Gomory-Chvátal cuts

Reminder

- $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ with $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$.
- For $\lambda \in [0, 1)^m$ such that $\lambda^T A \in \mathbb{Z}^n$, 
  
  \[(\lambda^T A)x \leq \lfloor \lambda^T b \rfloor\]

  is valid for all integral points in $P$.

Stable Sets

Definitions

- Let $G = (V, E)$ be an undirected graph.
- $S \subseteq V$ is stable if $\{\{u, v\} \in E : u, v \in S\} = \emptyset$.
- Stable sets are the integer solutions to:
  
  \[x_u + x_v \leq 1 \quad \text{for all } \{u, v\} \in E\]
  \[x_v \geq 0 \quad \text{for all } v \in V\]

- The stable set polytope is
  
  \[P_{\text{stab}}(G) = \text{conv}\{x \in \{0, 1\}^V : x_u + x_v \leq 1 \text{ for all } u, v \in E\}\]

Odd Cycle Inequalities

- An odd cycle $C$ in $G$ consists of an odd number of vertices $0, 1, \ldots, 2k$ and edges $\{i, i + 1\}$.
- The odd cycle inequality
  
  \[\sum_{v \in C} x_v \leq \frac{|C| - 1}{2}\]

  is valid for $P_{\text{stab}}(G)$.

- It has a cutting-plane proof that only needs one step of rounding.

- The separation problem for the class of odd cycle inequalities can be solved in polynomial time:
  
  - Let $y \in \mathbb{Q}^V$. 
• We may assume that \( y \geq 0 \) and \( y_u + y_v \leq 1 \) for all \( \{u, v\} \in E \).

• Define, for each edge \( e = \{u, v\} \in E \), \( z_e := 1 - y_u - y_v \).

• So \( z_e \geq 0 \) for all \( e \in E \).

• \( y \) satisfies all odd cycle constraints iff \( z \) satisfies

\[
\sum_{e \in C} z_e \geq 1 \text{ for all odd cycles } C.
\]

• If we view \( z_e \) as the “length” of edge \( e \), then \( y \) satisfies all odd cycle inequalities iff the length of a shortest odd cycle is at least 1.

### Shortest Odd Cycles

• A shortest odd cycle can be found in polynomial time:

• Split each node \( v \in V \) into two nodes \( v_1 \) and \( v_2 \).

• For each arc \( (u, v) \) create new arcs \( (u_1, v_2) \) and \( (u_2, v_1) \), both of the same length as \( (u, v) \).

• Let \( D' \) be the digraph constructed this way.

• For each \( v \in V \) find the shortest \( (v_1, v_2) \)-path in \( D' \).

• The shortest among these paths gives us the shortest odd cycle.

### Perfect Matchings

#### Definitions

• Let \( G = (V, E) \) be an undirected graph.

• A matching \( M \subseteq E \) is perfect if \( |M| = |V|/2 \).

• Perfect matchings are the integer solutions to:

\[
\sum_{e \in \delta(v)} x_e = 1 \quad \text{for all } v \in V
\]

\[
x_e \geq 0 \quad \text{for all } e \in E
\]

• The perfect matching polytope is

\[
P_{PM}(G) = \text{conv} \left\{ x \in \{0, 1\}^E : \sum_{e \in \delta(v)} x_e = 1 \text{ for all } v \in V \right\}
\]
Odd Cut Inequalities

- The following inequalities are valid for $P_{PM}(G)$:
  \[ \sum_{e \in \delta(U)} x_e \geq 1 \text{ for all } U \subset V, |U| \text{ odd} \]

- Each has a cutting-plane proof that requires rounding only once.

- The separation problem for this class of inequalities can be solved in polynomial time.

\{0,1/2\}-cuts

Definition

- Let
  \[ F_{1/2}(A, b) := \{ (\lambda^\top A)x \leq \lceil \lambda^\top b \rceil : \lambda \in \{0,1/2\}^m, \lambda^\top A \in \mathbb{Z}^n \} \]

  be the family of all \{0,1/2\}-cuts.

Question: Can one separate efficiently over $F_{1/2}(A, b)$?

NP-Hardness

**Theorem 1.** Let $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, and $y \in \mathbb{Q}^n$ with $Ay \leq b$. Checking whether $y$ violates some inequality in $F_{1/2}(A, b)$ is NP-complete.

Preliminaries

- Let $P = \{ x : Ax \leq b \}$ and $y \in P$.

  $y$ violates a \{0,1/2\}-cut iff there exists $\mu \in \{0,1\}^m$ such that
  - $\mu^\top A \equiv 0 \pmod{2}$,
  - $\mu^\top b \equiv 1 \pmod{2}$, and
  - $\mu^\top (b - Ax) < 1$.

  (Because $\mu^\top b = 2k + 1$ for some $k \in \mathbb{Z}$, and $\mu^\top A x \leq 2k$ can be written as $\mu^\top (b - Ax) \geq 1$.)

An NP-complete Problem

- Given $Q \in \{0,1\}^{r \times t}$, $d \in \{0,1\}^r$, and a positive integer $K$, decide whether there exists $z \in \{0,1\}^t$ with at most $K$ 1’s such that $Qz \equiv d \pmod{2}$. 

Reduction

- Let \( w := \frac{1}{K+1} \mathbf{1} \) and consider \( P = \{ x \in \mathbb{R}^n : Ax \leq b \} \) with:

\[
A := \begin{pmatrix} Q^T & 2I_{t+1} \end{pmatrix}, \quad b := \begin{pmatrix} 2 \cdot 1^t \\ 1 \end{pmatrix}, \quad y := \begin{pmatrix} 0^t \\ 1 \end{pmatrix}
\]

Proof Sketch

Step 1: Show \((A, b, y)\) is a valid instance.

