15.401 Finance Theory

MIT Sloan MBA Program

Andrew W. Lo
Harris & Harris Group Professor, MIT Sloan School

Lecture 13–14: Risk Analytics and Portfolio Theory

© 2007–2008 by Andrew W. Lo
Critical Concepts

- Motivation
- Measuring Risk and Reward
- Mean-Variance Analysis
- The Efficient Frontier
- The Tangency Portfolio

Readings:
- Brealey, Myers, and Allen Chapters 7 and 8.1
Motivation

What Is A Portfolio and Why Is It Useful?

- A portfolio is simply a specific combination of securities, usually defined by portfolio weights that sum to 1:

\[ \omega = \{ \omega_1, \omega_2, \ldots, \omega_n \} \]

\[ \omega_i = \frac{N_i P_i}{N_1 P_1 + \cdots + N_n P_n} \]

\[ 1 = \omega_1 + \omega_2 + \cdots + \omega_n \]

- Portfolio weights can sum to 0 (dollar-neutral portfolios), and weights can be positive (long positions) or negative (short positions).

- Assumption: Portfolio weights summarize all relevant information.
Motivation

Example:

- Your investment account of $100,000 consists of three stocks: 200 shares of stock A, 1,000 shares of stock B, and 750 shares of stock C. Your portfolio is summarized by the following weights:

<table>
<thead>
<tr>
<th>Asset</th>
<th>Shares</th>
<th>Price/Share</th>
<th>Dollar Investment</th>
<th>Portfolio Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>200</td>
<td>$50</td>
<td>$10,000</td>
<td>10%</td>
</tr>
<tr>
<td>B</td>
<td>1,000</td>
<td>$60</td>
<td>$60,000</td>
<td>60%</td>
</tr>
<tr>
<td>C</td>
<td>750</td>
<td>$40</td>
<td>$30,000</td>
<td>30%</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>$100,000</td>
<td>100%</td>
</tr>
</tbody>
</table>
Motivation

Example (cont):

- Your broker informs you that you only need to keep $50,000 in your investment account to support the same portfolio of 200 shares of stock A, 1,000 shares of stock B, and 750 shares of stock C; in other words, you can buy these stocks on margin. You withdraw $50,000 to use for other purposes, leaving $50,000 in the account. Your portfolio is summarized by the following weights:

<table>
<thead>
<tr>
<th>Asset</th>
<th>Shares</th>
<th>Price/Share</th>
<th>Dollar Investment</th>
<th>Portfolio Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>200</td>
<td>$50</td>
<td>$10,000</td>
<td>20%</td>
</tr>
<tr>
<td>B</td>
<td>1,000</td>
<td>$60</td>
<td>$60,000</td>
<td>120%</td>
</tr>
<tr>
<td>C</td>
<td>750</td>
<td>$40</td>
<td>$30,000</td>
<td>60%</td>
</tr>
<tr>
<td>Riskless Bond</td>
<td>−$50,000</td>
<td>$1</td>
<td>−$50,000</td>
<td>−100%</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td></td>
<td><strong>$50,000</strong></td>
<td><strong>100%</strong></td>
</tr>
</tbody>
</table>
Motivation

Example:

- You decide to purchase a home that costs $500,000 by paying 20% of the purchase price and getting a mortgage for the remaining 80%. What are your portfolio weights for this investment?

<table>
<thead>
<tr>
<th>Asset</th>
<th>Shares</th>
<th>Price/Share</th>
<th>Dollar Investment</th>
<th>Portfolio Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Home</td>
<td>1</td>
<td>$500,000</td>
<td>$500,000</td>
<td>500%</td>
</tr>
<tr>
<td>Mortgage</td>
<td>1</td>
<td>−$400,000</td>
<td>−$400,000</td>
<td>−400%</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>$100,000</td>
<td>100%</td>
</tr>
</tbody>
</table>

- What happens to your total assets if your home price declines by 15%?
**Motivation**

**Example:**

- You own 100 shares of stock A, and you have shorted 200 shares of stock B. Your portfolio is summarized by the following weights:

<table>
<thead>
<tr>
<th>Stock</th>
<th>Shares</th>
<th>Price/Share</th>
<th>Dollar Investment</th>
<th>Portfolio Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>100</td>
<td>$50</td>
<td>$5,000</td>
<td>???</td>
</tr>
<tr>
<td>B</td>
<td>−200</td>
<td>$25</td>
<td>−$5,000</td>
<td>???</td>
</tr>
</tbody>
</table>

- Zero net-investment portfolios do not have portfolio weights in percentages (because the denominator is 0)—we simply use dollar amounts instead of portfolio weights to represent long and short positions.
Motivation

Why Not Pick The Best Stock Instead of Forming a Portfolio?

