Problem Set Rules:

1. Each student should hand in an individual problem set.
2. Discussing problem sets with other students is permitted. Copying from another person or solution set is not permitted.
3. Late assignments will not be accepted. No exceptions.

Problem 1

(15 points) A local radio station is going to schedule commercials within 60 second blocks. Consider the following six commercials. number 1 is 12 seconds long, number 2 is 18 seconds long, number 3 is 22 seconds long, number 4 is 35 seconds long, number 5 is 40 seconds long and number 6 is 59 seconds long. What is the smallest number of 60 second blocks that the commercials fit into?

Solution. The key is to start with enough blocks for a trivial arrangement, in this case 5 obviously will work. Then we let \( y_j = 1 \) if block \( j \) is used and \( y_j = 0 \) otherwise and \( x_{ij} = 1 \) if commercial \( i \) is assigned to block \( j \) and \( x_{ij} = 0 \) otherwise. Based on this choice of decision variables, the formulation is as follows:

\[
\begin{align*}
\text{min} & \quad y_1 + y_2 + y_3 + y_4 + y_5 \\
\text{s.t.} & \quad 12x_{1j} + 18x_{2j} + 22x_{3j} + 35x_{4j} + 40x_{5j} + 59x_{6j} \leq 60y_j, \quad \text{for } j = 1, \ldots, 5 \\
& \quad x_{i1} + x_{i2} + x_{i3} + x_{i4} + x_{i5} = 1, \quad \text{for } i = 1, \ldots, 6 \\
& \quad x_{ij}, y_j \in \{0, 1\} \quad \text{for } j = 1, \ldots, 5 \text{ & } i = 1, \ldots, 6
\end{align*}
\]

The first set of constraints represent the condition that each possible programming block contains at most 60 seconds of commercial programming. The second set of constraints represent the condition that each commercial is assigned to exactly one programming block. We then minimize the total number of blocks. Additionally, we need all of our decision variables to be binary.

By inspection, however, the minimum number of blocks needed is 4; the sum of all commercial times is 186, meaning that more than 3 are needed, and there is a clear way to use only 4.

Problem 2

(25 points) A typical large oil and gas company operates many explorations and production projects, which involve several billion dollars every year. These companies are annually faced with the problem of where the capital should be spent and which combination of projects should
be selected from several possible project mixes. They have the difficult task of portfolio selection from a large number of competing projects for immediate or future operation under a limited amount of investments.

Consider a small firm with 4 competing oil production projects. Table 1 presents the production, capital, and the net present value for these projects. They contact you to help select the best combination of projects under a certain amount of investment, while fulfilling the firm’s goals.

(a) (5 points) Formulate an integer program to maximize the net present value (NPV) subject to a capital limit stating the firm can spend no more than $32 million (M$) and a production level stating the firm must produce at least 73 million barrels (Mbbl).

**Solution.** We define four binary variables $x_A, x_B, x_C, x_D$ as follows:

$$x_A := \begin{cases} 1 & \text{if Project A is selected,} \\ 0 & \text{otherwise.} \end{cases}$$

$$x_B := \begin{cases} 1 & \text{if Project B is selected,} \\ 0 & \text{otherwise.} \end{cases}$$

$$x_C := \begin{cases} 1 & \text{if Project C is selected,} \\ 0 & \text{otherwise.} \end{cases}$$

$$x_D := \begin{cases} 1 & \text{if Project D is selected,} \\ 0 & \text{otherwise.} \end{cases}$$

Then our formulation is:

$$\text{max} \quad 25x_A + 20x_B + 19x_C + 28x_D$$

$$\text{s.t.} \quad 11x_A + 9x_B + 14x_C + 17x_D \leq 32,$$

$$28x_A + 20x_B + 25x_C + 30x_D \geq 73,$$

$$x_A, x_B, x_C, x_D \in \{0, 1\}.$$  

(b) (9 points, 3 points each) Suppose that there are the following additional constraints:

(i) If Project A is selected, then Project B is also selected;
(ii) Either Project A is selected or Project C selected, but not both;
(iii) At least one of Projects A, B, and D is selected.

Extend your integer program to satisfy these constraints.

**Solution.** The additional constraints for each part are as follows:

(i) $x_A \leq x_B$ or equivalently $x_A - x_B \leq 0$;
(ii) $x_A + x_B = 1$.
(iii) $x_A + x_B + x_C \geq 1$.

(c) (5 points) Now assume that the selection must satisfy the production level as well as the budget limit over each of the next 3 years as indicated in Table 2. The production and capital for each project during the next 3 years are given in Table 3. Write an integer program to determine the most profitable selection.

