Lecture 07: Wave Equation and Standing Waves

In these notes we derive the wave equation for a string by considering the vertical displacement of a chain of coupled oscillators. In finding the general solution of the derived wave equation, we introduce the Fourier series and use the Fourier representation of a bounded string to derive a simple formula for the string's energy.

1 $N \to \infty$ and beyond

We previously motivated our discussion of coupled oscillators by arguing that oscillators rarely exist alone but are often connected to other oscillators. Our real-world examples were the oscillating degrees of freedom[1] in a table or in a bed, but to be faithful to these examples we would need to go well beyond the two and three particle systems we considered in the previous notes. Specifically, such systems contain a number of particles on the order of $\sim 10^{26}$ (Avogadro's number), and thus to more accurately describe the dynamics of such systems we need to learn how to study coupled oscillator systems where $N \gg 1$.

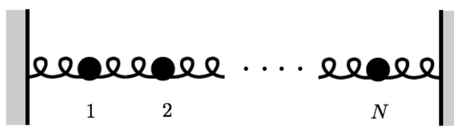

Figure 1: N oscillators coupled in series. What happens to the equation of motion for $N \to \infty$? Figure from [1]

Practically speaking, the dynamics of a system with a million particles is quite similar to the dynamics of a system with a billion particles and so on. So we find that taking $N \gg 1$ is tantamount to taking $N \to \infty$. Such $N \to \infty$ systems have an exceedingly large number of degrees of freedom and our task in this lesson is to find some way to model such coupled oscillator systems.

Thus our framing question is

Framing Question

What are the dynamics of coupled oscillator systems where the number of oscillators N goes to infinity?

We will find that the $N \to \infty$ limit of a coupled oscillator system is governed by a new dynamical equation which requires a new set of methods and techniques for analysis. This example represents the common motif of **emergence** in physics systems: considering a system with a large number of degrees of freedom often results in new properties not present with only a single degree of freedom.

2 Wave Equation for Transverse Oscillations

At the end of Lecture notes 06, we considered the system Fig. 1 in the $N \to \infty$ limit. There we found that if we changed variables and redefined constants in this limit, all the Newton's second law equations for all the

[1] Recall that the degrees of freedom are the positions of the particles in the system

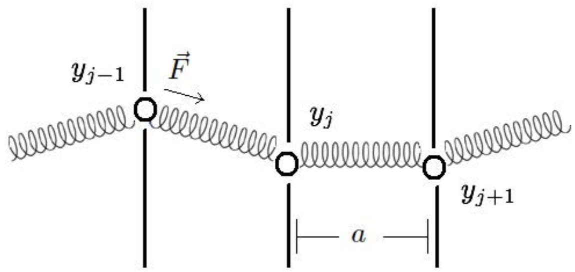

Figure 2: Transverse coupled oscillations: Each circle represents a particle of mass m and the springs are Hookean and have spring constant k. The rest length (not shown) of the spring is ℓ_R, and the masses are a horizontal distance a apart. The masses can only move in the vertical direction.

masses in Fig. 1 could be collectively reduced to a single second-order partial differential equation. In this section, we will derive this very same equation for a new system to demonstrate the equation's generality.

We begin with the situation shown in Fig. 2. We have a system of N masses arranged horizontally but constrained to move only vertically. The masses are separated by a horizontal distance a and are coupled to their neighbors by a spring of spring constant k and rest length ℓ_R. This system can be taken to represent a string on a very microscopic level. Our goal is to consider the system for an $N \to \infty$ number of masses and to determine the associated dynamical equation.

