2-person 0-sum
(or constant sum) game theory
Quotes of the Day

“My work is a game, a very serious game.”
-- M. C. Escher (1898 - 1972)

“Conceal a flaw, and the world will imagine the worst.”
-- Marcus Valerius Martialis (40 AD - 103 AD)

“Reveal your strategy in a game, and your outcome will be the worst.”
-- Professor Orlin (for the lecture on game theory)
Game theory is a very broad topic

6.254 Game Theory with Engineering Applications
14.12 Economic Applications of Game Theory
14.122 Microeconomic Theory II
14.126 Game Theory
14.13 Economics and Psychology
14.147 Topics in Game Theory
15.025 Game Theory for Strategic Advantage
17.881 Game Theory and Political Theory
17.882 Game Theory and Political Theory
24.222 Decisions, Games and Rational Choice
“Say you’re in a public library, and a beautiful stranger strikes up a conversation with you. She says: ‘Let’s show pennies to each other, either heads or tails. If we both show heads, I pay you $3. If we both show tails, I pay you $1. If they don’t match, you pay me $2.’

At this point, she is shushed. You think: ‘With both heads 1/4 of the time, I get $3. And with both tails 1/4 of the time, I get $1. So 1/2 of the time, I get $4. And with no matches 1/2 of the time, she gets $4. So it’s a fair game.’ As the game is quiet, you can play in the library.”

But should you? Should she?

submitted by Edward Spellman to Ask Marilyn on 3/31/02

Marilyn Vos Savant has a weekly column in Parade. She has the highest recorded IQ on record.
2-person 0-sum (or constant sum) Game Theory

- Two people make decisions at the same time.
- The payoff depends on the joint decisions.
- Whatever one person wins the other person loses (or the sum of their winnings is a constant).
  - Marilyn vos Savant answered the question incorrectly. [http://www.siam.org/siamnews/06-03/gametheory.pdf](http://www.siam.org/siamnews/06-03/gametheory.pdf)
Payoff (Reward) Matrix for Vos Savant’s Game

You (the Row Player) choose heads or tails

The beautiful stranger chooses heads or tails

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Heads</strong></td>
<td>3</td>
<td>-2</td>
</tr>
<tr>
<td><strong>Tails</strong></td>
<td>-2</td>
<td>1</td>
</tr>
</tbody>
</table>

Beautiful Stranger

This matrix is the payoff matrix for you, and the beautiful stranger gets the negative.
Payoff Matrix for Rock-Paper-Scissors

Row Player chooses a row: either $R_1$, $R_2$, or $R_3$. The three rows are referred to as strategies for the Row Player.

Column Player chooses a column: either $C_1$, $C_2$, or $C_3$, which are referred to as strategies for the Column Player.

<table>
<thead>
<tr>
<th></th>
<th>Rock</th>
<th>Paper</th>
<th>Scissors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rock</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>Paper</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>Scissors</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

This matrix is the payoff matrix for the Row Player, and the column player gets the negative.)
The Goal of today’s lecture

- Introduce some useful concepts in game theory.
- Focus on “guaranteed payoffs”.
- Determine optimal strategies for playing two-person constant-sum games.
- Show the connection with linear programming.
A payoff matrix

<table>
<thead>
<tr>
<th>Row Player chooses a row: either R₁, R₂, or R₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column Player chooses a column: either C₁, C₂, or C₃</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>C₁</th>
<th>C₂</th>
<th>C₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>R₁</td>
<td>-2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>R₂</td>
<td>2</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>R₃</td>
<td>1</td>
<td>0</td>
<td>-2</td>
</tr>
</tbody>
</table>

e.g., Row Player chooses R₃; Column Player chooses C₁

Row Player gets 1; Column Player gets -1.
The column player minimizes the payoff to the row player.

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<th>C₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>R₁</td>
<td>-2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>R₂</td>
<td>2</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>R₃</td>
<td>1</td>
<td>0</td>
<td>-2</td>
</tr>
</tbody>
</table>

Row Player gets 1; Column Player loses 1
Player strategies

- Player 1 (the row player) has three strategies: choose row 1, or row 2, or row 3.

- The column player has three strategies: choose column 1, or column 2, or column 3.

- We will later refer to these as pure strategies, for reasons that will become apparent when we describe mixed strategies.
A guaranteed payoff for the Row Player.

Floor($R_i$) is the min payoff in the row $R_i$.

<table>
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<tr>
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<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$</td>
<td>-2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$R_2$</td>
<td>2</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$R_3$</td>
<td>1</td>
<td>0</td>
<td>-2</td>
</tr>
</tbody>
</table>

Floor($R_1$) = min {-2, 1, 2} = -2.
Floor($R_2$) = min{2, -1, 0} = -1.
Floor($R_3$) = min{1, 0, -2} = -2.

