15.093 Optimization Methods

Lecture 6: Duality Theory II
1 Outline

- Geometry of duality
- The dual simplex algorithm
- Farkas lemma
- Duality as a proof technique

2 The Geometry of Duality

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad a_i^T x \geq b_i, \quad i = 1, \ldots, m
\end{align*}
\]

\[
\begin{align*}
\max & \quad p^T b \\
\text{s.t.} & \quad \sum_{i=1}^{m} p_i a_i = c \\
& \quad p \geq 0
\end{align*}
\]

3 Dual Simplex Algorithm

3.1 Motivation

- In simplex method $B^{-1}b \geq 0$
- Primal optimality condition

\[c' - c_B' B^{-1} A \geq 0'\]

same as dual feasibility
• Simplex is a **primal algorithm**: maintains **primal feasibility** and works towards **dual feasibility**

• **Dual algorithm**: maintains **dual feasibility** and works towards **primal feasibility**

<table>
<thead>
<tr>
<th>$-c'_B x_B$</th>
<th>$\bar{c}_1$</th>
<th>\ldots</th>
<th>$\bar{c}_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{B(1)}$</td>
<td>\vdots</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_{B(m)}$</td>
<td>$B^{-1} A_1$</td>
<td>\ldots</td>
<td>$B^{-1} A_n$</td>
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</tbody>
</table>

• Do not require $B^{-1} b \geq 0$

• Require $\bar{c} \geq 0$ (dual feasibility)

• Dual cost is

$$p'b = c'_B B^{-1} b = c'_B x_B$$

• If $B^{-1} b \geq 0$ then both dual feasibility and primal feasibility, and also same cost ⇒ **optimality**

• Otherwise, change basis

### 3.2 An iteration

1. Start with basis matrix $B$ and all reduced costs $\geq 0$.

2. If $B^{-1} b \geq 0$ optimal solution found; else, choose $l$ s.t. $x_{B(l)} < 0$. 

...
3. Consider the $l$th row (pivot row) $x_{B(l)}, v_1, \ldots, v_n$. If $\forall i \; v_i \geq 0$ then dual optimal cost $= +\infty$ and algorithm terminates.

4. Else, let $j$ s.t.
\[
\bar{e}_j = \min_{\{i | v_i < 0\}} \frac{\bar{e}_i}{|v_i|}
\]

5. Pivot element $v_j$: $A_j$ enters the basis and $A_{B(l)}$ exits.

### 3.3 An example

\[
\begin{align*}
\text{min} & \quad x_1 + x_2 \\
\text{s.t.} & \quad x_1 + 2x_2 \geq 2 \\
& \quad x_1 \geq 1 \\
& \quad x_1, x_2 \geq 0 \\
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad x_1 + x_2 \\
\text{s.t.} & \quad x_1 + 2x_2 - x_3 = 2 \\
& \quad x_1 - x_4 = 1 \\
& \quad x_1, x_2, x_3, x_4 \geq 0 \\
\end{align*}
\]

\[
\begin{align*}
\text{max} & \quad 2p_1 + p_2 \\
\text{s.t.} & \quad p_1 + p_2 \leq 1 \\
& \quad 2p_1 \leq 1 \\
& \quad p_1, p_2 \geq 0 \\
\end{align*}
\]

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
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<tbody>
<tr>
<td>$x_3$</td>
<td>-2</td>
<td>-1</td>
<td>-2$^*$</td>
<td>1</td>
</tr>
<tr>
<td>$x_4$</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
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<tbody>
<tr>
<td>$x_2$</td>
<td>-1</td>
<td>1/2</td>
<td>0</td>
<td>1/2</td>
</tr>
<tr>
<td>$x_4$</td>
<td>1</td>
<td>1/2</td>
<td>1</td>
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\text{max} & \quad 2p_1 + p_2 \\
\text{s.t.} & \quad p_1 + p_2 \leq 1 \\
& \quad 2p_1 \leq 1 \\
& \quad p_1, p_2 \geq 0 \\
\end{align*}
\]
4 Duality as a proof method

4.1 Farkas lemma

Theorem:
Exactly one of the following two alternatives hold:

1. \( \exists x \geq 0 \text{ s.t. } Ax = b \).

2. \( \exists p \text{ s.t. } p^T A \geq 0' \text{ and } p^T b < 0 \).

4.1.1 Proof

\( \Rightarrow \): If \( \exists x \geq 0 \text{ s.t. } Ax = b \), and if \( p^T A \geq 0' \), then \( p^T b = p^T Ax \geq 0 \)

\( \Leftarrow \): Assume there is no \( x \geq 0 \text{ s.t. } Ax = b \)

\[ \begin{array}{c|cccc}
& x_1 & x_2 & x_3 & x_4 \\
\hline
x_2 = & -3/2 & 0 & 0 & 1/2 & 1/2 \\
x_1 = & 1/2 & 0 & 1 & -1/2 & 1/2 \\
\end{array} \]

\( (P) \max \begin{array}{c}
0'x \\
\text{s.t. } Ax = b \\
x \geq 0
\end{array} \quad (D) \min \begin{array}{c}
p'b \\
\text{s.t. } p^T A \geq 0' \\
x \geq 0
\end{array} \]

(\( P \) infeasible \( \Rightarrow \) (\( D \)) either unbounded or infeasible
Since \( p = 0 \) is feasible \( \Rightarrow \) (\( D \)) unbounded
\( \Rightarrow \exists p : \ p^T A \geq 0' \text{ and } p^T b < 0 \)