RES.TLL-004 STEM Concept Videos, Fall 2013

Transcript – An Ode to ODEs

Here you see MIT's Walter Lewin, about to swing on a pendulum. We can write a function for how the angle theta 1 depends on time, which completely describes his motion.

But what if Walter were to start swinging from here, instead of there? We'd need a different function, theta 2 of t.

And what if one of his students were to give him a little push? That's right—we'd need a third function, theta 3 of t.

It would be great if we could write a single rule, based on physical principles, that governs the behavior of the pendulum. And every function that describes pendulum motion would have to satisfy this rule.

This video will show you how to write rules like these using the language of differential equations.

This video is part of the Differential Equations video series. Laws that govern a system's properties can be modeled using differential equations.

Hi, my name is Peter Dourmashkin. And I am a senior physics lecturer at MIT.

Differential equations are very important in modeling physical phenomena. They describe rules that constrain the behavior of many complex physical and social systems.

Before watching this video, you should be familiar with drawing free body force diagrams and applying Newton's second law in polar coordinates. You should remember how to write a Taylor series expansion of a function about a point, and understand what it means.

After watching this video, you should be able to do the following 3 things:

Understand that the physical laws governing a system's properties can be modeled using differential equations,

Explain that the solution to a differential equation is a family of functions,

And recognize that specifying initial conditions determines a particular solution function to a differential equation.

Let's see how differential equations arise as a natural language to model the pendulum. To develop this model, we will use our knowledge of physics.

First, what is a pendulum? A pendulum is typically an object, called a bob, which hangs from one end of a string. The other end is fixed to a pivot point.

Let's pause the video. Try to describe what you think are the important properties of pendulum motion. You might have said that the behavior is periodic, that the same motion repeats over and over. We say that the pendulum exhibits oscillations, or harmonic motion. Maybe you also noted that the motion is rotational, moving in a circular path about the pivot point.

Finally, you may know that the motion eventually stops, that is, the oscillations experience a damping force. Overall we describe the motion as "damped harmonic motion."

Now, let's develop a model for the pendulum.
A good first step is to choose a coordinate system. Because the pendulum rotates about a fixed pivot point, it makes sense to use polar coordinates. Theta is the angle the bob makes with the vertical, and r is the distance of the bob from the pivot point.

Remember that when using vector in polar coordinates, r-hat is the unit vector that points radially outward from the pivot point, and theta-hat is the unit vector that points perpendicular to r-hat, tangential to the motion of the pendulum.

Whenever you model a system, it's important to recognize that you are making some simplifying assumptions. We are going to start by making three:

1. The string has no mass.

2. The mass, m, of the bob is uniformly distributed.

3. The string has length L and does not stretch or shrink.

This means that all motion happens in the tangential direction; none in the radial direction.

Now let's figure out what we need to know to describe the motion of the pendulum.

At the speeds we are dealing with, we can ignore relativistic effects; there are no electromagnetic forces, and the scale is large enough that quantum mechanics isn't necessary. So we'll use Newton's second law to describe the motion of the pendulum.

What forces do you think we need to consider in this problem?
Pause the video here.

Let's draw a free body diagram for an instant where the bob is to the right of vertical, and moving downwards.

There is the gravitational force acting on the bob, due to the interaction between the Earth and the bob, which points straight down with magnitude mg. We treat this force as acting on the center of mass of the bob.

The string exerts a force on the bob, which you know as tension. We don't know what the magnitude of this force is, so we'll just write it as T, pointing inwards towards the pivot point.

Finally, the motion will eventually stop, so there must be damping forces acting on the system. The damping forces are complicated. For example, there is friction at the pivot point, and air resistance on the bob. Let's make an assumption that the damping force due to friction at the pivot point is negligible because the string is small and lubricated. Then the damping force is almost entirely from the drag due to air resistance.

The bob doesn't experience any drag when it is stationary, and experiences larger amounts of drag the faster it is moving. Because the speeds of the pendulum are relatively slow, that is we do not expect there to be turbulence, we can model the drag force as being linearly proportional to velocity, and opposing the direction of motion. Because we assume that all motion occurs in tangential direction, we can draw the damping force like so.

Recall that Newton's second law is a vector equation. Because force and acceleration are both vector quantities, we need to decompose them into r-hat and theta-hat components. Remember, acceleration can be decomposed into two component vectors, one pointing radially inward called centripetal acceleration, and a tangential component vector called angular acceleration.
We also need to decompose our force vectors. Notice that the tension force is already pointing radially inward, in the negative r-hat direction. And the damping force is pointing in the theta-hat direction, opposite the direction of the angular velocity vector.

So all we have to do is decompose the gravitational force. We project that vector onto the r-hat and theta-hat component vectors. In the r-hat direction the magnitude is mg cos(theta) and points outwards. In the theta-hat direction, the magnitude is mg sin(theta) and points in the negative theta-hat direction.

Newton's second law can now be written as two component equations. One in the radial direction, mg cosine theta minus tension is equal to m times centripetal acceleration.

And the other in the tangential direction, negative mg sine theta minus k times angular velocity is equal to m times the angular acceleration.

This system of two equations has two unknowns: the angle theta and the tension. Remember that we are trying to find an expression for how the angle theta depends on time. This second equation is more useful to us, because we can express both the angular velocity and the angular acceleration in terms of theta. Let's see how.

