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2. **Correct The Mistake**
   Determine whether these equations are false, and if so write the correct answer.

   (a) $\ln\left(\frac{x}{2}\right) + \ln\left(\frac{x}{2}\right) = \ln(x)$

   (b) $e^x e^y = e^{xy}$

   (c) $\frac{d}{dx}\cos(4x^2) = \sin(4x^2)$

   (d) $\int_0^\infty dx\, xe^{-x^2} = \infty$

   Solution:

   (a) This is an incorrect application of the rule for adding logarithms. When we add two logarithms with the same base, we obtain a new logarithm whose argument is a product of the arguments of the original two logarithms. The correct calculation is

$$\ln\left(\frac{x}{2}\right) + \ln\left(\frac{x}{2}\right) = \ln\left(\frac{x^2}{2}\right). \tag{1}$$

   (b) This is an incorrect application of the rule for multiplying exponentials. When we multiply two logarithms, we obtain a new exponential whose argument is a sum of the arguments of the original two exponentials. The correct calculation is

$$e^x e^y = e^{x+y}. \tag{2}$$

   (c) This calculation has the incorrect derivative for $\cos(u)$ and does not apply the chain rule. The correct calculation is

$$\frac{d}{dx}\cos(4x^2) = -8x\sin(4x^2). \tag{3}$$

   (d) A definite integral over an infinite domain is not necessarily infinite. Instead, we need to use u-substitution to evaluate this integral. Given that $\frac{d}{dx}e^{-x^2}/2 = -xe^{-x^2}$, we have

$$\int_0^\infty dx\, xe^{-x^2} = -\frac{e^{-x^2}}{2}\bigg|_0^\infty = \frac{1}{2}. \tag{4}$$

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3. **Probability**
We have a large number of atoms in a system. Each atom decays at a rate λ such that after a time t, there is the probability λt for the atom to have decayed. The probability that n atoms have decayed in time t is
$$p_n = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \tag{5}$$
where $n! = n(n-1)\cdots 2 \cdot 1$. What is $\langle n \rangle$, the average number of atoms that have decayed in a time t? *Hint: The Taylor series of e^x is $e^x = \sum_{n=0}^{\infty} x^n/n!$.*

Solution:

By the definition of average, we find that the average number of atoms that have decayed in a time t is

$$\begin{aligned}
\langle n \rangle &= \sum_{n=0}^{\infty} n p_n \\
&= \sum_{n=0}^{\infty} n \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\
&= \lambda t \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t} \\
&= \lambda t e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}.
\end{aligned} \tag{6}$$

In the third equality, we factored λt from the expression, used $n/(n!) = 1/(n-1)!$, and began our summation from $n = 1$ because the $n = 0$ term vanishes. Using the Taylor series for the exponential, we then find
$$\langle n \rangle = \lambda t e^{-\lambda t} e^{\lambda t} = \lambda t. \tag{7}$$

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4. **Probability, II**
   We have a random number generator in which each number from 0 to 999 is equally likely to occur. What is the probability of getting 547?

   We change the random number generator so that there is a probability of $1/2$ of getting a 9 in the hundreds digit and a probability of $1/2 \times 1/9$ of getting any particular other number (e.g., a probability of $1/18$ of getting a 2) in the hundreds digit. The probability distribution for the tens and ones digits remain unchanged. What is the probability of getting 547?

   Solution:

   If we have a random number generator in which each number from 0 to 999 is equally likely to occur, then the probability of getting 547 is the inverse of the total number of possible numbers. Namely,

$$p_{547} = \frac{1}{1000}. \tag{8}$$

   If we have a random number generator in which the probability of getting a 9 in the hundreds digit is $1/2$ and the probability of getting any other particular number is $1/2 \times 1/9 = 1/18$, (with the probability distribution of getting other numbers unchanged), the probability of obtaining 547 is

$$p_{547} = \frac{1}{18} \times \frac{1}{10} \times \frac{1}{10} = \frac{1}{1800}. \tag{9}$$

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5. **Information**

Our friend performs some random trial which yields a random number. We define "information" needed to specify the number as the average number of binary-valued questions that need to be answered to determine the hidden number. Determine the "information" needed to specify the number given that our friend conducted the following random trials

(a) Selecting a card from a deck of cards with numbers from 1 to 100 (each number occurring once)

(b) Selecting a card from a deck of cards with numbers from 1 to 100 where even numbers are twice as likely as odd numbers.

Solution:

(a) In Lecture Notes 03 "Entropy and Information", we determined that if there is a probability p_i of getting a number i out of a set of numbers $\{i\}$, then the average number of questions we will need to guess the number (under the given conditions of the "Guess that number" game) is

$$\langle \text{\# of Qs} \rangle = -\sum_{\{i\}} p_i \log_2 p_i. \tag{10}$$

If we have a deck of cards numbered 1 to 100, and we select one card, there is a uniform probability of $1/100$ for getting any particular card. Thus the amount of information needed to specify the selected card is

