15.401 Recitation
6: Portfolio Choice
Learning Objectives

- Review of Concepts
  - Portfolio basics
  - Efficient frontier
  - Capital market line

- Examples
  - XYZ
  - Diversification
  - Sharpe ratio
  - Efficient frontier
Review: portfolio basics

- A portfolio is a collection of $N$ assets ($A_1, A_2, \ldots, A_N$) with weights ($w_1, w_2, \ldots, w_N$) that satisfy
  \[ \sum_{i=1}^{N} w_i = 1 \]

- Each asset $A_i$ has the following characteristics:
  - Return: $\bar{r}_i$ (random variable)
  - Mean return: $\bar{r}_i$
  - Variance and std. dev. of return: $\sigma_i^2, \sigma_i$
  - Covariance with $A_j$: $\sigma_{ij}$
Review: portfolio basics

- The return of a portfolio is
  \[ \bar{r}_p = \sum_{i=1}^{N} w_i \bar{r}_i \]

- The mean/expected return of a portfolio is
  \[ E(r_p) = \bar{r}_p = \sum_{i=1}^{N} w_i \bar{r}_i \]

- The variance of a portfolio is
  \[ \sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} ; \quad \sigma_p = \sqrt{\sigma_p^2} \]

- Note: \( \sigma_{ii} \equiv \sigma_i^2 ; \quad \sigma_{ij} = \rho_{ij} \sigma_i \sigma_j \)
Example 1: XYZ

<table>
<thead>
<tr>
<th></th>
<th>E(r)</th>
<th>Variance-Covariance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>X</td>
<td>15%</td>
<td>0.090</td>
</tr>
<tr>
<td>Y</td>
<td>10%</td>
<td></td>
</tr>
<tr>
<td>Z</td>
<td>20%</td>
<td></td>
</tr>
</tbody>
</table>

- What is the expected return and variance of a portfolio of ...
  a. (X, Y) with weights (0.4, 0.6)?
  b. (X, Y, Z) with weights (0.2, 0.5, 0.3)?
  c. (X, Y, Z) with weights (1/3, 1/3, 1/3)?
Example 1: XYZ

Answer:

a. $E(r_p) = 12\%; \sigma_p^2 = 0.08880; \sigma_p = 29.80\%$

b. $E(r_p) = 14\%; \sigma_p^2 = 0.10133; \sigma_p = 31.83\%$

c. $E(r_p) = 15\%; \sigma_p^2 = 0.13567; \sigma_p = 36.83\%$
Example 1: XYZ

- What is the minimum possible variance of a portfolio with only Y and Z?

<table>
<thead>
<tr>
<th></th>
<th>E(r)</th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>X</td>
<td>Y</td>
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<tr>
<td>X</td>
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<td>0.090</td>
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<tr>
<td>Y</td>
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<td>0.040</td>
</tr>
<tr>
<td>Z</td>
<td>20%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example 1: XYZ

Answer:
Let \((w, 1-w)\) be the weights for \((Y, Z)\), then

\[
\arg\min_w \left[ w^2 \cdot 0.04 + 2w(1-w)(-0.036) + (1-w)^2 \cdot 0.625 \right]
\]

First-order condition:

\[
2w \cdot 0.04 + 2(1-2w)(-0.036) - 2(1-w) \cdot 0.625 = 0
\]

\[w^* = 0.8969\]

The minimum variance portfolio is

\((0.8969, 0.1031)\)
Example 2: diversification

- Suppose that your portfolio consists of $N$ equally weighted identical assets in the market, each of which has the following properties:
  - Mean $= 15\%$
  - Std dev $= 20\%$
  - Covariance with any other asset $= 0.01$

- What is the expected return and std dev of return of your portfolio if...
  - $N = 2$?
  - $N = 5$?
  - $N = 10$?
  - $N = \infty$?
Example 2: diversification

Answer:

⊙ Expected return

\[ E(r_p) = \sum_{i=1}^{N} \frac{1}{N} \cdot 0.15 = 0.15 \]