We begin by determining the forces acting on the jth mass. For the jth mass the magnitude of the force exerted by the $j+1$ neighbor should be equal to (by Hooke's law) k times the quantity defining the distance between the oscillators minus the rest length. Given that the distance between the two oscillators is $\sqrt{\Delta y^2 + a^2}$ where Δy is the difference in their y coordinates, we find that the magnitude of this force is

$$|\vec{F}_{j+1 \text{ on } j}| = k\left(\sqrt{(y_{j+1}-y_j)^2 + a^2} - \ell_R\right). \tag{1}$$

Since the masses can only move in the vertical direction, only the vertical component (i.e., the sine component) of this force is dynamically relevant. By trigonometry, we can show that the sine of the angle between the horizontal axis and the spring connecting the j and $j+1$ masses is

$$\sin\theta_j = \frac{y_{j+1}-y_j}{\sqrt{(y_{j+1}-y_j)^2 + a^2}}. \tag{2}$$

Therefore, the vertical component of the force exerted on j from $j+1$ is

$$\begin{aligned} F_{j+1 \text{ on } j,\, y} &= |\vec{F}_{j+1 \text{ on } j}|\sin\theta_j \\ &= k\left(\sqrt{(y_{j+1}-y_j)^2 + a^2} - \ell_R\right)\cdot\frac{(y_{j+1}-y_j)}{\sqrt{(y_{j+1}-y_j)^2 + a^2}} \qquad (3) \\ &= k(y_{j+1}-y_j)\left[1 - \frac{\ell_R}{a}\frac{1}{\sqrt{1+(y_{j+1}-y_j)^2/a^2}}\right], \qquad (4) \end{aligned}$$

where we factored out an a in the square root in the final line. Now, we will take the "strong coupling" approximation where the rest length ℓ_R of the spring is much less than the distance a between the oscillators.

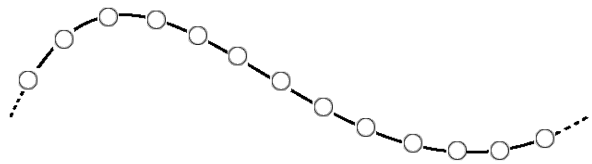

Figure 3: Transverse coupled oscillations approximate a string in the large N limit

Taking $\ell_R \ll a$ in Eq.(4), we find that the force becomes

$$F_{j+1 \text{ on } j,\, y} = k(y_{j+1} - y_j) \tag{5}$$

By a roughly identical argument and with the same approximation, we can show that the vertical component of the force on the jth mass due to the $j-1$ mass is

$$F_{j-1 \text{ on } j,\, y} = -k(y_j - y_{j-1}). \tag{6}$$

Now, with all the forces acting on the jth mass resolved, we can write the vertical equations of motion. By Newton's 2^{nd} law we have

$$m\ddot{y}_j = F_{j+1 \text{ on } j,\, y} + F_{j-1 \text{ on } j,\, y} = k(y_{j+1} - y_j) - k(y_j - y_{j-1}). \tag{7}$$

From here, we take the **continuum limit** of our system of N discrete masses. To take this limit we promote the $y_j(t)$ functions to $y(x,t)$, in which the horizontal position of the jth mass is replaced by the continuous variable x. In essence we are taking ja (where a is the spacing between the masses) to x. Eq.(7) then becomes

$$m\ddot{y}(x,t) = k\left(y(x+a,t) - y(x,t)\right) - k\left(y(x,t) - y(x-a,t)\right). \tag{8}$$

Next we make m and k implicitly dependent on the lattice spacing a such that as a becomes smaller, m gets smaller as well but k gets larger. We can then define a tension T (with units of force) as

$$T = \lim_{a\to 0} ka, \tag{9}$$

and a mass density μ (with units of mass per length) as

$$\mu = \lim_{a\to 0} \frac{m}{a}. \tag{10}$$

Dividing Eq.(8) by a and taking the limit as $a \to 0$, we then find

$$\begin{aligned}
\lim_{a\to 0} \frac{m}{a}\,\ddot{y}(x,t) &= \lim_{a\to 0} \frac{k}{a}\left[\left(y(x+a,t) - y(x,t)\right) - \left(y(x,t) - y(x-a,t)\right)\right] \\
\mu\,\ddot{y}(x,t) &= \lim_{a\to 0} ka\frac{1}{a}\left[\frac{y(x+a,t) - y(x,t)}{a} - \frac{y(x,t) - y(x-a,t)}{a}\right] \\
&= \lim_{a\to 0} ka \cdot \lim_{a\to 0} \frac{1}{a}\left[\frac{y(x+a,t) - y(x,t)}{a} - \frac{y(x,t) - y(x-a,t)}{a}\right] \\
&= Ty''(x,t)
\end{aligned} \tag{11}$$