- We don’t know which stock is best!
- Portfolios provide *diversification*, reducing unnecessary risks.
- Portfolios can enhance performance by focusing bets.
- Portfolios can customize and manage risk/reward trade-offs.

How Do We Construct a “Good” Portfolio?

- What does “good” mean?
- What characteristics do we care about for a given portfolio?
  - Risk and reward
  - Investors like higher expected returns
  - Investors dislike risk
Measuring Risk and Reward

- Reward is typically measured by return
- Higher returns are better than lower returns.
- But what if returns are unknown?
- Assume returns are random, and consider the distribution of returns.

Several possible measures:
- Mean: central tendency.
- 75%: upper quartile.
- 5%: losses.
Measuring Risk and Reward

- How about risk?
- Likelihood of loss (negative return).
- But loss can come from positive return (e.g., short position).
- A symmetric measure of dispersion is variance or standard deviation.

Variance Measures Spread:
- Blue distribution is “riskier”.
- Extreme outcomes more likely.
- This measure is symmetric.
Measuring Risk and Reward

Assumption

- Investors like high expected returns but dislike high volatility
- Investors care only about the expected return and volatility of their overall portfolio
  - Not individual stocks in the portfolio
  - Investors are generally assumed to be well-diversified

Key questions: How much does a stock contribute to the risk and return of a portfolio, and how can we choose portfolio weights to optimize the risk/reward characteristics of the overall portfolio?
Mean-Variance Analysis

Objective

- Assume investors focus only on the expected return and variance (or standard deviation) of their portfolios: higher expected return is good, higher variance is bad
- Develop a method for constructing optimal portfolios
Mean-Variance Analysis

Basic Properties of Mean and Variance For Individual Returns:

\[
\text{Mean} = \mathbb{E}[R_i] = \mu_i \\
\text{Variance} = \text{Var}[R_i] = \mathbb{E}[(R_i - \mu_i)^2] = \sigma_i^2 \\
\text{Standard Deviation} = \sqrt{\text{Var}[R_i]} = \sigma_i
\]

Basic Properties of Mean And Variance For Portfolio Returns:

\[
R_p = \omega_1 R_1 + \omega_2 R_2 + \cdots + \omega_n R_n \\
\mathbb{E}[R_p] = \omega_1 \mu_1 + \omega_2 \mu_2 + \cdots + \omega_n \mu_n \\
= \mu_p \text{ (Weighted Average)}
\]
Mean-Variance Analysis

Variance Is More Complicated:

\[
\begin{align*}
\text{Var}[R_p] &= \mathbb{E}[(R_p - \mu_p)^2] \\
&= \mathbb{E}\left[ \left( \omega_1(R_1 - \mu_1) + \omega_2(R_2 - \mu_2) + \cdots + \omega_n(R_n - \mu_n) \right)^2 \right] \\
\mathbb{E}[\omega_i\omega_j(R_i - \mu_i)(R_j - \mu_j)] &= \omega_i\omega_j\text{Cov}[R_i, R_j] \\
&= \omega_i\omega_j\sigma_{ij} \\
&= \omega_i\omega_j\sigma_i\sigma_j\rho_{ij}
\end{align*}
\]
# Mean-Variance Analysis

Portfolio variance is the weighted sum of all the variances and covariances:

\[
\begin{array}{|c|c|c|c|}
\hline
 & \omega_1(R_1 - \mu_1) & \omega_2(R_2 - \mu_2) & \cdots & \omega_n(R_n - \mu_n) \\
\hline
\omega_1(R_1 - \mu_1) & \omega_1^2\sigma_1^2 & \omega_1\omega_2\sigma_{12} & \cdots & \omega_1\omega_n\sigma_{1n} \\
\omega_2(R_2 - \mu_2) & \omega_2\omega_1\sigma_{21} & \omega_2^2\sigma_2^2 & \cdots & \omega_2\omega_n\sigma_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\omega_n(R_n - \mu_n) & \omega_n\omega_1\sigma_{n1} & \omega_n\omega_2\sigma_{n2} & \cdots & \omega_n^2\sigma_n^2 \\
\hline
\end{array}
\]