**Solution.** Suppose that $x_A, x_B, x_C, x_D$ are the binary variables as defined in Part A. Here we have a budget constraint and a production constraint for each year. Therefore,
Table 1: Portfolio optimization problem Data

<table>
<thead>
<tr>
<th>Project</th>
<th>NPV (M$)</th>
<th>Capital (M$)</th>
<th>Production (Mbbl)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>25</td>
<td>11</td>
<td>28</td>
</tr>
<tr>
<td>B</td>
<td>20</td>
<td>9</td>
<td>20</td>
</tr>
<tr>
<td>C</td>
<td>19</td>
<td>14</td>
<td>25</td>
</tr>
<tr>
<td>D</td>
<td>28</td>
<td>17</td>
<td>30</td>
</tr>
</tbody>
</table>

Table 2: Budget limitation and production level over the next 3 years

<table>
<thead>
<tr>
<th>Year</th>
<th>Budget (M$)</th>
<th>Production (Mbbl)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>18</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>30</td>
</tr>
</tbody>
</table>

the formulation is:

\[
\begin{align*}
\text{max} & \quad 25x_A + 20x_B + 19x_C + 28x_D \\
\text{s.t.} & \quad 5x_A + 4x_B + 6x_C + 8x_D \leq 18, \\
& \quad 4x_A + 3x_B + 5x_C + 5x_D \leq 10, \\
& \quad 3x_A + 3x_B + 4x_C + 5x_D \leq 7, \\
& \quad 5x_A + 5x_B + 6x_C + 8x_D \geq 20, \\
& \quad 5x_A + 7x_B + 9x_C + 10x_D \geq 25, \\
& \quad 8x_A + 8x_B + 10x_C + 12x_D \geq 30, \\
& \quad x_A, x_B, x_C, x_D \in \{0, 1\}.
\end{align*}
\]

(d) (6 points) Generalize your integer program to a general setting: Assume that you are given a set of \(n\) projects and your task is to maximize the net present value (NPV), subject to a capital limit stating that we can spend no more that \(C_i\) and a production limit stating that we must produce at least \(P_i\) million barrels over each of the next \(L\) year. Assume that \(n_{pj}\) represents the NPV of \(jth\) asset and \(p_{i,j}\) and \(c_{i,j}\) represent the production and capital for the \(j^{th}\) project in the \(i^{th}\) year, respectively. Write the corresponding integer program.

**Solution.** We define the decision variables \(x_1, x_2, \ldots, x_n\) as follows:

\[
x_j := \begin{cases} 
1 & \text{if Project } j \text{ is selected,} \\
0 & \text{otherwise.}
\end{cases}
\]
Then the problem is formulated as the following integer program:

\[
\begin{align*}
\text{max} & \quad \sum_{j=1}^{n} npv_{j} x_{j} \\
\text{s.t.} & \quad \sum_{j=1}^{n} c_{ij} x_{j} \leq C_{i}, \quad \text{for } i = 1, \ldots, L, \\
& \quad \sum_{j=1}^{n} p_{ij} x_{j} \geq P_{i}, \quad \text{for } i = 1, \ldots, L, \\
& \quad x_{j} \in \{0,1\}, \quad \text{for } j = 1, \ldots, n.
\end{align*}
\]

Table 3: Portfolio optimization problem data over the next 3 years

<table>
<thead>
<tr>
<th>Project</th>
<th>first year</th>
<th>second year</th>
<th>third year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capital (M$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>5</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>C</td>
<td>6</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>D</td>
<td>8</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Production (Mbbl)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
</tr>
<tr>
<td>B</td>
</tr>
<tr>
<td>C</td>
</tr>
<tr>
<td>D</td>
</tr>
</tbody>
</table>

Do not be scared by the fact that we have parameters \( n, C_{i}, P_{i}, c_{ij}, p_{ij} \) instead of numbers! You can treat them just as you would treat numbers. Start by defining the decision variables.

**Problem 3**

(24 points) We have two binary variables \( x_{1}, x_{2} \in \{0,1\} \). We want to represent the outcome of the three logical operations AND, OR, XOR applied on \( x_{1} \) and \( x_{2} \). The definition of these three operations is as follows:

- Let \( w = (x_{1} \text{ AND } x_{2}) \). \( w \) is 1 if and only if both \( x_{1} \) and \( x_{2} \) are 1, and is 0 otherwise.
- Let \( y = (x_{1} \text{ OR } x_{2}) \). \( y \) is 1 if and only if at least one of the variables \( x_{1} \) and \( x_{2} \) is 1, and is 0 otherwise.
- Let \( z = (x_{1} \text{ XOR } x_{2}) \). \( z \) is 1 if and only if the variables \( x_{1} \) and \( x_{2} \) have different values, and is 0 otherwise.

In Table 4, we give the value of each variable \( w, y, z \) in terms of the value of \( x_{1} \) and \( x_{2} \). The definition in the table is equivalent to the one given above.

Your goal in this problem is to define \( w, y, z \) using linear constraints only. For the AND and OR operations, you are not allowed to introduce additional variables.
Table 4: Value of $w, y, z$ as a function of $x_1, x_2$.

(a) (8 points) Write a set of linear constraints that define $w = (x_1 \text{ AND } x_2)$. The constraints should only involve the variables $w, x_1, x_2$.

Solution. There are multiple acceptable formulations; one is the following:

\[
\begin{align*}
w & \leq x_1 \\
& \leq x_2 \\
& \geq x_1 + x_2 - 1
\end{align*}
\]

(b) (8 points) Write a set of linear constraints that define $y = (x_1 \text{ OR } x_2)$. The constraints should only involve the variables $y, x_1, x_2$.

