15.401 Recitation
5: Options
Learning Objectives

- Review of Concepts
  - Payoff profile
  - Put-call parity
  - Valuation of options
  - Binomial tree

- Examples
  - Payoff replication
  - Arboreal Corporation
Review: elements of a call/put option

- **Type:**
  - Call: holder has the right but not the obligation to buy
  - Put: holder has the right but not the obligation to sell

- **Quantity of the underlying asset:**
  - Usually one share of stock with current price $S$

- **Strike/exercise price ($K$)**

- **Expiration date ($T$)**

- **Style:**
  - European: can only be exercised at $T$
  - American: can be exercised at any time between 0 and $T$. 
Review: payoff profile

<table>
<thead>
<tr>
<th></th>
<th>Call</th>
<th>Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long</td>
<td><img src="Call_Long.png" alt="Payoff" /></td>
<td><img src="Put_Long.png" alt="Payoff" /></td>
</tr>
<tr>
<td>Short</td>
<td><img src="Call_Short.png" alt="Payoff" /></td>
<td><img src="Put_Short.png" alt="Payoff" /></td>
</tr>
</tbody>
</table>
Review: payoff profile

- The payoff of a portfolio of options is the sum of payoffs of the individual components:

\[ \text{1 put} + \text{1 call} = \text{Straddle} \]

\[ \text{1 call} @ K_1 + \text{-2 call} @ K_1 + \text{1 call} @ K_3 = \text{Butterfly spread} \]
Review: put-call parity

- Two portfolios with identical payoffs

- 1 call @ $K$

- 1 put @ $K$

- 1 stock

- Bond w/ FV=$K$

$$K$$
Review: put-call parity

- No arbitrage implies that the two portfolios must have the same cost:

\[ C + PV(K) = P + S \]

\[ C + \frac{K}{(1+r)^T} = P + S \]

- This is the **put-call parity**.
- Note: the call and put must have the same exercise price (K).
Review: value of an option

<table>
<thead>
<tr>
<th></th>
<th>Value of call</th>
<th>Value of put</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strike price (K)</td>
<td>Decrease</td>
<td>Increase</td>
</tr>
<tr>
<td>Price of underlying asset (S)</td>
<td>Increase</td>
<td>Decrease</td>
</tr>
<tr>
<td>Volatility of the underlying asset (σ)</td>
<td>Increase</td>
<td>Increase</td>
</tr>
<tr>
<td>Maturity (T)</td>
<td>Increase</td>
<td>Increase</td>
</tr>
<tr>
<td>Interest rate (r)</td>
<td>Increase</td>
<td>Decrease</td>
</tr>
</tbody>
</table>
Review: binomial tree

- Idea: if there are only two states of the world next period, we can price options given the underlying asset and a risk-free asset ("bond") by replication:

```
Underlying Asset          Bond                  Call
S                      p                     S_u
                       1-p                   B
                        S_d                 B/(1+r)
                        B                    B
                        C                   C_u
                        C_d
```
Review: binomial tree

- Replication:

<table>
<thead>
<tr>
<th></th>
<th>CF at t = 0</th>
<th>CF at t=1 (“up” state)</th>
<th>CF at t=1 (“down” state)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A shares of underlying asset</td>
<td>-A x S</td>
<td>A x $S_u$</td>
<td>A x $S_d$</td>
</tr>
<tr>
<td>Bond (FV=B)</td>
<td>- B/(1+r)</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>-A x S - B/(1+r)</td>
<td>A x $S_u$ + B</td>
<td>A x $S_d$ + B</td>
</tr>
<tr>
<td><strong>Replication</strong></td>
<td>= -C</td>
<td>= $C_u$</td>
<td>= $C_d$</td>
</tr>
</tbody>
</table>

- $A = (C_u - C_d) / (S_u - S_d)$
- $B = C_u - A x S_u$
- $C = A x S + B/(1+r)$
Review: binomial tree

- Equivalently, we can solve for the risk-neutral probability, $q$:
  \[ S = \frac{qS_u + (1-q)S_d}{1+r} \]

- Then,
  \[ C = \frac{qC_u + (1-q)C_d}{1+r} \]

- Note: $q$ is not related to the state probability $p$. In fact, $p$ is not used in the pricing of $C$. 
Example 1: payoff replication

How would you replicate the following payoff profile using only call and put options?

a) 

b)
Example 1: payoff replication

Answer:

a) Long 1 call (K=10)
   Short 1 call (K=15)
   Short 1 call (K=25)
   Long 1 call (K=30)

b) Long 1 put (K=8)
   Short 1 call (K=8)
   Long 2 calls (K=12)
   Short 1 call (K=20)
Example 2: Arboreal Corporation

- Arboreal Corporations stock price is currently $102. At the end of 3 months it will be either $120 or $90. The 3-month spot rate is 2%. What is the value of a 3-month European call option with a strike price of $110?

![Stock and Call Diagram]

\[ S = 102, \quad K = 110, \quad r = 0.02, \quad \Delta t = 0.25 \]

- The value of the call option can be calculated using the Black-Scholes formula or a binomial model. The formula for the call option value is:

\[ C = S - Ke^{-r\Delta t} \]

- Substituting the values:

\[ C = 102 - 110e^{-0.02 \times 0.25} = 102 - 110 \times 0.995 = 102 - 109.475 = 2.525 \]

- Therefore, the value of the call option is approximately $2.525.
Example 2: Arboreal Corporation

- The call can be replicated with:
  - Long 1/3 stock: costs $34
  - Short bond with FV=30: costs \(-\frac{30}{1+2\%}\) = -$29.41

- The price of the call must be
  \[ C = 34 - 29.41 = 4.59 \]

- Alternatively, we can solve for the risk-neutral probability:
  \[ \frac{120q + 90(1-q)}{1+2\%} = 102 \Rightarrow q = 0.468 \]

- The price of the call is then
  \[ C = \frac{10(0.468) + 0(1-0.468)}{1+2\%} = 4.59 \]