15.401 Recitation
1: Present Value
Learning Objectives

- **Review of Concepts**
  - Compounding/discounting
  - PV/FV
  - Real vs. nominal rate
  - Annuities and perpetuities

- **Examples**
  - CD
  - Auto loan
  - Scholarship fund
  - Project planning
Review: Compounding / Discounting

- We can...
  - move money forward in time by **compounding**.
  - move money backward in time by **discounting**.

\[ x(1+r)^{m-n} \]

\[ (1+r)^{m-n} \cdot Y \]

\[ x(1+r)^{s-n} \]

\[ (1+r)^{s-n} \cdot Y \]

\[ t = m \]

\[ n \]

\[ Y \]

\[ s \]

- **Note:**
  - Only relative time matters
  - Multiplying by \((1+r)^{m-n}\) = dividing by \((1+r)^{n-m}\).
Review: APR vs. EAR

- **Annual percentage rate (APR) vs. equivalent annual return (EAR):**

  \[
  \text{EAR} = \left(1 + \frac{\text{APR}}{N}\right)^N - 1 \quad (N = \text{comp. freq.})
  \]

- **Note:**
  - *always* use the EAR when compounding and discounting
  - Due to interest compounding, the EAR is higher than the APR whenever the compounding frequency is higher than once a year.
Continuous Compounding (optional)

- Given a fixed APR, higher compounding frequency leads to higher EAR. Suppose we take compounding frequency to infinity, then

\[
\text{EAR}_\infty = \lim_{n \to \infty} \left(1 + \frac{\text{APR}}{N}\right)^N - 1 = e^{\text{APR}} - 1.
\]

\[\left(e = 2.71828183\ldots\right)\]

- The continuously compounded EAR is the highest possible EAR for a given APR.
Review: PV / FV

- **Cash flow:**

  \[ C_0 \quad C_1 \quad C_2 \quad \ldots \quad C_{T-1} \quad C_T \quad C_T \quad \ldots \]

  \[ 0 \quad 1 \quad 2 \quad \ldots \quad T-1 \quad T \quad T+1 \quad \ldots \text{ periods} \]

  - Present value (PV):

    \[ PV_0 = C_0 + \frac{C_1}{(1+r)^1} + \frac{C_2}{(1+r)^2} + \ldots \]

  - Future value (FV):

    \[ FV_T = C_0(1+r)^T + C_1(1+r)^{T-1} + \ldots \]

    \[ + C_T(1+r)^0 + C_{T+1}(1+r)^{-1} + \ldots \]
Review: Nominal vs. Real Interest Rate

- Nominal-real interest rate conversion:
  \[ 1 + r_{\text{real}} = \frac{1 + r_{\text{nominal}}}{1 + i} \]

- Nominal-real cash flow conversion:
  \[ C_{\text{real}} = \frac{C_{\text{nominal}}}{1 + i} \]

- When you discount or compound,
  - Either use the **nominal** cash flow and the **nominal** interest rate
  - Or use the **real** cash flow and the **real** interest rate
  - **Do not** mix and match
Review: Annuity/Perpetuity

- **Annuity:**

  \[
  PV_0 = \frac{C}{r} \left[ 1 - \frac{1}{(1+r)^T} \right]
  \]

- **Perpetuity:**

  \[
  PV_0 = \frac{C}{r}
  \]
Review: Growing Annuity/Perpetuity

- **Growing Annuity:**
  
  
  \[
  \begin{align*}
  &\quad C \quad C(1+g) \quad \ldots \quad C(1+g)^{T-2} \quad C(1+g)^{T-1} \\
  &0 \quad 1 \quad 2 \quad \ldots \quad T-1 \quad T \text{ periods}
  \end{align*}
  \]
  \[
  PV_0 = \frac{C}{r-g} \left[ 1 - \frac{(1+g)^T}{(1+r)^T} \right].
  \]

- **Growing Perpetuity** \((r > g)\):
  
  
  \[
  \begin{align*}
  &\quad C \quad C(1+g) \quad C(1+g)^2 \quad \ldots \ldots \\
  &0 \quad 1 \quad 2 \quad 3 \quad \ldots \ldots \text{periods}
  \end{align*}
  \]
  \[
  PV_0 = \frac{C}{r-g}.
  \]
Example 1: CD

- You can invest $10,000 in a CD offered by your bank. The CD matures in 5 years and the bank quotes you a rate of 4.5%. How much will you have in 5 years, if the 4.5% is
  a) EAR
  b) Quarterly APR
  c) Monthly APR
Example 1: CD

☐ Answer:

a) $10,000 \times (1.045)^5 = $12,461.82

b) $r_{\text{EAR}} = \left(1 + \frac{0.045}{4}\right)^4 = 1.04576$

$10,000 \times (1.04576)^5 = $12,507.51

c) $r_{\text{EAR}} = \left(1 + \frac{0.045}{12}\right)^{12} = 1.04594$

$10,000 \times (1.04594)^5 = $12,517.96
Example 2: Auto Loan

You would like to buy a new car for $22,000. The dealer requires a down payment of $10,000 and offers you 6% APR financing (compounded monthly) for 5 years for the remaining balance. What is your monthly payment?
Example 2: Auto Loan

Answer: let $C$ be the monthly payment, then

$$22000 = \frac{C}{0.06/12} \left[ 1 - \frac{1}{(1 + 0.06/12)^{12\times5}} \right] + 10000.$$ 

$C = \$231.99.$
Example 3: Scholarship Fund

- You would like to establish a scholarship fund that will help outstanding students with financial difficulties pay their college tuition.
  - Starting today, you hope to give 50 students $20,000 each in today’s money (i.e., adjusted for inflation) every year.
  - The effective nominal interest rate is 5%/yr.
  - Inflation is 2%/yr.

- How much money do you need now if you want the fund to last forever?
Example 3: Scholarship Fund

Answer:
- Method 1: nominal amount + nominal interest rate
  \[ 1m + \frac{1m \times 1.02}{1.05 - 1.02} = 35m. \]

- Method 2: real amount + real interest rate
  \[ r_{\text{real}} = \frac{1.05}{1.02} - 1 = 2.9412\% \quad \Rightarrow \quad 1m + \frac{1m}{0.029412} = 35m. \]

- Note: same answer!

You need $35 million today.
Example 4: Project Planning

GeneriCorp is considering whether or not to expand into a new market. The company faces the following cash flow (in $million) if it decides to expand:

- $-200$ in year 0
- $-400$ in year 1
- $-300$ in year 2
- $+100$ in year 3
- $+500$ in year 4
- $+600$ in year 5

A committee appointed by the CEO determined that the appropriate discount rate is 9%. Should the company take on the expansion project?
Example 4: Project Planning

- **Answer:**

\[
\text{NPV} = -200 - \frac{400}{1.09} - \frac{300}{1.09^2} + \frac{100}{1.09^3} + \frac{500}{1.09^4} + \frac{600}{1.09^5}
\]

\[
= $1.91m.
\]

- **Positive NPV = take the project; though NPV is dangerously close to zero.**