Physics III: Final Solutions (Individual and Group)

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(Tuesday July 18, 9 AM)

First and Last Name: ________________________________

Exam Instructions:
This is an open-notebook exam, so feel free to use the notes you have transcribed throughout the summer and problem sets you have completed, but cellphones, laptops, and any notes written by someone else are prohibited. You will have **1 hour and 20 minutes** to complete this exam.

Since this is a timed exam, your solutions need not be as "organized" as are your solutions to assignments. Short calculations and succinct explanations are acceptable, and you can state (without derivation) the standard solutions to the equations of motion we derived in class. However, you should also recognize that you cannot receive partial credit for derivations/explanations you do not provide.

Problem 1: ________________

Problem 2: ________________

Problem 3: ________________

Problem 4: ________________

Problem 5: ________________

Problem 6: ________________

Total:

1. **Morse Potential (30 points)**
The interaction energy between two atoms in a diatomic molecule can be approximated by what is known as the Morse Potential:

$$U(x) = U_0 \left(1 - \beta e^{-\alpha x}\right)^2, \tag{1}$$

where $\alpha > 0$, $\beta > 0$, and $U_0 > 0$.

This potential is plotted in Fig. 1

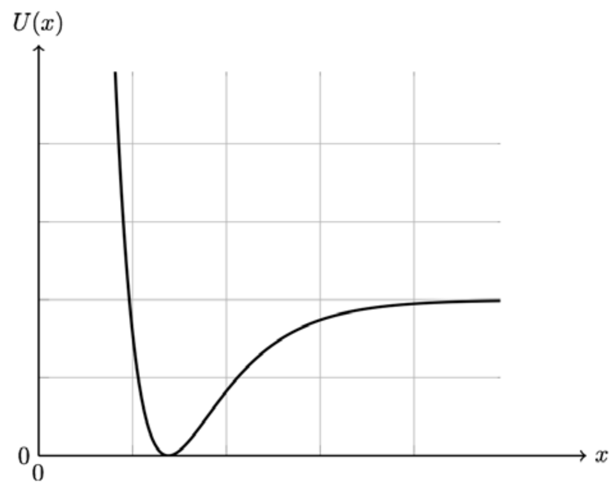

Figure 1

(a) (5 points) What are the units of U_0, β, and α?

(b) (10 points) At what value of x does $U(x)$ have a stable equilibrium?

(c) (5 points) What is the frequency of small-oscillations for a mass m, near this stable equilibrium? Given your answers in (a), show that the units of this result make sense.

(d) (10 points) Let's say the mass begins at the equilibrium position found in (b) with a velocity v_0. Assuming validity of the small-oscillation approximation, what is $x(t)$ in terms of the parameters of the system?

Solution:

(a) $U(x)$ is a potential energy, and so it has the units of Joules (hereafter denoted "J"). U_0, as the coefficient of the function defining $U(x)$, must also have these units: $[U_0] =$ J. The parameter β is being subtracted from a number so it must have no units: $[\beta] = 0$. Finally, the argument of an exponential must be without units, so given that $[x] =$ meters, we have $[\alpha] =$ (meters)$^{-1}$.

(b) We can find the stable equilibrium of the potential energy, by finding the value of x where $U'(x) = 0$, and where $U''(x) > 0$. Solving for this first condition we find

$$U'(x) = 2U_0 \left(1 - \beta e^{-\alpha x}\right) \beta \alpha e^{-\alpha x}, \tag{2}$$

which when set to zero for $x = x_{\text{eq}}$ yields

$$\boxed{x_{\text{eq}} = \frac{1}{\alpha} \ln \beta.} \tag{3}$$

Checking that this yields a stable equilibrium, we find

$$\begin{aligned} U''(x)\Big|_{x=x_{\text{eq}}} &= \frac{d}{dx}\left[2U_0\beta\alpha\left(e^{-\alpha x} - \beta e^{-2\alpha x}\right)\right]\Big|_{x=x_{\text{eq}}} \\ &= 2U_0\beta\alpha\left[-\alpha e^{-\alpha x_{\text{eq}}} + 2\alpha\beta e^{-2\alpha x_{\text{eq}}}\right] \\ &= 2U_0\beta\alpha\left[-\frac{\alpha}{\beta} + \frac{2\alpha\beta}{\beta^2}\right] = 2U_0\alpha^2 > 0. \end{aligned} \tag{4}$$

Thus, with $U''(x_{\text{eq}}) > 0$, we indeed find that Eq.(3) is the stable equilibrium.