where we used the limit definition of a derivative in the last line and used "primes" to signify derivatives with respect to x. Since $y(x,t)$ is a function of two variables, we should write Eq.(11) more explicitly in terms of partial derivatives as

$$\frac{\partial^2}{\partial t^2}y(x,t) = \frac{T}{\mu}\frac{\partial^2}{\partial x^2}y(x,t), \tag{12}$$

which is formally known as the one-dimensional wave equation. We note that in taking the $N \to \infty$ limit of our transversely oscillating coupled system, we have moved from a system of discrete oscillators to a continuous string. Thus Eq.(12) defines the dynamics of motion for the string shown in Fig. 3.

Longitudinal versus Transverse waves: At the end of Lecture notes 06, we derived a wave equation identical to Eq.(12), but we had begun by trying to describe a rather different system. Namely, we considered the system Fig. 1 (which is quite different from the system in Fig. 2) in the case of $N \to \infty$, and we also found the wave equation.

Why is this? On one level, the fact that Eq.(12) applies to two different contexts points to the ubiquity of the wave equation in describing spatially propagating oscillations, but the difference between the contexts deserves to be highlighted.

In Fig. 1, we were considering oscillations along an axis coincident with the axis of masses, while in Fig. 2 we are considering oscillations along an axis perpendicular to the axis of masses. When the wave displacement is in the same direction as its direction of propagation (i.e., the axis defining the position of the wave amplitude) , we say the wave is a **longitudinal wave** while when the wave displacement is perpendicular to the direction of propagation we say the wave is a **transverse wave** (See Fig. 4). Thus the wave equations derived from the cases in Fig. 1 and Fig. 2 were, respectively, modeling longitudinal and transverse waves. This distinction may seem academic, but it is in fact very important in understanding the properties of sound waves (which are longitudinal) and electromagnetic waves (which are transverse).

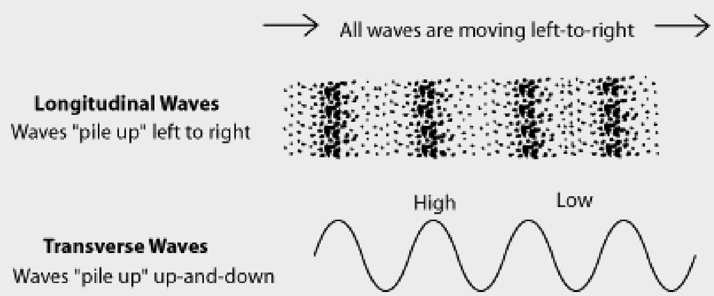

Figure 4: Longitudinal versus transverse waves.
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2.1 Dimensional Analysis

In Eq.(12) we have our desired dynamical equation for a coupled oscillator system where the number of oscillators N is taken to infinity. In the next section, we will begin seeking solutions to Eq.(12) but before we do so we will use dimensional analysis to get a sense of the significance of the parameter ratio in the equation. From our definitions of T and μ we know that have units of force and linear mass density, respectively. Therefore the units of their ratio is

$$\left[\frac{T}{\mu}\right] = \frac{\mathrm{N}}{\mathrm{kg/m}} = \frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}^2}\times\frac{\mathrm{m}}{\mathrm{kg}} = \frac{\mathrm{m}^2}{\mathrm{s}^2}. \tag{13}$$

Thus, we see that T/μ has units of velocity squared. This turns out to not be a coincidence as we will later discover; the quantity $\sqrt{T/\mu}$ will define the speed of wave propagation along the string.

