15.082J and 6.855J and ESD.78J

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The Multicommodity Flow Problem
Lecture overview

- Notation
- A small illustrative example
- Some applications of multicommodity flows
- Optimality conditions
- A Lagrangian relaxation algorithm
On the Multicommodity Flow Problem
O-D version

K origin-destination pairs of nodes
\((s_1, t_1), (s_2, t_2), \ldots, (s_K, t_K)\)

Network \(G = (N, A)\)

\(d_k = \) amount of flow that must be sent from \(s_k\) to \(t_k\).

\(u_{ij} = \) capacity on \((i,j)\) shared by all commodities

\(c_{ij}^k = \) cost of sending 1 unit of commodity \(k\) in \((i,j)\)

\(x_{ij}^k = \) flow of commodity \(k\) in \((i,j)\)
A Linear Multicommodity Flow Problem

Quick exercise: determine the optimal multicommodity flow.
A Linear Multicommodity Flow Problem

5 units good 1

1 → 4: 5 units good 1

2 → 5: 2 units good 2

3 → 6: 2 units good 2

1 → 2: $1

2 → 3: $1

3 → 5: $6

4 → 5: $1

5 → 6: $1

5 units good 1

\[ u_{25} = 5 \]

\[ x_{32}^2 = x_{25}^2 = x_{56}^2 = 2 \]

\[ x_{12}^1 = x_{25}^1 = x_{54}^1 = 3 \]

\[ x_{14}^1 = 2 \]
The Multicommodity Flow LP

Min \[ \sum_{(i,j) \in A} \sum_k c_{ij}^k x_{ij}^k \]

\[ \sum_j x_{ij}^k - \sum_j x_{ji}^k = \begin{cases} d_k & \text{if } i = s_k \\ -d_k & \text{if } i \in t_k \\ 0 & \text{otherwise} \end{cases} \]

\[ \sum_k x_{ij}^k \leq u_{ij} \quad \text{for all } (i, j) \in A \]

\[ x_{ij}^k \geq 0 \quad \forall (i, j) \in A, \ k \in K \]

Supply/demand constraints

Bundle constraints
Assumptions (for now)

- **Homogeneous goods.** Each unit flow of commodity $k$ on $(i, j)$ uses up one unit of capacity on $(i,j)$.

- **No congestion.** Cost is linear in the flow on $(i, j)$ until capacity is totally used up.

- **Fractional flows.** Flows are permitted to be fractional.

- **OD pairs.** Usually a commodity has a single origin and single destination.
## Application areas

<table>
<thead>
<tr>
<th>Type of Network</th>
<th>Nodes</th>
<th>Arcs</th>
<th>Flow</th>
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<td>Communic. Networks</td>
<td>O-D pairs for messages</td>
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<td>Distribution Networks</td>
<td>plants warehouses,...</td>
<td>highways railway tracks etc.</td>
<td>trucks, trains, etc</td>
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Fact: The internet protocol in most use today was developed in 1981.
On Fractional Flows

- In general, linear multicommodity flow problems have fractional flows, even if all data is integral.

- The integer multicommodity flow problem is difficult to solve to optimality.
A fractional multicommodity flow

\[ u_{ij} = 1 \text{ for all arcs} \]
\[ c_{ij} = 0 \text{ except as listed.} \]

1 unit of flow must be sent from \( s_i \) to \( t_i \) for \( i = 1, 2, 3 \).
A fractional multicommodity flow

\[ u_{ij} = 1 \text{ for all arcs} \]
\[ c_{ij} = 0 \text{ except as listed.} \]

1 unit of flow must be sent from \( s_i \) to \( t_i \) for \( i = 1, 2, 3 \).

Optimal solution: send \( \frac{1}{2} \) unit of flow in each of these 15 arcs.
Total cost = $3.
Decomposition based approaches

Price directed decomposition.

Focus on prices or tolls on the arcs. Then solve the problem while ignoring the capacities on arcs.

Resource directive decomposition.