(c) From the theory of oscillations near equilibria, we know that the frequency of small oscillations of a mass m about an equilibrium x_{eq} for a potential $U(x)$ is

$$\omega = \sqrt{\frac{U''(x_{\text{eq}})}{m}}, \tag{5}$$

given our result in (b), we thus have

$$\boxed{\omega = \sqrt{\frac{2U_0\alpha^2}{m}}.} \tag{6}$$

We note that the units of this result are consistent with our expectations: We expect ω to have units of s^{-1} and we find

$$\left[\sqrt{\frac{2U_0\alpha^2}{m}}\right] = \left[\frac{2U_0}{m}\right]^{1/2} \times [\alpha] = \left(\frac{\text{kg}\cdot\text{m}^2}{\text{kg}\cdot\text{s}^2}\right)^{1/2} \times \frac{1}{\text{m}} = \frac{1}{s}, \tag{7}$$

as expected.

(d) For a simple harmonic oscillator system, with a stable equilibrium x_{eq} and a frequency ω, the equation of motion is

$$\ddot{x} + \omega^2(x - x_{\text{eq}}) = 0. \tag{8}$$

The general solution to this equation of motion is

$$x = x_{\text{eq}} + A\cos(\omega t) + B\sin(\omega t). \tag{9}$$

Given the condition $x(t=0) = x_{\text{eq}}$, and $\dot{x} = v_0$, we find, respectively, $A = 0$ and $B = v_0/\omega$. Given Eq.(3) and Eq.(6), we then find

$$\boxed{x(t) = \frac{1}{\alpha}\ln\beta + v_0\sqrt{\frac{m}{2U_0\alpha^2}}\sin\left(t\sqrt{\frac{m}{2U_0\alpha^2}}\right).} \tag{10}$$

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2. **Saturn's Rings (25 points)**
Before James Clerk Maxwell consolidated the equations of electromagnetism, he studied Saturn's rings. In this problem, we study a simple aspect of one of the dynamical motions he studied.

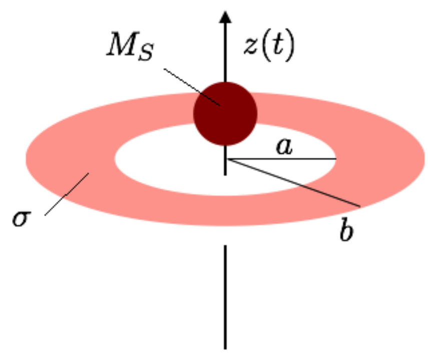

Figure 2: Depiction of Saturn-Ring system: $z(t)$ denotes the vertical position of the planet Saturn above the plane of the rings. We assume the planet is confined to move only along the z axis.

Let's say that planet saturn exhibits small oscillations about the center of the plane of its rings. Assuming the rings are defined as a disk with mass-per-area density σ, inner radius a, and outer radius b, we find that for $z(t)$ sufficiently small, the potential energy of the system[1] is

$$U(z) = U_0 + \pi G M_S \sigma \left(\frac{b-a}{ab} \right) z^2 + \mathcal{O}(z^4/a^4), \tag{11}$$

where U_0 is a constant and G is Newton's Gravitational constant.

(a) (5 points) What is the equation of motion of the mass M_S assuming it is confined to move only in the z direction? What important equation of motion is this result equivalent to?

(b) (5 points) Let's say that the mass M_S now moves through a dense cloud of particles such that it experiences a drag force

$$F_{\text{drag}} = -2\gamma M_S \, \dot{z} \tag{12}$$

for some γ. What is the equation of motion now?

(c) (10 points) We now take the motion of Saturn about the center of the plane of the rings to be an underdamped oscillator. Saturn's motion has the initial amplitude of A_0. Given the fact that Saturn begins from rest (i.e., zero velocity), determine the initial position $z(t=0)$.
(*Hint: Think of the general solution to the type of equation of motion in (b)*)

(d) (5 points) Assume the motion is *very weakly* damped. In terms of the parameters in Eq.(11) and γ, determine $E(t)$ the energy of the oscillator as a function of time.

