An Ode to ODEs

Differential Equations Series

Instructor's Guide

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Developed by the Teaching and Learning Laboratory at MIT
for the Singapore University of Technology and Design

Introduction

When to Use this Video

- In Soph 301, in recitation, before Lecture 4: Applications to Differential Equations
- Prior knowledge: drawing free body force diagrams, applying Newton's second law in polar coordinates, and how to write a Taylor series expansion of a function about a point.

Key Information

Duration: 18:18
Narrator: Peter Dourmashkin, Ph.D.
Materials Needed:
• Paper
• Pencil

Learning Objectives

After watching this video students will be able to:

- Understand that the physical laws governing a system's properties can be modeled using differential equations.
- Explain that the solution to a differential equation is a family of functions.
- Recognize that specifying initial conditions determines a particular solution function to a differential equation.

Motivation

- Students have a difficult time understanding that solutions to differential equations are functions, and not points, or values of a function at a point.
- Textbooks tend not to fully explain why *n* initial conditions must be specified to determine a solution for a *n*th order differential equation. In this video, we make a concrete argument tied to the Taylor Series, which students are familiar with from Calculus.

Student Experience

It is highly recommended that the video is paused when prompted so that students are able to attempt the activities on their own and then check their solutions against the video.

During the video, students will:

- Describe the important forces acting on a swinging pendulum.
- Discuss whether or not differential equations have unique solutions.
- Predict whether two pendulums swinging from the same initial position will have the same behavior.
- Determine how many initial conditions are required to specify a solution for a 3rd order differential equation.

Video Highlights

This table outlines a collection of activities and important ideas from the video.

Time	Feature	Comments
0:05	Walter Lewin on a pendulum	
1:19	Prerequisites and Learning objectives	
2:04	Chapter 1: Describing Pendulum Motion	The damped pendulum is modeled as an ordinary differential equation using polar coordinates.
2:28	Activity	Brainstorm a list of pendulum properties.
3:59	Simplifying assumptions	Assumptions used to simplify the model are described.
4:48	Activity	Identify relevant forces acting on the pendulum bob and their direction.
11:26	Chapter 2: Solution Functions	Solutions to the model are discussed. Solutions to ODEs are infinite families of functions that depend on initial conditions. Our model requires that we specify two initial conditions.
12:17	Activity	Discussion question: is there a unique solution to our differenitial equation?
13:38	Activity	Determine the difference between two solutions, both with the same initial position, but different initial velocities.
14:50	Taylor Series	Proof of the need for two initial conditions is shown via use of the Taylor Series.
16:50	Activity	How many initial conditions would be required to specify the solution to a 3rd order, ordinary, differential equation?
17:03	To Review	

Video Summary

This video leads students through modeling the regular, non-linear pendulum with a differential equation. We explore why solutions to a differential equation are an infinite family of functions. We show that to determine a specific solution to this second order differential equation, two initial conditions must be specified. Proof of the need for two initial conditions is shown via use of the Taylor Series.

Soph 301 Materials

Pre-Video Materials

When appropriate, this guide is accompanied by additional materials to aid in the delivery of some of the following activities and discussions.

The first three problems are concept questions developed by MIT for 8.01 in order to test student understanding of centripetal motion.

1. Centripetal Acceleration (Appendix A1)

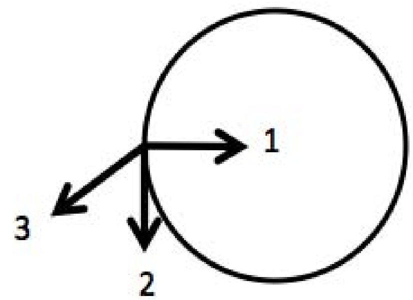
As an object speeds up along a circular path in a counterclockwise direction, shown below, its accleration points:

(a) toward the center of the circular path

(b) in a direction tangent to the circular path

(c) outward

(d) none of the above



2. Centripetal Acceleration (Appendix A2)

An object moves counterclockwise along a circular path shown below. As it moves along this path, its acceleration vector continuously points towards the point S.

(a) The object speeds up at P, Q, and R.

