Problem 1. Consider the space \( C[0, T] \) of continuous functions \( x : [0, T] \rightarrow \mathbb{R} \), endowed with the uniform metric \( \rho(x, y) = \| x - y \| = \sup_{0 \leq t \leq T} | x(t) - y(t) | \). Construct an example of a closed bounded set \( K \subset C[0, T] \) which is not compact. (A set \( K \subset C[0, T] \) is bounded if there exists a large enough \( r \) such that \( K \subset B(0, r) \), where 0 is a function which is identically zero on \([0, T]\)).

Problem 2. Given two metric spaces \( (S_1, \rho_1), (S_2, \rho_2) \) show that a function \( f : S_1 \rightarrow S_2 \) is continuous if and only if for every open set \( O \subset S_2 \), \( f^{-1}(O) \) is an open subset of \( S_1 \).

Problem 3. Establish that the space \( C[0, T] \) is complete with respect to \( \| x - y \| \) metric and the space \( D[0, T] \) is complete with respect to the Skorohod metric.

Problem 4. Problem 1 from Lecture 2. Additionally to the parts a)-c), construct an example of a random variable \( X \) with a finite mean and a number \( x_0 > \mathbb{E}[X] \), such that \( I(x_0) < \infty \), but \( I(x) = \infty \) for all \( x > x_0 \). Here \( I \) is the Legendre transform of the random variable \( X \).

Problem 5. Establish the following fact, (which we have used in proving the upper bound part of the Cramér’s theorem for general closed sets \( F \)): given two strictly positive sequences \( x_n, y_n > 0 \), show that if \( \lim \sup_n (1/n) \log x_n \leq I \), \( \lim \sup_n (1/n) \log y_n \leq I \), then \( \lim \sup_n (1/n) \log (x_n + y_n) \leq I \).

Problem 6. Suppose \( M(\theta) < \infty \) for all \( \theta \). Show that \( I(x) \) is a strictly convex function.

\textit{Hint.} Give a direct proof of convexity of \( I \) and see where inequality may turn into equality. You may use the following fact which we have established in the class: for every \( x \) there exists \( \theta_0 \) such that \( x = \hat{M}(\theta_0)/M(\theta_0) \).