Solution:

(a) By Newton's 2nd Law, the dynamics of M_S in the z direction are defined by the equation

$$M_S \ddot{z} = F_{\text{net},z}. \tag{13}$$

[1] If you take a college-level mechanics course, you would learn how to compute this result.

The only force in the z direction arises from the force the disk exerts on the planet. By the relationship between force and potential energy, we find

$$\begin{aligned} M_S \ddot{z} &= -\frac{d}{dz} U(z) \\ &= -2\pi G M_S \sigma \left(\frac{b-a}{ab} \right) z + \mathcal{O}(z^3/a^3), \end{aligned} \tag{14}$$

or, upon dividing by M_S and neglecting the higher order terms,

$$\boxed{\ddot{z} + \omega_0^2 z = 0,} \tag{15}$$

which is the **simple harmonic oscillator equation of motion** with

$$\omega_0^2 = 2\pi G \sigma \left(\frac{b-a}{ba} \right). \tag{16}$$

(b) If the planet experiences a drag force $F_{\text{drag}} = -2\gamma M_S \dot{z}$, then Newton's 2nd Law gives us the equation

$$M_S \ddot{z} = F_{\text{net},z} = -2\pi G M_S \sigma \left(\frac{b-a}{ab} \right) z - 2\gamma M_S \dot{z} + \mathcal{O}(z^3/a^3), \tag{17}$$

which yields the equation of motion

$$\boxed{\ddot{z} + 2\gamma \dot{z} + \omega_0^2 z = 0,} \tag{18}$$

where ω_0 is defined in Eq.(16).

(c) The general solution to Eq.(18) is

$$z(t) = A_0 e^{-\gamma t} \cos(\Omega t - \phi), \tag{19}$$

where $\Omega = \sqrt{\omega_0^2 - \gamma^2}$ and A_0 is the initial amplitude of the motion. If the planet begins with zero velocity, then we have

$$\begin{aligned} 0 &= \dot{z}(t) \\ &= -A_0 e^{-\gamma t} \Big[\Omega \sin(\Omega t - \phi) + \gamma \cos(\Omega t - \phi) \Big] \Big|_{t=0} \\ &= -A_0 e^{-\gamma t} \Big[-\Omega \sin(\phi) + \gamma \cos(\phi) \Big], \end{aligned} \tag{20}$$

from which we can infer

$$\tan\phi = \frac{\gamma}{\Omega} = \frac{\gamma}{\sqrt{\omega_0^2 - \gamma^2}}. \tag{21}$$

From this result we find that $\cos\phi$ is

$$\cos\phi = \frac{\sqrt{\omega_0^2 - \gamma^2}}{\omega_0}. \tag{22}$$

Thus, returning to Eq.(19), we find

$$\boxed{z(t=0) = A_0 \cos(\phi) = A_0 \sqrt{1 - \frac{\gamma^2}{\omega_0^2}}.} \tag{23}$$

(d) We know for a very weakly damped oscillator with no energy offset, the energy as a function of time is

$$E(t) = \frac{1}{2}kA_0^2 e^{-2\gamma t}, \tag{24}$$

where k is the spring constant of the motion. In our case, we have from Eq.(11) an energy offset of U_0 and our spring constant can be inferred from Eq.(14). Doing so, we find $k = 2\pi G M_s \sigma (b-a)/ba$. Therefore, given the amplitude A_0 defined in the prompt, we find that the energy as a function of time is

$$\boxed{E(t) = U_0 + \pi G M_S \left(\frac{b-a}{ba}\right) A_0^2 e^{-2\gamma t}.} \tag{25}$$

3. **Masses together (30 points)**
Say we have a system of two masses M and m which are joined by a spring of spring constant k. On its other end the mass M is attached to a spring of spring constant K_0 which is attached to a wall. We define the mass M to be at the position X and we take the mass m to be at the position x. The spring attached to the wall is in equilibrium when $X = 0$, and the spring joining the two masses is in equilibrium when $x - X = 0$.

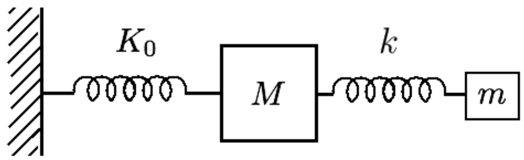

Figure 3: Two coupled masses

(a) (5 points) What are the equations of motion of the system?

