15.093 Optimization Methods

Lecture 24: Semidefinite Optimization
1 Outline

1. Minimizing Polynomials as an SDP
2. Linear Difference Equations and Stabilization
3. Barrier Algorithm for SDO

2 SDO formulation

2.1 Primal and dual

\[ (P): \min \ C \cdot X \]
\[ \text{s.t.} \ A_i \cdot X = b_i \quad i = 1, \ldots, m \]
\[ X \geq 0 \]

\[ (D): \max \sum_{i=1}^{m} y_i b_i \]
\[ \text{s.t.} \ C - \sum_{i=1}^{m} y_i A_i \geq 0 \]

3 Minimizing Polynomials

3.1 Sum of squares

- A polynomial \( f(x) \) is a \textit{sum of squares} (SOS) if
  \[ f(x) = \sum g_j^2(x) \]
  for some polynomials \( g_j(x) \).
- A polynomial satisfies \( f(x) \geq 0 \) for all \( x \in \mathcal{R} \) if and only if it is a sum of squares.
- \textbf{Not} true in more than one variable!

3.2 Proof

- \((\Leftarrow)\) Obvious. If \( f(x) = \sum g_j^2(x) \) then \( f(x) \geq 0 \).
3.3 SOS and SDO

- Let $zx = (1, x, x^2, \ldots, x^k)'$.
- $f(x) = z(x)'Qz(x)$ is a sum of squares if and only if
  
  $$f(x) = z(x)'Qz(x),$$

  where $Q \succeq 0$, i.e., $Q = L'L$.

- Then, $f(x) = z(x)'L'L(x) = ||Lz(x)||^2$.

3.4 Formulation

- Consider $\min f(x)$.

- Then, $f(x) \geq \gamma$ if and only if $f(x) - \gamma = zx'Qzx$ with $Q \succeq 0$. This implies linear constraints on $\gamma$ and $Q$.

- Reformulation

  $$\max \gamma$$

  $$\text{s.t.} \begin{cases} f(x) - \gamma = z(x)'Qz(x) \\ Q \succeq 0 \end{cases}$$

3.5 Example

3.5.1 Reformulation

$$\min f(x) = 3 + 4x + 2x^2 + 2x^3 + x^4.$$ $$f(x) - \gamma = 3 - \gamma + 4x + 2x^2 + 2x^3 + x^4 = (1, x, x^2)'Q(1, x, x^2).$$

$$\max \gamma$$

$$\text{s.t.} \begin{cases} 3 - \gamma = q_{11} \\ 4 = 2q_{12}, \quad 2 = 2q_{13} + q_{22} \\ 2 = 2q_{23}, \quad 1 = q_{33} \\ Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \succeq 0 \end{cases}$$

Extensions to multiple dimensions.
4 Stability

- A linear difference equation
  \[ x(k + 1) = Ax(k), \quad x(0) = x_0 \]
- \( x(k) \) converges to zero iff \(|\lambda_i(A)| < 1, i = 1, \ldots , n\)
- Characterization:
  \[ |\lambda_i(A)| < 1 \quad \forall i \iff \exists P > 0 \quad A'PA - P < 0 \]

4.1 Proof

- \((\implies)\) Let \( Av = \lambda v \). Then,
  \[ 0 > v' (A'PA - P)v = (|\lambda|^2 - 1) v' P v, \]
  and therefore \(|\lambda| < 1\)

- \((\impliedby)\) Let \( P = \sum_{i=0}^{\infty} A'^i QA^i \), where \( Q > 0 \). The sum converges by the eigenvalue assumption. Then,
  \[ A'PA - P = \sum_{i=1}^{\infty} A'^i QA^i - \sum_{i=0}^{\infty} A'^i QA^i = -Q < 0 \]

4.2 Stabilization

- Consider now the case where \( A \) is not stable, but we can change some elements, e.g., \( A(L) = A + LC \), where \( C \) is a fixed matrix.
- Want to find an \( L \) such that \( A + LC \) is stable.
- Use Schur complements to rewrite the condition:
  \[
  (A + LC)' P (A + LC) - P < 0, \quad P > 0
  \]
  \[
  \begin{bmatrix}
  P & (A + LC)' P \\
  P (A + LC) & P
  \end{bmatrix} > 0
  \]
  Condition is nonlinear in \((P, L)\)

4.3 Changing variables

- Define a new variable \( Y := PL \)
  \[
  \begin{bmatrix}
  P & A'P + C'Y' \\
  PA + YC & P
  \end{bmatrix} > 0
  \]
- This is linear in \((P, Y)\).
- Solve using SDO, recover \( L \) via \( L = P^{-1} Y \)
5 Primal Barrier Algorithm for SDO

- $X \succeq 0 \iff \lambda_1(X) \geq 0, \ldots, \lambda_n(X) \geq 0$

- Natural barrier to repel $X$ from the boundary $\lambda_1(X) > 0, \ldots, \lambda_n(X) > 0$:

$$- \sum_{j=1}^{n} \log(\lambda_j(X)) =$$

$$- \log(\prod_{j=1}^{n} \lambda_j(X)) = - \log(\det(X))$$

- Logarithmic barrier problem

$$\min \quad B_\mu(X) = C \bullet X - \mu \log(\det(X))$$

s.t. $A_i \bullet X = b_i, \ i = 1, \ldots, m,$

$X > 0$

- Derivative: $\nabla B_\mu(X) = C - \mu X^{-1}$

Follows from

$$\log \det(X + H) \approx \log \det(X) + \text{tr}(X^{-1} H) + \cdots$$

- KKT conditions

$$A_i \bullet X = b_i, \ i = 1, \ldots, m, \quad C - \mu X^{-1} = \sum_{i=1}^{m} y_i A_i,$$

$$X > 0,$$

- Given $\mu$, need to solve these nonlinear equations for $X, C, y_i$

- Apply Newton’s method until we are “close” to the optimal

- Reduce value of $\mu$, and iterate until the duality gap is small

5.1 Another interpretation

- Recall the optimality conditions:

$$A_i \bullet X = b_i, \ i = 1, \ldots, m, \quad \sum_{i=1}^{m} y_i A_i + S = C,$$

$$X, S \succeq 0,$$

$$XS = 0$$

- Cannot solve directly. Rather, perturb the complementarity condition to $XS = \mu I$.

- Now, unique solution for every $\mu > 0$ (the “central path”)

- Solve using Newton, for decreasing values of $\mu$. 
6 Differences with LO

- Many different ways to linearize the nonlinear complementarity condition

\[ X S = \mu I \]

- Want to preserve symmetry of the iterates
- Several search directions.

7 Convergence

7.1 Stopping criterion

- The point \((X, y_i)\) is feasible, and has duality gap:

\[ C \cdot X - \sum_{i=1}^{m} y_i b_i = \mu X^{-1} \cdot X = n \mu \]

- Therefore, reducing \(\mu\) always decreases the duality gap
- Barrier algorithm needs \(O\left(\sqrt{n} \log \frac{\epsilon_0}{\epsilon}\right)\) iterations to reduce duality gap from \(\epsilon_0\) to \(\epsilon\)

8 Conclusions

- SDO is a powerful modeling tool
- Barrier and primal-dual algorithms are very powerful
- Many good solvers available: ScDuMi, SDPT3, SDPA, etc.
- Pointers to literature and solvers:
  
  www-user.tu-chemnitz.de/~helmberg/semidef.html