- There are \( n \) variances, and \( n^2 - n \) covariances
- Covariances dominate portfolio variance
- Positive covariances increase portfolio variance; negative covariances decrease portfolio variance (diversification)
Consider the special case of two assets:

\[ R_p = \omega_a R_a + \omega_b R_b \]

\[
\begin{align*}
E[R_p] &= \omega_a \mu_a + \omega_b \mu_b \\
\text{Var}[R_p] &= \omega_a^2 \sigma_a^2 + \omega_b^2 \sigma_b^2 + 2\omega_a \omega_b \text{Cov}[R_a, R_b] \\
&= \omega_a^2 \sigma_a^2 + \omega_b^2 \sigma_b^2 + 2\omega_a \omega_b \sigma_a \sigma_b \rho_{ab}
\end{align*}
\]

Because \( \rho_{ab} \equiv \frac{\text{Cov}[R_a, R_b]}{\sigma_a \sigma_b} \)

\[ \text{Cov}[R_a, R_b] = \sigma_a \sigma_b \rho_{ab} \]

- As correlation increases, overall portfolio variance increases
Mean-Variance Analysis

Example: From 1946 – 2001, Motorola had an average monthly return of 1.75% and a std dev of 9.73%. GM had an average return of 1.08% and a std dev of 6.23%. Their correlation is 0.37. How would a portfolio of the two stocks perform?

\[
\begin{align*}
E[R_p] &= \omega_{GM} 1.08 + \omega_{MOT} 1.75 \\
\text{Var}[R_p] &= \omega_{GM}^2 6.23^2 + \omega_{MOT}^2 9.73^2 + \\
&\quad 2\omega_{GM}\omega_{MOT} (0.37 \times 6.23 \times 9.73)
\end{align*}
\]

<table>
<thead>
<tr>
<th>(w_{MOT})</th>
<th>(w_{GM})</th>
<th>(E[R_p])</th>
<th>(\text{var}(R_p))</th>
<th>(\text{stdev}(R_p))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1.08</td>
<td>38.8</td>
<td>6.23</td>
</tr>
<tr>
<td>0.25</td>
<td>0.75</td>
<td>1.25</td>
<td>36.2</td>
<td>6.01</td>
</tr>
<tr>
<td>0.50</td>
<td>0.50</td>
<td>1.42</td>
<td>44.6</td>
<td>6.68</td>
</tr>
<tr>
<td>0.75</td>
<td>0.25</td>
<td>1.58</td>
<td>64.1</td>
<td>8.00</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1.75</td>
<td>94.6</td>
<td>9.73</td>
</tr>
<tr>
<td>1.25</td>
<td>-0.25</td>
<td>1.92</td>
<td>136.3</td>
<td>11.67</td>
</tr>
</tbody>
</table>
Mean-Variance Analysis

Mean/SD Trade-Off for Portfolios of GM and Motorola

- Expected Return vs. Standard Deviation of Return

GM
Motorola

- GM: 0.5%, 1.7%, 2.1%
- Motorola: 0.9%, 1.3%, 1.7%
Mean-Variance Analysis

Example (cont): Suppose the correlation between GM and Motorola changes. What if it equals –1.0? 0.0? 1.0?

\[
E[R_p] = \omega_{GM} 1.08 + \omega_{MOT} 1.75 \\
Var[R_p] = \omega_{GM}^2 6.23^2 + \omega_{MOT}^2 9.73^2 + 2\omega_{GM}\omega_{MOT}(\rho_{GM,MOT} \times 6.23 \times 9.73)
\]

<table>
<thead>
<tr>
<th>(w_{Mot})</th>
<th>(w_{GM})</th>
<th>(E[R_p])</th>
<th>(corr = -1)</th>
<th>(corr = 0)</th>
<th>(corr = 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1.08%</td>
<td>6.23%</td>
<td>6.23%</td>
<td>6.23%</td>
</tr>
<tr>
<td>0.25</td>
<td>0.75</td>
<td>1.25</td>
<td>2.24</td>
<td>5.27</td>
<td>7.10</td>
</tr>
<tr>
<td>0.50</td>
<td>0.50</td>
<td>1.42</td>
<td>1.75</td>
<td>5.78</td>
<td>7.98</td>
</tr>
<tr>
<td>0.75</td>
<td>0.25</td>
<td>1.58</td>
<td>5.74</td>
<td>7.46</td>
<td>8.85</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1.75</td>
<td>9.73</td>
<td>9.73</td>
<td>9.73</td>
</tr>
</tbody>
</table>
Mean-Variance Analysis

Mean/SD Trade-Off for Portfolios of GM and Motorola

Expected Return vs. Standard Deviation of Return for different correlation coefficients (corr=1, corr=0, corr=-0.5, corr=-1).

