15.083J/6.859J Integer Optimization

Lecture 7: Ideal formulations III
1 Outline

- Minimal counterexample
- Lift and project

2 Matching polyhedron

\[ P_{\text{matching}} = \left\{ x \mid \sum_{e \in \delta(i)} x_e = 1, \; i \in V, \right. \]

\[ \sum_{e \in \delta(S)} x_e \geq 1, \; S \subset V, \; |S| \text{ odd}, \; |S| \geq 3, \]

\[ 0 \leq x_e \leq 1, \; e \in E \}. \]

- \( F \) set of perfect matchings in \( G \).
- Theorem: For the perfect matching problem \( P_{\text{matching}} = \text{conv}(F) \).

2.1 Proof Outline

- \( \text{conv}(F) \subset P_{\text{matching}} \).
- For reverse: Assume \( G = (V, E) \) is a graph such that \( P_{\text{matching}} \not\subset \text{conv}(F) \), and \( |V| + |E| \) is the smallest.
- \( x \) be an extreme point of \( P_{\text{matching}} \) not in \( \text{conv}(F) \).
- For each edge \( e = \{u, v\}, \; x_e > 0 \), otherwise we could delete \( e \) from \( E \).
- \( x_e < 1 \), otherwise we could replace \( V \) by \( V \setminus \{u, v\} \) and \( E \) by all edges in \( E \) incident to \( V \setminus \{u, v\} \).
- \( |E| > |V| \); otherwise, either \( G \) is disconnected (in this case one of the components of \( G \) will be a smaller counterexample), or \( G \) has a node of degree one (in this case the edge \( e \) incident to \( v \) satisfies \( x_e = 1 \)), or \( G \) is the disjoint union of cycles (in this case the theorem holds trivially).
- \( x \) extreme point of \( P_{\text{matching}} \), there are \( |E| \) linearly independent tight constraints.
- There exists a \( S \subset V \) with \( |S| \) odd, \( |S| \geq 3, \; |V \setminus S| \geq 3 \), and

\[ \sum_{e \in \delta(S)} x_e = 1. \]

- Contract \( V \setminus S \) to a single new node \( u \), to obtain \( G' = (S \cup \{u\}, E') \).
- \( x'_e = x_e \) for all \( e \in E(S) \), and for \( v \in S \),

\[ x'_{\{u,v\}} = \sum_{\{ j \in V \setminus S, (v,j) \in E \}} x_{\{v,j\}}. \]

\( x' \) satisfies constraints with respect to \( G' \).
• As \( G \) is a smallest counterexample, \( x' \) belongs to the convex hull of matchings on \( G' \),

\[ x' = \sum_{M'} \lambda_{M'} x^{M'} \]

• Contract \( S \) to a single new node \( t \) we obtain a graph \( G'' = ((V \setminus S) \cup \{t\}, E'') \) and a vector \( x'' \):

\[ x'' = \sum_{M''} \mu_{M''} x^{M''} \]

• “Glue together” perfect matchings \( M' \) and \( M'' \)

\[ x = \sum_{e \in \delta(S)} \sum_{M' \cap \delta(S) = (e)} \frac{\lambda_{M'} \mu_{M''}}{x_e} x^M \]

3 Lift and project

• \( S = \{ x \in \mathbb{Z}^n \mid Ax \leq b \} \).

• (Lift) Multiply \( Ax \leq b \) by \( x_j \) and \( 1 - x_j \)

\[ (Ax)x_j \leq bx_j \quad (*) \]

\[ (Ax)(1 - x_j) \leq b(1 - x_j) \]

and substitute \( y_{ij} = x_i x_j \) for \( i, j = 1, \ldots, n, i \neq j \) and \( x_j = x_j^2 \). Let \( L_j(P) \) be the resulting polyhedron.

• (Project) Project \( L_j(P) \) back to the \( x \) variables by eliminating variables \( y \).

Let \( P_j \) be the resulting polyhedron, i.e., \( P_j = (L_j(P))_x \).

3.1 Theorem

\( P_j = \text{conv}(P \cap \{ x \in \mathbb{R}^n \mid x_j \in \{0, 1\} \}) \)

Proof:

• \( x' \in P \cap \{ x \in \mathbb{R}^n \mid x_j \in \{0, 1\} \} \) and \( y'_{ij} = x'_i x'_j \).

• Since \( x'_j = (x'_j)^2 \) and \( Ax' \leq b \), \( (x', y') \in L_j(P) \) and thus \( x' \in P_j \). Hence,

\[ \text{conv}(P \cap \{ x \in \mathbb{R}^n \mid x_j \in \{0, 1\} \}) \subseteq P_j \]

• If \( P \cap \{ x \in \mathbb{R}^n \mid x_j = 0 \} = \emptyset \), then from the Farkas lemma there exists \( u \geq 0 \), such that \( u' A = -e_j \) and \( u' b = -1 \). Thus, for all \( x \) satisfying (*) we have

\[ u' Ax(1 - x_j) \leq u' b(1 - x_j). \]

Hence, for all \( x \in P_j \)

\[ -e'_j x(1 - x_j) = -x_j(1 - x_j) \leq -(1 - x_j). \]