(b) (10 points) What are the normal mode frequencies of this system?

(c) (5 points) We now take the spring constant $K_0 \gg k$, so that the motion of the mass m does not affect the motion of the mass M (but the converse is not true). Under this approximation, what are the two equations of motion of the system? *Hint: Assume the positions X and x are of the same order of magnitude.*

(d) (10 points) **[Given the situation in (c)]** Say the mass M begins from rest at a position $X(0) = X_0$, and the mass m begins from rest at the position $x(0) = 0$. Under the approximate equations of motion, what is the position of M and the position of m as functions of time? That is, determine $X(t)$ and $x(t)$, respectively. *(Hint: You should solve the X equation of motion first.)*

(e) (5 points) **[Given the situation in (d)]** What would the value of M have to be in order to drive the mass m at resonance?

Solution:

(a) Given the system in Fig. 3, we find the equations of motion

$$M\ddot{X} = -K_0 X + k(x - X) \tag{26}$$
$$m\ddot{x} = -k(x - X). \tag{27}$$

(b) To find the normal mode frequencies of this system, we must first write the above system as a matrix equation. Doing so we have

$$\begin{pmatrix} \ddot{X} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} -\Omega^2 & \omega_M^2 \\ \omega_0^2 & -\omega_0^2 \end{pmatrix} \begin{pmatrix} X \\ x \end{pmatrix}, \tag{28}$$

where we defined

$$\omega_0^2 \equiv \frac{k}{m}, \qquad \omega_M^2 \equiv \frac{k}{M}, \qquad \Omega^2 \equiv \frac{K_0 + k}{M}. \tag{29}$$

From here, we need to solve the eigenvalue-eigenvector equation

$$\begin{pmatrix} -\Omega^2 & \omega_M^2 \\ \omega_0^2 & -\omega_0^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \alpha^2 \begin{pmatrix} A \\ B \end{pmatrix} \tag{30}$$

and compute $|\text{Im}[\alpha]|$ (absolute value is needed to ensure positive frequencies). Determining the characteristic equation, we have

$$\begin{aligned} 0 &= \det \begin{pmatrix} -\Omega^2 - \alpha^2 & \omega_M^2 \\ \omega_0^2 & -\omega_0^2 - \alpha^2 \end{pmatrix} \\ &= \alpha^4 + \alpha^2(\omega_0^2 + \Omega^2) + \Omega^2\omega_0^2 - \omega_0^2\omega_M^2. \end{aligned} \tag{31}$$

Using the quadratic formula, we find the solutions

$$\begin{aligned} \alpha_\pm^2 &= \frac{-(\omega_0^2 + \Omega^2) \pm \sqrt{(\omega_0^2 + \Omega^2)^2 - 4\left(\Omega^2\omega_0^2 - \omega_0^2\omega_M^2\right)}}{2}. \\ &= \frac{-(\omega_0^2 + \Omega^2) \pm \sqrt{(\omega_0^2 - \Omega^2)^2 + 4\omega_0^2\omega_M^2}}{2} \end{aligned} \tag{32}$$

Thus we find

$$\begin{aligned} \alpha_+ &= i\left(\frac{\omega_0^2 + \Omega^2 - \sqrt{(\omega_0^2 - \Omega^2)^2 + 4\omega_0^2\omega_M^2}}{2}\right)^{1/2} \\ \alpha_- &= i\left(\frac{\omega_0^2 + \Omega^2 + \sqrt{(\omega_0^2 - \Omega^2)^2 + 4\omega_0^2\omega_M^2}}{2}\right)^{1/2}, \end{aligned} \tag{33}$$

where α_+ and α_- can also be equal to the negatives of the stated values. Computing $|\text{Im}[\alpha]|$, we find the normal mode frequencies

$$\boxed{\begin{aligned} \omega_+ &= \left(\frac{\omega_0^2 + \Omega^2 - \sqrt{(\omega_0^2 - \Omega^2)^2 + 4\omega_0^2\omega_M^2}}{2}\right)^{1/2} \\ \omega_- &= \left(\frac{\omega_0^2 + \Omega^2 + \sqrt{(\omega_0^2 - \Omega^2)^2 + 4\omega_0^2\omega_M^2}}{2}\right)^{1/2}, \end{aligned}}$$

where ω_0, Ω, and ω_M are defined in Eq.(29).