Solution. Again, there are multiple acceptable formulations; one is the following:

\[
\begin{align*}
y & \geq x_1 \\
& \geq x_2 \\
& \leq x_1 + x_2
\end{align*}
\]

(c) (8 points) Write a set of linear constraints that define $z = (x_1 \text{ XOR } x_2)$. You are allowed to introduce an auxiliary variable in this case. Thus, the constraints should involve the variables $z, x_1, x_2$, and possibly an additional variable $a$.

Solution. Again, there are multiple acceptable formulations; one is the following:

\[
\begin{align*}
z & = x_1 + x_2 - 2a \\
a & \leq x_1 \\
& \leq x_2 \\
& \geq x_1 + x_2 - 1 \\
a & \in \{0,1\}
\end{align*}
\]

Problem 4

(36 points) We are given an integer program defined as follows:

\[
\begin{align*}
\max & \quad 10x_1 + 22x_2 + 5x_3 + 15x_4 + 17x_5 + 12x_6 + 4x_7 \\
\text{s.t.:} & \quad 5x_1 + 3x_2 + 8x_3 + 9x_4 + 16x_5 + 5x_6 + 10x_7 \leq 700 \\
& \quad \forall i = 1, 2, 3 \\
\quad & \quad \forall i = 4, 5, 6, 7 \\
& \quad 0 \leq x_i \leq 200.
\end{align*}
\]

For each of the parts below, you are to add constraint(s) and possibly variables to ensure that the logical condition is satisfied by the integer program. Each part is independent; that is,
no part depends on the parts preceding it. You do not need to repeat the integer programming objective or constraints given above. You may use the big $M$ method for formulating constraint when it is appropriate. If you need to use the big $M$ method, choose the answer that corresponds to the best possible value for the big $M$ coefficient that appears in the formulation. (By “best possible value” we mean the smallest possible value such that the logical constraints modeled through the big $M$ remain valid.)

(a) (3 points) Write a single linear constraint that is equivalent to the statement “If $x_2 = 1$ is selected, then $x_1 = 0$”

Solution. $x_1 + x_2 \leq 1$

(b) (3 points) Write a single linear constraint that is equivalent to the statement “$x_1$ and $x_3$ cannot both be 1”.

Solution. $x_1 + x_3 \leq 1$

(c) (4 points) Add a single integer variable $w_4$ and a constraint that ensures that $x_8$ is divisible by 3 but not divisible by 6. (The remainder when dividing by 6 must be 3).

Solution. $x_8 - 6w_4 = 3$.

(d) (4 points) Add three binary variables $w_5$, $w_6$, and $w_7$ and two constraints that ensures that $x_5 = 9$ or 15 or 20.

Solution. $x_5 = 9w_5 + 15w_6 + 20w_7$ and $w_5 + w_6 + w_7 = 1$.

(e) (7 points) Add 3 binary variables $w_1$, $w_2$, and $w_3$ and at most 4 constraints so as to ensure that at least two of the following constraints is satisfied: (i) $x_4 \geq 50$, (ii) $x_5 \leq 25$, (iii) $x_6 + x_7 \leq 100$.

Solution. $x_4 \geq 50 - M(1 - w_1)$, $x_5 \leq 25 + M(1 - w_2)$, $x_6 + x_7 \leq 100 + M(1 - w_3)$ and $w_1 + w_2 + w_3 \geq 2$, with $M \geq 300$.

(f) (7 points) Add variable(s) and constraint(s) that ensure either $2x_4 + x_5 \leq 50$ or $4x_4 - x_5 \geq 20$, but not both.

Solution. $2x_4 + x_5 \leq 50 + M(1 - w_8)$, $4x_4 - x_5 \geq 20 - M(1 - w_9)$, $w_8 + w_9 = 1$, with $M \geq 450$.

(g) (8 points) Add variable(s) and constraint(s) that model the cost of $x_4$ as $f_4(x_4)$, which is defined as follows: If $0 \leq x_4 \leq 50$, then $f_4(x_4) = 20x_4$. If $51 \leq x_4 \leq 100$, then $f_4(x_4) = 1000$. If $101 \leq x_4 \leq 200$, then $f_4(x_4) = -500 + 15x_4$. 


Solution.

New Obj Fn: \( \max \)  
\[
10x_1 + 22x_2 + 5x_3 + f_4(x) + 17x_5 + 12x_6 + 4x_7
\]
\[
w_1 + w_2 + w_3 = 1 \\
y_1 + y_2 + y_3 = x_4 \\
20y_1 + 1000w_2 - 500w_3 + 15y_3 = f_4(x_4)
\]
\[
0w_1 \leq y_1 \leq 50w_1 \\
51w_2 \leq y_2 \leq 100w_2 \\
101w_3 \leq y_3 \leq 200w_3
\]
\[
w_1, w_2, w_3 \in \{0, 1\}
\]
\[
0 \leq y_1, y_2, y_3 \leq 200.
\]