⊙ Variance

\[
\sigma(r_p) = \sum_{i=1}^{N} \frac{0.2^2}{N^2} + \sum_{i=1}^{N} \sum_{j \neq i} \frac{0.01}{N^2} = N\left(\frac{0.2^2}{N^2}\right) + N(N-1)\frac{0.01}{N^2}
\]

\[ = \frac{0.04}{N} + \left(1 - \frac{1}{N}\right)0.01 = 0.01 + \frac{0.03}{N} \]
Example 2: diversification

Answer:

- $N = 2$:
  \[ E(r_p) = 15\%; \sigma_p^2 = 0.0250; \sigma_p = 15.81\% \]

- $N = 5$:
  \[ E(r_p) = 15\%; \sigma_p^2 = 0.0160; \sigma_p = 12.65\% \]

- $N = 10$:
  \[ E(r_p) = 15\%; \sigma_p^2 = 0.0130; \sigma_p = 11.40\% \]

- $N = \infty$:
  \[ E(r_p) = 15\%; \sigma_p^2 = 0.0100; \sigma_p = 10.00\% \]
Example 2: diversification

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Review: diversification

Idiosyncratic risk can be diversified away; investors are not compensated for such risk.

Systematic risk cannot be diversified away; investors are compensated with higher expected returns.
Review: efficient frontier

- Given two assets, we can form portfolios with weights \((w, 1-w)\). As we vary \(w\), we can plot the path of the mean return and standard deviation of return of the resulting portfolio.
- The shape of the path depends on the correlation between the two assets.
- When the correlation is low, a large portion of asset return variation comes from idiosyncratic risk that can be diversified away.
Review: efficient frontier

- $\rho = 1$
  - perfectly correlated
  - no risk reduction potential
- $-1 < \rho < 1$
  - imperfectly correlated
  - some risk reduction potential
- $\rho = -1$
  - perfectly negatively correlated
  - most risk reduction potential

Image by MIT OpenCourseWare.
Review: efficient frontier

We can repeat the previous exercise for $N$ assets:
Review: efficient frontier

- The efficient frontier can be described by a function $\sigma^*(r_p)$, which minimizes the portfolio std dev given an expected return:

$$
\sigma^*(r_p) \equiv \min_{\{w_i\}} \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}} \quad \text{s.t.} \quad \begin{cases}
\sum_{i=1}^{N} w_i = 1 \\
\sum_{i=1}^{N} w_i \bar{r}_i = r_p
\end{cases}
$$

- Analytical solution for $\sigma^*(r_p)$ is possible but difficult to derive.
Review: capital market line

- Efficient frontier + risk-free asset = CML

Image by MIT OpenCourseWare.
Example 3: Sharpe ratio

- The Sharpe ratio measures the reward-risk tradeoff of an asset or a portfolio. It is defined as

\[ S = \frac{\bar{r} - r_f}{\sigma} \]

- The higher Sharpe ratio, the more desirable an asset / a portfolio is. Suppose \( r_f = 5\% \). What is the portfolio of (A, B) with the highest Sharpe ratio?

<table>
<thead>
<tr>
<th></th>
<th>E(r)</th>
<th>COV-VAR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>A</td>
</tr>
<tr>
<td>A</td>
<td>15%</td>
<td>0.090</td>
</tr>
<tr>
<td>B</td>
<td>10%</td>
<td></td>
</tr>
</tbody>
</table>
Example 3: Sharpe ratio

Answer:

\[
\max_w S_p = \max_w \frac{wr_A + (1-w)r_B - r_f}{\sqrt{w^2 \sigma_A^2 + 2w(1-w)\sigma_{AB} + (1-w)^2 \sigma_B^2}}
\]