- corr=1
- corr=0
- corr=-0.5
- corr=-1

The graph shows the trade-off between mean return and standard deviation for various correlation coefficients.
Mean-Variance Analysis

Example: In 1980, you were thinking about investing in GD. Over the subsequent 10 years, GD had an average monthly return of 0.00% and a std dev of 9.96%. Motorola had an average return of 1.28% and a std dev of 9.33%. Their correlation is 0.28. How would a portfolio of the two stocks perform?

\[
\begin{align*}
E[R_p] &= \omega_{GD} 0.00 + \omega_{MOT} 1.28 \\
Var[R_p] &= \omega_{GD}^2 9.96^2 + \omega_{MOT}^2 9.33^2 + 2\omega_{GD}\omega_{MOT} (0.28 \times 9.96 \times 9.33)
\end{align*}
\]

<table>
<thead>
<tr>
<th>(w_{Mot})</th>
<th>(w_{GD})</th>
<th>(E[R_P])</th>
<th>(\text{var}(R_P))</th>
<th>(\text{stdev}(R_P))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.00</td>
<td>99.20</td>
<td>9.96</td>
</tr>
<tr>
<td>0.25</td>
<td>0.75</td>
<td>0.32</td>
<td>71.00</td>
<td>8.43</td>
</tr>
<tr>
<td>0.50</td>
<td>0.50</td>
<td>0.64</td>
<td>59.57</td>
<td>7.72</td>
</tr>
<tr>
<td>0.75</td>
<td>0.25</td>
<td>0.96</td>
<td>64.92</td>
<td>8.06</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1.28</td>
<td>87.05</td>
<td>9.33</td>
</tr>
</tbody>
</table>
Mean-Variance Analysis

Mean/SD Trade-Off for Portfolios of GD and Motorola

- GD
- Motorola

Expected Return vs. Standard Deviation of Return

© 2007–2008 by Andrew W. Lo
Mean-Variance Analysis

**Example:** You are trying to decide how to allocate your retirement savings between Treasury bills and the stock market. The T-Bill rate is 0.12% monthly. You expect the stock market to have a monthly return of 0.75% with a standard deviation of 4.25%.

\[
\begin{align*}
E[R_p] &= \omega_{\text{TBill}} 0.12 + \omega_{\text{STK}} 0.75 \\
\text{Var}[R_p] &= \omega_{\text{TBill}}^2 0.02 + \omega_{\text{STK}}^2 4.25^2 + 2\omega_{\text{TBill}}\omega_{\text{STK}} (0.00 \times 0.00 \times 4.25) \\
\sigma_p &\equiv \sqrt{\text{Var}[R_p]} = \omega_{\text{STK}} 4.25
\end{align*}
\]

<table>
<thead>
<tr>
<th>$w_{\text{STK}}$</th>
<th>$w_{\text{Tbill}}$</th>
<th>$E[R_p]$</th>
<th>$\text{var}(R_p)$</th>
<th>$\text{stdev}(R_p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.12</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.33</td>
<td>0.67</td>
<td>0.33</td>
<td>1.97</td>
<td>1.40</td>
</tr>
<tr>
<td>0.67</td>
<td>0.33</td>
<td>0.54</td>
<td>8.11</td>
<td>2.85</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0.75</td>
<td>18.06</td>
<td>4.25</td>
</tr>
</tbody>
</table>
Mean-Variance Analysis

Mean/SD Trade-Off for Portfolios of T-Bills and The Stock Market

Expected Return vs. Standard Deviation of Return for different portfolios.
Mean-Variance Analysis

Summary

\[
\begin{align*}
\mathbb{E}[R_p] & = \omega_a \mu_a + \omega_b \mu_b \\
\text{Var}[R_p] & = \omega_a^2 \sigma_a^2 + \omega_b^2 \sigma_b^2 + 2\omega_a \omega_b \sigma_a \sigma_b \rho_{ab}
\end{align*}
\]

