Lecture 05: Driven Oscillations

In these notes, we derive the properties of both an undamped and damped harmonic oscillator under the influence of a sinusoidal-in-time driving force. This discussion is supplemented by the important physics concepts of beats and resonance.

1 Pushing your Friend on a Swing

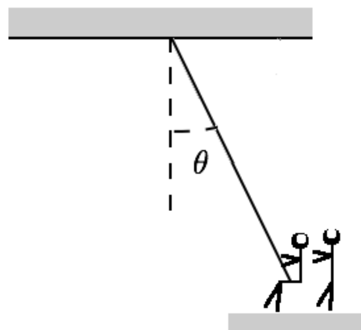


Figure 1: Pushing your friend on a swing

Say you go to the playground and jump on the swing. If you walk yourself to an initial angle away from the vertical and release, your motion (assuming you stay pretty still in your seat) would be well described by the equation

$$\ddot{\theta} + \frac{b}{m}\dot{\theta} + \frac{g}{\ell}\sin\theta = 0, \tag{1}$$

where b is your drag coefficient and ℓ is the length of the swing. If your initial angle is sufficiently small, your motion would be described by the even simpler equation

$$\ddot{\theta} + \frac{b}{m}\dot{\theta} + \frac{g}{\ell}\,\theta = 0, \tag{2}$$

Given what we previously, learned we know that the motion of the swing is underdamped, and thus, over time, your swing amplitude will gradually get smaller and smaller until you're making barely perceptible oscillations around the equilibrium position. This would be a sad situation indeed, so you ask your friend to combat the insidious forces of air drag by pushing you on the swing. But now comes the question of technique: How should your friend push you so that you achieve the highest swing?

If you've ever been on a swing, you probably believe from experience that your friend should give you a strong push whenever you reach the apex of your swing. In physics-speak, this is tantamount to your friend pushing you with the same frequency and in the same phase as your swing. That is, whenever you complete one period of motion your friend applies a force, and this force is maximum whenever you're at your maximum angle. Given what we know about oscillators we should be able to explain why this is the case (or even *if* it is the case). That is our objective in this lesson.

We will specify our forces to be ”periodic” forces because periodic forces 1) best model the force your friend applies to you on the swing and 2) are relatively simple forces to analyze for the harmonic oscillators. In what follows we will further specify the label ”periodic” to be ”sinusoidally periodic” because sinusoidal functions are the simplest periodic functions and are therefore convenient entry points into a more general analysis. Moreover, due to the magic of Fourier analysis, by considering sinusoidal forces we are in fact laying the ground work to consider all types of time-dependent forces. Thus our framing question is

Framing Question

What are the properties of harmonic oscillators driven by an external sinusoidal force?

2 Driven Undamped Oscillator

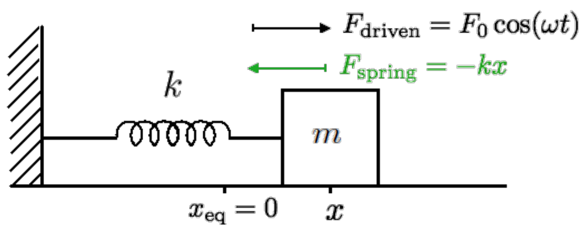


Figure 2: Driven Undamped Harmonic Oscillator

Our ultimate objective is to determine the properties of a damped harmonic oscillator driven by an external sinusoidal force. But before we explore this desired case, we will consider the relatively simpler system of a driven undamped oscillator. In other words, we return to our simple harmonic oscillator system from the second lecture and apply an external sinusoidal force to the oscillating block. The situation is depicted in Fig. 2.

We have a block of mass m acted upon by a spring of spring constant k (with an equilibrium position at $x = 0$) and by an external force $F_0 \cos(\omega t)$. By Netwon's second law, the equation of motion of the block is

$$m\ddot{x} = F_{\text{net}} = -kx + F_0 \cos(\omega t), \tag{3}$$

or, upon adding kx to both sides and dividing by m,

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t), \tag{4}$$

where $\omega_0 = \sqrt{k/m}$. Eq.(4) is the equation of motion of this system, and in order to more precisely characterize this system we need to solve it. To solve Eq.(4) we will guess a particular solution and then manipulate its parameters in order to make it consistent with Eq.(4). Given the "$\cos(\omega t)$" term on the right-hand-side of Eq.(4), we make the informed guess that its solution should be of the form