Method 1: grid search

1. Set up a grid for \(w\), e.g., \(w = 0, 0.1, 0.2, ..., 1.0\)
   The finer the grid, the more accurate the result

2. Calculate the Sharpe ratio for each \(w\)

3. Find the maximum Sharpe ratio.
Example 3: Sharpe ratio

- **Method 1: grid search**

<table>
<thead>
<tr>
<th>( w )</th>
<th>( 1-w )</th>
<th>( r_p - r_f )</th>
<th>( \sigma_p )</th>
<th>( S_p )</th>
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<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.0500</td>
<td>0.2000</td>
<td>0.2500</td>
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<td>0.0600</td>
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<tr>
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<td>0.0650</td>
<td>0.1844</td>
<td>0.3525</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
<td>0.0700</td>
<td>0.1897</td>
<td>0.3689</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.0750</td>
<td>0.2000</td>
<td>0.3750</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4</td>
<td>0.0800</td>
<td>0.2145</td>
<td>0.3730</td>
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<tr>
<td>0.7</td>
<td>0.3</td>
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<tr>
<td>1</td>
<td>0</td>
<td>0.1000</td>
<td>0.3000</td>
<td>0.3333</td>
</tr>
</tbody>
</table>
Example 3: Sharpe ratio

Maximum:
\[ w^* = 0.52 \]
\[ E(r) = 12.60\% \]
\[ \sigma = 20.26\% \]
\[ \text{Max } S = 0.3752 \]
Example 3: Sharpe ratio

- **Method 2: Excel Solver**

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>E(r)</td>
<td>Asset A</td>
<td>Asset B</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Asset A</td>
<td>=B3</td>
<td>=B4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Asset A</td>
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<td>0.015</td>
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<tr>
<td>4</td>
<td>Asset B</td>
<td>=1–B3</td>
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<td>0.015</td>
<td>0.04</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ r_p - r_f \quad \sigma_p \quad S \]

- f: `SUMPRODUCT(B3:B4, C3:C4) - 0.05`
- g: `SQRT(B3*D2*D3+B3*E2*E3+B4*D2*D4+B4*E2*E4)`

**Solver**

- Set Target Cell: `$E$7`
- Equal To: `Max`
- By Changing Cell: `$B$3`
Example 3: Sharpe ratio

- Method 2: Excel Solver

<table>
<thead>
<tr>
<th>A</th>
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<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>E(r)</td>
<td>Asset A</td>
<td>Asset B</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.52</td>
<td>0.48</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Asset A</td>
<td>0.15</td>
<td>0.09</td>
<td>0.015</td>
</tr>
<tr>
<td>3</td>
<td>Asset B</td>
<td>0.48</td>
<td>0.1</td>
<td>0.015</td>
</tr>
<tr>
<td>4</td>
<td>0.076</td>
<td>0.202583</td>
<td>0.375154</td>
<td></td>
</tr>
</tbody>
</table>

\[ r_p - r_f \quad \sigma_p \quad S \]

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Example 3: Sharpe ratio

- Method 3: analytical solution
  - Full derivation:

\[
\frac{\partial S}{\partial w} = \frac{(\bar{r}_A - \bar{r}_B)\left(\sigma_p^2\right)^{\frac{1}{2}} - \frac{1}{2} \left(\sigma_p^2\right)^{\frac{1}{2}} \left(2w \sigma_A^2 + 2(1-2w) \sigma_{AB} - 2(1-w) \sigma_B^2\right) (\bar{r}_p - r_f)}{\left(\sigma_p^2\right)^{\frac{1}{2}}}
\]

\[
= \frac{(\bar{r}_A - \bar{r}_B)\left(w^2 \sigma_A^2 + 2w(1-w) \sigma_{AB} + (1-w)^2 \sigma_B^2\right) - \left(w \sigma_A^2 + (1-2w) \sigma_{AB} - (1-w) \sigma_B^2\right) (w\bar{r}_A + (1-w)\bar{r}_B - r_f)}{\sigma_p^2}
\]

\[= 0\]

\[0 = (\bar{r}_A - \bar{r}_B)\left(w^2 \sigma_A^2 + 2w(1-w) \sigma_{AB} + (1-w)^2 \sigma_B^2\right) - \left(w \sigma_A^2 + (1-2w) \sigma_{AB} - (1-w) \sigma_B^2\right) (w\bar{r}_A + (1-w)\bar{r}_B - r_f)
\]

