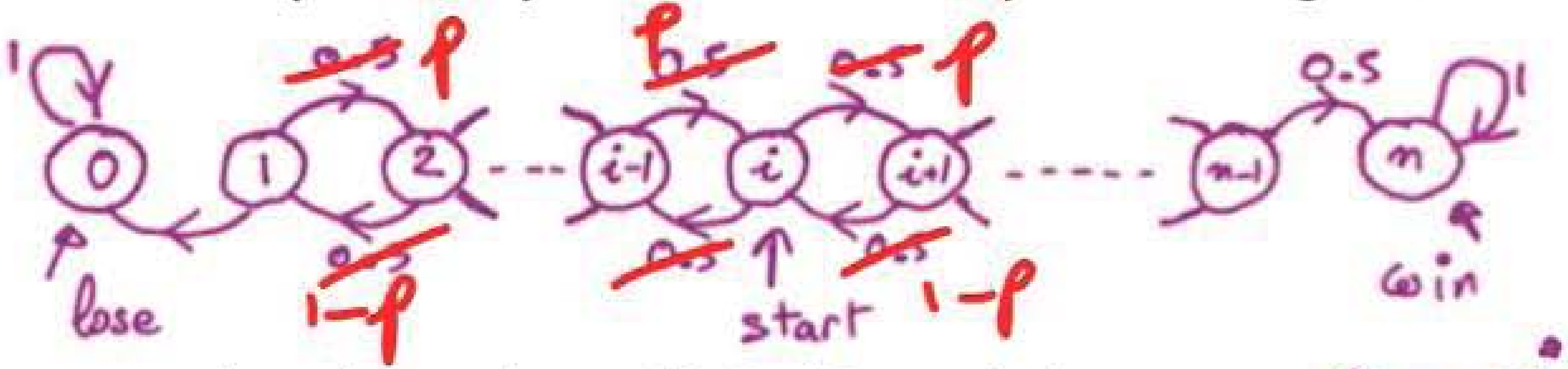
$t_9^* = 0.2\, t_3 + 0.6 \cdot t_5 + 0.2 \cdot \cancel{t_9} = 0 + 1$





gambler's example

- a gambler starts with i dollars; each time, she bets \$1 in a fair game, until she either has 0 or n dollars.
- what is the probability a_i that she ends up with having n dollars?



$i = 0,\ a_i = 0$ ✓ $\quad i = n,\ a_i = 1$ ✓

$0 < i < n,\ a_i = ?$

$$a_i = \cancel{0.5}^{\,p}\, a_{i+1} + \cancel{0.5}^{\,(1-p)}\, a_{i-1}$$

$$\Downarrow$$

$$\boxed{a_i = i/n} \to 0 \quad n \to \infty$$

- expected wealth at the end? $0 \cdot (1 - a_i) + n \cdot a_i = n \times i/n = i$!!

- how long does the gambler expect to stay in the game?
  - $\mu_i =$ expected number of plays, starting from i
  - for $i = 0, n$: $\mu_i = 0$
  - in general

$$\mu_i = 1 + \sum_j p_{ij}\mu_j$$

$1 < i < n \quad \mu_i = 1 + \cancel{0.5}^{\,p}\,\mu_{i+1} + \cancel{0.5}^{\,1-p}\,\mu_{i-1}$

$$\Downarrow$$

$$\boxed{\mu_i = i(n-i)}$$

- in case of unfavorable odds? $p \neq 0.5$

$r = \frac{1-p}{p} \qquad a_i = \frac{1-r^i}{1-r^n} \qquad \mu_i = \left(\frac{r+1}{r-1}\right)\left(i - n \times \frac{1-r^i}{1-r^n}\right)$