(b) The object slows downa at P, Q, and R.

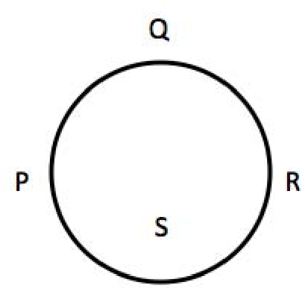
(c) The object speeds up at P and slows down at R.

(d) The object slows down at P and speeds up at R.

(e) The object speeds up at Q.

(f) The object slows down at R.

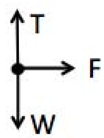
(g) No object can execute such a motion.



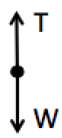
3. Centripetal Acceleration: Pendulum (Appendix A3)

A pendulum bob swings down and is moving fast at the lowest point in its swing. T is the tension in the string, and W is the gravitational force exerted on the pendulum bob. Which free-body diagram best represents the forces exerted on the pendulum bob at the lowest point? The lengths of the arrows represent the relative magnitude of the forces.

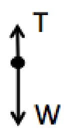
(a) T=W



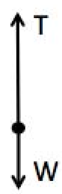
(b) T=W



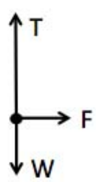
(c) T<W



(d) T>W



(e) T>W



4. Which of the following are true about the Taylor series.

(a) The Taylor Series write a function as an infinite sum, you can replace the function with this sum anytime and anywhere.

(b) The Taylor Seires is an infinite sum that tells you the value of a function at a point based on the value of all of the derivatives of that function at some nearby point.

(c) The Taylor Series is an infinite sum that tells you the value of a function on some interval based on the value of all of the derivatives of that function at some nearby point.

(d) By considering the first 3 terms of a Taylor Series for a function, you obtain a polynomial whose value, first, and second derivatives all agree with the function at one point.

(e) By considering the first 3 terms of a Taylor Series for a function, you obtain a polynomial whose value, first, and second derivatives all agree with the function on some small interval.

Post-Video Materials

1. We know that the motion of a pendulum is constrained such that the velocity is always tangent to the circle of motion. Writing the velocity in polar coordinates, this means that the velocity always points in the tangential direction, with no velocity component pointing in the radial direction. However, we know that the accleration contains both radial and tangential components. Where do they come from? In particular, explain why the unit vector pointing in the tangential direction as a time derivative that points in the radial direction.

2. Find a differential equation that describes the tension in the pendulum. How is this differential equation related the the equation we discussed in the video?

3. In the video, we show that we need 2 pieces of initial data to find a particular solution to a second order ODE. We chose the initial (angular) position, and the initial angular velocity. The following discussion questions are intended to have students think more about what is necessary.

(a) Does it matter what two initial conditions we start with? For example, can we determine a solution from an initial angular velocity and an intitial angular acceleration? Why or Why not?

(b) What could happen if we started with 3 pieces of information? Think about how this is similar or dissimilar to an over-determined linear system.

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Additional Resources

Going Further

Differential equations are the language that describes the governing rules that constrain the physical world around us. As such, a basic understanding of differential equations is important for anyone interested in pursuing basic scientific knowledge.

This video is intended to help students explore the basic principles and properties of Ordinary Differential Equations. Solution methods for differential equations will be explored in Soph 301, and numerical methods for solving differential equations will be explored in Soph 302. This video provides a nice starting point for both courses.

References

The following educational articles describe student difficulties, student misconceptions, and approaches to teaching differential equations. These resources may provide inspiration in creating your own materials.

- Arslan, S. (2010). Do students really understand what a differential equation is? *Int. J. of Math. Ed. in Sci. Technol., 41*(7), 873–888.
- Boyce, W. E. (1994). New directions in elementary differential equations. *The Coll. Math. J.*, *25*(5), 364–371.
- Davis, P. (1994). Asking good questions about differential equations. *The Coll. Math. J., 25*(5), 394–400.
- Hubbard, J. H. (1994). What it means to understand a differential equation. *The Coll. Math. J., 25*(5), 372–384.
- Pennell, S., Avitabile, P., & White, J. (2009). An engineering-oriented approach to the introductory differential equations course. *PRIMUS, 19*(1), 88–99.