Replacing \( x_j^2 \) by \( x_j \), we obtain that \( x_j \geq 1 \) is valid for \( P_j \). Since, in addition, \( P_j \subseteq P \), we conclude that

\[ P_j \subseteq P \cap \{ x \in \mathbb{R}^n \mid x_j = 1 \} = \text{conv}(P \cap \{ x \in \mathbb{R}^n \mid x_j \in \{0, 1\} \}) \].
• Similarly, if \( P \cap \{ x \in \mathbb{R}^n \mid x_j = 1 \} = \emptyset \), then
  \[
P_j \subseteq \text{conv}(P \cap \{ x \in \mathbb{R}^n \mid x_j \in \{0, 1\} \}).
  \]
• Suppose \( P \cap \{ x \in \mathbb{R}^n \mid x_j = 0 \} \neq \emptyset, P \cap \{ x \in \mathbb{R}^n \mid x_j = 1 \} \neq \emptyset \).
  
• We prove that all valid inequalities for \( \text{conv}(P \cap \{ x \in \mathbb{R}^n \mid x_j \in \{0, 1\} \}) \) are also valid for \( P_j \).
• \( a'x \leq \alpha \) a valid inequality for \( \text{conv}(P \cap \{ x \in \mathbb{R}^n \mid x_j \in \{0, 1\} \}) \).
• \( x \in P \). If \( x_j = 0 \), then for all \( \lambda \in \mathcal{R} \) \( a'x + \lambda x_j = a'x \leq \alpha \).
• If \( x_j > 0 \), then there exists \( \lambda \leq 0 \), such that for all \( x \in P \),
  \[
a'x + \lambda x_j \leq \alpha.
  \]
• Analogously, since \( a'x \leq \alpha \) is valid for \( P \cap \{ x \in \mathbb{R}^n \mid x_j = 1 \} \), there exists some \( \nu \leq 0 \) such that for all \( x \in P' \),
  \[
a'x + \nu(1 - x_j) \leq \alpha.
  \]
• For all \( x \) satisfying (*),
  \[
  (1 - x_j)(a'x + \lambda x_j) \leq (1 - x_j)\alpha
  
  x_j(a'x + \nu(1 - x_j)) \leq x_j\alpha.
  \]
• Hence,
  \[
a'x + (\lambda + \nu)(x_j - x_j^2) \leq \alpha.
  \]
• After setting \( x_j^2 = x_j \) we obtain that for all \( x \in P_j \), \( a'x \leq \alpha \), thus all valid inequalities for \( \text{conv}(P \cap \{ x \in \mathbb{R}^n \mid x_j \in \{0, 1\} \}) \) are also valid for \( P_j \), and thus \( P_j \subseteq \text{conv}(P \cap \{ x \in \mathbb{R}^n \mid x_j \in \{0, 1\} \}) \).

3.2 Example

\( P = \{(x_1, x_2) \mid 2x_1 - x_2 \geq 0, 2x_1 + x_2 \leq 2, x_1 \geq 0, x_2 \geq 0\} \).

\[
\begin{align*}
2x_1^2 - x_1x_2 & \geq 0 \\
2x_1(1 - x_1) - x_2(1 - x_1) & \geq 0 \\
2x_1^2 + x_1x_2 & \leq 2x_1 \\
2x_1(1 - x_1) + x_2(1 - x_1) & \leq 2(1 - x_1) \\
x_1^2 & \geq 0 \\
x_1(1 - x_1) & \geq 0 \\
x_2x_1 & \geq 0 \\
x_2(1 - x_1) & \geq 0.
\end{align*}
\]
\[ y = x_1 x_2, \quad x_1^2 = x_1 \]

\[
\begin{align*}
2x_1 - y & \geq 0 \\
-x_2 + y & \geq 0 \\
y & \leq 0 \\
x_2 - y & \leq 2 - 2x_1 \\
x_1 & \geq 0 \\
0 & \geq 0 \\
y & \geq 0 \\
x_2 - y & \geq 0.
\end{align*}
\]

This implies that \( y = 0, \)

\[
\begin{align*}
x_1 & \geq 0 \\
-x_2 & \geq 0 \\
x_2 & \leq 2 - 2x_1 \\
x_1 & \geq 0 \\
x_2 & \geq 0,
\end{align*}
\]

which leads to

\[
P_i = \{(x_1, x_2) | 0 \leq x_1 \leq 1, \ x_2 = 0\} = \text{conv}(P \cap \{(x_1, x_2) | x_1 \in \{0, 1\}\}).
\]

### 3.3 Convex hull

- \( P_{i_1, i_2, \ldots, i_t} = (P_{i_1})_{i_2 \ldots i_t} \).
- Theorem: The polyhedron \( P_{i_1, i_2, \ldots, i_t} \) satisfies:
  \[
P_{i_1, \ldots, i_t} = \text{conv}(P \cap \{x \in \mathbb{R}^n | x_i \in \{0, 1\}, \ i \in \{i_1, \ldots, i_t\}\}).
\]
- \( P_{1, \ldots, n} = P_1. \)