$$\langle \text{\# of Qs} \rangle = \log_2(100) \approx 6.64. \tag{11}$$

(b) If even numbers are twice as likely as odd numbers, then the probability of getting an even number is $2/3$ and the probability of getting an odd number is $1/3$. Since there are 50 even numbers and 50 odd numbers for a deck of cards from 1 to 100, the probability of getting any particular even number is $2/3 \times 1/50 = 1/75$ and the probability of getting any particular odd number is $1/3 \times 1/50 = 1/150$. From these probabilities, we find that the amount of information needed to specify a drawn card from this deck is

$$\begin{aligned}
\langle \text{\# of Qs} \rangle &= -\sum_{\{i\}} p_i \log_2 p_i \\
&= -\sum_{\text{even \#s}} \frac{1}{75} \log_2 \frac{1}{75} - \sum_{\text{odd \#s}} \frac{1}{150} \log_2 \frac{1}{150} \\
&= -\frac{50}{75} \log_2 \frac{1}{75} - \frac{50}{150} \log_2 \frac{1}{150} \\
&= \frac{2}{3} \log_2 75 + \frac{1}{3} \log_2 150 \\
&= \frac{1}{3} \log_2(75^2 \times 150) \approx 6.56.
\end{aligned} \tag{12}$$

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6. **Entropy of a Macrostate**
   We have a spin system with N spins of which $n_\uparrow$ spins are pointing up and $N - n_\uparrow$ spins are pointing down. What is the entropy of the macrostate characterized by a given $n_\uparrow$, given that each associated microstate is equally likely? *Hint: Entropy is k_B times the natural logarithm of the number of microstates for a particular macrostate.*

   Solution:

   For a spin system with N spins of which $n_\uparrow$ spins are pointing up and $N - n_\uparrow$ spins are pointing down, then the number of microstates associated with this macrostate is

$$\Omega = \binom{N}{n_\uparrow}. \tag{13}$$

   Each of the microstates with a common N and $n_\uparrow$ are equally likely, so the entropy of the macrostate (or set of microstates) with N and n_{uparrow} is

$$S = k_B \ln \Omega = k_B \ln \binom{N}{n_\uparrow}. \tag{14}$$

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6. **Maximization with constraint**
   The perimeter of a rectangle with side lengths x and y is given $P = 2x + 2y$. The area of the rectangle is given by $A = xy$. In terms, of P for what value of x is A maximized? Given this x, what is y? *Note: You might be able to guess the answer but I also want you to show it analytically.*

   Solution:

   We will find the maximum area by writing A exclusively as a function of one of the sides of the rectangle and then implementing the local maximization algorithm. First, we note that if $P = 2x + 2y$, then

   $$y = \frac{1}{2}(P - 2x), \tag{15}$$

   where P is the perimeter of the rectangle. The area as an exclusive function of A is then

   $$A = xy = \frac{x}{2}\left(P - 2x\right). \tag{16}$$

   We are now, looking for the value of x, which maximizes A. Namely, the x such that $A'(x) = 0$ but $A''(x) < 0$. First computing $A'(x)$, we have

   $$A'(x) = \frac{P}{2} - 2x. \tag{17}$$

   Setting $A'(x) = 0$, we find that $A(x)$ *might* have a maximum at $x = P/4$ and (by Eq.(15)) $y = P/4$. We affirm the local maximum condition by noting that

   $$A''(x) = -2 < 0, \tag{18}$$

   Therefore, $x = y = P/4$, subject to the constraint $P = 2x + 2y$, indeed maximizes the area $A = xy$.

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7. **Gaussian integral**
   Compute the integral

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_N \, \exp\left(-\sum_{i=1}^{N} \lambda_i x_i^2\right).$$

Write the final result using the product symbol $\prod$.

Solution:

To evaluate the given multi-dimensional integral, we first recall the identity

$$\int_{-\infty}^{\infty} dx \, e^{-\lambda x^2} = \sqrt{\frac{\pi}{\lambda}}. \tag{19}$$

Now, turning the the multi-dimensional integral, we have

$$\begin{aligned} \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_N \, \exp\left(-\sum_{i=1}^{N} \lambda_i x_i^2\right) &= \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_N \, \prod_{i=1}^{N} \exp\left(-\lambda_i x_i^2\right) \\ &= \prod_{i=1}^{N} \int_{-\infty}^{\infty} dx_i \, e^{-\lambda_i x_i^2} \\ &= \prod_{i=1}^{N} \sqrt{\frac{\pi}{\lambda_i}} \end{aligned} \tag{20}$$

or

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_N \, \exp\left(-\sum_{i=1}^{N} \lambda_i x_i^2\right) = \frac{\pi^{N/2}}{\prod_{i=1}^{N} \lambda_i^{1/2}} \tag{21}$$

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8. **Local Minimum and Exponential Functions**
   Assume that the function $e^{-g(x)}$ has a local maximum at $x = x_1$. What can we say about the values of the first derivative of $g(x)$ and the second derivative of $g(x)$ both evaluated at $x = x_1$?

   Solution: The function $e^{-g(x)}$ is exclusively positive and decreases to zero as the argument of the exponential gets larger. Therefore, if $e^{-g(x)}$ has a maximum, then $g(x)$ must be at a minimum. Namely, for $e^{-g(x)}$ having a local maximum at $x = x_1$, we have

$$g'(x = x_1) = 0, \qquad \text{and} \qquad g''(x = x_1) > 0. \tag{22}$$

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Resource: Introduction to Statistical Physics
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