3 Solving the Wave Equation

Now that we have the wave equation Eq.(12), our next task is to try solve it. There are a few ways to solve Eq.(12), but the one method which will prove most useful to us is one that you've encountered before: **separation of variables**. In separation of variables, we begin with the dynamical equation

$$\frac{\partial^2}{\partial t^2}y(x,t) = v^2\frac{\partial^2}{\partial x^2}y(x,t), \tag{14}$$

where for notational simplicity we incorporated the parameter v defined as

$$v = \sqrt{\frac{T}{\mu}}. \tag{15}$$

We note that in Eq.(12) the partial derivatives defined with respect to different variables are on opposite sides of the equal sign. From the form of Eq.(14), we can then postulate that the solutions should take the form

$$y(x,t) = f(t)g(x), \tag{16}$$

where $f(t)$ and $g(x)$ are exclusively functions of t and x, respectively. This is the essential step in the separation of variables procedure. You might think searching for solutions of the form Eq.(16) artificially constrains the possible solutions we can find, and this is true[2]. But it turns out that continuing with this guess and eventually employing superposition, we will be able to fully construct the general solution to Eq.(14).

Inserting Eq.(16) into Eq.(14), we find

$$g(x)\frac{d^2f(t)}{dt^2} = v^2 f(t)\frac{d^2g(x)}{dx^2} \tag{17}$$

In order to complete the step of separating variables, we divide both sides by Eq.(16), to obtain

$$\frac{1}{f(t)}\frac{d^2f(t)}{dt^2} = v^2\frac{1}{g(x)}\frac{d^2g(x)}{dx^2}. \tag{18}$$

Now because f is exclusively a function of t and g is exclusively a function of x, Eq.(18) can schematically be written as

$$\text{(some function of } t) = \text{(some function of } x). \tag{19}$$

Equating two functions each of which is defined by a variable not found in the other does not make sense, unless both functions are zero or, more generally, constants. Thus, in order to move from Eq.(18) towards a solution we need to equate both sides of the equality to a constant. Anticipating the desired properties of our solution (namely, we want sinusoidal solutions) we will take this constant to be $-\omega^2$ where ω is yet to be given a physical interpretation. Thus, we have

$$\frac{1}{f(t)}\frac{d^2f(t)}{dt^2} = -\omega^2 = v^2\frac{1}{g(x)}\frac{d^2g(x)}{dx^2} \tag{20}$$

[2] I am cheating a bit in writing Eq.(16) because I know the form the solution to this problem should take and choosing $y(x,t)$ to match that form. This is not a necessary choice for solutions to Eq.(14). In the next lecture/lesson we will posit a different form for $y(x,t)$ consistent with the properties we're considering in a new situation.

or, the system of equations

$$\frac{d^2 f(t)}{dt^2} = -\omega^2 f(t) \tag{21}$$

$$\frac{d^2 g(x)}{dx^2} = -k^2 g(x), \tag{22}$$

where, again for notational simplicity, we introduced a new parameter k defined as

$$k = \frac{\omega}{v}. \tag{23}$$

Given all of the experience we've built in this direction, solving Eq.(21) and Eq.(22) should be quite simple by now. Writing down their solutions, we find

$$f(t) = A\cos(\omega t) + B\sin(\omega t) \quad \text{and} \quad g(x) = C\cos(kx) + D\sin(kx), \tag{24}$$

where A, B, C, and D are arbitrary constants. ByEq.(16), these results imply that the solution $y(x,t)$ is

$$y(x,t) = \Big(A\cos(\omega t) + B\sin(\omega t)\Big)\Big(C\cos(kx) + D\sin(kx)\Big). \tag{25}$$

With Eq.(25), we have apparently managed to solve Eq.(14), but we are not quite done. Unlike the general solutions we previously studied, we cannot easily interpret Eq.(25) because it is defined in terms of parameters whose physical significance is not yet clear. The most we could say is that Eq.(25) represents a wave motion which is oscillating in time and space, but what does ω—seemingly introduced as part of a calculational trick—actually represent? What does k, which is defined in terms of ω, represent? To answer these questions we would need to start from Eq.(25) and introduce an additional constraint which is not specific to all solutions to Eq.(14) but is appropriate to the scenario depicted in Fig. 2.