(c) If we make the approximation $K_0 \gg k$, the equations of motion Eq.(26) and Eq.(27) become

$$M\ddot{X} = -K_0 X \tag{34}$$
$$m\ddot{x} = -k(x - X), \tag{35}$$

which implies X oscillates independently of x.

(d) Given that X begins at rest from the position $X = 0$, we can solve Eq.(34) to find

$$\boxed{X(t) = X_0 \cos(\Omega_0 t),} \tag{36}$$

where we defined $\Omega_0 = \sqrt{K_0/M}$. With Eq.(36), Eq.(35) becomes

$$\ddot{x} + \omega_0^2 x = \omega_0^2 X_0 \cos(\Omega_0 t) \tag{37}$$

where we maintained the definition $\omega_0 = \sqrt{k/m}$. Eq.(39) is the equation of motion for a forced oscillator. Applying the standard method of solution for such, we find that the general solution for $x(t)$ is a sum of the homogeneous and particular solution:

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) + \frac{\omega_0^2 X_0}{\omega_0^2 - \Omega_0^2}\cos(\Omega_0 t). \tag{38}$$

Imposing the condition that $x(0) = 0$ and $\dot{x}(0) = 0$, we find

$$\boxed{x(t) = \frac{\omega_0^2 X_0}{\omega_0^2 - \Omega_0^2}\Big[\cos(\Omega_0 t) - \cos(\omega_0 t)\Big].} \tag{39}$$

(e) In order for this forced oscillator system to be in resonance, we need the denominator of Eq.(39) to go to zero. That is we need $\Omega_0^2 = \omega_0^2$. Given the definition of these frequencies, we find that this is valid if M satisfies

$$\boxed{M = \frac{K_0}{k} m.} \tag{40}$$

4. **A bent string (20 points)**
A string of length L is fixed at both ends. At $t = 0$, the string is at rest and it is distorted as shown in the figure below and then released.

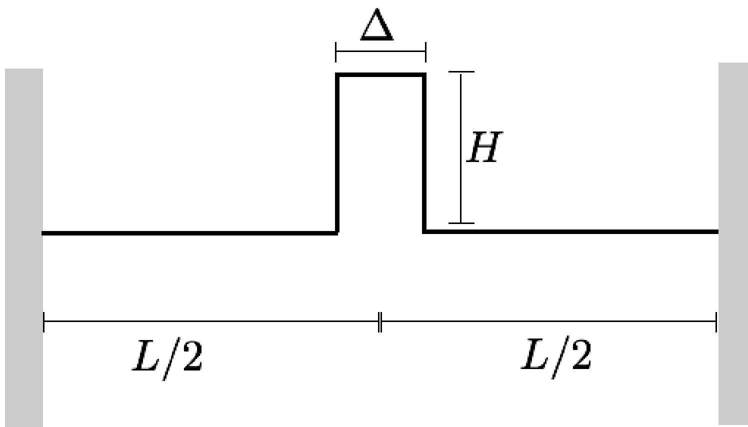

Figure 4: Bent String

(a) (5 points) Given the depiction above, what is $y(x, 0)$?

(b) (10 points) Derive an expression for the amplitude of the mth harmonic of this string. (You should be able to write your answer as a single term)

(c) (5 points) Show that for $L \gg \Delta$, the amplitude of the first few harmonics (i.e., the first few values of m) is independent of m.

Solution:

(a) From Fig. 4, we find that $y(x, 0)$ is

$$y(x,0) = \begin{cases} 0 & \text{for } 0 \le x < \frac{L}{2} - \frac{\Delta}{2} \\ H & \text{for } \frac{L}{2} - \frac{\Delta}{2} \le x < \frac{L}{2} + \frac{\Delta}{2} \\ 0 & \text{for } \frac{L}{2} + \frac{\Delta}{2} < x \le L. \end{cases} \tag{41}$$