Observations

- \( \mathbb{E}[R_p] \) is a weighted average of stocks’ expected returns
- \( \text{SD}(R_p) \) is smaller if stocks’ correlation is lower. It is less than a weighted average of the stocks’ standard deviations (unless perfect correlation)
- The graph of portfolio mean/SD is nonlinear
- If we combine T-Bills with any risky stock, portfolios plot along a straight line
Mean-Variance Analysis

The General Case:

\[ \mathbb{E}[R_p] = \omega_1 \mu_1 + \cdots + \omega_n \mu_n \]

\[ \text{Var}[R_p] = \sum_{i=1}^{n} \omega_i^2 \sigma_i^2 + \sum_{i \neq j} \omega_i \omega_j \text{Cov}[R_i, R_j] \]

- Portfolio variance is the sum of weights times entries in the covariance matrix

<table>
<thead>
<tr>
<th></th>
<th>( \omega_1 )</th>
<th>( \omega_2 )</th>
<th>( \cdots )</th>
<th>( \omega_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_1 )</td>
<td>( \sigma_1^2 )</td>
<td>( \text{Cov}[R_1, R_2] )</td>
<td>( \cdots )</td>
<td>( \text{Cov}[R_1, R_n] )</td>
</tr>
<tr>
<td>( \omega_2 )</td>
<td>( \text{Cov}[R_2, R_1] )</td>
<td>( \sigma_2^2 )</td>
<td>( \cdots )</td>
<td>( \text{Cov}[R_2, R_n] )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \ddots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( \omega_n )</td>
<td>( \text{Cov}[R_n, R_1] )</td>
<td>( \text{Cov}[R_n, R_2] )</td>
<td>( \cdots )</td>
<td>( \sigma_n^2 )</td>
</tr>
</tbody>
</table>
Mean-Variance Analysis

The General Case:

- Covariance matrix contains $n^2$ terms
  - $n$ terms are variances
  - $n^2 - n$ terms are covariances
- In a well-diversified portfolio, covariances are more important than variances
- A stock’s covariance with other stocks determines its contribution to the portfolio’s overall variance
- Investors should care more about the risk that is common to many stocks; risks that are unique to each stock can be diversified away
Mean-Variance Analysis

Special Case:
- Consider an equally weighted portfolio:

\[ \omega_i = \frac{1}{n} \]

\[ \text{Var}[R_p] = \sum_{i=1}^{n} \frac{\sigma_i^2}{n^2} + \frac{1}{n^2} \sum_{i \neq j} \text{Cov}[R_i, R_j] \]

\[ = \frac{1}{n} \times \text{Average Variance} + \frac{n-1}{n} \times \text{Average Covariance} \]

\[ \approx \text{Average Covariance} \]

- For portfolios with many stocks, the variance is determined by the average covariance among the stocks.
Mean-Variance Analysis

Example: The average stock has a monthly standard deviation of 10% and the average correlation between stocks is 0.40. If you invest the same amount in each stock, what is variance of the portfolio? What if the correlation is 0.0? 1.0?

\[
\text{Cov}[R_i, R_j] = \rho_{ij} \times \sigma_i \sigma_j = 0.40 \times 0.10 \times 0.10 = 0.004
\]

\[
\text{Var}[R_p] = \frac{1}{n} 0.10^2 + \frac{n-1}{n} 0.004 \approx 0.004 \text{ if } n \text{ large}
\]

\[
\sigma_p \approx \sqrt{0.004} = 6.3\%
\]
Mean-Variance Analysis

Example (cont):

- SD of Portfolio vs. Number of Stocks for different values of $\rho$:
  - $\rho = 1.0$ (horizontal line at 10%)
  - $\rho = 0.4$ (dotted line at 6%)
  - $\rho = 0.0$ (solid line)

- The SD of the portfolio decreases as the number of stocks increases, especially for lower values of $\rho$.
Mean-Variance Analysis

Eventually, Diversification Benefits Reach A Limit:

- Remaining risk known as **systematic** or **market risk**
- Due to common factors that cannot be diversified
- Example: S&P 500
- Other sources of systematic risk may exist:
  - Credit
  - Liquidity
  - Volatility
  - Business Cycle
  - Value/Growth
- Provides motivation for **linear factor models**
The Efficient Frontier

Given Portfolio Expected Returns and Variances:

\[ \mathbb{E}[R_p] = \omega_1 \mu_1 + \cdots + \omega_n \mu_n \]

\[ \text{Var}[R_p] = \sum_{i=1}^{n} \omega_i^2 \sigma_i^2 + \sum_{i \neq j} \omega_i \omega_j \text{Cov}[R_i, R_j] \]

How Should We Choose The Best Weights?