3.1 Boundary Conditions

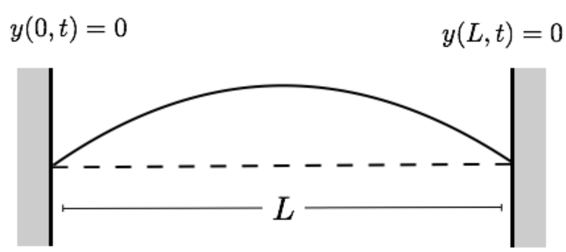


Figure 5: String with fixed end points a distance L apart. The string's motion obeys Eq.(14) with y at $x = 0$ and $x = L$ fixed at zero.

The solution Eq.(25) is actually too general, and in order to make use of it we need to incorporate constraints imposed by our particular system of study. We previously noted that Eq.(14) describes the motion of waves on a string, but in these notes we will further specify that we are looking for solutions to this equation corresponding to a "bounded string" such as that depicted in Fig. 5. Such a depiction is consistent with our original transverse oscillator picture of the string in Fig. 2; we can imagine having a collection of oscillating masses spanning a fixed length L, and then taking $N \to \infty$ by reducing the lattice spacing a and inserting more and more oscillators into our fixed extent. In this scenario, the masses at the point $x = 0$ and $x = L$

will remain essentially fixed at $y = 0$ while the intervening oscillators would have unconstrained values of y.

Therefore, in order for Eq.(25) to be a solution we need to impose the **boundary conditions**

$$y(x = 0, t) = 0 \quad \text{and} \quad y(x = L, t) = 0. \tag{26}$$

Conceptually, boundary conditions are to space what initial conditions are to time. They dictate the value of the function when one of its independent values is fixed, and thereby take us from a general solution to a more specific solution. Imposing the first boundary condition in Eq.(26) on Eq.(25), we find

$$y(x = 0, t) = \Big[A\cos(\omega t) + B\sin(\omega t)\Big]C = 0 \tag{27}$$

which implies $C = 0$. Imposing the second boundary condition, gives us

$$y(x = L, t) = \Big[A\cos(\omega t) + B\sin(\omega t)\Big]D\sin(kL) = 0, \tag{28}$$

implying that that $\sin(kL) = 0$. Since L is fixed by the setup of the problem, this condition determines the value of our previously obscure parameter k. In order for Eq.(28) to be valid we must have

$$kL = n\pi \tag{29}$$

where n can be any positive or negative integer[3]. In order to have positive wavenumbers, we will only allow positive integers. In this sense k, can take on a countably infinite number of values. To denote which k we're talking about it makes sense to attach the subscript n to k and define

$$k_n = \frac{n\pi}{L} \quad \text{where } n = 1, 2, \ldots. \tag{30}$$

By Eq.(23), this solution of k also gives us an infinite number of ω values. With the relation $kv = \omega$ (where v is defined Eq.(15)), we find that the ω_n corresponding to the k_n in Eq.(30) is

$$\omega_n = \frac{n\pi}{L}v \quad \text{where } n = 1, 2, \ldots. \tag{31}$$

The index n defines particular solutions to the wave equation, and each of these solutions are associated with their own coefficients A and B in Eq.(25). Thus incorporating these new index-dependent expressions for ω, k, and their coefficients we have the solution (for a particular n)

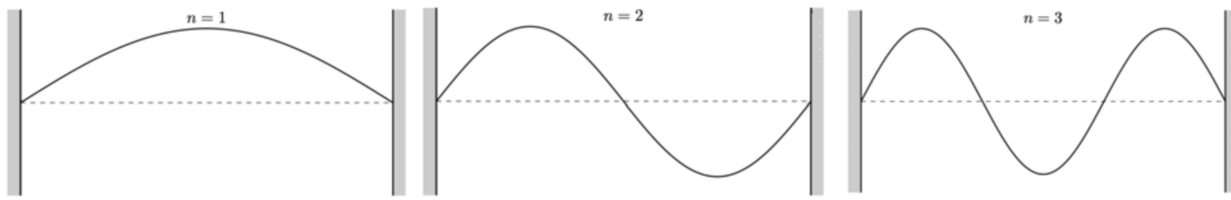
$$y_n(x, t) = \Big[A_n\cos(\omega_n t) + B_n\sin(\omega_n t)\Big]\sin\left(\frac{n\pi}{L}x\right), \tag{32}$$