(b) The general solution to the wave equation for a bounded string with condition $y(x, 0) = y(x, L) = 0$, is

$$y(x,t) = \sum_{n=1}^{\infty} \Big[\alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t)\Big] \sin\left(\frac{n\pi}{L} x\right), \tag{42}$$

where $\omega_n = n\pi v/L$. The coefficients α_n and β_n are computed from the initial conditions of this system:

$$\alpha_n = \frac{2}{L} \int_0^L dx\, y(x,0)\, \sin\left(\frac{n\pi}{L} x\right) \tag{43}$$

$$\beta_n = \frac{2}{L\omega_n}\int_0^L dx\, \dot{y}(x,0)\, \sin\left(\frac{n\pi}{L}x\right). \tag{44}$$

Because the string begins from rest, we have $\dot{y}(x,0) = 0$, and thus $\beta_n = 0$. Computing α_n given Eq.(41), we find

$$\begin{aligned}
\alpha_n &= \frac{2}{L}\int_0^L dx\, \sin\left(\frac{n\pi}{L}x\right) \times \begin{cases} 0 & \text{for } 0 \le x < \frac{L}{2} - \frac{\Delta}{2} \\ H & \text{for } \frac{L}{2} - \frac{\Delta}{2} \le x < \frac{L}{2} + \frac{\Delta}{2} \\ 0 & \text{for } \frac{L}{2} + \frac{\Delta}{2} < x \le L. \end{cases} \\
&= \frac{2H}{L}\int_{L/2-\Delta/2}^{L/2+\Delta/2} dx\, \sin\left(\frac{n\pi}{L}x\right) \\
&= -\frac{2H}{L}\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\Big|_{L/2-\Delta/2}^{L/2+\Delta/2} \\
&= -\frac{2H}{n\pi}\Big[\cos\left(\frac{n\pi}{2} + \frac{n\pi\Delta}{2L}\right) - \cos\left(\frac{n\pi}{2} - \frac{n\pi\Delta}{2L}\right)\Big] \\
&= \frac{4H}{n\pi}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi\Delta}{2L}\right).
\end{aligned} \tag{45}$$

Given that the amplitude is exclusively positive, we find that the amplitude A_m of the mth harmonic is

$$\boxed{A_m = \frac{4H}{m\pi}\left|\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{m\pi\Delta}{2L}\right)\right|.} \tag{46}$$

(c) For $L \gg \Delta$, we have $\Delta/L \ll 1$. Thus provided we are looking at the first few harmonics (i.e., low values of m) we can make the approximation

$$\sin\left(\frac{m\pi\Delta}{2L}\right) \simeq \frac{m\pi\Delta}{2L}. \tag{47}$$

In this approximation, Eq.(46) becomes

$$\boxed{A_m \simeq \frac{2H\pi\Delta}{L}\left|\sin\left(\frac{m\pi}{2}\right)\right| = 2H\frac{\pi\Delta}{L} \quad \text{[for } m \text{ odd]},} \tag{48}$$

which is indeed independent of m.

5. **Rolling in bowl (30 points)**
A spherical ball of radius r and mass M, moving under the influence of gravity, rolls back and forth without slipping across the center of a bowl which is itself spherical with a larger radius R (Fig. 5). The position of the ball can be described by the angle θ between the vertical and a line drawn from the center of curvature of the bowl to the center of mass of the ball.

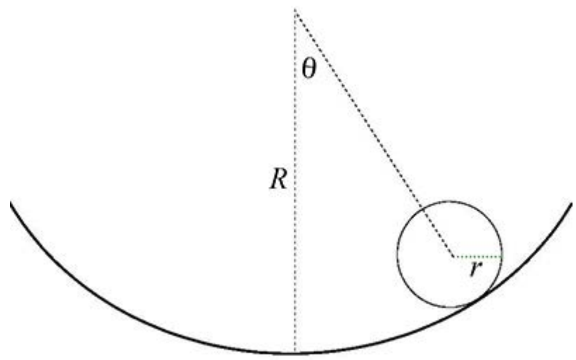

Figure 5: Ball in Bowl

The total energy of the system is

$$E_{\text{tot}} = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}Mv^2 + Mg(R-r)(1-\cos\theta), \tag{49}$$

where $I = \frac{2}{5}Mr^2$ is the moment of inertia of the ball, and $v = r\dot{\theta}$ is the velocity of the sphere. All quantities except θ are time-independent constants.