$$x(t) \stackrel{?}{=} A\cos(\omega t), \tag{5}$$

where we note that ω is the angular frequency of the driving force and is not ω_0, the natural frequency of the oscillator. Inserting Eq.(5) into Eq.(4), we find

$$\begin{aligned} \frac{F_0}{m}\cos(\omega t) &= \ddot{x}(t) + \omega_0^2 x(t) \\ &= -\omega^2 A\cos(\omega t) + \omega_0^2 A\cos(\omega t) \end{aligned}$$

$$= (\omega_0^2 - \omega^2) A\cos(\omega t). \tag{6}$$

The first and last line of Eq.(6) are only equivalent if $A = (F_0/m)/(\omega_0^2 - \omega^2)$. Therefore, given Eq.(5), we find that a particular solution to Eq.(4) is

$$x(t) = \frac{F_0/m}{\omega_0^2 - \omega^2}\cos(\omega t). \tag{7}$$

However, there is a problem with Eq.(7): It doesn't contain any free parameters. All of the parameters in Eq.(7) are determined by the properties of the block and driving, and so we cannot impose initial conditions on the system in order to obtain motion for our specific setup of the system.

We can resolve this problem by recognizing that Eq.(7) is only *part* of the solution to Eq.(4). Specifically, if we have a function $x_h(t)$, which has two free parameters and which satisfies the differential equation

$$\ddot{x}_h(t) + \omega_0^2 x_h(t) = 0, \tag{8}$$

then the solution

$$x(t) = x_h(t) + x_p(t), \tag{9}$$

where x_p is given by Eq.(7) would also have two free parameters but would satisfy Eq.(4). We can see this as follows:

$$\begin{aligned}\ddot{x}(t) + \omega_0^2 x(t) &= \ddot{x}_h(t) + \ddot{x}_p(t) + \omega_0^2 x_h(t) + \omega_0^2 x_p(t)\\ &= \Big(\ddot{x}_h(t) + \omega_0^2 x_h(t)\Big) + \Big(\ddot{x}_p(t) + \omega_0^2 x_p(t)\Big)\\ &= 0 + \frac{F_0}{m}\cos(\omega t) = \frac{F_0}{m}\cos(\omega t).\end{aligned} \tag{10}$$

Thus, the *general* solution to Eq.(4) is given by Eq.(9), where $x_h(t)$ is the general solution to Eq.(8). From the second lecture, we already know the general solution of Eq.(8), so we find that Eq.(9) is

$$x(t) = B\cos(\omega_0 t) + C\sin(\omega_0 t) + \frac{F_0/m}{\omega_0^2 - \omega^2}\cos(\omega t), \tag{11}$$

where B and C are arbitrary constants. Alternatively, the general solution can be written as

$$x(t) = A_0\cos(\omega_0 t - \phi) + \frac{F_0/m}{\omega_0^2 - \omega^2}\cos(\omega t), \tag{12}$$

where Eq.(12) and Eq.(11) are related by $\sqrt{B^2 + C^2} = A_0$ and $\tan^{-1} C/B = \phi$.

Eq.(12) is our desired kinematic equation for the situation depicted in Fig. 2. We can develop an ever deeper physical understanding of this situation by considering the motion for specific initial conditions. Say our oscillator starts at a position $x(t = 0) = 0$ m with velocity $\dot{x}(t = 0) = 0$ m/s. Imposing these conditions on Eq.(12), we find

$$\phi = 0 \quad \text{and} \quad A_0 = -\frac{F_0/m}{\omega_0^2 - \omega^2}, \tag{13}$$

which leads to the solution

$$x(t) = \frac{F_0/m}{\omega_0^2 - \omega^2}\Big[\cos(\omega t) - \cos(\omega_0 t)\Big]. \tag{14}$$

One thing we notice about Eq.(14) is that it doesn't seem to be defined for $\omega = \omega_0$; as $\omega \to \omega_0$ the coefficient of the second term goes to infinity. No matter for the moment. What does the motion look like when $\omega \neq \omega_0$? For arbitrarily chosen parameters, we plot Eq.(14) in Fig. 3. The motion seems to be a strange amalgamation of two oscillations: There is a fast oscillation which seems to be occurring within the envelope of a much slower oscillation. In other words, we have a sinusoid whose amplitude is itself oscillating sinusoidally!