where we absorbed the coefficient D into a redefinition of A and B. Now, Eq.(32) is just *one* solution to the wave equation satisfying Eq.(26). How would we find the *most general* solution? Linear combinations. If $y_n(x, t)$ is one solution to Eq.(14), then the general solution is

$$\begin{aligned} y(x, t) &= \sum_{n=1}^{\infty} c_n y_n(x, t) \\ &= \sum_{n=1}^{\infty}\Big[\alpha_n\cos(\omega_n t) + \beta_n\sin(\omega_n t)\Big]\sin\left(\frac{n\pi}{L}x\right), \end{aligned} \tag{33}$$

where we defined $\alpha_n = c_n A_n$ and $\beta_n = c_n B_n$. We are at last done. Eq.(33) represents our most general solution to Eq.(14) given the boundary conditions Eq.(26). We note that this solution is specific to these

[3] We cannot allow n to be 0 because this would lead to the trivial $y(x, t) = 0$ result.



(a) First harmonic (b) Second harmonic (c) Third harmonic

Figure 6: Plot of harmonics of Eq.(33). The ends of the string are at $x = 0$ and $x = L$. We note that higher harmonics cross the x axis more often than lower harmonics and have shorter wavelengths (defined in the next section)

boundary conditions and would not be valid if either end of the string had a different condition.

As a comment about nomenclature, the functions of position in Eq.(33) are called **harmonics** as denoted by the integer n which defines them. For example, the $n = 1$ solution is termed the first harmonic; the $n = 2$ solution the second harmonic and so on. The first three harmonics are shown in Fig. 6.

3.1.1 Wavenumber, Dispersion Relation, and Wavelength

With Eq.(33), we are now in a position to interpret the parameters ω and k we previously introduced. By dimensional analysis and ω's appearance in the general solution, we see that ω_n defined in Eq.(31) is the angular frequency of our motion. The quantity k_n defined in Eq.(30) is termed the **wave number**. The relationship between the wavenumber and the angular frequency is called the **dispersion relation**, and for the classical one-dimensional wave the dispersion relation is given by Eq.(23) or by

$$kv = \omega. \tag{34}$$

The interpretation of the wavenumber most naturally follows from defining a new quantity called the **wavelength** which is to space what the period is to time. We know that the angular frequency is related to the period T of an oscillation through

$$T = \frac{2\pi}{\omega} \tag{35}$$

The period gives the amount of time it takes the oscillating-in-time behavior represented by a solution in Eq.(33) to repeat. Similarly, we define the wavelength as

$$\lambda = \frac{2\pi}{k}, \tag{36}$$

The wavelength gives the distance one would have to translate a solution in Eq.(33) in order to find the same solution. For the bounded string, Eq.(30) tells us the wavelength has the value $\lambda_n = 2L/n$ for positive n which means higher values of n (and thus, by Eq.(31), higher frequencies) correspond to shorter wavelengths of motion. This is exemplified by the harmonics in Fig. 6.

With the general solution Eq.(33) obtained, we have essentially completed the analysis of the coupled oscillator in the $N \to \infty$ limit. But for completeness, we now need to discuss how to apply this solution to specific situations. Doing so would require us to develop a general algorithm for computing the α_n and β_n coefficients in Eq.(33) and will lead us to develop a mathematical framework which is widely employed throughout science and engineering.

3.2 Fourier Series and Specific Solutions

Now that we have our general solution Eq.(33), our next task is to determine a way to find specific solutions. The process is somewhat less straight forward than in the simpler cases of oscillators, so we will work through it in depth.

