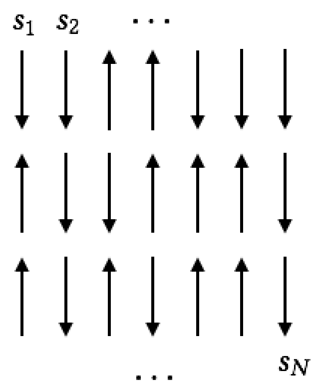
Physics III – Workshop Problems – Introduction to Statistical Physics

Mean-Field Ising Model

Week Summary

In this problem sheet, you will work in your group in order to make a small contribution to a larger problem. We will consider what is known as the **mean-field Ising model**. An **Ising model** is generally any model involving spins (i.e., quantities that can take on a $+1$ or -1 value). A **mean-field** Ising model is one in which each spin interacts with the mean field created by all other spins.

We use this model as our first quantitative example of how temperature can define the **phase** of a system. In particular, we will show how the macrostate of this system (defined by the average-spin) varies as a function of temperature and allows us to define a **disordered** and **ordered** phase for the system.



$$E = -\frac{J}{2N} \sum_{i,j=1}^{N} s_i s_j$$

Figure 1: Particular microstate of mean-field Ising system. The general form of the energy is included to the right.

We consider a system with arbitrary dimension and size. Inside this system there are many magnetic dipoles which we represent schematically as arrows in Fig. 1. In quantum mechanics, the magnetic dipoles of electrons arise from the angular momentum spin of these electrons. Therefore, we simply call these dipoles "spins" and require them to either point up (associated with a $+1$ value) or down (associated with a -1) value[1]. Specifically, we have N of these spins each of which is denoted s_i for the ith spin. Each spin s_i can have the value $+1$ or -1.

[1]From quantum mechanics, electron spin, when measured, can point either up or down relative to some coordinate axis.

1. **Free Energy of Mean-Field Ising Model**
   Our ultimate goal for this problem is to write the free energy of the system as a function of the mean spin $m = \sum_{i=1}^{N} s_i/N$.

   (a) For a collection of spins $\{s_i\}$ in a magnetic field H, where each spin has magnetic dipole moment μ, the energy of the system is

   $$E = -\mu H \sum_{i=1}^{N} s_i. \tag{1}$$

   For the mean-field Ising model, we take the magnetic field H to be generated entirely by the mean of the spins $\{s_i\}$ (hence the name "mean-field"). In particular we define,

   $$H = \frac{H_0}{N} \sum_{j=1}^{N} s_j \qquad \text{[Assumption of mean-field Ising model]}, \tag{2}$$

   where H_0 is some constant with units of magnetic field. Given that we can write Eq.(1) as

   $$E = -\frac{J}{2N} \sum_{i,j=1}^{N} s_i s_j, \tag{3}$$

   (where summations over both indices run from 1 to N) what is J?

   (b) We define the mean spin for this system as

   $$m = \frac{1}{N} \sum_{i=1}^{N} s_i. \tag{4}$$

   Write the energy explicitly as a function of m. We denote this energy now as $E_N(m)$.

   (c) Say our system has $n_\uparrow$ spins and $n_\downarrow$ spins. What is m in terms of $n_\uparrow$ and $n_\downarrow$?

   (d) Write the entropy of this system as a function of m with N included as a constant parameter. Denote this entropy as $S_N(m)$. (*Your final answer will include factorials*).

   (e) What is the free energy $F_N(m, T)$ of this system? Your final result should have m, J, N, T (temperature), and k_B.

2. **Stirling's Approximation of Free Energy**
The goal of this part of the problem is to eliminate the factorials from the expression for the free energy.

As a function of the mean spin

$$m = \frac{1}{N}\sum_{i=1}^{N} s_i, \tag{5}$$

the free energy of the system of our mean-field Ising model can be written as

$$F_N(m,T) = -\frac{JN}{2}m^2 - k_B T \ln \Omega_N(m), \tag{6}$$

where T is temperature and where

$$\ln \Omega_N(m) = \ln N! - \ln\left[\tfrac{N}{2}(1+m)\right]! - \ln\left[\tfrac{N}{2}(1-m)\right]!. \tag{7}$$

(a) Using the further simplified form of Stirling's approximation[2]

$$\ln N! = N \ln N - N, \tag{8}$$

show that Eq.(7) can be written as

$$\ln \Omega_N(m) = N c_0 - \frac{N}{2} g(m) - \frac{N}{2} h(m), \tag{9}$$

where c_0 is an m-independent constant, and $g(m)$ and $h(m)$ are two different functions of m. Determine c_0, $g(m)$, and $h(m)$.

Hint: The best way to solve this problem is to apply Eq.(8) *to Eq.*(7) *and keep simplifying until you get a result of the form Eq.*(9).

[2]Of course, the nature of an approximation is that it is an *approximation* and not an equality, but we use the equality here because it is simplest to do so.

3. **Local Minimum of Mean-Field Ising Model**
The goal of this problem is to find the value of m that defines the local minimum of the free energy $F_N(m,T)$ of our system.

(a) As a function of the mean spin, defined as

$$m = \frac{1}{N}\sum_{i=1}^{N} s_i, \tag{10}$$

the free energy of our system, can be written as

$$F_N(m,T) = \frac{N}{2}\left[-Jm^2 + k_B T\left(\ln(1-m^2) + m\ln\frac{1+m}{1-m}\right)\right]. \tag{11}$$

In terms of the parameters of the free energy, determine the **two conditions** that $\overline{m}$ must satisfy in order to be a local minimum of $F_N(m,T)$.

(b) We can define the **hyperbolic tangent** as

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}. \tag{12}$$

Show that if we have

$$\tanh(x) = y, \tag{13}$$

that

$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right). \tag{14}$$

(c) Use Eq.(14) to write one of the conditions from (a) in terms of the hyperbolic tangent function.

4. **Plot of Free Energy Function and Minimum**
The goal of this problem is to represent graphically, the local minimum of the free energy of the Free energy of our system.

   (a) We can write the free energy of this system in terms of a function $f(m)$ where

$$f(m) = -m^2 + \lambda \left[\ln(1-m^2) + m \ln \frac{1+m}{1-m} \right], \tag{15}$$

   and λ is a tunable parameter. If the local minimum of Eq.(15) occurs at $m = \overline{m}$ then $\overline{m}$ obeys equation

$$\overline{m} = \tanh\left(\frac{\overline{m}}{\lambda}\right), \tag{16}$$

   where $\tanh(x)$ is the hyperbolic tangent defined as

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}. \tag{17}$$

   (b) Plot Eq.(15) for various choices of λ. What value of of λ separates the plots where $f(m)$ has two local minima from those where $f(m)$ has only one local minimum?

   (c) Plot the left and the right hand side of Eq.(16) (on the same plot) for various choices of λ. What value of λ separates the plots where Eq.(16) has three solutions from those where it has only one solution?

   (d) Using the result from (b), plot $\overline{m}$ as a function of λ. Make sure your plot shows all of the solutions of Eq.(16).

Resource: Introduction to Statistical Physics
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