15.083J/6.859J Integer Optimization

Lecture 6: Ideal formulations II
1 Outline

- Randomized rounding methods

2 Randomized rounding

- Solve $c^*x$ subject to $x \in P$ for arbitrary $c$.
- $x^*$ be optimal solution.
- From $x^*$ create a new random integer solution $x$, feasible in $P$: $E[c'x] = Z_{LP} = c^*x^*$.
- $Z_{LP} \leq Z_{IP} \leq E[Z_{IP}] = Z_{IP}$.
- Hence, $P$ integral.

2.1 Minimum $s - t$ cut

\[
\begin{align*}
\text{minimize} & \quad \sum_{(u,v) \in E} c_{uv}x_{uv} \\
\text{subject to} & \quad x_{uv} \geq y_u - y_v, \quad \{u, v\} \in E, \\
& \quad x_{uv} \geq y_v - y_u, \quad \{u, v\} \in E, \\
& \quad y_s = 1, \\
& \quad y_t = 0, \\
& \quad y_u, x_{uv} \in \{0, 1\}.
\end{align*}
\]

2.1.1 Algorithm

- Solve linear relaxation. Position the nodes in the interval $(0, 1)$ according to the value of $y_u^*$.
- Generate a random variable $U$ uniformly in the interval $[0, 1]$.
- Round all nodes $u$ with $y_u^* \leq U$ to $y_u = 0$, and all nodes $u$ with $y_u^* > U$ to $y_u = 1$. Set $x_{uv} = |y_u - y_v|$ for all $\{u, v\} \in E$.

2.2 Theorem

For every nonnegative cost vector $c$,

\[ E[Z_{IP}] = Z_{IP} = Z_{LP}. \]
\[ y_u \text{ is rounded to } 0 \quad y_v \text{ is rounded to } 1 \]

\[
\begin{align*}
Z_{IP} & \leq E[Z_H] = E \left[ \sum_{(u,v) \in E} c_{uv}x_{uv} \right] \\
& = \sum_{(u,v) \in E} c_{uv}P \left( \min \left( y_u^*, y_v^* \right) \leq U < \max \left( y_u^*, y_v^* \right) \right) \\
& = \sum_{(u,v) \in E} c_{uv} |y_u^* - y_v^*| \\
& = Z_{LP} \leq Z_{IP}
\end{align*}
\]

2.3 Stable matching

- \( n \) men \( \{m_1, \ldots, m_n\} \) and \( n \) women \( \{w_1, \ldots, w_n\} \), with each person having a list of strict preference order.
- Find a stable perfect matching \( M \) of the men to women:
- There does not exist a man \( m \) and a woman \( w \) who are not matched under \( M \), but prefer each other to their assigned mates under \( M \).

2.3.1 Formulation

- \( w_1 \succ_m w_2 \) if man \( m \) prefers \( w_1 \) to \( w_2 \).
- \( m_1 \succ_w m_2 \) if woman \( w \) prefers \( m_1 \) to \( m_2 \).
- Decision variables

\[ x_{ij} = \begin{cases} 
1, & \text{if } m_i \text{ is matched to } w_j, \\
0, & \text{otherwise.}
\end{cases} \]
\[ N = \{1, \ldots, n\} \]
\[
\sum_{j=1}^{n} x_{ij} = 1, \quad i \in N, \\
\sum_{i=1}^{n} x_{ij} = 1, \quad j \in N, \\
x_{ij} \in \{0, 1\}, \quad i, j \in N, \\
x_{ij} + \sum_{k \mid w_k < m_i w_j} x_{ik} + \sum_{k \mid m_k < m_i} x_{kj} \leq 1, \quad i, j \in N.
\]

### 2.3.2 Proposition

If \( x_{ij} > 0 \), then
\[
x_{ij} + \sum_{k \mid w_k < m_i w_j} x_{ik} + \sum_{k \mid m_k < m_i} x_{kj} = 1.
\]

### 2.3.3 Proof

\[
\min \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} \\
\text{s.t. } x \in P_{SM}
\]

**Dual**
\[
\max \sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{n} \beta_j - \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} \\
\text{s.t. } \alpha_i + \beta_j - \sum_{k \mid w_k > m_i w_j} \gamma_{ik} - \sum_{k \mid m_k > m_i} \gamma_{kj} \leq 1, \quad i, j \in N, \\
\gamma_{ij} \geq 0.
\]

\( x \in P_{SM} \). Set
\[
\alpha_i = \sum_{j=1}^{n} \gamma_{ij}, \quad \beta_j = \sum_{i=1}^{n} \gamma_{ij} \text{ and } \gamma_{ij} = x_{ij} \text{ for all } i, j \in N.
\]

- **Dual:**
\[
\gamma_{ij} + \sum_{k \mid w_k < m_i w_j} \gamma_{ik} + \sum_{k \mid m_k < m_i} \gamma_{kj} \leq 1, \quad \forall i, j \in N,
\]
feasible if \( \gamma_{ij} = x_{ij} \) and \( x \in P_{SM} \).

- **Objective**
\[
\sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{n} \beta_j - \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij}.
\]

- **Complementary slackness of optimal primal and dual solutions.**
2.4 Key Theorem

\[ P_{SM} = \text{conv}(S). \]

2.4.1 Randomization

- Generate a random number \( U \) uniformly in \([0,1]\).
- Match \( m_i \) to \( w_j \) if \( x_{ij} > 0 \) and in the row corresponding to \( m_i \), \( U \) lies in the interval spanned by \( x_{ij} \) in \([0,1]\). Accordingly, match \( w_j \) to \( m_i \) if in the row corresponding to \( w_j \), \( U \) lies in the interval spanned by \( x_{ij} \) in \([0,1]\).
- Key property: \( x_{ij} > 0 \), then the intervals spanned by \( x_{ij} \) in rows corresponding to \( m_i \) and \( w_j \) coincide in \([0,1]\).
- The matching is stable: \( w_k \) who is preferred by \( m_i \) to his mate \( w_j \) under the assignment, i.e., the interval spanned by \( x_{ik} \) is on the right of the interval spanned by \( x_{ij} \) in the row corresponding to \( m_i \), is assigned a mate whom she strictly prefers to \( m_i \), since in the row corresponding to \( w_k \) the random number \( U \) lies strictly to the left of the interval \( x_{ik} \).
- \( x_{ij}^U = 1 \) if \( m_i \) and \( w_j \) are matched.

\[ \mathbb{E}[x_{ij}^U] = P(U \text{ lies in the interval spanned by } x_{ij}) = x_{ij}. \]

- \( x_{ij} = \int_0^1 x_{ij}^u \, du \): \( x \) can be written as a convex combination of stable matchings \( x^u \) as \( u \) varies over the interval \([0,1]\).