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2. Simple Harmonic Motion

A particle undergoing simple harmonic motion has the position

$$x(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t).$$

This position can also be written as $x(t) = E \cos(\omega_0 t - \phi)$ where $E > 0$. What are E and ϕ in terms of A and B ?

Solution: Using the sum of angles formula we find

$$\begin{aligned} x(t) &= E \cos(\omega_0 t - \phi) \\ &= E \cos \phi \cos(\omega_0 t) + E \sin \phi \sin(\omega_0 t). \end{aligned} \tag{1}$$

Equating this result to the given position, we find

$$E \cos \phi \cos(\omega_0 t) + E \sin \phi \sin(\omega_0 t) = A \cos(\omega_0 t) + B \sin(\omega_0 t), \tag{2}$$

which implies that $E \cos \phi = A$ and $E \sin \phi = B$. Using trigonometric identities and the condition $E > 0$, we then find

$$E = \sqrt{A^2 + B^2}, \quad \phi = \tan^{-1} \frac{B}{A}. \tag{3}$$

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3. **Effective spring constant**

Two springs with spring constants k_1 and k_2 are connected in parallel as shown in Fig. 1. What is the

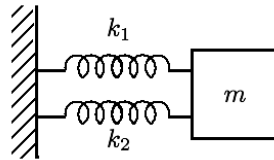


Figure 1

effective spring constant k_{eff} ? In other words, if the mass is displaced by x , find the k_{eff} for which the force equals $F = -k_{\text{eff}}x$.

Solution: If the mass is displaced a distance x from its equilibrium position, the spring with spring constant k_1 exerts a restoring force $-k_1x$. Similarly, the spring with spring constant k_2 exerts a restoring force $-k_2x$ on the mass. Thus the net force on the mass is

$$F_{\text{net}} = -k_1x - k_2x = -(k_1 + k_2)x. \quad (4)$$

Therefore the effective spring constant is $k_{\text{eff}} = k_1 + k_2$.

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4. Superposition

Let $x_1(t)$ and $x_2(t)$ be solutions to the differential equation

$$\ddot{x}(t) = b x(t).$$

where b is a constant. Is the linear combination $x(t) = c_1 x_1(t) + c_2 x_2(t)$, where c_1 and c_2 are constants, a solution to the differential equation? (Provide a calculation showing why or why not).

Solution: If x_1 and x_2 solve the differential equation $\ddot{x}(t) = b x(t)$, then we have

$$\ddot{x}_1^2(t) = b x_1(t) \quad \ddot{x}_2^2(t) = b x_2(t). \quad (5)$$

For $x(t) = c_1 x_1(t) + c_2 x_2(t)$ to be a solution to the differential equation, it too must satisfy the equations above. Checking whether it does, we have

$$\begin{aligned} \left[\frac{d^2}{dt^2} (c_1 x_1(t) + c_2 x_2(t)) \right]^2 &\stackrel{?}{=} b [c_1 x_1(t) + c_2 x_2(t)] \\ (c_1 \ddot{x}_1 + c_2 \ddot{x}_2)^2 &\stackrel{?}{=} b c_1 x_1 + b c_2 x_2 \\ c_1^2 \ddot{x}_1^2 + 2c_1 c_2 \ddot{x}_1 \ddot{x}_2 + c_2^2 \ddot{x}_2^2 &\stackrel{?}{=} b c_1 x_1 + b c_2 x_2 \\ c_1^2 b x_1 + 2c_1 c_2 \ddot{x}_1 \ddot{x}_2 + c_2^2 b x_2 &\neq b c_1 x_1 + b c_2 x_2 \end{aligned} \quad (6)$$

where in the last line we used Eq.(5). Thus the linear combination $c_1 x_1(t) + c_2 x_2(t)$ does not satisfy the differential equation and is not a solution to it.

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5. **An Anti-Hookean oscillator**

A particle of mass m and at a position x is subject to a force

$$F(x) = +kx,$$

with $k > 0$.

- (a) What is the most general form of $x(t)$ in terms of the parameters of the system?
- (b) The particle begins at x_0 and after long times (i.e., $t \rightarrow \infty$), the particle is at $x = 0$. What is the particle's initial velocity?