\[= (\bar{r}_A - \bar{r}_B)\left(w \sigma_{AB} + (1-w) \sigma_B^2\right) - \left(w \sigma_A^2 + (1-2w) \sigma_{AB} - (1-w) \sigma_B^2\right) (w\bar{r}_A + (1-w)\bar{r}_B - r_f)
\]

\[= (\bar{r}_A - \bar{r}_B)\left(\sigma_{AB} - \sigma_B^2\right) [\bar{r}_B - r_f] - [w \sigma_A^2 + (1-2w) \sigma_{AB} - (1-w) \sigma_B^2] [w \bar{r}_A + (1-w)\bar{r}_B - r_f]
\]

\[= \left[\left(\bar{r}_A - r_f\right) \sigma_B^2 - \left(\bar{r}_B - r_f\right) \sigma_{AB}\right] + \left[\left(\bar{r}_A - r_f\right) \sigma_A^2 + \left(\bar{r}_B - r_f\right) \sigma_{AB}\right] w
\]

\[w^* = \frac{\left(\bar{r}_A - r_f\right) \sigma_A^2 - \left(\bar{r}_B - r_f\right) \sigma_{AB}}{\left(\bar{r}_A - r_f\right) \sigma_A^2 + \left(\bar{r}_B - r_f\right) \sigma_{AB}}
\]

\[= 0.52\]
Example 3: Sharpe ratio

- Method 3: analytical solution
  - Result only:
    The general solution for the 2-asset Sharpe ratio maximization problem is

\[
W^* = \frac{\left(\overline{r}_A - r_f\right)\sigma_B^2 - \left(\overline{r}_B - r_f\right)\sigma_{AB}}{\left(\overline{r}_A - r_f\right)\left(\sigma_B^2 - \sigma_{AB}\right) + \left(\overline{r}_B - r_f\right)\left(\sigma_A^2 - \sigma_{AB}\right)}
\]
Example 4: efficient frontier

Given the risky assets A and B in the previous question, what is the efficient frontier?

<table>
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<tr>
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<th>E(r)</th>
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<tbody>
<tr>
<td></td>
<td></td>
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</tr>
<tr>
<td>A</td>
<td>15%</td>
<td>0.090</td>
</tr>
<tr>
<td>B</td>
<td>10%</td>
<td></td>
</tr>
</tbody>
</table>

Given 5% risk-free rate, what is the capital market line?
Example 4: efficient frontier

Table from the previous question:

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>$w$</td>
<td>$1-w$</td>
<td>$r_p$</td>
<td>$\sigma_p$</td>
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</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0.1500</td>
<td>0.3000</td>
</tr>
</tbody>
</table>
Example 4: efficient frontier

- Scatter plot of $(r_p, \sigma_p)$ pairs:
Example 4: efficient frontier

- **Capital market line:**

  - **Tangency portfolio:**
    - $w = 0.52$
    - $E(r) = 12.60\%$
    - $\sigma = 20.26\%$
    - $Max\ S = 0.3752$

  CML is the line passing through (0, 0.05) and tangent to the efficient frontier.
Example 4: efficient frontier

- The moral of the story:
  - The CML is tangent to the efficient frontier at the tangency portfolio.
  - The tangency portfolio is the portfolio of risky assets that maximizes the Sharpe ratio.
  - The slope of the CML is the maximum Sharpe ratio.
  - Rational investors always hold a combination of the tangency portfolio and the risk-free asset. The proportion depends on investors’ risk preferences.
Sneak Peak: CAPM

- The **tangency portfolio** is the **market portfolio**.
- An asset’s **systematic risk** is measured by **beta**, which is defined as the **correlation** of its return and the market return, normalized by the variance of market return:

\[
\beta_i = \frac{\sigma_{im}}{\sigma_m^2}
\]

- Since investors are only compensated for **systematic risk**, asset return is an increasing function of beta:

\[
E(\tilde{r}_i) = r_f + \beta_i(\tilde{r}_i - r_f)
\]