Allocate flow capacity among commodities and solve

Simplex based approaches

Try to speed up the simplex method by exploiting the structure of the MCF problem.
Minimize

\[ \sum_{1 \leq k \leq K} c^k x^k \]  

subject to

\[ \sum_{1 \leq k \leq K} x^k_{ij} \leq u_{ij} \quad \text{for all } (i, j) \in A \]  

\[ Nx^k = b^k \quad \text{for } k = 1, 2, \ldots, K \]  

\[ 0 \leq x^k_{ij} \leq u^k_{ij} \quad \text{for all } (i, j) \in A \text{ for } k = 1, 2, \ldots, K \]
Theorem. The multicommodity flow \( x = (x^k) \) is an optimal multicommodity flow for (17) if there exists non-negative prices \( w = (w_{ij}) \) on the arcs so that the following is true:

1. If \( w_{ij} > 0 \), then \( \sum_k x_{ij}^k = u_{ij} \)

2. The flow \( x^k \) is optimal for the \( k \)-th commodity if \( c^k \) is replaced by \( c^{w,k} \), where

\[
c_{ij}^{w,k} = c_{ij}^k + w_{ij}
\]

Recall: \( x^k \) is optimal for the \( k \)-th commodity if there is no negative cost cycle in the \( k \)th residual network.
A Linear Multicommodity Flow Problem

5 units good 1

1

$5

4

5 units good 1

2 units good 2

2

$1

3

$6

6

2 units good 2

Set $w_{2,5} = 2$

Create the residual networks

$u_{25} = 5$

$x^2_{32} = x^2_{25} = x^2_{56} = 2$

$x^1_{12} = x^1_{25} = x^1_{54} = 3$

$x^1_{14} = 2$
The residual network for commodity 1

Set $w_{2,5} = \$2$

There is no negative cost cycle.
The residual network for commodity 2

Set $w_{2,5} = \$2$

There is no negative cost cycle.
One can also define node potentials $\pi$ so that the reduced cost

$$c_{ij}^{\pi,k} = c_{ij}^k + w_{ij} - \pi_i^k + \pi_j^k \geq 0$$

for all $(i, j) \in A$ and $k = 1, \ldots, K$

This combines optimality conditions for min cost flows with the partial dualization optimality conditions for multicommodity flows.
According to NPD Fashion World, what percentage of lingerie is returned to the store?

50%

What is the average life span for a taste bud?

10 days

Charles Osborne set the record for the longest case of the hiccups. How long did they last?

68 years. An outside source estimated that Osborne hiccupsed 430 million times over the 68-year period. He also fathered 8 children during this time period.
Outside a barber’s shop, there is often a pole with red and white stripes. What is the significance of the red stripes?

It represents the bloody bandages used in blood-letting wrapped around a pole.

How many digestive glands are in the human stomach?

Around 35 million

What is the surface area of a pair of human lungs?

Around 70 meters$^2$. Approximately the same size as a tennis court.
Lagrangian relaxation for multicommodity flows

\[ \text{Min} \sum_{(i,j) \in A} \sum_k c_{ij}^k x_{ij}^k \]

\[ \sum_j x_{ij}^k - \sum_j x_{ji}^k = \begin{cases} 
  d_k & \text{if } i = s_k \\
  -d_k & \text{if } i \in t_k \\
  0 & \text{otherwise} 
\end{cases} \]

\[ \sum_k x_{ij}^k \leq u_{ij} \quad \text{for all } (i, j) \in A \]

\[ x_{ij}^k \geq 0 \quad \forall (i, j) \in A, \ k \in K \]

Supply/demand constraints

Bundle constraints
Lagrangian relaxation for multicommodity flows

Minimize \[ \sum_{(i,j) \in A} \sum_k c_{ij}^k x_{ij}^k + \sum_{(i,j) \in A} \sum_k w_{ij}( \sum_k x_{ij}^k - u_{ij} ) \]

\[ \sum_j x_{ij}^k - \sum_j x_{ji}^k = \begin{cases} d_k & \text{if } i = s_k \\ -d_k & \text{if } i = t_k \\ 0 & \text{otherwise} \end{cases} \]

\[ \sum_k x_{ij}^k \leq u_{ij} \quad \text{for all } (i,j) \in A \]

\[ x_{ij}^k \geq 0 \quad \forall (i,j) \in A, \quad k \in K \]

Penalize the bundle constraints.