Solution: The equation of motion of this system is $m\ddot{x} - kx = 0$, or

$$\ddot{x} - \frac{k}{m}x = 0. \tag{7}$$

Finding the general solution by guessing a general exponential and fixing its parameters, we find

$$x(t) = Ae^{t\sqrt{k/m}} + Be^{-t\sqrt{k/m}}. \tag{8}$$

If the particle begins at x_0 , we have $A + B = x_0$. If the particle is at $x = 0$ when $t \rightarrow \infty$, then we have $A = 0$. Thus, the specific solution is

$$x(t) = x_0e^{-t\sqrt{k/m}}, \tag{9}$$

and the particle's initial velocity is $v_0 = -x_0\sqrt{k/m}$.

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6. Exponential force

A particle of mass m is attached to a spring of spring constant k and is subject to another external force $F(t) = F_0 e^{-bt}$. The equation of motion of the system is

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} e^{-bt}, \quad (10)$$

where $\omega_0^2 = k/m$. Guess a solution of the form $x(t) = Ae^{\alpha t}$ and determine what A and α should be in order for this solution to satisfy the above differential equation.

Solution: Guessing a solution of the form $x(t) = Ae^{\alpha t}$ and plugging it into the equation of motion we find

$$(\alpha^2 + \omega_0^2)Ae^{\alpha t} = \frac{F_0}{m} e^{-bt}. \quad (11)$$

In order for this equation to be true for all time, we need to take $\alpha = -b$ and $A = F_0/[m(\alpha^2 + \omega_0^2)] = F_0/[m(b^2 + \omega_0^2)]$. We therefore find the particular solution to the given differential equation is

$$x(t) = \frac{F_0/m}{\omega_0^2 + b^2} e^{-bt}. \quad (12)$$

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7. Matrix algebra

We have the following system of two equations of motion

$$\begin{aligned}m\ddot{x}_1 &= -k_L x_1 + k_M(x_2 - x_1) \\ m\ddot{x}_2 &= -k_R x_2 - k_M(x_2 - x_1),\end{aligned}$$

where x_1 and x_2 are position variables and k_L , k_M , and k_R are spring constants. Say we want to write the two equations as a matrix equation where a 2×2 matrix multiplies a 2×1 matrix to yield a 2×1 matrix:

$$m \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

What should the values of the question marks be?

Solution: The equations of motion for x_1 and x_2 are

$$\begin{aligned}m\ddot{x}_1 &= -(k_L + k_M)x_1 + k_M x_2 \\ m\ddot{x}_2 &= -(k_R + k_M)x_2 + k_M x_1,\end{aligned}$$

Writing the above equation of motion as a matrix, we have

$$m \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -k_L - k_M & k_M \\ k_M & -k_R - k_M \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

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8. **Second derivatives**

Say we have the function

$$f(x) = x^2 + \sin(2x).$$

Compute the value of

$$\lim_{a \rightarrow 0} \frac{f(x+a) - 2f(x) + f(x-a)}{a^2}.$$

Solution: The given limit is the definition of a second derivative. Thus for the provided function we have

$$\lim_{a \rightarrow 0} \frac{f(x+a) - 2f(x) + f(x-a)}{a^2} = \frac{d^2 f}{dx^2} = 2 - 4 \sin(2x). \quad (13)$$

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9. Introduction to Fourier Series

Using the identity

$$\sin(\alpha)\sin(\beta) = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)],$$

compute the integral

$$\int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx$$

where n and m are both integers. Consider two cases: (i) $n = m$ and (ii) $n \neq m$

Solution: Using the given identity, we find

$$\int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx = \frac{1}{2} \int_0^L \left[\cos\left(\frac{(n-m)\pi}{L}x\right) - \cos\left(\frac{(n+m)\pi}{L}x\right) \right] dx \quad (14)$$

when $n = m$, we have

$$\begin{aligned} \int_0^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx &= \frac{1}{2} \int_0^L \left[1 - \cos\left(\frac{2m\pi}{L}x\right) \right] dx \\ &= \frac{L}{2} + \frac{L}{2m\pi} \sin\left(\frac{2m\pi}{L}x\right) \Big|_0^L = \frac{L}{2}. \end{aligned} \quad (15)$$

For $n \neq m$, we have

$$\begin{aligned} \int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx &= \frac{1}{2} \int_0^L \left[\cos\left(\frac{(n-m)\pi}{L}x\right) - \cos\left(\frac{(n+m)\pi}{L}x\right) \right] dx \\ &= \frac{1}{2} \left[\frac{L}{(n-m)\pi} \sin\left(\frac{(n-m)\pi}{L}x\right) - \frac{L}{(n+m)\pi} \sin\left(\frac{(n+m)\pi}{L}x\right) \right]_0^L \\ &= 0. \end{aligned} \quad (16)$$

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10. Solutions to Wave Equation

We have the wave equation

$$\frac{\partial^2 y(x, t)}{\partial t^2} = v^2 \frac{\partial^2 y(x, t)}{\partial x^2},$$

where v is a parameter with units of velocity, and x and t are independent position and time variables, respectively. Which of the following functions could be solutions to this wave equation.