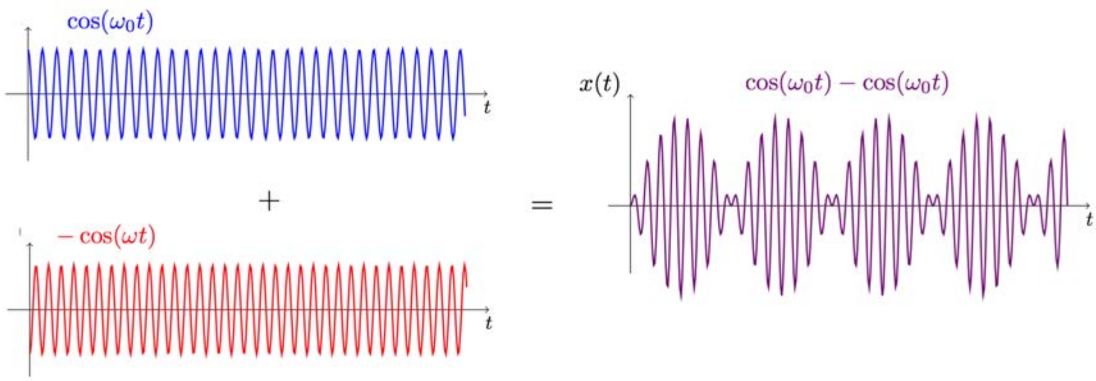


Figure 3: The superposition of two sinusoids which are close in frequency leads to beating

This phenomena is termed a **beat phenomena** or simply **beating**. We can better understand it by applying a trigonometric identity to Eq.(14). With the identity

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}, \tag{15}$$

we find Eq.(14) becomes

$$x(t) = \frac{2F_0/m}{\omega^2-\omega_0^2}\sin\left[\frac{(\omega-\omega_0)t}{2}\right]\sin\left[\frac{(\omega+\omega_0)t}{2}\right] \tag{16}$$

Eq.(16) is precisely the behavior we observe in Fig. 3: A sinusoid oscillating at the fast frequency $(\omega+\omega_0)/2$ whose amplitude is modulated at the slower frequency $(\omega-\omega_0)/2$. By convention, we define the **beat frequency** as the frequency at which the more slowly envelope goes to zero. Thus the beat frequency is twice the frequency of the $\sin((\omega-\omega_0)t/2)$ factor, that is

$$f_{\text{beat}} = \frac{\omega_{\text{beat}}}{2\pi} = \frac{1}{2\pi}\times 2\times\frac{|\omega-\omega_0|}{2} = \frac{|\omega-\omega_0|}{2\pi}. \tag{17}$$

What's more, Eq.(16) predicts that as the driving frequency ω gets closer and closer to the natural frequency ω_0, the slow frequency oscillations go to zero and only the fast frequency oscillations remain. But we can't make this prediction yet, because the denominator in Eq.(16) goes to zero in this limit as well. We consider this problem in the next section.

2.1 Resonance

In the previous section, we were able to quantitatively describe the undamped and driven oscillator for the case where $\omega \neq \omega_0$, but what happens when $\omega \to \omega_0$? We begin with our solution to the driven simple harmonic oscillator for the condition $x_0 = 0$ and $v_0 = 0$:

$$x(t) = \frac{2F_0/m}{\omega^2-\omega_0^2}\sin\left[\frac{(\omega-\omega_0)t}{2}\right]\sin\left[\frac{(\omega+\omega_0)t}{2}\right]. \tag{18}$$

To determine what happens as $\omega \to \omega_0$, we define $\Delta\omega$ by the equation

$$\omega = \omega_0 + \Delta\omega, \tag{19}$$

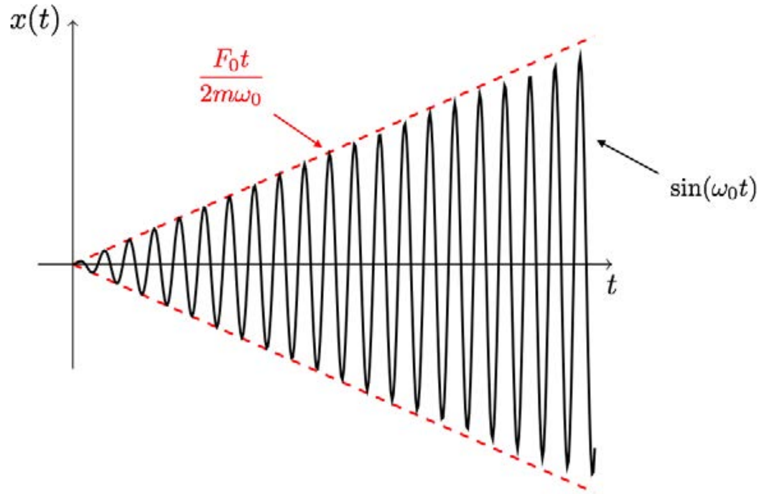


Figure 4: Driven Simple Harmonic Oscillator at Resonance

and take the limit $\Delta\omega/\omega_0 \ll 1$. With the definition Eq.(19), we find