Say we are given the functions $y(x,0)$ and $\dot{y}(x,0)$ defining the transverse displacement and the time-derivative of the transverse displacement at $t=0$. Then, by Eq.(33), we have

$$y(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi}{L}x\right) \tag{37}$$

$$\dot{y}(x,0) = \sum_{n=1}^{\infty} \beta_n \omega_n \sin\left(\frac{n\pi}{L}x\right), \tag{38}$$

In order to find the specific solution, we need to determine α_n and β_n from Eq.(37) and Eq.(38). But we cannot solve for these constants algebraically because they occur within a summation. Instead, in order to isolate these coefficients we make use of an integration identity. For sinusoidal functions, we have the formula

$$\frac{2}{L}\int_0^L dx\, \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right) = \begin{cases} 1 & \text{if } n=m \\ 0 & \text{if } n\neq m \end{cases}. \tag{39}$$

By defining the quantity δ_{nm} (termed the **Kronecker delta**) as

$$\delta_{nm} = \begin{cases} 1 & \text{if } n=m \\ 0 & \text{if } n\neq m \end{cases}, \tag{40}$$

we can write Eq.(39) more succinctly as

$$\frac{2}{L}\int_0^L dx\, \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right) = \delta_{nm}. \tag{41}$$

One of the important properties of δ_{nm} is that when we sum over n, we pick out terms where $n=m$. For example, if we have a set of constants F_n then Eq.(40) implies

$$\sum_{n=1}^{\infty} F_n\delta_{nm} = \sum_{n=1}^{\infty} F_n \times \begin{cases} 1 & \text{if } n=m \\ 0 & \text{if } n\neq m \end{cases} = F_m. \tag{42}$$

Eq.(42) provides the trick we need to isolate the α_n and β_n coefficients . Starting with Eq.(37), we multiply both sides by $2\sin(m\pi x/L)/L$ and integrate the result from $x=0$ to $x=L$. We then find

$$\begin{aligned} \frac{2}{L}\int_0^L dx\, y(x,0)\sin\left(\frac{m\pi}{L}x\right) &= \int_0^L dx\, \sum_{n=1}^{\infty}\alpha_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right) \\ &= \frac{2}{L}\sum_{=1}^{\infty}\alpha_n \int_0^L dx\, \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right), \end{aligned} \tag{43}$$

where in the second line we moved the integration into the summation in order to integrate the x dependent quantities. Next, using Eq.(41), we have

$$\frac{2}{L}\int_0^L dx\, y(x,0)\sin\left(\frac{m\pi}{L}x\right) = \sum_{n=1}^{\infty}\alpha_n\delta_{nm}, \tag{44}$$

which by Eq.(42) implies

$$\alpha_m = \frac{2}{L}\int_0^L dx\, y(x,0)\sin\left(\frac{m\pi}{L}x\right). \tag{45}$$

By a very similar calculation we can show that β_m is given by

$$\beta_m = \frac{2}{\omega_m L}\int_0^L dx\, \dot{y}(x,0)\sin\left(\frac{m\pi}{L}x\right). \tag{46}$$

Fourier Series: Both Eq.(37) and Eq.(38) are formally known as **Fourier series**. The basic idea behind Fourier series is that any function which is defined on a fixed domain can be expressed as a linear combination of sines and cosines. When we find the coefficients defining this linear combination, as in Eq.(45), we are said to have found the "Fourier decomposition" of the bounded function. In Eq.(37) we only have sine functions due to our $y(0,t) = y(L,t) = 0$ boundary conditions, but for more general boundary conditions we would have to include cosine functions as well.

Since a periodic function is simply a function defined within a fixed domain which is repeated many times across the real line (sine and cosine functions defined within $0 \leq x \leq 2\pi$ are classic examples of this), we can use Fourier series to generally represent any periodic function.

$$\text{Any periodic function} = \text{Linear combination of sines and cosines} \tag{47}$$

Indeed, in most applications an investigator receives a time-dependent periodic signal which is then decomposed into sines and cosines in order to learn something about the frequency properties of the signal. If you stay in science or engineering you will become intimately familiar with the techniques used to accomplish this.