(a) (10 points) Assume that the ball begins from rest at an angle θ_0 away from the vertical. Using conservation of energy, derive an expression for the period of the ball (i.e., the time it takes the ball to move from θ_0 to $-\theta_0$ and back to θ_0.)

(b) (10 points) By conservation of energy, E_{tot} must be independent of time. Using this fact, Eq.(49), and what you know about derivatives derive an equation of motion for θ.

(c) (10 points) Take the small-angle approximation for the equation derived in (b). What should the result in (a) reduce to in this approximation?

Solution:

(a) Using the definition of the moment of inertia and velocity, we find that Eq.(49) becomes

$$E_{\text{tot}} = \frac{7}{10}Mr^2\dot{\theta}^2 + Mg(R-r)(1-\cos\theta). \tag{50}$$

If the ball begins from rest at an angle θ_0, then by conservation of energy we have

$$Mg(R-r)(1-\cos\theta_0) = \frac{7}{10}Mr^2\dot{\theta}^2 + Mg(R-r)(1-\cos\theta). \tag{51}$$

Solving for $\dot{\theta}$ gives us

$$\dot{\theta} = \sqrt{\frac{10g}{7r^2}(R-r)(\cos\theta - \cos\theta_0)}. \tag{52}$$

Noting that the ball travels $1/4$ of a period as it moves from 0 to θ_0, we find that the period of this system is

$$\begin{aligned}\frac{T}{4} &= \int_0^{\theta_0} \frac{1}{\dot{\theta}} d\theta \\ &= \sqrt{\frac{7r^2}{10g(R-r)}} \int_0^{\theta_0} d\theta \, \frac{1}{\sqrt{\cos\theta - \cos\theta_0}} \end{aligned} \tag{53}$$

or

$$\boxed{T = 2\sqrt{\frac{14r^2}{5g(R-r)}} \int_0^{\theta_0} d\theta \, \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}.} \tag{54}$$

(b) By conservation of energy, Eq.(50) should be constant in time. Therefore, its time derivative should be zero. Differentiating Eq.(50) with respect to time and setting the result to zero, we have

$$\begin{aligned}0 &= \frac{d}{dt} E_{\text{tot}} \\ &= \left(\frac{7}{5} M r^2 \ddot{\theta} + Mg(R-r)\sin\theta \right) \dot{\theta}, \end{aligned} \tag{55}$$

which, upon dividing by the coefficient of the $\ddot{\theta}$ terms, gives us the equation of motion

$$\boxed{\ddot{\theta} + \frac{5g(R-r)}{7r^2} \sin\theta = 0.} \tag{56}$$

(c) In the small-angle approximation, we can take $\sin\theta \simeq \theta$. Thus Eq.(56), would become

$$\ddot{\theta} + \omega^2 \theta = 0, \tag{57}$$

where the angular frequency ω is

$$\omega = \sqrt{\frac{5g(R-r)}{7r^2}}. \tag{58}$$

In (a) we computed the period of the ball rolling in the bowl. In the small-angle approximation, this period is 2π times the inverse of the angular frequency. So for θ_0 small, we expect the result of (a) to reduce to

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{7r^2}{5g(R-r)}}.} \tag{59}$$

6. **Coupled Oscillator (30 points)**
Three masses are coupled through two springs of spring constant as shown in the figure below.

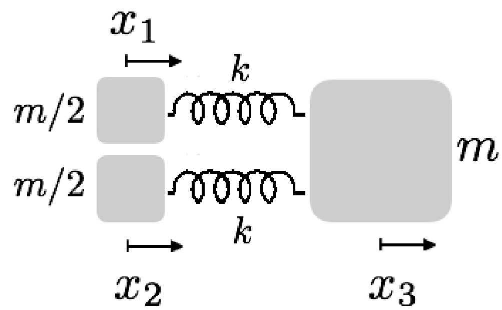

Figure 6: Three Oscillators

The two smaller masses each have mass $m/2$, the larger mass has mass m, and all the springs have spring constant k.

(a) (10 points) The system has three normal modes. One of these modes consists of the all of the masses moving to the right at constant speed, i.e.,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{60}$$

Using symmetry and the fact that the center of mass of the two other types of motion remains constant, write the other two normal modes as vectors of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}, \tag{61}$$

where you should determine A, B, and C.

(b) (10 points) Write the equation of motion of the above system as a matrix equation.