- (a) $y(x, t) = h(x)g(vt)$
- (b) $y(x, t) = g(x - vt)$
- (c) $y(x, t) = h(x) + g(vt)$
- (d) $y(x, t) = f(x + vt)$

Solution: (a), (b), (c), and (d) are all possible solutions to the wave equation. We can see this by guessing trial solutions.

For (a), $h(x) = Ae^{ikx}$ and $g(vt) = Be^{ikvt}$ yield $y(x, t) = Ce^{ik(x+vt)}$ (where $C = AB$) which is indeed a valid solution to the wave equation.

For (b), plugging in $g(x - vt)$ into the wave equation yields

$$\begin{aligned}\frac{\partial^2}{\partial t^2} y(x, t) &= v^2 \frac{\partial^2}{\partial x^2} y(x, t) \\ \frac{\partial^2}{\partial t^2} g(x - vt) &= v^2 \frac{\partial^2}{\partial x^2} g(x - vt) \\ (-v)^2 g(x - vt) &= v^2 g(x - vt) \\ g(x - vt) &= g(x - vt).\end{aligned}\tag{17}$$

Thus $g(x - vt)$ is a solution.

For (c), we find $y(x, t) = h(x) + g(vt)$ is a solution if $h(x)$ and $g(vt)$ satisfy

$$h(x) = h_0 + xh_1 + \frac{1}{2}x^2\delta\tag{18}$$

$$g(vt) = g_0 + vtg_1 + \frac{1}{2}v^2t^2\delta,\tag{19}$$

for arbitrary coefficients h_0, g_0, h_1 , and g_1 and a constant δ all of which can be set with initial/boundary conditions.

For (d), we find $y(x, t) = f(x + vt)$ is a solution by a similar calculation to that in (b).

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11. **Electromagnetic Waves**

The electric field in an electromagnetic wave is given by

$$E_y(z, t) = E_0 \sin(kz - \omega t).$$

This electric field is related to the magnetic field, through the Maxwell equation

$$\frac{\partial B_x(z, t)}{\partial z} = \frac{1}{c^2} \frac{\partial E_y(z, t)}{\partial t},$$

where c is the speed of light. Assuming constants of integration are irrelevant, what is $B_x(z, t)$?

Solution: Differentiating the electric field with respect to time we have

$$\frac{\partial E_y(z, t)}{\partial t} = -\omega E_0 \cos(kz - \omega t). \quad (20)$$

Integrating, this result (and ignoring the constants of integration) with respect z , gives us

$$c^2 B_x(z, t) = \int dz \frac{\partial E_y(z, t)}{\partial t} = -\omega E_0 \int dz \cos(kz - \omega t) = -\frac{\omega}{k} E_0 \sin(kz - \omega t). \quad (21)$$

Thus

$$B_x(z, t) = -\frac{\omega}{kc^2} E_0 \sin(kz - \omega t) = -\frac{1}{c} E_0 \sin(kz - \omega t), \quad (22)$$

where we used $kc = \omega$.

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12. **The Predators and the Prey**

The following system of differential equations provides a very simple model for the number of rabbits R and the number of foxes F in a population:

$$\begin{aligned}\frac{dR}{dt} &= \alpha R(t) - \beta R(t)F(t) \\ \frac{dF}{dt} &= \delta R(t)F(t) - \gamma F(t),\end{aligned}$$

where $\alpha, \beta, \gamma,$ and δ are constants. At what values of R and F do the number of rabbits and the number of foxes remain constant? (Find all such possible values)

Solution: For the number of rabbits and number of foxes to remain constant, we need $\dot{R}(t) = 0$ and $\dot{F}(t) = 0$. Thus

$$\begin{aligned}0 &= \alpha R(t) - \beta R(t)F(t) = R(t)(\alpha - \beta F(t)) \\ 0 &= \delta R(t)F(t) - \gamma F(t) = F(t)(\delta R(t) - \gamma),.\end{aligned}$$

A trivial, time-independent solution is $R(t) = 0$ and $F(t) = 0$. The non-trivial solutions yielding constant R and F are

$$F(t) = \frac{\alpha}{\beta}, \quad R(t) = \frac{\gamma}{\delta}. \quad (23)$$

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