Relax the bundle constraints.
Lagrangian relaxation for multicommodity flows

\[ L(w) = \text{Min} \sum_{(i,j) \in A} \sum_{k} (c_{ij}^k + w_{ij}) x_{ij}^k - \sum_{(i,j) \in A} w_{ij} u_{ij} \]

\[ \sum_{j} x_{ij}^k - \sum_{j} x_{ji}^k = \begin{cases} d_k & \text{if } i = s_k \\ -d_k & \text{if } i \in t_k \\ 0 & \text{otherwise} \end{cases} \]

\[ \sum_{k} x_{ij}^k \leq u_{ij} \text{ for all } (i, j) \in A \]

\[ x_{ij}^k \geq 0 \quad \forall (i, j) \in A, \ k \in K \]

Simplify the objective function.
Choose an initial value $w^0$ of the “tolls” $w$, and find the optimal solution for $L(w)$. 

\[ u_{25} = 5 \quad \text{and} \quad u_{32} = 2 \] 

\[ \text{5 units good 1} \quad \text{5 units good 1} \]

\[ \text{3 units good 2} \quad \text{3 units good 2} \]
Subgradient Optimization for solving the Lagrangian Multiplier Problem

The flow on (2,5) = 8 > u_{25} = 5.

The flow on (3,2) = 3 > u_{32} = 2.
Choosing a search direction

\[ r^+ = \max (0, r) \]

\[ y_{ij} = \sum_k x_{ij}^k \quad = \text{flow in arc } (i,j) \]

\[ w_{ij}^{q+1} = \left[ w_{ij}^q + \theta_q (y_{ij} - u_{ij}) \right]^+ \]

(y-u)^+ is called the search direction.

\[ w_{25}^1 = \left[ w_{25}^0 + \theta_0 (8 - 5) \right]^+ = 3\theta_0 \]

\[ w_{32}^1 = \left[ w_{32}^0 + \theta_0 (3 - 2) \right]^+ = \theta_0 \]

θ_q is called the step size.

So, if we choose \( \theta_0 = 1 \), then \( w_{25}^1 = 3 \) and \( w_{32}^1 = 1 \)

Then solve \( L(w^1) \).
Solving $L(w^1)$

If $\theta^1 = 1$, then $w^2 = 0.$

\[ w^2_{25} = [w^1_{25} + \theta_1(0 - 5)]^+ = [3 - 5\theta_1]^+ \]

\[ w^2_{32} = [w^1_{32} + \theta_1(0 - 2)]^+ = [1 - 2\theta_1]^+ \]
Comments on the step size

- The search direction is a good search direction.

- But the step size must be chosen carefully.

- Too large a step size and the solution will oscillate and not converge

- Too small a step size and the solution will not converge to the optimum.
The step size $\theta_q$ should be chosen so that

$$ \lim_{q \to \infty} \theta_q = 0 \quad \text{and} \quad \sum_{q=1}^{\infty} \theta_q = \infty \quad (1) $$

e.g., take $\theta_q = 1/q$.

**Theorem.** If the step size is chosen as on the previous slides, and if $(\theta_q)$ satisfies (1), then the $w^q$ converges to the optimum for the Lagrangian dual.
The optimal multipliers and flows.

\[ \lim_{q \to \infty} w_{32}^q = 1 \]

\[ \lim_{q \to \infty} w_{25}^q = 2 \]
Suppose that $w_{32} = 1.001$ and $w_{25} = 2.001$

Conclusion: Near Optimal Multipliers do not always lead to near optimal (or even feasible) flows.
Summary of MCF

- Applications

- Optimality Conditions

- Lagrangian Relaxation
  - subgradient optimization

- Next Lecture: Column Generation and more