- All feasible portfolios lie inside a bullet-shaped region, called the **minimum-variance boundary or frontier**
- The **efficient frontier** is the top half of the minimum-variance boundary (why?)
- Rational investors should select portfolios from the efficient frontier
The Efficient Frontier

- Efficient Frontier
- Minimum-Variance Boundary

Expected Return vs. Standard Deviation of Return
The Efficient Frontier

Example: You can invest in any combination of GM, IBM, and MOT. What portfolio would you choose?

<table>
<thead>
<tr>
<th>Stock</th>
<th>Mean</th>
<th>Std dev</th>
<th>Variance / covariance</th>
</tr>
</thead>
<tbody>
<tr>
<td>GM</td>
<td>1.08</td>
<td>6.23</td>
<td>38.80</td>
</tr>
<tr>
<td>IBM</td>
<td>1.32</td>
<td>6.34</td>
<td>16.13</td>
</tr>
<tr>
<td>Motorola</td>
<td>1.75</td>
<td>9.73</td>
<td>22.43</td>
</tr>
</tbody>
</table>
The Efficient Frontier

Example: You can invest in any combination of GM, IBM, and MOT. What portfolio would you choose?

<table>
<thead>
<tr>
<th>Stock</th>
<th>Mean</th>
<th>Std dev</th>
<th>Variance / covariance</th>
</tr>
</thead>
<tbody>
<tr>
<td>GM</td>
<td>1.08</td>
<td>6.23</td>
<td>GM: 38.80</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>IBM: 16.13</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Motorola: 22.43</td>
</tr>
<tr>
<td>IBM</td>
<td>1.32</td>
<td>6.34</td>
<td>GM: 16.13</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>IBM: 40.21</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Motorola: 23.99</td>
</tr>
<tr>
<td>Motorola</td>
<td>1.75</td>
<td>9.73</td>
<td>GM: 22.43</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>IBM: 23.99</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Motorola: 94.63</td>
</tr>
</tbody>
</table>
Example: You can invest in any combination of GM, IBM, and MOT. What portfolio would you choose?

<table>
<thead>
<tr>
<th>Stock</th>
<th>Mean</th>
<th>Std dev</th>
<th>Variance / covariance</th>
</tr>
</thead>
<tbody>
<tr>
<td>GM</td>
<td>1.08</td>
<td>6.23</td>
<td>GM  38.80  IBM  16.13  Motorola  22.43</td>
</tr>
<tr>
<td>IBM</td>
<td>1.32</td>
<td>6.34</td>
<td>GM  16.13  IBM  40.21  Motorola  23.99</td>
</tr>
<tr>
<td>Motorola</td>
<td>1.75</td>
<td>9.73</td>
<td>GM  22.43  IBM  23.99  Motorola  94.63</td>
</tr>
</tbody>
</table>

\[ E[R_p] = (w_{GM} \times 1.08) + (w_{IBM} \times 1.32) + (w_{Mot} \times 1.75) \]
The Efficient Frontier

**Example:** You can invest in any combination of GM, IBM, and MOT. What portfolio would you choose?

<table>
<thead>
<tr>
<th>Stock</th>
<th>Mean</th>
<th>Std dev</th>
<th>GM</th>
<th>IBM</th>
<th>Motorola</th>
</tr>
</thead>
<tbody>
<tr>
<td>GM</td>
<td>1.08</td>
<td>6.23</td>
<td>38.80</td>
<td>16.13</td>
<td>22.43</td>
</tr>
<tr>
<td>IBM</td>
<td>1.32</td>
<td>6.34</td>
<td>16.13</td>
<td>40.21</td>
<td>23.99</td>
</tr>
<tr>
<td>Motorola</td>
<td>1.75</td>
<td>9.73</td>
<td>22.43</td>
<td>23.99</td>
<td>94.63</td>
</tr>
</tbody>
</table>

\[
E[R_p] = (w_{GM} \times 1.08) + (w_{IBM} \times 1.32) + (w_{Mot} \times 1.75)
\]

\[
\text{var}(R_p) = (w_{GM}^2 \times 38.80) + (w_{IBM} \times 40.21) + (w_{Mot}^2 \times 94.23) + \\
(2 \times w_{GM} \times w_{IBM} \times 16.13) + (2 \times w_{GM} \times w_{Mot} \times 22.43) + \\
(2 \times w_{IBM} \times w_{Mot} \times 23.99)
\]
The Efficient Frontier

Example: You can invest in any combination of GM, IBM, and MOT. What portfolio would you choose?

