Algebra Before Numbers

When you encounter a physics problems which defines physical quantities in terms of their numerical values there are two approaches to obtaining a solution: you can substitute the numbers into the problem immediately and manipulate the numerical quantities to find the final result; or you can solve the problem analytically *first* (that, is solve it using the symbolic representation of the physical quantities) and substitute in the numerical quantities only after you have obtained the final result.

It is almost always[1] preferable to do the latter, namely it's better to perform the algebra before you substitute in numbers. Even if there are no variables given in the problem, it is generally better to define numerical quantities in terms of variables and then work with the variables in lieu of the numbers. There are three main reasons for adopting this "algebra before numbers" strategy.

- **Cleaner Solution:** Without numbers cluttering your work, you typically end up with a cleaner written solution which is easier to subsequently analyze.
- **Can Check Your Result:** With an analytic formula as your end result, you can apply intuitive checks (like dimensional considerations and limiting cases) to check your solution.
- **Fewer Numerical mistakes:** By only substituting numbers at the end of the calculation, you avoid the potential numerical mistakes arising from the repeated substitution and manipulation of numerical quantities.

We can more precisely illustrate these advantages with an example problem

1 Example: Momentum Transfer

We will construct the solution to the following problems by first going through the "numbers first" strategy, and then going through the "algebra first" first strategy. We will then compare both strategies to determine which informs us more about the nature of our solution.

> Tanvi (who is 35 kg) is sitting in a stationary 10 kg wagon on level ground. The wagon can roll free without friction. Tanvi wants to propel the wagon forward without touching the ground. Conveniently she is carrying one 10 kg stone in the wagon.
>
> She throws the stone horizontally off the back of the wagon, and the stone flies out of her hand with a speed of 8 m/s relative to Tanvi's velocity *after* the stone is released. How fast is the wagon moving forward afterwards?

1.1 Numbers Before Algebra

Throughout this calculation we will prioritize immediate substitution of numbers.

This is a momentum conservation problem. The system is isolated so the total momentum must always be zero. This is clear before Tanvi throws the stones off the wagon, but it must also be true after she throws them.

[1] I say "almost always" because considering numerical quantities first may give you a sense of what type of analytical calculation (an approximation, for example) you need to do. Also if a quantity is numerically zero, you can usually substitute in this value at the beginning.

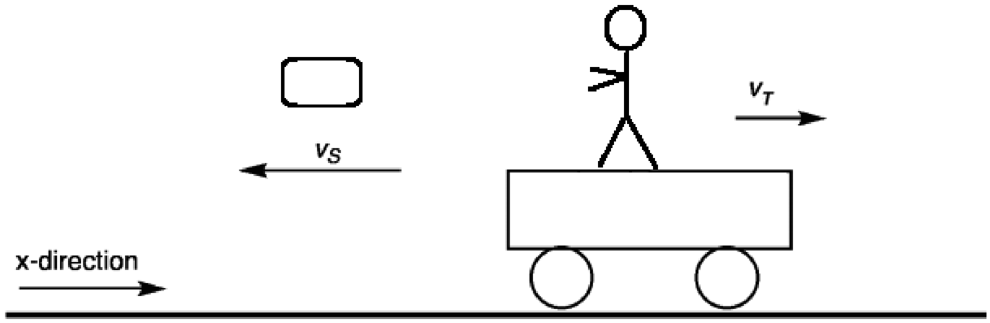

Figure 1

We define Tanvi's mass to be $M_T = 35$ kg, the wagon's mass to be $M_W = 35$ kg, and the stone's mass to be $M_S = 10$ kg. We also define the final velocity of Tanvi (and the wagon) relative to the ground to be V_{T-G}; the final velocity of the stone relative to Tanvi (and the wagon) to be $V_{S-T} = -8$ m/s; the final velocity of the stone relative to the ground to be V_{S-G}.

Our goal is to find V_{T-G}. By conservation of momentum we have

$$\begin{aligned} 0 &= (M_T + M_W)V_{T-G} + M_S V_{S-G} \\ &= (45\text{ kg})V_{T-G} + (10\text{ kg})V_{S-G}. \end{aligned} \tag{1}$$

By the definition of relative velocity, we know that the velocity of the stone relative to the ground is

$$\begin{aligned} V_{S-G} &= V_{S-T} + V_{T-G} \\ &= -8\text{ m/s} + V_{T-G}. \end{aligned} \tag{2}$$

With Eq.(1) and Eq.(2) we have a system of two independent equations in two variables, and so we should be able to solve for all variables. Inserting Eq.(1) into Eq.(1), we find

$$0 = (45\text{ kg})V_{T-G} + (10\text{ kg})(-8\text{ m/s} + V_{T-G}), \tag{3}$$

and upon solving for V_{T-G} we obtain

$$V_{T-G} = 1.45\text{ m/s}, \tag{4}$$

which is the desired result. But how do we know if this answer is correct. We'd like to believe we performed all of the relevant steps correctly and did not make a mistake in logic, but how can we be more sure? This is the main disadvantage with substituting in numbers before working through the algebra. You end up with a result which is conceptually impenetrable.

1.2 Algebra Before Numbers

We will now go through the previous calculation again, but we'll delay substituting in numbers until the very end.

Our goal is to find V_{T-G} and we start off with the momentum conservation equation

$$0 = (M_T + M_W)V_{T-G} + M_S V_{S-G}. \tag{5}$$

By the definition of relative velocity, we know

$$V_{S-G} = V_{S-T} + V_{T-G}. \tag{6}$$

Substituting Eq.(6) into Eq.(5) yields

$$\begin{aligned} 0 &= (M_T + M_W)V_{T-G} + M_S(V_{S-T} + V_{T-G}) \\ &= (M_T + M_W + M_S)V_{T-G} + M_S V_{S-T}, \end{aligned} \tag{7}$$

which upon solving for V_{T-G} gives

$$V_{T-G} = -\frac{M_S}{M_T + M_W + M_S} V_{S-T}. \tag{8}$$

If we were to substitute in the relevant numerical quantities now, we would indeed confirm Eq.(4), but let's pause to consider how this analytical result informs our understanding of this problem.

First we note that the minus sign in Eq.(8) suggests that the velocity of Tanvi is always in the opposite direction of the velocity of the stone. This makes sense; since Tanvi is throwing the stone in one direction, momentum conservation would clearly force her to move in the opposite direction.

We also note that if we were to set M_S to 0 kg, then Tanvi's velocity would become zero. Intuitively this is tantamount to saying that Tanvi doesn't throw anything off the wagon, and hence there is no momentum transfer to propel her forward.

Similarly, if we made $M_W \gg M_S$ (say $M_W = 1000$ kg with $M_S = 10$ kg) then we will again find that Tanvi's final velocity is essentially zero. This result should be intuitively clear because the heavier the wagon is, the more difficult it would be to propel it by throwing much lighter stones.

Finally, we note that the dimensions of the left-hand side of Eq.(8) are length/time (i.e., the units of velocity), and the units of the right-hand side are mass/mass $\times$ length/time = length/time.

Thus both in terms of limiting cases and dimensional analysis Eq.(8) is a reasonable answer. It was impossible to judge this reasonableness with the numerical result Eq.(4) alone. This is why it is generally better to work through the *algebra before* inserting in the relevant *numerical quantities*.

Resource: Introduction to Oscillations and Waves
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