$$\sin\left[\frac{(\omega+\omega_0)t}{2}\right] = \sin\left(\omega_0 t + \Delta\omega\, t/2\right) = \sin(\omega_0 t) + \mathcal{O}(\Delta\omega) \tag{20}$$

$$\sin\left[\frac{(\omega-\omega_0)t}{2}\right] = \sin\left(\Delta\omega\, t/2\right) = \Delta\omega\, t/2 + \mathcal{O}\left((\Delta\omega)^3\right) \tag{21}$$

$$\omega^2 - \omega_0^2 = (\omega+\omega_0)(\omega-\omega_0) = 2\omega_0\Delta\omega + \mathcal{O}\left((\Delta\omega)^2\right). \tag{22}$$

Thus Eq.(18) to lowest order in $\Delta\omega$ is

$$\begin{aligned} x(t) &= \frac{2F_0/m}{2\omega_0\Delta\omega + \mathcal{O}\left((\Delta\omega)^2\right)}\Big(\Delta\omega\, t/2 + \mathcal{O}\left((\Delta\omega)^3\right)\Big)\Big(\sin(\omega_0 t) + \mathcal{O}(\Delta\omega)\Big) \\ &= \frac{F_0}{2m\omega_0} t\sin(\omega_0 t) + \mathcal{O}(\Delta\omega). \end{aligned} \tag{23}$$

Currently, with $\Delta\omega \ll \omega_0$, our driven frequency ω is close (but not quite equal) to ω_0. Taking the $\Delta\omega \to 0$ limit of Eq.(23), is tantamount to taking $\omega \to \omega_0$ and doing so yields

$$x(t) = \frac{F_0}{2m\omega_0} t\sin(\omega_0 t) \tag{24}$$

When the driving frequency matches the natural frequency of an oscillator we say that the system is in **resonance**. One of the main features of resonance is that the amplitude of oscillation is at a value higher than it would be if the system were not in resonance. For the simple harmonic oscillator, the resonance solution Eq.(24) exhibits an amplitude which increases linearly in time (See Fig. 4). Thus for long times the harmonic oscillator amplitude appears to increase without bound. However, the result Eq.(24) is not true to reality because for large enough amplitudes of motion the simple harmonic oscillator equation of motion no longer applies. Namely, with large for large amplitudes we would need to incorporate nonlinear or energy dissipating terms into the harmonic oscillator equation of motion. We consider the effect of energy dissipating terms in the next section.

3 Driven Damped Oscillator

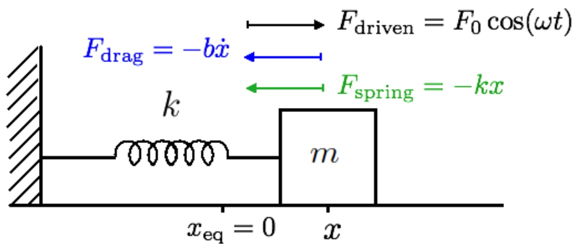


Figure 5: Driven Damped Harmonic Oscillator

At resonance, the velocity of the driven undamped harmonic oscillator is the time derivative of Eq.(24):

$$\dot{x}(t) = \frac{F_0}{2m\omega_0}\sin(\omega_0 t) + \frac{F_0}{2m} t\cos(\omega_0 t) \tag{25}$$

The second term of Eq.(25) increases without bound as t increases thus indicating that the driven simple harmonic oscillator goes faster and faster as time goes on. If this were to occur in a real system (and not just an idealized model) then, eventually, velocity dependent forces like the drag force $F_{\text{drag}} = -b\dot{x}$ would become relevant. Thus, to better understand how real oscillating systems are affected by driven oscillations, it is better to consider drag forces from the start. We can analyze such a driven and damped oscillator by including a drag force in Eq.(3):

$$m\ddot{x} = F_{\text{net}} = -kx - b\dot{x} + F_0\cos(\omega t), . \tag{26}$$

The resulting equation of motion is then

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t), \tag{27}$$

where $\gamma = b/2m$ and $\omega_0^2 = k/m$. Eq.(27) is so far the most general equation we've considered for harmonic motion. It is a damped and sinusoidally driven harmonic oscillator. As always, our goal is to solve this equation. To do so we will make use of a simple trick. We will define $x(t)$ as the real part of a complex variable $z(t)$:

$$x(t) = \text{Re}[z(t)] \tag{28}$$

and say the equation of motion of $z(t)$ is

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = \frac{F_0}{m}e^{i\omega t}, \tag{29}$$