If the Row Player selects Row j, her payoff will be at least Floor($R_j$).

The value of the game for the Row Player is at least $\max \{\text{Floor}(R_j): j = 1, \ldots, m\}$. -1
The Column Player’s guarantee

**Ceiling**\((C_i)\) is the max payoff in the column \(C_j\).

<table>
<thead>
<tr>
<th></th>
<th>(C_1)</th>
<th>(C_2)</th>
<th>(C_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R_1)</td>
<td>-2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>(R_2)</td>
<td>2</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>(R_3)</td>
<td>1</td>
<td>0</td>
<td>-2</td>
</tr>
</tbody>
</table>

- \(\text{Ceiling} (C_1) = \max \{-2, 2, 1\} = 2\).
- \(\text{Ceiling} (C_2) = \max \{1, -1, 0\} = 1\).
- \(\text{Ceiling} (C_3) = \max \{2, 0, -2\} = 2\).

If the Column Player selects Column \(C_j\), the Row Player’s payoff will be at most \(\text{Ceiling}(C_j)\).

The value of the game for the Row Player is at most \(\min \{\text{Ceiling}(C_j) : j = 1, \ldots, n\}\). (1)
The two guarantees

- The row player can guarantee a payoff of at least -1.

- The column player can guarantee that the payoff to the row player is at most 1.
If you are the row player, what row (strategy) would you choose?

1. **R₁**
2. **R₂**
3. **R₃**

<table>
<thead>
<tr>
<th></th>
<th>C₁</th>
<th>C₂</th>
<th>C₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>R₁</td>
<td>100</td>
<td>50</td>
<td>5</td>
</tr>
<tr>
<td>R₂</td>
<td>50</td>
<td>30</td>
<td>10</td>
</tr>
<tr>
<td>R₃</td>
<td>25</td>
<td>20</td>
<td>15</td>
</tr>
</tbody>
</table>
We say that column $i$ dominates column $C_j$ if $C_i \leq C_j$.

Column $C_3$ dominates $C_1$ and $C_2$. Since the column player is rational, she will choose $C_3$.

If the Column Player chooses $C_3$, then the row player chooses $R_3$.

It is called a saddle point. If either player chooses a different strategy, the payoff is worse for that player.
Suppose that there is a row \( R_i \) and a column \( C_j \) such that Floor\((R_i)\) = Ceiling\((C_j)\).

Then the element \( a_{ij} \) is a saddlepoint of the game, and the value of the game for the row player is \( a_{ij} \).

The value of this game is 15 for the Row Player. The Row Player will choose \( R_3 \), and the column Player will choose \( C_3 \). Any switching of strategy lowers the value of the game to the player.
Next: 2 volunteers

Row Player puts out 1, 2 or 3 fingers.

Column Player simultaneously puts out 1, 2, or 3 fingers

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<tr>
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<td>1</td>
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RP tries to maximize his or her total

CP tries to minimize R’s total.

We will run the game for 5 trials.
Who has the advantage, R or C?

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1. R
2. C
3. neither
Mental Break

Rock
beats scissors

Paper
beats rock

Scissors
beats paper

Image by MIT OpenCourseWare.
Random (mixed) strategies

Suppose we permit the Row Player to choose a mixed strategy, that is, the strategy is one in which she chooses rows with probabilities. (Her strategy is randomized).

Example: Suppose that Strategy 1 consists of choosing \( R_1 \) with probability \( \frac{1}{2} \), and \( R_3 \) with probability \( \frac{1}{2} \).

The row player flips a coin. If it is heads, she chooses \( R_1 \). If tails, she chooses \( R_3 \).
The Row Player can guarantee receiving a payoff of at least  \(-\frac{1}{2}\).
The Row Player can guarantee an expected payoff $\geq 0$. 

Expected Payoff

<table>
<thead>
<tr>
<th>R₁</th>
<th>R₂</th>
<th>R₃</th>
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<tbody>
<tr>
<td>-2</td>
<td>1</td>
<td>2</td>
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<tr>
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Probabilities for Strategy $S₂$

Probabilities:

- $R₁$: 1/3
- $R₂$: 1/3
- $R₃$: 1/3

Expected Payoff: [ ] [ ] [ ]
The Row Player’s Problem

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Choose $x$ so that

$$x₁ + x₂ + x₃ = 1$$

and

$$x₁, x₂, x₃ ≥ 0.$$
Row Player’s Problem

Choose \( x \) so that

\[ x_1 + x_2 + x_3 = 1 \]

\( x_1, x_2, x_3 \geq 0 \).