- \( y \in P \): Observe that \( b - Ay = (w_1, \ldots, w_t, 0)^\top \geq 0 \).

Proof sketch

Step 2: Equivalence of “Yes”-instances.

- \( \exists \mu \in \{0, 1\}^m \) with \( \mu^\top A \equiv 0 \pmod{2}, \mu^\top b \equiv 1 \pmod{2} \) iff \( \exists z \in \{0, 1\}^t \) such that \( Qz \equiv d \pmod{2} \):

\[
A := \begin{pmatrix} Q^T & 2I_{t+1} \end{pmatrix}, \quad b := \begin{pmatrix} 2 \cdot 1^t \\ 1 \end{pmatrix}
\]

Proof sketch

Step 2: Equivalence of “Yes”-instances.

- \( \exists \mu \text{ s.th. } \mu^\top (b - Ay) < 1 \text{ iff } \exists z \text{ s.th. } w^\top z < 1 \iff 1^\top z \leq K \):

\[
\mu^\top (b - Ay) = \mu^\top (w_1, \ldots, w_t, 0)^\top
\]

Primal Separation

The Primal Separation Problem

- Let \( P \) be a 0/1-polytope.

- Given a point \( y \in \mathbb{Q}^n \) and a vertex \( \hat{x} \in P \), find \( c \in \mathbb{Z}^n \) and \( d \in \mathbb{Z} \) such that \( cx \leq d \) for all \( x \in P \), \( c\hat{x} = d \), and \( cy > d \), if they exist.

Theorem 2. For 0/1-polytopes, optimization and primal separation are polynomial-time equivalent.
Perfect Matchings

- Let $\hat{x}$ be the incidence vector of a perfect matching $M$.
- Let $y \in \mathbb{Q}_+^E$ be a point satisfying the node-degree equations.
- We have to find a min-weight odd cut (w.r.t. the edge weights given by $y$) among those that intersect $M$ in exactly one edge.
- Let $\{s, t\} \in M$ be an arbitrary edge of $M$.
- Let $G_{\{s, t\}}$ be the graph obtained from $G$ by contracting the end nodes of all edges $e \in M \setminus \{\{s, t\}\}$.
- The minimum weight odd cut among those that contain exactly the edge $\{s, t\}$ of $M$ can be computed by finding a min-weight $\{s, t\}$-cut in $G_{\{s, t\}}$.

**Theorem 3.** The primal separation problem for the perfect matching polytope of a graph $G = (V, E)$ can be solved with $|V|/2$ max-flow computations.

**Corollary 4.** A minimum weight perfect matching can be computed in polynomial time.

**Proof Sketch**

Primal Separation

\[\downarrow\]

Verification

\[\downarrow\]

Augmentation

\[\downarrow\]

Optimization

**The Verification Problem**

- Let $P \subseteq \mathbb{R}^n$ be a 0/1-polytope.
- Given an objective function $c \in \mathbb{Z}^n$ and a vertex $\hat{x} \in P$, decide whether $\hat{x}$ minimizes $cx$ over $P$. 
Primal Separation ⇒ Verification
• Let $C$ be the cone defined by the linear inequalities of $P$ that are tight at $\hat{x}$.
• By LP duality, $\hat{x}$ minimizes $cx$ over $P$ iff $\hat{x}$ minimizes $cx$ over $C$.
• By the equivalence of optimization and separation, minimizing $cx$ over $C$ is equivalent to solving the separation problem for $C$.
• One can solve the separation problem for $C$ by solving the primal separation problem for $P$ and $\hat{x}$.

The Augmentation Problem
• Let $P \subseteq \mathbb{R}^n$ be a 0/1-polytope.
• Given an objective function $c \in \mathbb{Z}^n$ and a vertex $x \in P$, find a vertex $x' \in P$ such that $cx' < cx$, if one exists.

Verification ⇒ Augmentation
• We may assume that $x = 1$.
• Use “Verification” to check whether $x$ is optimal. If not:

\[
M := \sum_{i=1}^{n} |c_i| + 1;
\]
for $i = 1$ to $n$ do
\[
c_i := c_i - M;
\]
call the verification oracle with input $x$ and $c$;
if $x$ is optimal then
\[
y_i := 0;
\]
c_i := c_i + M
else
\[
y_i := 1
\]
return $y$. 

Augmentation ⇒ Optimization
• We may assume that $c \geq 0$.
• Let $C := \max\{c_i : i = 1, \ldots, n\}$, and $K := \lfloor \log C \rfloor$.
• For $k = 0, \ldots, K$, define $c^k$ by $c^k_i := \lfloor c_i / 2^{K-k} \rfloor$, $i = 1, \ldots, n$.
for $k = 0, 1, \ldots, K$ do
\[
\text{while} \ x^k \text{ is not optimal for } c^k \text{ do}
\]
x^k := AUG($x^k$, $c^k$)
x^{k+1} := x^k
return $x^{K+1}$. 


Running Time

- $O(\log C)$ many phases.
- At the end of phase $k - 1$, $x^k$ is optimal with respect to $c^{k-1}$, and hence for $2c^{k-1}$.
- Moreover, $c^k = 2c^{k-1} + c(k)$, for some 0/1-vector $c(k)$.
- If $x^{k+1}$ denotes the optimal solution for $c^k$ at the end of phase $k$, we obtain
  
  $c^k(x^k - x^{k+1}) = 2c^{k-1}(x^k - x^{k+1}) + c(k)(x^k - x^{k+1}) \leq n.$

- Thus, the algorithm determines an optimal solution by solving at most $O(n \log C)$ augmentation problems.