Taking the real part of Eq.(29) and using Eq.(28), we find Eq.(27). Therefore, if we find the general solution to $z(t)$ and take the real part, we should find the general solution of $x(t)$. To find the solution to Eq.(29), we follow a method analogous to that in the previous section. We take the general solution to be

$$z(t) = z_h(t) + z_p(t) \tag{30}$$

where $z_h(t)$ is the homogeneous solution to Eq.(29) (i.e., the solution for $F_0 = 0$), and $z_p(t)$ is the solution we find after using a $z_p(t) = Ae^{i\omega t}$ guess.

First, considering the homogeneous solution, we know from the previous lesson that the general solution to Eq.(29) when $F_0 = 0$ is

$$z_h(t) = \begin{cases} A_+ e^{\alpha_+ t} + A_- e^{\alpha_- t} & \text{when } \alpha_+ \neq \alpha_- \\ (B + Ct)e^{-\gamma t} & \text{when } \alpha_+ = \alpha_- \end{cases} \tag{31}$$

where

$$\alpha_\pm = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}, \tag{32}$$

and $A_\pm$, B, and C are arbitrary parameters which are specified by the initial conditions.

Now, considering the particular solution, we insert the guess

$$z_p(t) = Ae^{i\omega t} \tag{33}$$

into Eq.(29) and determine what A should be in order for the equation to be valid. Doing so we find

$$\begin{aligned} \frac{F_0}{m} e^{i\omega t} &= \ddot{z} + 2\gamma \dot{z} + \omega_0^2 z \\ &= -\omega^2 A e^{i\omega t} + 2i\gamma\omega A e^{i\omega t} + \omega_0^2 A e^{i\omega t} \end{aligned} \tag{34}$$

which implies A must be the complex quantity

$$A = \frac{F_0/m}{\omega_0^2 - \omega^2 + 2i\gamma\omega}, \tag{35}$$

writing out the particular solution in terms of A we find

$$z_p(t) = \frac{F_0/m}{\omega_0^2 - \omega^2 + 2i\gamma\omega} e^{i\omega t}. \tag{36}$$

Noting $a + bi = \sqrt{a^2 + b^2}(\cos\theta + i \sin\theta)$ where $\tan\theta = b/a$, we can write

$$\omega_0^2 - \omega^2 + 2i\gamma\omega = \sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}\,(\cos\phi_{\text{dr}} + i \sin\phi_{\text{dr}}) \tag{37}$$

where

$$\tan\phi_{\text{dr}} \equiv \frac{2\gamma\omega}{\omega_0^2 - \omega^2}. \tag{38}$$

Therefore with Eq.(37) the particular solution becomes

$$\begin{aligned} z_p(t) &= \frac{F_0/m}{\omega_0^2 - \omega^2 + 2i\gamma\omega} e^{i\omega t} \\ &= A_{\text{dr}} e^{i(\omega t - \phi_{\text{dr}})}. \end{aligned} \tag{39}$$

where

$$A_{\text{dr}} \equiv \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}. \tag{40}$$

Combining Eq.(31) and Eq.(39) into Eq.(30) and taking the real part, we find the solution

$$x(t) = x_h(t) + A_{\text{dr}} \cos(\omega t - \phi_{\text{dr}}), \tag{41}$$

where A_{dr} and ϕ_{dr} are given in Eq.(40) and Eq.(38), respectively, and where the homogeneous solution is

(again from the previous lesson)

$$x_h(t) = \begin{cases} A_0 e^{-\gamma t}\cos(\Omega t - \phi) & \text{when } \gamma < \omega_0 \\ e^{-\gamma t}\left(A_+ e^{\Gamma t} + A_- e^{-\Gamma t}\right) & \text{when } \gamma > \omega_0 \\ (B + Ct)e^{-\gamma t} & \text{when } \gamma = \omega_0 \end{cases} \tag{42}$$

with $\Omega = \sqrt{\omega_0^2 - \gamma^2}$ and $\Gamma = \sqrt{\gamma^2 - \omega_0^2}$. Eq.(41) is the desired form of the solution to the driven and damped harmonic oscillator. From this solution, we see that for long times the homogeneous solution goes to zero due to the $e^{-\gamma t}$ factor, and we are left with the particular solution:

$$x(t) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}\cos(\omega t - \phi_{\text{dr}}) \qquad \text{[For long times, i.e., for } t \gg 1/\gamma\text{]} \tag{43}$$