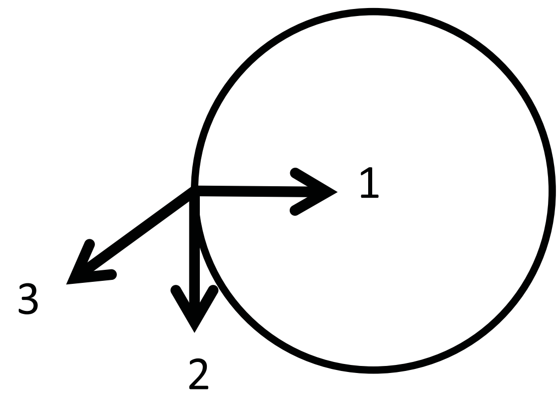
The following article models the simple and non-linear pendulum, and suggests a student activity that allow students to explore the effects of simplifying assumptions on the model, and compares model to experimental data collected by the student.

- Reid, T. S. & King, S. C., (2009). Pendulum motion and differential equations. *PRIMUS, 19*(2), 205–217.

These video references discuss pendulum behavior, the first from a physics perspective, the second from a mathematical perspective.

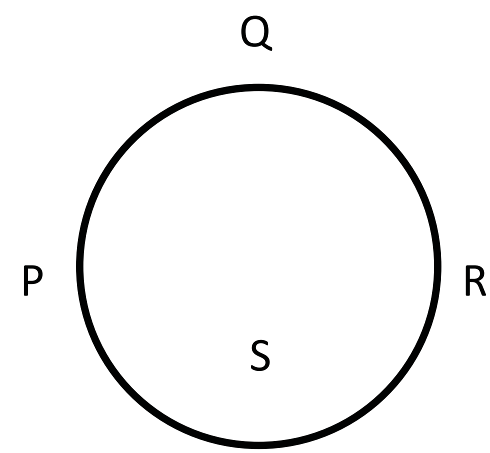
- Lewin, W., *8.01 Classical Mechanics*, Fall 1999. (Massachusetts Institute of Technology: MIT OpenCouseWare), http://ocw.mit.edu (Accessed November 27, 2012). License: Creative Commons BY-NC-SA
  -Lecture 10: Hook's Law and Simple Harmonic Motion
- Mattuck, A., *18.03 Differential Equationss*, Spring 2003. (Massachusetts Institute of Technology: MIT OpenCouseWare), http://ocw.mit.edu (Accessed November 27, 2012). License: Creative Commons BY-NC-SA
  -Lectures 31: Non-linear Autonomous Systems

As an object speeds up along the circular path in a counterclockwise direction, which of the following arrows best represents its acceleration?



1. toward the center of the circle
2. in a direction tangential to the circle
3. outward
4. none of the above

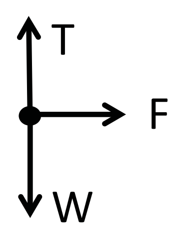
As an object moves counterclockwise along the circular path shown below. As it moves along the path, its acceleration vector continuously points towards point *S.*



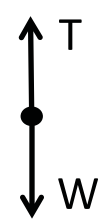
a) The object speeds up at P, Q, and R.
b) The object slows down at P, Q, and R.
c) The object speeds up at P, and slows down at R.
d) The object slows down at P, and speeds up at at R.
e) The object speeds up at Q.
f) The object slows down at Q.
g) No object can execute such a motion.

A pendulum bob swings down and is moving fast at the lowest point in its swing. T is the tension. W is the gravitational force exerted on the pendulum bob. F is a dissipation force. Which free-body diagram below best represents the forces exerted on the pendulum bob at its lowest point? The lengths of the arrows represent the relative magnitudes.

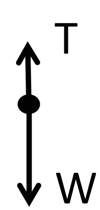
a) T=W



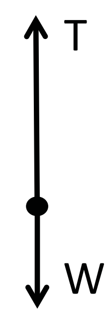
b) T=W



c) T<W



d) T>W



e) T>W