<table>
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<th>C_1</th>
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<th>C_3</th>
</tr>
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<tbody>
<tr>
<td>Prob.</td>
<td>x_1</td>
<td>x_2</td>
</tr>
</tbody>
</table>

C_1
C_2
C_3
Row Player’s Problem

Max $z$

\[
\begin{align*}
z & \leq -2x_1 + 2x_2 + x_3 & \text{E}(C_1) \\
z & \leq x_1 - x_2 & \text{E}(C_2) \\
z & \leq 2x_1 - 2x_3 & \text{E}(C_3) \\
x_1 + x_2 + x_3 & = 1 \\
x_1, x_2, x_3 & \geq 0.
\end{align*}
\]

Max $z = \text{Min} \{ \text{E}(C_1), \text{E}(C_2), \text{E}(C_3) \}$

s.t. $x_1 + x_2 + x_3 = 1$

$x_1, x_2, x_3 \geq 0$

This is the called the \textit{maximin problem}
An *optimal* mixed strategy for the Row Player.

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<td>0</td>
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<tr>
<td>R₃</td>
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<td>0</td>
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**Expected Payoff**

|     | 1/9  | 1/9  | 1/9  |

**Prob.**

- \(x₁ = 7/18\)
- \(x₂ = 5/18\)
- \(x₃ = 1/3\)

**Strategy** \(x_{\text{opt}}\)

The guaranteed (minimax) payoff is 1/9.
Suppose that the column player chooses a mixed strategy.

If \( v \) is the guaranteed payoff, then \( v \) is the minimum value s.t.,

\[
\begin{align*}
E(R_1) &= \frac{1}{3} \\
E(R_2) &= \frac{1}{3} \\
E(R_3) &= -\frac{1}{3}
\end{align*}
\]

The column player can guarantee that \( R \) is payed at most \( \frac{1}{3} \).
Choose a mixed (randomized) strategy $y$ that minimizes the guaranteed payoff to the Row Player.

This is the \textit{minimax payoff}.

$$\begin{array}{c|c|c|c}
\text{C}_1 & \text{C}_2 & \text{C}_3 \\
\hline
\text{R}_1 & -2 & 1 & 2 \\
\text{R}_2 & 2 & -1 & 0 \\
\text{R}_3 & 1 & 0 & -2 \\
\end{array}$$

$$\begin{align*}
\min \ v \\
v \geq E_y(R_1) \\
v \geq E_y(R_2) \\
v \geq E_y(R_3)
\end{align*}$$
An \textit{optimal} randomized strategy for the Column Player

\begin{center}
\begin{tabular}{c|c|c|c}
 & \textbf{C}_1 & \textbf{C}_2 & \textbf{C}_3 \\
\hline
\textbf{R}_1 & -2 & 1 & 2 \\
\textbf{R}_2 & 2 & -1 & 0 \\
\textbf{R}_3 & 1 & 0 & -2 \\
\end{tabular}
\end{center}

\begin{itemize}
\item \textbf{Exp. payoff}
\begin{align*}
1/9 \\
1/9 \\
1/9 \\
\end{align*}
\item \textbf{Prob.}
\begin{align*}
y_1 &= 1/3 \\
y_2 &= 5/9 \\
y_3 &= 1/9 \\
\end{align*}
\end{itemize}

\[ v = \max \{ E_y(R_1), E_y(R_2), E_y(R_3) \} \]
\[ = \max \{ 1/9, 1/9, 1/9 \} = 1/9. \]

That is, CP can play to guarantee at most 1/9 for R.
**On the Optimal Expected Payoffs**

Version 1. You are the row Player and you choose the strategy given earlier. The most intelligent person on earth is playing against you with the aid of the most powerful computer.

Your expected payoff is $\frac{1}{9}$ on average.

Version 2. You are the Row Player and have access to the most powerful computer on Earth and a brilliant game theorist. You are playing against column Player who is using the simple randomized strategy given earlier.

Your expected payoff is at most $\frac{1}{9}$ on average.
Fundamental Theorem of 2-person 0-sum Game Theory.

Let $X$ be the set of feasible mixed strategies for the Row Player.

$$z = \max_{x \in X} \{ \min \{ E_x(C_1), E_x(C_2), \ldots, E_x(C_m) \} \}$$

$$= \max_{x \in X} \{ \min \text{ of the expected column payoffs} \}$$

Let $Y$ be the set of mixed strategies for the Column Player.

$$v = \min_{y \in Y} \{ \max \{ E_y(R_1), E_y(R_2), \ldots, E_y(R_n) \} \}$$

$$= \min_{y \in Y} \{ \max \text{ of the expected row payoffs} \}$$

Theorem. For any 2-person 0-sum game, the maximin value is equal to the minimax value.

The maximin value is the minimax value.
Developers of Game Theory

John Von Neumann
Oskar Morgenstern

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15.053 Optimization Methods in Management Science
Spring 2013

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