This long time solution is interesting given that the oscillator is driven by the force $F(t) = F_0\cos(\omega t)$. Comparing the driving force to the solution, we see that the solution lags behind the driving force by a phase ϕ_{dr}. Also, although it is not precisely clear from the solution Eq.(43), the frequency at which the amplitude of the position is maximized is *not* the frequency of the undamped oscillator ω_0. We analyze these features behavior more in depth in the following section.

3.1 Amplitude and Resonance

In analyzing the long time solution Eq.(43) the two quantities which are of particular importance are the amplitude and phase.

The amplitude tells us the the **resonant frequency** of our system. We define the resonant frequency as the frequency at which the amplitude of the long-time oscillation is maximized. Given Eq.(43), we see that this amplitude is maximized if the denominator is minimized. Noting that the square root of a function is optimized when the function itself is optimized, we can find the resonant frequency using the standard algorithm:

$$0 = \frac{d}{d\omega}\Big((\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2\Big) = 2(\omega_0^2 - \omega^2)(-2\omega) + 8\gamma^2\omega, \tag{44}$$

which gives the critical points $\omega = 0$ and $\omega = \sqrt{\omega_0^2 - 2\gamma^2}$. Performing the second derivative test, we find

$$\frac{d^2}{d\omega^2}\Big((\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2\Big) = -4(\omega_0^2 - \omega^2) + 8\omega^2 + 8\gamma^2, \tag{45}$$

which is negative (i.e., yields a local minimum) for $\omega = 0$, and is positive (i.e., yields a local maximum) for $\omega = \sqrt{\omega_0^2 - 2\gamma^2}$, both contingent on the assumption $\omega_0^2 > 2\gamma^2$; if $\omega_0 < 2\gamma^2$ then the resonant frequency is zero. Therefore, (for $\omega_0 > \gamma\sqrt{2}$) we find that the resonant frequency of the damped harmonic oscillator system is

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2\gamma^2}, \tag{46}$$

which, interestingly, is slightly smaller than the frequency $\Omega = \sqrt{\omega_0^2 - \gamma^2}$ of the under damped motion. This implies that a sinusoidally various force produces the largest amplitude motion for the damped harmonic oscillator if the force is oscillating slightly slower than the un-driven frequency of the oscillator.

In spite of the interesting nature of Eq.(46), for our future work it will prove easiest to consider our system in the very weakly damped scenario in which $\omega_0 \gg \gamma$. In this very weakly damped limit, the resonance frequency Eq.(46) simplifies to

$$\omega_{\text{res}} \simeq \omega_0 \simeq \Omega. \tag{47}$$

We plot A_{dr} in this limit of $\gamma/\omega \ll 1$ in Fig. 6a. In Fig. 6b we plot A_{dr} in the case where very weak damping does not apply.

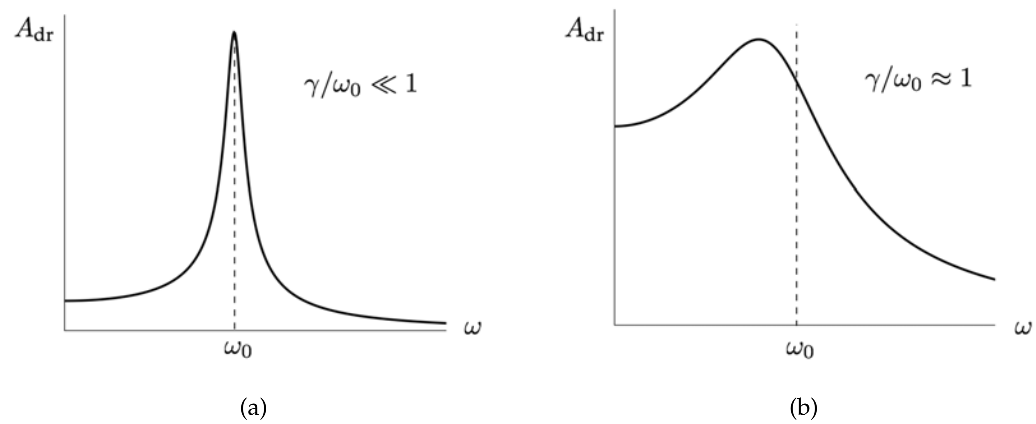


Figure 6: Amplitude of driven damped oscillator for two regimes of γ/ω_0

What happens to the phase Eq.(38) at resonance for $\omega_0 \gg \gamma$? Taking $\omega = \omega_{\rm res} \simeq \omega_0$ we find