<table>
<thead>
<tr>
<th>Stock</th>
<th>Mean</th>
<th>Std dev</th>
<th>GM</th>
<th>IBM</th>
<th>Motorola</th>
</tr>
</thead>
<tbody>
<tr>
<td>GM</td>
<td>1.08</td>
<td>6.23</td>
<td>38.80</td>
<td>16.13</td>
<td>22.43</td>
</tr>
<tr>
<td>IBM</td>
<td>1.32</td>
<td>6.34</td>
<td>16.13</td>
<td>40.21</td>
<td>23.99</td>
</tr>
<tr>
<td>Motorola</td>
<td>1.75</td>
<td>9.73</td>
<td>22.43</td>
<td>23.99</td>
<td>94.63</td>
</tr>
</tbody>
</table>

\[
E[R_p] = (w_{GM} \times 1.08) + (w_{IBM} \times 1.32) + (w_{Mot} \times 1.75)
\]

\[
\text{var}(R_p) = (w_{GM}^2 \times 38.80) + (w_{IBM} \times 40.21) + (w_{Mot}^2 \times 94.63) + \\
(2 \times w_{GM} \times w_{IBM} \times 16.13) + (2 \times w_{GM} \times w_{Mot} \times 22.43) + \\
(2 \times w_{IBM} \times w_{Mot} \times 23.99)
\]
The Efficient Frontier

Example (cont): Feasible Portfolios

Expected Return

Standard Deviation of Return

Motorola

IBM

GM
The Tangency Portfolio

- If there is also a riskless asset (T-Bills), all investors should hold exactly the same stock portfolio!
- All efficient portfolios are combinations of the riskless asset and a unique portfolio of stocks, called the tangency portfolio.*
  - In this case, efficient frontier becomes straight line

* Harry Markowitz, Nobel Laureate
The Tangency Portfolio

The Tangency Portfolio

GM
IBM
Motorola

0.0% 2.0% 4.0% 6.0% 8.0% 10.0% 12.0% 14.0% 16.0%

Standard Deviation of Return

0.0% 0.6% 1.2% 1.8% 2.4%

Expected Return

Tbill

P
The Tangency Portfolio

Expected Return

Tangency portfolio

GM

IBM

Motorola

GM

IBM

Motorola

Tbill

Standard Deviation of Return

0.0% 2.0% 4.0% 6.0% 8.0% 10.0% 12.0% 14.0% 16.0%

0.0% 0.6% 1.2% 1.8% 2.4%
The Tangency Portfolio

Sharpe ratio
A measure of a portfolio’s risk-return trade-off, equal to the portfolio’s risk premium divided by its volatility:

\[
\text{Sharpe Ratio} \equiv \frac{\mathbb{E}[R_p] - r_f}{\sigma_p} \quad \text{(higher is better!)}
\]

- The tangency portfolio has the highest possible Sharpe ratio of any portfolio
- Aside: **Alpha** is a measure of a mutual fund’s risk-adjusted performance. The tangency portfolio also maximizes the fund’s alpha.
The Tangency Portfolio

Slope = Sharpe Ratio (why?)

Expected Return vs. Standard Deviation of Returns

- IBM
- GM
- Motorola
- Tbill

Tangency portfolio

Slope = Sharpe Ratio (why?)
Key Points

- **Diversification reduces risk.** The standard deviation of a portfolio is always less than the average standard deviation of the individual stocks in the portfolio.

- **In diversified portfolios, covariances among stocks are more important than individual variances.** Only systematic risk matters.

- **Investors should try to hold portfolios on the efficient frontier.** These portfolios maximize expected return for a given level of risk.

- **With a riskless asset, all investors should hold the tangency portfolio.** This portfolio maximizes the trade-off between risk and expected return.
Additional References