(c) (10 points) What are the normal mode **frequencies** of the system? *Hint: One of these frequencies is very simple*

Solution:

(a) Given that the we have two objects of mass $m/2$ each joined in parallel to a larger object of mass m, the system can only move in two ways while still maintaining the same center of mass. For this system, the center of mass is

$$X_{\text{cm}} = \frac{m/2x_1 + m/2x_2 + mx_3}{m/2 + m/2 + m} = \frac{1}{2}\Big(\frac{x_1 + x_2}{2} + x_3\Big). \tag{62}$$

For X_{cm} to be stationary upon motion in the system, we can have dynamics which follow (1) $x_1 \to x_1 + \Delta x$, $x_1 \to x_2 - \Delta x$, and $x_3 \to x_3$ (for some arbitrary Δx), or (2) $x_1 \to x_1 + \Delta x$, $x_1 \to x_2 + \Delta x$, and $x_3 \to x_3 - \Delta x$. The motion defined by (1) looks like alternating pistons, while the motion defined by (2) looks like an accordion. These motions are represented, respectively, by the normal modes

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_+ = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_- = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}. \tag{63}$$

(b) Given the system in Fig. 6, we find the system of equations

$$(m/2)\ddot{x}_1 = k(x_3 - x_1) \tag{64}$$

$$(m/2)\ddot{x}_2 = k(x_3 - x_2) \tag{65}$$

$$m\ddot{x}_3 = -k(x_3 - x_1) - k(x_3 - x_2). \tag{66}$$

Dividing each equation by the corresponding mass term, and writing the result as a matrix equation, we find

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = \begin{pmatrix} -2\omega_0^2 & 0 & 2\omega_0^2 \\ 0 & -2\omega_0^2 & 2\omega_0^2 \\ \omega_0^2 & \omega_0^2 & -2\omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{67}$$

(c) To find the normal mode frequencies of this system, we need to solve the eigenvalue-eigenvector equation

$$\begin{pmatrix} -2\omega_0^2 & 0 & 2\omega_0^2 \\ 0 & -2\omega_0^2 & 2\omega_0^2 \\ \omega_0^2 & \omega_0^2 & -2\omega_0^2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \alpha^2 \begin{pmatrix} A \\ B \\ C \end{pmatrix} \tag{68}$$

for α^2 and then compute $|\mathrm{Im}[\alpha]|$ for each possible value. We could implement the "characteristic equation" method, but given the normal modes found in (a) and the fact that normal modes are eigenvectors of the angular frequency matrix, we can implement a simpler method.
From the eigenvector stated in the prompt, we have

$$\begin{pmatrix} -2\omega_0^2 & 0 & 2\omega_0^2 \\ 0 & -2\omega_0^2 & 2\omega_0^2 \\ \omega_0^2 & \omega_0^2 & -2\omega_0^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0, \tag{69}$$

which by Eq.(68), implies that one normal mode frequency is

$$\boxed{\omega_0 = 0.} \tag{70}$$

From the first eigenvector found in (a), we have

$$\begin{pmatrix} -2\omega_0^2 & 0 & 2\omega_0^2 \\ 0 & -2\omega_0^2 & 2\omega_0^2 \\ \omega_0^2 & \omega_0^2 & -2\omega_0^2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2\omega_0^2 \\ 2\omega_0^2 \\ 0 \end{pmatrix}, \tag{71}$$

which by Eq.(68), implies that $\alpha_+^2 = -2\omega_0^2$ and (computing $|\mathrm{Im}[\alpha]|$) we have

$$\boxed{\omega_+ = \omega_0\sqrt{2}.} \tag{72}$$

Finally, from the last eigenvector found in (a), we have

$$\begin{pmatrix} -2\omega_0^2 & 0 & 2\omega_0^2 \\ 0 & -2\omega_0^2 & 2\omega_0^2 \\ \omega_0^2 & \omega_0^2 & -2\omega_0^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4\omega_0^2 \\ -4\omega_0^2 \\ 4\omega_0^2 \end{pmatrix}, \tag{73}$$

which by Eq.(68), implies that $\alpha_-^2 = -4\omega_0^2$ and (computing $|\mathrm{Im}[\alpha]|$) we have

$$\boxed{\omega_- = 2\omega_0.} \tag{74}$$