$$\tan\phi_{\rm dr} \simeq \pm\infty. \tag{48}$$

Given that $\sin\phi_{\rm dr} = 2\gamma\omega/A_{\rm dr} > 0$, we then find that $\phi_{\rm dr} \simeq \pi/2$ at resonance. Therefore, at resonance the long time solution Eq.(43) becomes

$$x(t) \simeq \frac{F_0}{2m\gamma\omega_0}\cos(\omega t - \pi/2) = \frac{F_0}{2m\gamma\omega_0}\sin(\omega t) \qquad \text{[Resonant solution in the } \omega_0 \gg \gamma \text{ limit]}, \tag{49}$$

which implies that the position is $90°$ or $\pi/2$ out of phase with the force $F(t) = F_0\cos(\omega t)$. In particular, the position *lags* behind the force by a quarter of a period, such that when the force is at a maximum or minimum the position is zero and vice versa. For example, when the oscillator is moving past its equilibrium position $x = 0$, the magnitude of the driving force is greatest, and when the position is at its amplitude, the driving force is zero.

In and out of phase: What do we mean when we say two oscillating functions are ”in phase” or ”out of phase”?

First, we can only speak of two oscillating functions as being in or out of phase if they have the same frequency. Second, we say that two oscillating functions are ”in phase” when then their peaks and valleys occur at the same time; otherwise the functions are out of phase by the amount needed to shift them into phase. Here are some examples

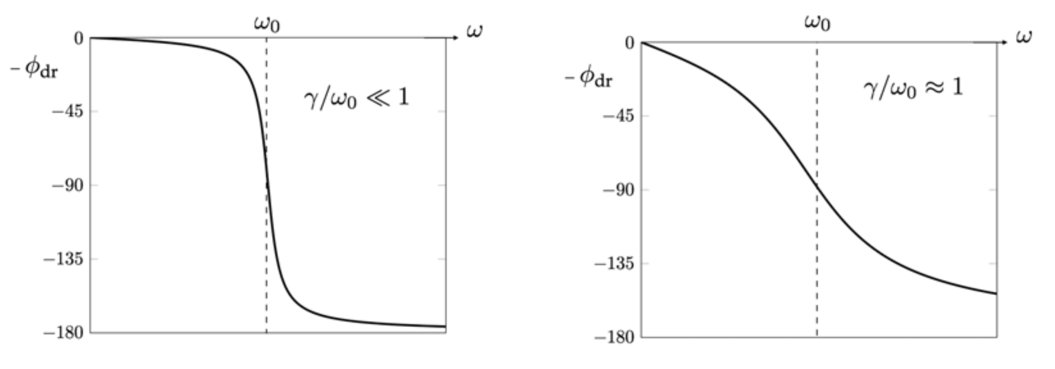
- $\cos(\omega t)$ vs $\cos(2\omega t)$: Different frequencies, so the functions are neither in or out of phase
- $\cos(\omega t)$ vs $4\cos(\omega t)$: Same frequency with constant offset functions, so the functions are in phase
- $\cos(\omega t)$ vs $\sin(\omega t)$: Same frequency, but $\sin(\theta) = \cos(\theta - \pi/2)$ so the functions are $90°$ or $\pi/2$ radians out of phase.

The reason that resonance leads to a $90°$ phase difference between the position and force can be explained using energy[1] In order to achieve resonance, we want to maximize the amplitude of our oscillator motion. This occurs when the driving force supplies as much energy as possible to the oscillator, or equivalently does as much work as possible on the oscillator. With the definition $W = F\Delta x$ we can infer that the maximum

[1] This explanation is transcribed from [1].

work is done on the system when the largest force is applied over the greatest distance. Given a fixed interval of time, the oscillator is covering the greatest distance when it is moving the fastest, namely when it is at its equilibrium position. Correspondingly, the oscillator is covering the smallest distance when it is moving the slowest, namely at the ends, or amplitudes, of its motion. Thus in order to efficiently supply the maximum amount of work, the force should be at its max when the oscillator is near its equilibrium and should be at its min when the oscillator is at its amplitude.