With Eq.(45) and Eq.(46) we now have the apparatus to determine a specific solution of Eq.(33). To show how this is achieved in practice, we work through a simple example. Say, we have a string which is initially at rest (i.e., $\dot{y}(x,0) = 0$) and in the shape given by the function

$$y(x,0) = \frac{1}{2}\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right) + \frac{1}{4}\sin\left(\frac{3\pi x}{L}\right). \tag{48}$$

By a trigonometric identity we can write this function as

$$y(x,0) = \frac{1}{4}\sin\left(\frac{2\pi x}{L}\right) + \frac{1}{4}\sin\left(\frac{3\pi x}{L}\right) \tag{49}$$

and thus by Eq.(45) and Eq.(40), we find

$$\begin{aligned}\alpha_m &= \frac{2}{L}\int_0^L dx\,\frac{1}{4}\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{m\pi}{L}x\right) + \frac{2}{L}\int_0^L dx\,\frac{1}{4}\sin\left(\frac{3\pi x}{L}\right)\sin\left(\frac{m\pi}{L}x\right)\\ &= \frac{1}{4}\delta_{2m} + \frac{1}{4}\delta_{3m}.\end{aligned} \tag{50}$$

Then with $\beta_n = 0$ (from the fact that $\dot{y}(x,0) = 0$), we find that Eq.(33) states that $y(x,t)$ is

$$y(x,t) = \frac{1}{4}\left[\cos(\omega_2 t)\sin\left(\frac{2\pi x}{L}\right) + \cos(\omega_3 t)\sin\left(\frac{3\pi x}{L}\right)\right]. \tag{51}$$

4 Energy of a Wave

We have successfully explored some dynamical and kinematical aspects of waves on a finite length of string. There are many other questions we can ask about such waves—such as their nonlinear behavior and their properties when we impose other types of boundary conditions—but we will satisfy ourselves for now by determining the energy of such waves.

For waves on a finite domain, the total mechanical energy is given by

$$E_{\text{tot}} = \int_0^L dx\, \left[\frac{\mu}{2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2\right]. \tag{52}$$

We can derive this result by beginning with a discrete (i.e., pre-continuum limit) expression for the energy of N coupled oscillators and then taking $N \to \infty$ while making the standard replacements of the continuum limit. We seek to evaluate Eq.(52) in terms of the α_n and β_n coefficients of Eq.(33). Since the finite string is not a dissipative system, the total energy is conserved and is thus the same for all times. So to compute the energy, we evaluate Eq.(52) at the convenient time $t = 0$. For $t = 0$, we find

$$\frac{\partial y(x,0)}{\partial t} = \sum_{n=1}^{\infty} \beta_n \omega_n \sin\left(\frac{n\pi}{L}x\right), \tag{53}$$

evaluating the first term in Eq.(52), we find

$$\begin{aligned}\int_0^L dx\, \frac{\mu}{2}\left(\frac{\partial y}{\partial t}\right)^2 &= \frac{\mu}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \beta_n\omega_n\beta_m\omega_m \int_0^L dx\, \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\\ &= \frac{\mu}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \beta_n\omega_n\beta_m\omega_m \frac{L}{2}\delta_{nm}\\ &= \frac{\mu L}{4}\sum_{n=1}^{\infty}\beta_n^2\omega_n^2, \end{aligned} \tag{54}$$

where to get the final line, we used Eq.(40) to sum over the m. By a very similar calculation, we can evaluate the second term in Eq.(52) at $t = 0$. We ultimately find that the total energy is given by

$$E_{\text{tot}} = \frac{L}{2}\sum_{n=1}^{\infty}\left[\frac{\mu}{2}\beta_n^2\omega_n^2 + \frac{T}{2}\alpha_n^2 k_n^2\right] = \frac{L}{2}\sum_{n=1}^{\infty}\frac{\mu}{2}\left(\beta_n^2 + \alpha_n^2\right)\omega_n^2, \tag{55}$$

where we used $T = v^2\mu$ (from Eq.(15)) and $kv = \omega$ in the last equality. Thus, we see that the mechanical energy of a wave can be a reduced to a sum over the frequencies squared weighted by the corresponding Fourier coefficients.

References

[1] D. Morin, *Introduction to classical mechanics: with problems and solutions*. Cambridge University Press, 2008.

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