The circular motion of the pendulum is a 1-dimensional motion, parameterized by time t. The displacement of the pendulum away from center is the arc length, which is L times theta of t. Because the angular velocity is tangent to the circle of motion, it can be found simply by differentiating the displacement with respect to the parameter t, which gives us that the angular velocity is L times the time derivative of theta. The angular acceleration is just the time derivative of the angular velocity, and is given by L times the second time derivative of theta.

Observe that this works because we are only looking for the angular velocity and angular acceleration which point tangent to our 1-dimensional motion. In fact, we see that the velocity is everywhere tangent to the motion, if we think of the velocity as the limit of the displacement vector with respect to a small change in time.

However, the acceleration has two components. It has an angular component that we computed, but it also has a radial component, the centripetal acceleration. The tangential component comes from how the velocity vector changes in magnitude, and the radial component comes from how the velocity vector changes in direction.

So now we've found expressions for the magnitudes of the angular velocity and angular acceleration:
We can substitute these equations into Newton's Second Law in the theta-hat direction; we obtain the following equation.

This is a second order ordinary differential equation or 2nd order ODE. It is ordinary because there is only one independent variable, time, that the angle theta depends on. It is second order because the highest degree derivative that appears in the equation is a second derivative.

Any function describing the pendulum motion must satisfy this differential equation.
Keep in mind that there are other forces we ignored, such as friction at the pivot point. This is a very simple but nontrivial model for the non-linear pendulum.

Notice that the first derivative term is our velocity dependent drag force. If we decide that this term is negligible, we would get this equation. It is still complicated, but you might notice that it looks somewhat familiar.

If we assume that the displacement of the angle is very small, we can approximate sin theta as the angle theta, which gives us this equation. This is the equation of the simple harmonic oscillator. So you see that this piece connecting the term sin theta and the second derivative of theta produces the oscillatory behavior, and the first derivative term is responsible for the damping of the motion.

We've seen that we can apply our knowledge of physical laws to model a non-linear pendulum with a differential equation.
How does this differential equation describe the motion of a pendulum?

A differential equation is an equation where the unknown element is a function, rather than a number or a single value. The differential equation is a rule that describes how the system evolves from any time t to a time t plus Delta t. In particular, it describes the relationship between the angle theta and its first and second derivatives as the system evolves over time.

Remember that to describe the motion of the pendulum, we need an expression for the angle theta as a function of time.

Do you think there is a unique solution to this differential equation? Pause the video here and discuss.

You probably have some sense that the motion of the pendulum depends on how and where you release it—conditions we refer to as initial conditions.

For example, if we barely move the pendulum away from center, and release it from rest, the oscillation will be small.
Larger displacements will give larger oscillations, even though eventually they all damp down to zero.
All of these pendulum motions satisfy the same differential equation!

Since every starting position will give a different dependence of theta on time, the solution is not unique. We say that the solution to a differential equation is an infinite family of functions.

But what if we were are interested in finding one solution in particular?

In the examples you just saw, we specified the initial angle from which the pendulum was released. Is this the only piece of information we need to pick out a specific solution from the infinite family?

Let's test that. Here you see two identical pendulums, each starting from the same initial angle. Are we guaranteed that the motion will be the same?

Pause the video here to make a prediction.

Clearly not. What was the difference between the initial conditions for these two pendulums?
Pause the video here.

The initial angles of both pendulums were the same. However, one was released from rest, and the other had an initial push. In other words, the initial angular velocities were different. Remember, our differential equation only holds after the pendulum is released because it doesn't include the action of the pushing force.

Now we see the importance of specifying two initial conditions—the initial angle and the initial angular velocity.
Remember, the differential equation is the rule for how the system evolves over time. In order to solve this differential equation, we are going to use the Taylor series expansion for the function theta about t=0.

Recall that this is a polynomial series, whose coefficients are determined by all of the derivatives of the function theta at time t=0. So how can we find all of these derivatives of theta?

Let's use the differential equation to find the second derivative at time t=0. Notice that it depends on both of our initial conditions: initial angle and initial angular velocity. This is why we need these initial conditions.

We can differentiate the second derivative to find the third derivative at time t=0. Notice again that the third derivative depends on the initial angle, the initial angular velocity, and the second derivative at time t=0.

We can continue this process and find recursive formulas for the all of the higher order derivatives of theta at time t=0! That's how the two initial conditions allow us to pick out a solution, by specifying all of the coefficients for the Taylor series at t=0.
Our Taylor series solution completely describes the function theta. But it only works on the interval of convergence. If the series converges everywhere, we're done! But if it doesn't converge everywhere, it must converge on some interval.

So you see, a differential equation can determine a specific function if you have enough pieces of initial data to determine all the Taylor coefficients for the function. In our case, with a second order ordinary differential equation, we just needed two initial conditions.

Pause the video. If you had a third order differential equation, how many initial conditions would you need to specify one solution? [Pause] That's right, three!

We wanted to describe the motion of a pendulum. We did this by making a model, where we made some simplifying assumptions and applied Newton's Second Law.

This gave us the differential equation, which is the rule that any function of pendulum motion and its derivatives must satisfy.

We saw there is an infinite family of solution functions for a differential equation.

Because our differential equation was second order, we needed 2 initial conditions to specify one solution.

The initial conditions allowed us to determine the coefficients of the Taylor series for the angle theta as a function of time, which solved the differential equation.