3.2 Phase and Frequency



(a) (b)

Figure 7: Phase of driven damped oscillator for two regimes of γ/ω_0

We previously found the phase of the solution Eq.(43) when the frequency was at resonance. What are the properties of this phase, for other frequencies?

- $\boldsymbol{\omega \simeq \omega_0 \ \rightarrow\ \phi = \pi/2}$: Oscillator and driving force are $90°$ out of phase; Resonance condition discussed above.
- $\boldsymbol{\omega \simeq 0 \ \rightarrow\ \phi = 0}$: Oscillator and driving force are in phase; $x \sim F_0 \cos(\omega t)$.

  Explanation: With a small driving frequency, Eq.(43) tells us that $\ddot{x}$ and $\dot{x}$ are very small. Therefore the equation of motion $m\ddot{x} = -kx - b\dot{x} + F_0 \cos(\omega t)$ becomes approximately $0 \simeq -kx + F_0 \cos(\omega t)$ which implies $x(t) \simeq F_0/k \cos(\omega t)$.
- $\boldsymbol{\omega \simeq \infty \ \rightarrow\ \phi = \pi}$: Oscillator and driving force are $180°$ out of phase; $x \sim -F_0 \cos(\omega t)$.

  Explanation: By Eq.(43), the $\omega \rightarrow \infty$ limit implies the $x(t)$ and $\dot{x}(t)$ are small and close to zero. This implies that the particle remains near the origin and is mostly stationary. Therefore the net force $F_{\text{net}} = -kx - b\dot{x} + F_0 \cos(\omega t)$ can be approximated as $F_{\text{net}} \simeq F_0 \cos(\omega t)$. By newton's second law, we then have $\ddot{x} \simeq F_0/m \cos(\omega t)$ which in turn implies $x \simeq -F_0/m\omega^2 \cos(\omega t)$.

We depict this phase behavior as a function of ω for various values of γ/ω_0 in Fig. 7a. In Fig. 7b we see how these phase limits manifest when the damed system is near critical damping.

4 Swing Revisited

Question: With what frequency and with what phase should your friend push you on a swing in order to give you your maximum displacement?

Having developed the theory of forced and damped oscillations we are now better prepared to analyze the scenario which motivated this lesson: Your friend pushing you on a swing. We first asked at what frequency your friend should push you in order to give you your maximum amplitude? If we take the swing to be very weakly damped (which is arguably a good approximation for a pendulum), and we approximate your friends pushing force as a sinusoidal function (which is admittedly not a very good approximation for a pushing force) then Eq.(47) tells us that what we guessed before: your friend should push you at the same frequency at which you're swinging.

$$\omega_{\text{push}} \simeq \omega_0. \tag{50}$$

But our discussion of phase indicates there is another relevant variable to consider here: At what point during your swing should your friend apply this force?

If you have ridden on a swing before and been pushed by someone else, likely that person applied their force when your were at the top (i.e., the maximum position) of your motion. But, in fact, this is not the way to go.

Our previous theoretical works (specifically Eq.(49)) suggests that, assuming your friend is applying a fixed force for a fixed duration of time, then he should apply that force when you are at the *bottom of your motion* rather than at the top. Namely, in order for you to swing as high as possible, your friend should apply a maximum force at the point when you are moving quickly at your minimum angle rather than when you are relatively stationary at your maximum angles. Mathematically, this means that the phase at which your friend should push you should be $\pi/2$ ahead of the oscillation of your swing.

$$\phi_{\text{push}} \simeq +\pi/2 \tag{51}$$

This "optimal pushing configuration" is depicted in Fig. 8

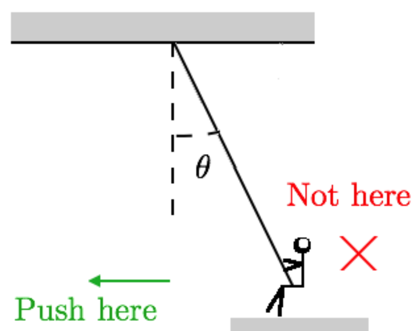


Figure 8: Most efficient way to push your friend on a swing

The reason people rarely push their friends this way in practice (other than the fact that people don't know the general physics of the situation) is that it is rather difficult for someone to safely apply a force to an object which is moving really quickly.

References

[1] D. Morin, *Introduction to classical mechanics: with problems and solutions*. Cambridge University Press, 2008.

Resource: Introduction to Oscillations and Waves
Mobolaji Williams